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ASYMPTOTIC BEHAVIOUR OF STOCHASTIC SEMIGROUPS

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1. INTRODUCTION

The problem to be treated in this note is concerned with the asymptotic behaviour of stochastic semigroups, as the time becomes very large. The subject are largely motived by the Theory of Markov processes. Stochastic semigroups usually arise from pure probabilistic problems such as random walks stochastic differential equations and many others.

An outline of the paper is as follows. Section one deals with the basic definitions relative to K-positivity and stochastic semigroups. Asymptotic behaviour of the iterations of stochastic operators is analyzed in the second section. And then it's applied to study asymptotic properties of stochastic semigroups.

2. DEFINITIONS AND NOTATION

As important auxiliary concepts, we need the notions of positivity respect a cone K and stochastic semigroups.

Let X be a real Riesz space and it's assume that there exists a normal cone K which generates X; i. e. K satisfies (i) $\alpha K+\beta K \subset K$ for $\alpha \geq 0$ and $\beta \geq 0$, (ii) $K \cap (-K)=\{0\}$, (iii) K is closed, (iv) there is $\delta>0$ such that for all $x,y \in K$, $\|x + y\| \geq \delta \|x\|$, and (v) $X = K-K$. A partial ordering is introduced into X by setting $x \leq y$ whenever $y-x \in K$.

An operator $T \in \mathfrak{B}(X)$ is called positive (more precisely K-positive) if $TK \subset K$.

A K- positive operator $T \in \mathfrak{B}(X)$ will be a K-stochastic (stochastic, to abreviate) if there is a quasi-interior element $e \in K$ and a K-total set H' such that

$$(Te,x') = (e,x'),\ \forall\ x' \in H'.$$

A subset H'of dual cone is called K-totol if $(x,x') \geq 0$ for all $x' \in H'$ implies that $x \in K$.

A family of stochastic operators $\{ T(t) : t \geq 0 \}$ will be called stochastic semigroup if

1. $T(t+s) = T(t)\,T(s)$, for $t \geq 0$ and $s \geq 0$
2. $T(0)\,x= x$, for all $x \in X$.

3. MAIN RESULTS

Here is presented a spectral decomposition theorem for stochastic operators. These properties will be used to analyszed the asymptotic behaviour of their iterations.

Let (X,K) be a Riesz space where K is a normal and generating cone in X. To be able to formulate the promised theorem we consider a special class of K-stochastic operators. Denote by $\mathcal{A}$ the class of K-positive operators in $\mathcal{B}(X)$ whose spectral radius, is a pole of the resolvent operator with the residue, in the Laurent expansion, having finite-dimensional range.

<u>THEOREM 3.1.</u> Let T an operator , $T \in \mathcal{A}$ with spectral radius r(T). Then, r(T) = 1 or T can be decomposited as T= U + V , where U is a compact permutator K-positive and V is a bounded operator such that VU=UV=0 and r(V) < 1.

In order to prove the theorem we used some results of the Perron-Frobenious Theory of positive operators ([5],[8]) and the following auxiliar lemma.

<u>LEMMA 3.1.</u> Let

$$U_m = \sum_{1}^{p} \lambda_i^m P_i , \quad m= 0, 1, 2,...$$

where λ_i, i=1, ... , p, are the eigenvalues of peripherical spectrum of T whose are poles of the resolvent having finite multiplicity. And P_i denotes the proyection operator on associated eigensubspace to λ_i for each i=1, ... , p.

Then it is followed:

1. $U_m \, U_l = U_{m+l}$, $m,l \in Z$.
2. $U_m K \subset U_m K_o \subset K_o \subset K$, with $K_o=U_oK$
3. $\| U_o x \| = \|x\|$ and $\| U_m x \| = \|x\|$ for all $x \in X_o = U_o X$.
4. For every quasi-interior element $e \in K$, $U_m e$ and Ue are quasi-interior elements of K, $m \in Z$.

From Theroem 3. 1 it follows that the sequence of the n-th powers T^n of T is asymptotically periodic. We recall that if p=1, $\{T^n : n \in IN\}$ is asymptotically stable.

Let $\{ T(t) : t \geq 0 \}$ be a stochastic semigroup with generator A, defined on $D(A) \subset X$. It's known that $\sigma(A) = \{ \lambda : Re(\lambda) < \lambda_0 \}$ and $r(T(t)) = e^{\lambda_0 t}$, $t \geq 0$, with $\lambda_0 = \sup \{Re(\lambda) : \lambda \in \sigma(A) \}$.In the case of stochastic semigroups $\lambda_0 \leq 0$. If $\lambda_0 < 0$, it's clear that $\{ T(t) : t \geq 0 \}$ is asimptotically stable. So it'll be analyzed the case $\lambda_0 = 0$.

It's easy to prove the following theorem, which emphasizes the role of discrete time semigroups.

<u>THEOREM 3.2.</u> Let $\{T(t): t \geq 0\}$ be a stochastic semigroup and let $t_o > 0$ (fixed). Then the asymptotical stability of $\{T(t) : t \geq 0 \}$ is equivalent to the asymptotical stability of the discrete time semigroup $\{T(t_o n): n \in IN\}$.

So, by using the previous results, it's concluded the following corollary.

<u>COROLLARY 3.1</u> Let $\{T(t) : t \geq 0\}$ be a stochastic semigroup such that there exists $t_o > 0$ with $T(to) \in \mathcal{A}$. Then, $\{T (t) : t \geq 0 \}$ is asymptotically periodic.

REFERENCES

[1].ARENDT, W. ; GREINER, G. : The spectral mapping theorem for one-parameter groups of positive operators on Co(X). *Semigroup Forum* 30 (1984) , no. 3, 297-330.

[2].GOLDSTAIN, J.A. : *Semigroups of Linear Operators and Applications.* Oxford University Press. Oxford, New York . 1985.

[3].GREINER, G. : A spectral decomposition of strongly continkuous groups of positive operators. *Quart. J. Math.* Oxford Serv. (2) 35 (1984), no. 137, 37-47.

[4].LASOTA, A. : Asymptotic behaviour of solutions:statistical stability and chaos. *Warszawa* (1983), 1255-1267.

[5].MAREK, I. : Frobenius theory of positive operators: comparison theorems and applications, *SIAM J. Appl. Math.* 19 (1970), no. 3, 607-628.

[6].NAGEL, R. : What can positivity do for stability?. *Functional analysis: surveys and recent results*, III (Paderbon, 1983), 145-154, North Holland Math. Stud . 90, North-Holland, Amsterdam-New York, 1984.

[7].SCHAEFER, H. H. : On the stability of Co-semigroups. *Math. Ann.*no. 1, 280 ; 20 (1988), 8, 161-164.

[8].SCHAEFER, H. H. : *Banach Lattices and Positive Operators.* Springer-Verlag; Berlin, Heidelberg, New York, 1974.