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INFINITE-DIMENSIONAL SETS OF CONSTANT WIDTH AND THEIR APPLICATIONS.

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1. *Introduction.*

Set of constant width appear as a curiosity in the context of finite-dimensional euclidean spaces. These sets are convex bodies of such an space with the property that the distance between any two distinct parallel supporting hyperplanes is constant. The most easy example of a set of constant width which is not a ball is the so called "Reuleaux triangle" in the euclidean plane. This is the intersection of three closed discs of radius r, whose centers are the vertices of an equilateral triangle of side length r.

The aim of this talk is to show how, once finite-dimensional euclidean spaces are replaced by arbitrary Banach spaces, the resulting concept of set of constant width becomes interesting in relation with several question on the geometry of Banach spaces, mainly in what concerns to the L-M theory. First of all, it must be remarked that constant width will depend on the prefixed norm on the given Banach space. Thus the closed unit ball for a given norm, which of course will be of constant width relative to this norm, cannot be of constant width relative to another norm unless the second norm is a multiple of the former. In the converse direction, every convex body in a real Banach space is of constant width when the space is suitably equivalently renormed. Note however that, as we will see later, sets of constant width in some infinite-dimensional Banach spaces may have empty interior without being trivial, where trivial means reduced to a point.

Now we give the precise definition of sets of constant width. If K is a bounded subset of a Banach space X and g is a norm-one element in X^*, the diameter of $Reg(K)$ measures the "width" of K in the direction determined by g. It follows from the Hahn-Banach theorem that the diameter of K is given by

$$diam(K)=\sup \{ diam(Re_g(K)) : g \in X^*, \|g\|=1 \}.$$

By *set of constant width* in X we mean a bounded closed convex subset K of X such that $diam(Re_g(K))$ remains constant when g runs over the unit sphere of X^*, equivalently

$$diam(Re_f(K))=\|f\| diam(K)$$

for all f in X^*. Interesting reformulations of this concept are contained in the following

PROPOSITION 1 [14].- *Let K be a bounded closed convex subset of a Banach space, with diameter ∂. Then the following three assertions are equivalent.*

i) K is a set of constant width in X.

ii) The closure of $K-K$ in X equals ∂B_X (where, as usual, B_X denotes the closed unit ball of X).

iii) $K-K$ contains the interior of ∂B_X.

With this proposition, the above mentioned facts about the dependence of constant width on the norm are easily verified, and we have even more: a non trivial set of constant width in a Banach space determines the norm up to a positive multiple. On the other hand, the definition itself of sets of constant width has some almost direct consequences which we collect in the following

Remark 1. i) Any set of constant width in a Banach space is "diametrically complete" that is: no point can be added to such a set without increasing its diameter, equivalently the set (say K) is the intersection of the closed balls centered at its points with radius the diameter of K.

ii) Any closed convex subset of a Banach space X, whose w*-closure in the bidual X^{**} is a ball, is a set of constant width in X. Such sets were introduced and studied in [3] under the name of *pseudoballs* (actually in [3] the additional requirement that the diameter is two is taken, but for convenience we do not assume this restriction).

2. *Constant width and metric projection onto subspaces.*

Certain types of subspaces of Banach spaces can be characterized by means of the behaviour of the metric projection onto them. The most simple case is the one of M-summands. Recall that a subspace X of a Banach space Y is called an *M-summand* of Y if there is a linear projection π from Y onto X such that

$$\|y\| = Max\ \{\ \|\pi(y)\|,\ \|y-\pi(y)\|\ \}$$

for all y in Y. It is an easy exercise to prove the following

PROPOSITION 2.- *A subspace X of a Banach space Y is an M-summand of Y if and only if, for each y in Y, the set $P_X(y)$ of best approximants for y in X is a ball with diameter $2\|y+X\|$.*

M-ideals and semi-M-ideals are consecutive generalizations of M-summands. Their definitions require the ones of L-summands and semi-L-summands, respectively. A subspace X of a Banach space Y is said to be a *semi-L-summand* of Y if each element y in Y has a unique best approximant $\pi(y)$ in X satisfying

$$\|y\| = \|\pi(y)\| + \|y-\pi(y)\|.$$

If in addition π is a linear mapping, X is called an *L-summand* of Y. *Semi-M-ideals* (resp.: *M-ideals*) of Y are defined as those closed subspaces of Y whose polars are semi-L-summands (resp.: L-summands) of the dual space Y^*. Now sets of constant width becomes relevant in the following nice characterization of semi-M-ideals.

THEOREM 1 [9].-*A subspace X of a Banach space Y is a semi-M-ideal of Y if and only if , for each y in Y, $P_X(y)$ is a set of constant width in X with diameter $2\|y+X\|$.*

Using a suitable characterization of pseudoballs obtained in [3], another result by A. Lima reads as follows.

THEOREM 2 [9].- *A subspace X of a Banach space Y is an M-ideal of Y if and only if, for each y in Y, $P_X(y)$ is a pseudoball in X with diameter $2\|y+X\|$.*

Call a Banach space X over $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) a *proper M-ideal* (resp.: *proper semi-M-ideal*) if there is a larger Banach space over $\mathbb{K}$ containing it as an M-ideal not M-summand (resp.: semi-M-ideal not

M-ideal), and analogously call a pseudoball (resp.: set of constant width) *proper* if it is not a ball (resp.: pseudoball). It follows from the above results that proper M-ideals (resp.: proper semi-M-ideals) must contain proper pseudoballs (resp.: proper set of constant width). Although our terminology of proper semi-M-ideals and proper sets of constant width is not universally accepted, it is suitable to state the following theorems.

THEOREM 3 [3].- *A Banach space over* $\mathbb{K}$ *is a proper M-ideal if and only if it contains a proper pseudoball.*

THEOREM 4 [17].- *A Banach space over* $\mathbb{R}$ *is a proper semi-M-ideal if and only if it contains a proper set of constant width.*

Remark 2. i) Theorem 4 is not true in the complex case, since $\mathbb{C}$ has proper sets of constant width and it cannot be a proper complex semi-M-ideal.

ii) Actually the existence of proper complex semi-M-ideals is in doubt, this problem being perhaps the most important one which remains open in the L-M theory. It is equivalent to the existence of an asymmetric compact convex set K in $\mathbb{C}^2$ with the property that $f(K)$ is a disk in $\mathbb{C}$ for every linear functional f on $\mathbb{C}^2$ [17]. Of course, if such a set exists, it is a proper set of constant width for suitable norm on $\mathbb{C}^2$.

iii) Let K be either a pseudoball in a complex Banach space, or merely a set of constant width in a real Banach space. Then K satisfies the following property:

(*) for all natural number n, all $x_1,\dots,x_n$ in K, and all scalars $\alpha_1,\dots,\alpha_n$ with $\sum|\alpha_i|\leq 1$ and $\sum\alpha_i=0$, we have $\|\sum\alpha_i x_i\|\leq\frac{\partial}{2}$, where ∂ denotes the diameter of K.

This fact can be easily verified, it is the common point in the proofs of Theorems 3 and 4, and shows why Theorem 4 does not holds in the complex case. It follows that a complex Banach space is a proper (complex) semi-M-ideal if and only if it contains a proper set of constant width satisfying (*).

iv) Note that, while any proper M-ideal must be non reflexive, there exist finite-dimensional proper semi-M-ideals (for example, $\mathbb{C}$

regarded as a real Banach space).

Next we discuss the existence of proper pseudoballs in a C*-algebra.

PROPOSITION 3.- *For a C*-algebra A, the following assertions are equivalent:*

i) A has not a unit.

ii) A contains a non trivial pseudoball with empty interior.

iii) A contains a proper pseudoball.

Proof. (i)⇒(ii).- Two well known facts are applied. The first one that, for every F in the bidual of a Banach space X, $P_X(F)$ has empty interior in X, and the second one that the bidual of a C*-algebra A is a C*-algebra with unit 1 containing A as a subalgebra. Therefore, if A has not a unit, $P_A(1)$ has empty interior in A and it is a non trivial pseudoball in A because A is a closed two-sided ideal of the C*-algebra $A+\mathbb{C}1$ (so an M-ideal [15,16]) and Theorem 2 applies.

(ii)⇒(iii).- This is clear: every non trivial pseudoball with empty interior is proper.

(iii)⇒(i).- This follows from the general fact that every unital Banach algebra does not contain proper pseudoballs (apply [7; Propositions 3.3 and 3.1], [3; Theorem 4.3] and Theorem 3).

Remark 3.- The above proposition is also true in the more general context of A being a noncommutative Jordan C*-algebra (in particular a JB*-algebra) because the results on associative C*-algebras used in the proof have been extended to noncommutative Jordan C*-algebras [13].

Sets of constant width are also relevant to characterize "absolute subspaces". In some sense, absolute subspaces are the smallest natural class of subspaces containing semi-M-ideals, semi-L-summands and L^p-summands. To see this, consider the following natural joint generalization of semi-L-summands and L^p-summands ($1<p\leq\infty$): for a subspace X of a Banach space Y over $\mathbb{K}$, we may require the existence of a mapping π from Y onto X satisfying

$$\pi(\lambda y+x)=\lambda\pi(y)+x$$

for all λ in $\mathbb{K}$, y in Y, and x in X, and

$$\|y\| = |(\|\pi(y)\|, \|y-\pi(y)\|)|$$

for all y in Y, where $|.|$ denotes a fixed absolute norm on $\mathbb{R}^2$. Such a subspace is then called a *semi-|.|-summand* of Y (a *|.|-summand,* if in addition π is a linear mapping). *Semi-|.|-ideals* (resp.: *|.|-ideals*) of Y are those closed subspaces of Y whose polars are semi-$|.|^*$-summands (resp.: $|.|^*$-summands) of Y^*, where

$$|(r,s)|^* := Max \{ |rb+sa| : |(a,b)|=1 \}.$$

In this way |.|-summands are |.|-ideals. In our general context, the above concepts are not enough to admit predualization, so we are obliged to introduce *semi-|.|-idealoids* as those closed subspaces of Y whose polars are semi-$|.|^*$-ideals in Y^*. The above defined five kinds of subspaces have been widely discussed in [10], where among others the following facts are proved:

- No further predualization is to be considered, because closed subspaces whose polars are semi-$|.|^*$-idealoids are nothing that semi-|.|-ideals.

- Closed subspaces whose polars are $|.|^*$-ideals are nothing that |.|-ideals, and therefore "|.|-idealoids" are not introduced.

- Every semi-|.|-summand is a semi-|.|-idealoid.

In the following picture we sumarize the relation between the five classes of subspaces and we motivate another unifying class for the study of the relevant common properties of subspaces defined above.

$$\begin{array}{ccc}
|.|\text{-summand} & \Rightarrow & \text{semi-}|.|\text{-summand} \\
\Downarrow & & \Downarrow \\
|.|\text{-ideal} & \Rightarrow & \text{semi-}|.|\text{-idealoid} \\
\Downarrow & & \Downarrow \\
\text{semi-}|.|\text{-ideal} & \Rightarrow & ?
\end{array}$$

To have a better motivation for the expected new concept, note that in general no implication in the picture is revertible, that there exist semi-idealoids which are neither semi-summands nor ideals, and that the following equivalences hold:

$$\text{summand} \Leftrightarrow \text{semi-summand} + \text{semi-ideal}$$
$$\text{ideal} \Leftrightarrow \text{semi-ideal} + \text{semi-idealoid}.$$

The picture above is completed by replacing the interrogation symbol by $|.|$-*subspace*, namely a subspace X of our Banach space Y which is a semi-$|.|$-ideal of $X+\mathbb{K}y$ for all y in Y (in short, X is "1-locally" a semi-$|.|$-ideal). In this respect it should be remarked that

$$\text{1-locally semi-}|.|\text{-summand} \Leftrightarrow \text{1-locally } |.|\text{-summand} \Leftrightarrow \text{semi-}|.|\text{-summand}$$

and

$$\text{1-locally semi-}|.|\text{-idealoid} \Leftrightarrow \text{1-locally } |.|\text{-ideal} \Leftrightarrow \text{semi-}|.|\text{-idealoid.}$$

Besides the unifying interest of the new concept (introduced and studied in [11]), it has the desirable property that subspaces whose polars are $|.|^*$-subspaces are nothing that $|.|$-subspaces.

A subspace X of Y will be called an *absolute subspace* of Y if there is an absolute norm $|.|$ on $\mathbb{R}^2$ such that X is a $|.|$-subspace of Y. Among the properties of absolute subspaces we have that they are *absolutely proximinal*, that is, X is proximinal in Y and the norm of a vector y in Y depends only on the distances $d(0,P_X(y))$ and $\|y+X\|$ (other independent examples of absolutely proximinal subspaces are those with the $1\frac{1}{2}$ *ball property*, which were introduced in [16] and can be characterized as those proximinal subspaces X of Y such that

$$\|y\|=d(0,P_X(y))+\|y+X\|$$

for all y in Y [5]). Moreover, if X is an absolute subspace of Y, there exists γ with $0\leq\gamma\leq1$ such that, for all y in Y, $P_X(y)$ is a set of constant width with diameter $2\gamma\|y+X\|$. Actually Theorem 2.6 in [11] together with Proposition 1.6 in [12] leads easily to the following

THEOREM 5.- *A subspace X of a real Banach space Y is an absolute subspace of Y if and only if it is absolutely proximinal in Y and there exists γ with $0\leq\gamma\leq1$ such that, for every y in Y, $P_X(y)$ is a set of constant width in X with diameter $2\gamma\|y+X\|$.*

Remark 4.- If we replace in the theorem "set of constant width" by "ball" (resp.: "pseudoball"), we obtain a characterization of semi-summands (resp.: semi-idealoids), and even then the result remains true in the complex case because both semi-summands and semi-idealoids are $\mathbb{R}$-determined [10; Corollary 1.15]. This characterization for semi-summands follows directly from Theorem 5 and [11; Corollary 2.2], while for the case of semi-idealoids it can be easy obtained from the one for semi-summands taking into account that, if X is an absolute subspace of Y, X^{oo} is an absolute subspace of Y^{**} with the same associated number γ (which depends only on the absolute norm $|.|$ for which X is a $|.|$-subspace [11; Theorem 2.5.(i)]), and therefore $P_{X^{oo}}(y)=w^*\text{-}clos(P_X(y))$ for all y in Y (by Remark 1(i)), hence $P_X(y)$ is a pseudoball in X for all y in Y, iff X is 1-locally a semi-$|.|$-idealoid of Y, iff X is a semi-$|.|$-idealoid of Y (by [11; Proposition 1.8]).

Theorem 5 is not true in the complex case. This is exhibited in the next example, which gives also the existence of "proper" absolute subspaces in the real case. Note that, if there are not proper complex M-ideals, then for complex Banach spaces

semi-summands ≡ summands,

and

ideals ≡ semi-idealoids ≡ semi-ideals ≡ absolute subspaces.

EXAMPLE 1 [11].- Let Y be $\mathbb{C}^2$ normed by

$$\|(z,w)\|:=Max\ \{\ |w|,\ Max\ \{|z+\lambda w|\ :\ \lambda\in K\}\ \},$$

where K is a proper set of constant width in $\mathbb{C}$ with diameter 1. Then $X=\mathbb{C}\times\{0\}$ is an absolute subspace of $Y_{\mathbb{R}}$ (the real Banach space underlying Y), but it is not an absolute subspace of Y. As a consequence, X cannot be a semi-ideal nor a semi-idealoid of $Y_{\mathbb{R}}$.

3. *Constant width and properly semi-L-embedded spaces.*

Among the several classes of subspaces considered above, the most general one is that of absolutely proximinal subspaces, containing as particular distinguished subclasses the one of subspaces with the $1\frac{1}{2}$ ball property and that of absolute subspaces. Absolute subspaces with the $1\frac{1}{2}$ ball property are exactly the $|.|_\gamma$-subspaces, where $0\leq\gamma\leq 1$ and

$$|(a,b)|_\gamma:=Max\ \{\ |b|,\ |a|+\gamma|b|\ \}.$$

As a consequence, semi-L-summands and semi-M-ideals satisfy the $1\frac{1}{2}$ ball property while, for $1<p<\infty$, L^p-summands do not satisfy it. The proper absolute subspace in Example 1 is a $|.|_{\frac{1}{2}}$-subspace, hence it satisfies the $1\frac{1}{2}$ ball property.

Once the relation between all considered types of subspaces has been clarified, one may ask for those Banach spaces which are of one of these types when regarded as subspaces of their biduals. Of course we look for non reflexive spaces, and in fact such Banach spaces exist, but only for every particular types of the above considered classes of subspaces. Thus, a Banach space X is absolutely proximinal in X^{**} (if and) only if it has the $1\frac{1}{2}$ ball property in X^{**} [12], while X is an absolute subspace of X^{**} (if and) only if it is either a semi-L-summand or an M-ideal in X^{**} [4]. Note also that, by Proposition 2, no non reflexive Banach space can be an M-summand in its bidual. Calling a Banach space *semi-L-embedded* (resp.: *L-embedded, M-embedded*) when it is a semi-L-summand (resp.: L-summand, M-ideal) of its bidual, the existence of non reflexive L-embedded or M-embedded spaces is well known: c_0 is M-embedded, while l_1 (as the dual of any M-embedded space) is L-embedded. Even the theory of L-embedded and M-embedded spaces has attained a wide development, as can be seen for example in the forthcoming book [8]. On the other hand, c has the $1\frac{1}{2}$ ball property in its bidual and it is neither M-embedded nor semi-L-embedded [16].

Now, the question arises if there exist *properly semi-L-embedded spaces*, that is, Banach spaces which are semi-L-summands but not L-summands in their biduals. The affirmative answer to this question is contained in the recent work by R. Payá and A. Rodríguez [14], whose fundamental lines will be explained in what follows.

Because I have worked intensively in numerical ranges, and the Banach spaces $A(K)$ (of all real valued continuous affine functions on a compact convex set K) are relevant in this theory, I attacked the above problem by possing the question of determining those compact convex sets K for which $A(K)$ is semi-L-embedded. In answering this question, sets of constant width become crucial. For, by Alaoglu theorem and Remark 1(i), every set of constant width in the dual of a Banach space is a compact convex set for the w*-topology, and conversely one can easily deduce from standard results in the theory of compact convex sets [1, 2] that every compact convex set is (affinely homeomorphic to) a set of constant width endowed with the w*-topology in the dual of a suitable (essentially unique) Banach space. Using this circle of ideas, we obtain

PROPOSITION 4.- *Given a compact convex set K, A(K) is semi-L-embedded* *if and only if K is (affinely homeomorphic to) a set of constant width in the dual of a semi-L-embedded space.*

At a first time the proposition can seen disappointing, because no properly semi-L-embedded space is still known, and one is tempted to replace "semi-L-embedded" by "L-embedded" in both equivalent assertions without disturbing the validity of the result, a fact that, in verifying the proposition, is clearly true for the "only if" part but (fortunately for our pourpose) it is far from being so clear for the "if" part. Therefore for the sequel only the following corollary is to be considered.

COROLLARY 1.- *If K is a set of constant width in the dual of an L-embedded space, then $A(K)$ is semi-L-embedded.*

Properly semi-L-embedded spaces will appear when we will be sure that the above corollary cannot be improved in general. Although, for K as in the corollary, we do not know a reasonably clear criterion to decide whether $A(K)$ is properly semi-L-embedded, we have found such a criterion in a relevant particular case, namely when the L-embedded space under consideration is the dual of an M-embedded space.

PROPOSITION 5.- *Let E be an M-embedded space and K a set of constant width in E^{**}. Then $A(K)$ is semi-L-embedded, and it is L-embedded if and only if the closure of $q(K)$ is a pseudoball, where $q:E^{**} \longrightarrow E^{**}/E$ denotes the quotient mapping (observe that the closure of $q(K)$ is always a set of constant width in E^{**}/E).*

Now the existence of properly semi-L-embedded $A(K)$-spaces will follows from the existence of a pair (E,K) as in the proposition, with the property that $\overline{q(K)}$ is a proper set of constant width in E^{**}/E. Therefore we review the construction we have made of such a pair. Consider any finite-dimensional compact convex set J and let K denote the abstract infinite countable cartesian product of copies of J, with its natural structure of compact convex set. Representing J as a set of constant width in a finite-dimensional Banach space Z, and writting $E:=c_0(Z)$, E is M-embedded [6; Theorem 3.4] and the representation of J

in Z induces naturally a representation of K as a set of constant width in $l_\infty(Z)=E^{**}$, so by the proposition $A(K)$ is semi-L-embedded. To discuss the possibility of $A(K)$ being L-embedded, the following two facts should be considered:

- E^{**}/E has no proper pseudoballs (so $A(K)$ is L-embedded iff $\overline{q(K)}$ is a ball). This is true because Z, as any finite-dimensional Banach space, satisfies the so called "intersection property", which is preserved by taking l_∞-summs of copies of a given Banach space and by passing to a quotient by an M-ideal, and implies the absence of proper pseudoballs [3,7].
- $\overline{q(K)}$ is a ball in E^{**}/E iff J is a ball in Z. Once we know that balls are characterized as those sets of constant width whose diameter is twice its outer radius, this follows from the obvious fact that $\overline{q(K)}$ and J have the same diameter and the less obvious one that they have also the same outer radius (see [14; Lemma 1.9] for details).

Now, since a set of constant width is a ball if and only if it has a centre of symmetry, we have

THEOREM 6.- *If K denotes the countable infinite product of copies of a finite-dimensional compact convex set J, then $A(K)$ is semi-L-embedded, and it is L-embedded if and only if J has a centre of symmetry.*

COROLLARY 2.- *If K denotes the countable infinite product of copies of the triangle, then $A(K)$ is a properly semi-L-embedded space.*

We finish this talk by remarking that it is not difficult to verify that all semi-L-embedded spaces appearing in Theorem 6 are dual renorming of l_1 (of course, the case of J being reduced to a point is excluded).

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