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ABOUT CERTAIN ISOMORPHIC PROPERTIES OF BANACH SPACES IN PROJECTIVE TENSOR PRODUCTS

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This note is an announcement of results contained in the papers [4],[5],[6] concerning isomorphic properties of Banach spaces in projective tensor products (for this definition and some property we refer to [1]).At the end,some new result is obtained,too.

In the sequel $L(E,F^*)$ (resp.$K(E,F^*)$) denotes the space of all operators (resp.compact operators) from a B.space E into the dual B.space F^*.The first result needs the definition of a new isomorphic property introduced in [4]:we say that a B. space X has the (DPrcP) if any Dunford-Pettis set,i.e. a bounded set M such that $\lim_n \sup_M |x^*_n(x)|=0$ for any w-null sequence $(x^*_n) \subset X^*$,is relatively compact.In [4] we obtained the following results

THEOREM 1([4]).A dual B.space X^* has the (DPrcP) iff X does not contain a copy of l^1.

THEOREM 2([4]).Let E,F be two B.spaces not containing l^1.If $L(E,F^*)=K(E,F^*)$,then $E\otimes_\pi F$ does not contain l^1.

The proof of Theorem 2 is based upon Theorem 1 and another characterization of B.spaces not containing l^1 proved in[3]. We note that Theorem 2 answers a question put by Ruess in[12].

The following two results we present are about Property (V) of Pelczynski ([10]) and the Reciprocal Dunford-Pettis Property (RDPP) ([8]).The first is from [5] ,the other from [6].

THEOREM 3.Let E be a B.space with Property (V) and F be a reflexive space.If $L(E,F^*)=K(E,F^*)$,then $E\otimes_\pi F$ has Property (V).

THEOREM 4.Let E,F be two B.spaces.Then the following are equivalent,provided $L(E,F^*)=K(E,F^*)$,

i) E and F possess the (RDPP) and l^1 doesn't embed into at least one of them

ii) $E\otimes_\pi F$ has the (RDPP).

In the proof of Theorem 3 we used a characterization of Pro=

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perty (V) contained in [10] and in the proof of Theorem 4 a characterization of the (RDPP) to be found in [9] and the already quoted result from [3]. We note that all of our papers contain remarks about the necessity of the assumption "$L(E,F^*)=K(E,F^*)$". In order to illustrate the techniques we used, based upon results about weak sequential compactness in $K(E,F^*)$ ([12]), we present a new result about the so-called Grothendieck Property (GrP): a B.space X has the (GrP) if w^*-null sequences in X^* are w-null ([1]).

THEOREM 5. Let E be a B.space with the (GrP) and F be a reflexive space. If $L(E,F^*)=K(E,F^*)$, then $E\otimes_\pi F$ has the (GrP).

Proof. Let (B_n) be a w^*-null sequence in $K(E,F^*)=(E\otimes_\pi F)^*$. Take $x^{**}\in E^{**}$ and $y\in F=F^{**}$. The operator mapping $B\in K(E,F^*)$ into $B^*(y)\in E^*$ is w^*-w^* sequentially continuous; indeed, if (T_n) is w^*-null and $x\in E$, we have $T_n^*(y)(x)=T_n(x\otimes y)\longrightarrow 0$ because $x\otimes y\in E\otimes_\pi F$ and (T_n) is w^*-null. Hence $B_n^*(y)\xrightarrow{w^*}\theta$ in E^*. But E has the (GrP) and so $B_n^*(y)\xrightarrow{w}\theta$. Hence, $B_n^{**}(x^{**})(y)\longrightarrow 0$ and this means that $B_n\xrightarrow{w}\theta$ ([12]). We are done.

THEOREM 6. Let E be a reflexive space and F be a B.space with the (GrP). If $L(E,F^*)=K(E,F^*)$, then $E\otimes_\pi F$ has the (GrP).

Proof. Since $E\otimes_\pi F$ is isomorphic to $F\otimes_\pi E$ it is enough to apply Theorem 5 to $F\otimes_\pi E$; so we need to prove that $L(F,E^*)=K(F,E^*)$. Take $T\in L(F,E^*)$; we have $T^*:E^{**}\longrightarrow F^*$ and $T^*_{|E}\in K(E,F^*)$. Since T^* is w^*-w^* continuous and B_E is w^*-dense in $B_{E^{**}}$, it is quite easy to prove that $\overline{T^*_{|E}(B_E)}\supset T^*(B_{E^{**}})$. We are done.

The hypothesis of reflexivity of E (or F) in the above results is not restrictive thanks to the following remark

REMARK. If $E\otimes_\pi F$ has the (GrP), then either E or F is reflexive.

Assume l^1 embeds into both E and F. A result in [11] gives that (L^1 and so) l^2 embeds into both E^* and F^*. Hence $l^2\otimes_\varepsilon l^2$ is a subspace of $L(E,F^*)$, that is weakly sequentially complete as a dual of a space with the (GrP) must be. Now, recall that c_0 lives inside of $l^2\otimes_\varepsilon l^2$; a contradiction. And so either E or F doesn't contain l^1; this means that either E or F is reflexive ([2]).

As a consequence, we note that the space $l^{\infty}\otimes_{\pi} l^{\infty}$ cannot have the (GrP), whereas $l^{\infty}\otimes_{\pi} l^{p}$, $2 < p < \infty$, has that property. Instead, $l^{\infty}\otimes_{\pi} l^{p}$, $1 < p \leq 2$, doesn't possess the (GrP), since its dual space contains c_0, as proved in [7].

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