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APPROXIMATION OF CONVEX BODIES BY POLYNOMIAL BODIES II: WIDTH AND RADIUS CASES

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1. In the terms of the above paper (of this journal), we have

PROPOSITION (1). Let $\mathbf{P},\mathbf{Q}\in\mathcal{P}_{2k}$, then

$$\begin{aligned} &\min\{r_P(\theta),r_Q(\theta)\} \leq \\ &\leq r_{tP+(1-t)Q}(\theta) \leq tr_P(\theta) + (1-t)r_Q(\theta) \leq \\ &\leq \max\{r_P(\theta),r_Q(\theta)\} \end{aligned}$$

for each $t\in(0,1)$ and $\theta\in[0,2\pi]$, with equality iff $r_P(\theta)=r_Q(\theta)$.

PROPOSITION (2). Let $\mathbf{P},\mathbf{Q}\in\mathcal{P}_{2k}$, with $\mathbf{P}\neq\mathbf{Q}$. Then if $\theta\in[0,2\pi]$ and $r_P(\theta)>r_Q(\theta)$, we have $r_{tP+(1-t)Q}(\theta)\uparrow r_P(\theta)$, when $t\uparrow 1$, and $r_{tP+(1-t)Q}(\theta)\downarrow r_Q(\theta)$, when $t\downarrow 0$.

Finally, as a consequence of **(1)**, **(2)** and a simple topological result, we have a method of constructing polynomials in $\mathcal{P}^e(B)$ or $\mathcal{P}^1(B)$.

PROPOSITION (3). Let $\mathbf{P}\in\mathcal{P}^e(B)$ $(\mathbf{P}\in\mathcal{P}^1(B))$, and $\mathbf{Q}\in\mathcal{P}_{2k}$ be such that

$$r_P(\theta) = r(\theta) \Rightarrow r_P(\theta) < r_Q(\theta) \quad (>)$$

Then there exists a $t\in(0,1)$ such that $\mathbf{Q}_t=t\mathbf{P}+(1-t)\mathbf{Q}\in\mathcal{P}^e(B)$ $(\mathbf{Q}_t\in\mathcal{P}^1(B))$, and $S_{Q_t}\cap S=\emptyset$.

2.-We say that $E\subset\mathbb{R}^2$ is a *star body* if

i) E is compact, and

ii) $x\in E$ implies $\lambda x\in Int(E)$, for $\lambda\in(-1,1)$.

Let $\mathcal{S}$ be the family of all star bodies in $\mathbb{R}^2$ and $\mathcal{C}_s$ the family of all convex star bodies, i.e. the family of all convex, compact, symmetric sets in $\mathbb{R}^2$ with nonempty interior.

For each star body $E\in\mathcal{S}$ we define the radius function of E by $r_E(\theta) = \sup\{t>0: t(\cos\theta,\sin\theta)\in E\}$ for $\theta\in[0,2\pi]$. This function is continuous (see [3], p.16).

Now we consider two types of convergence of star bodies.

Let $E,E_1,E_2,\ldots\in\mathcal{S}$. We say that $E_n\to E$ *uniformly* $(E_n\to_u E)$ iff $\|r_{E_n}-r_E\|_\infty\to 0$.

We say that $E_n\to E$ in the *Hausdorff metric* $(E_n\to_H E)$ if $d_H(E_n,E)\to 0$ where, for $A,C\in\mathcal{S}$, $d_H(A,C)=\inf\{t>0: A\subset C+tB,\ C\subset A+tB\}$.

THEOREM (4). Let $E,E_1,E_2,\ldots\in\mathcal{S}$. Then

a) $E_n\to_u E$ implies $E_n\to_H E$.

b) If $E,E_n\in\mathcal{C}_s$ then $E_n\to_u E$ iff $E_n\to_H E$.

3.-We generalize the *Loewner* result (see [1]) in two ways: first we consider 2k-polynomial spheres, and in the second we consider two criteria of approximation -width and radius-. (Actually we also consider the area criterion, but it is left for a later paper).

By **(3)** we can prove that the polynomial sphere of the best approximation width-exterior (radius-exterior) -with polynomials of degree 2k-, touches S in at least 2k+2 points -k+1 and their opposite-.

THEOREM (5). Let $B_Q\in\mathcal{B}_w^e(B)$ or $B_Q\in\mathcal{B}_r^e(B)$. Then $S_Q\cap S$ has at least 2k+2 points.

We have a complementary result of **(5)**, that we can call *draw-back theorem.* We can say loosely speaking, that the sphere of the best approximation touches S in at least 2k+2 points and alternately moves away from S reaching the "same" maximal distance at also 2k+2 points (at least).

THEOREM (6). (a) Let $B_Q\in\mathcal{B}_w^e(B)$. Then there exist 2k+2 different points $u_i\in S_Q$, such that for $i=1,\ldots,2k+2$, $w(\mathbf{Q})=\sup\{\|z\|:z\in S_Q\}=\|u_i\|$. (b) Let $B_Q\in\mathcal{B}_r^e(B)$. Then there exist distinct angles $\theta_0,\ldots,\theta_k\in$ $\in[0,\pi)$, such that for $i=0,\ldots,k$, $r_Q(\theta_i)-r(\theta_i)=\|r_Q-r\|_\infty$.

As a consequence of the contact **(5)**, and draw-back **(6)**, theorems we can prove the unicity of best exterior-approximation.

THEOREM (7). (a) There exists a unique $\mathbf{Q}\in\mathcal{P}^e(B)$, such that $B_Q\in\mathcal{B}_w^e(B)$. (b) There exists a unique $\mathbf{Q}\in\mathcal{P}^e(B)$ such that $B_Q\in\mathcal{B}_r^e(B)$.

From the preceding results we can define the *width* and *radius exterior best approximation operators*: $\mathcal{W}_e:\mathcal{C}_s\longrightarrow \mathrm{Int}[\mathcal{P}_{2k}]$ $\mathcal{R}_e:\mathcal{C}_s\longrightarrow \mathrm{Int}[\mathcal{P}_{2k}]$ such that for each $B\in\mathcal{C}_s$, $\mathcal{W}_e(B)=\mathbf{Q}\in\mathcal{P}^e(B)$, $B_Q\in\mathcal{B}_w^e(B)$ and $\mathcal{R}_e(B)=\mathbf{P}\in\mathcal{P}^e(B)$, $B_P\in\mathcal{B}_r^e(B)$.

THEOREM (8). Let $C,C_1,C_2,\ldots\in\mathcal{C}_s$ such that $C_n\to_H C$, then: (a) $\mathbf{P}_n=\mathcal{W}_e(C_n)\to\mathcal{W}_e(C)=$ $=\mathbf{P}$. (b) $B_{P_n}\to_u B_P$. (c)$B_{P_n}\to_H B_P$.

THEOREM (9). Let $C,C_1,C_2,\ldots\in\mathcal{C}_s$, such that $C_n\to_H C$. Then $\mathcal{R}_e(C_n)=\mathbf{P}_n\longrightarrow\mathcal{R}_e(C)=\mathbf{P}$, and then $B_{P_n}\to_u B_P$, and $B_{P_n}\to_H B_P$.

4.-It is interesting to observe that the similarity between width and radius cases is only aparent, as we saw in the existence theorems [2]. And that this is even more evident when we work in the interior approximation, rather than in the exterior approximation.

THEOREM (10). (a) Let $B_Q\in\mathcal{B}_r^1(B)$, then $S_Q\cap S$ has at least 2k+2 points. (b) Let $B_Q\in\mathcal{B}_r^1(B)$. Then there exist $\theta_1,\ldots,\theta_{k+1}\in[0,\pi)$ such that, for $i=1,\ldots,k+1$, $\|r-r_Q\|_\infty=r(\theta_i)-r_Q(\theta_i)$.

In the width-interior case the situation is really notable, because we have proved already both theorems -in the exterior case-. We have that the contact-interior theorem is the same (essentially) as the draw-back exterior theorem. Conversely the draw-back-interior theorem is the same as the contact-exterior one. And this is so because we have that the width-interior best approximation polynomial body B_Q, is supported, exteriorly by the unit ball B, and interiorly by the homotecy of B of ratio $w(Q)^{-1}$.

THEOREM (11). (a) Let $B_Q\in\mathcal{B}_w^1(B)$. Then $S_Q\cap S$ has at least 2k+2 different points. (b) Let $B_Q\in\mathcal{B}_w^1(B)$. Then there exist at least, $u_1,\ldots,u_{2k+2}\in S_Q$, such that $\|u_i\|=\inf\{\|u\|:u\in S_Q\}=w^{-1}(Q)$.

The unicity theorems are consequence, as in the exterior case, of the contact and draw-back theorems. We have

THEOREM (12). The sets $\mathcal{B}_w^1(B)$ and $\mathcal{B}_r^1(B)$ are singletons.

5.-Now we can define the *width* and *radius interior best approximation operators* $\mathcal{W}_1:\mathcal{C}_s\to\mathrm{Int}[\mathcal{P}_{2k}]$, $\mathcal{R}_1:\mathcal{C}_s\to\mathrm{Int}[\mathcal{P}_{2k}]$, such that for $B\in\mathcal{C}_s$, we have $\mathcal{W}_1(B)=\mathbf{Q}$, with $B_Q\in\mathcal{B}_w^1(B)$, and $\mathcal{R}_1(B)=\mathbf{P}$, with $B_P\in\mathcal{B}_r^1(B)$.

Now the continuity of the operator $\mathcal{W}_1$ is trivial, because we know that the width function $w:\mathcal{P}_{2k}\to[1,\infty]$, defined by $w(\mathbf{P})=[\|\mathbf{P}\|/\min\{\mathbf{P}(u):u\in S\}]^{1/2k}$ is continuous on $\mathrm{Int}(\mathcal{P}_{2k})$, and $\mathcal{W}_1(B)=w^{-2k}[\mathcal{W}_e(B)]\mathcal{W}_e(B)$. We have

THEOREM (13). (a) The operator $\mathcal{W}_1:\mathcal{C}_s\to\mathrm{Int}[\mathcal{P}_{2k}]$ is continuous. (b) The operator $\mathcal{R}_1:\mathcal{C}_s\to\mathrm{Int}[\mathcal{P}_{2k}]$ is continuous.

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