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HALL π-SUBGROUPS AND CONJUGACY CLASSES*

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In the following, G denotes a finite group, π a non-empty set of primes numbers, π' the complementary of π, and we use the standard notation of the theory of groups. If K is a π-subgroup of G, we will denote by $\nu_\pi^K(G)$ the number of Hall π-subgroups of G containing K. We say that G is a D_π-group, if G contains a unique conjugacy class of Hall π-subgroups. A D_π-group G is said to be an $\tilde{L}_\pi$-group, if all subgroups of G having Hall π-subgroups are D_π-groups. If G is π-separable or G has nilpotent Hall π-subgroups, then we say that G is an X_π-group. Evidently, all X_π-groups are $\tilde{L}_\pi$-groups.

In this work, we obtain precise information about the number $\nu_\pi^K(G)$ which is useful for the analysis of the number of conjugacy classes of elements of G. We analyse this number, in terms of the arithmetic structure of the order of G and, for some refinements of the results, its Hall structure. The method is elementary and, in particular, does not use character theory.

For each natural number n, we define the following numbers:

- $\pi(n)$ denotes the set of all prime numbers dividing n.
- $d(n) = g.c.d.(\, p-1 \mid p\in\pi(n))$.
- $\mu_\pi(n) = g.c.d.(p-1 \mid p\in \pi(n)-\pi)$.
- $e_G(\pi) = \begin{cases} 1 & \text{if } 2\notin\pi \text{ and } \mu_\pi(|G|)/d(|G|) \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$

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- $r(G)$ denotes the number of conjugacy classes of elements of G and if T is a non-empty subset of G, $r_G(T)$ denotes the number of conjugacy classes of elements of G which intersect T.

Finally, if G is an X_π-group and H is a Hall π-subgroup of G satisfying:

$$r(H)|H| \equiv |H|^2 \pmod{2^{e_G(\pi)} d(|G|)\mu_\pi(|G|)},$$

then we say that G is an Y_π-group. In particular, groups containing an abelian Hall π-subgroup are Y_π-groups.

Now we can state the main results of this paper:

(I) Let G be an X_π-group and K a π-subgroup of G. Then the following congruence holds:

$$\nu_\pi^K(G) \equiv 1 \pmod{\mu_\pi(|G|)}.$$

This result improves those of J.S. Clowes, given in [2].

(II) Let H be a Hall π-subgroup of G. The following congruences are obtained:

a) If G is $\tilde{L}_\pi$-group, then

$$r_G(C_G(H)) \equiv |C_G(H)| \pmod{d(|G|)\mu_\pi(|G|)}.$$

Note that $r_G(C_G(H))$ is the number of conjugacy classes whose cardinal is a π'-number.

b) If G is X_π-group, then

$$r_G(H) \equiv |H| \pmod{d(|G|)^2}.$$

Note that $r_G(H)$ is the number of conjugacy classes of π-elements of G.

Further, if H is abelian, then

$$r_G(H) \equiv |H| \pmod{d(|G|)\mu_\pi(|G|)}.$$

(III) Finally, we notice that $r(G)$ is closely connected with the number $r(H)$, where H is a Hall π-subgroup of G and G is an X_π-group. Indeed, we get the following congruence:

$$|G|(|G|-r(G)) \equiv |H|(|H|-r(H)) \pmod{2^{e_G(\pi)} d(|G|)\mu_\pi(|G|)}.$$

In addition, if $\mathcal{P} = \{\pi_1, \ldots, \pi_u\}$ is a partition of a subset of $\pi(|G|)$, so that G is Y_{π_i}-group for all $i = 1, \ldots, u$, and $\pi_1 \neq \pi(|G|)$ in case $u=1$, then the following congruence holds:

$$r(G) \equiv |G| \pmod{D_{\mathcal{P}}(|G|)/\text{g.c.d.}(D_{\mathcal{P}}(|G|),|G|)},$$

where $D_{\mathcal{P}}(|G|) = \text{g.c.d.}((p-1)\delta'_p \mid p\in\pi(|G|))$ and

$$\delta'_p = \begin{cases} \text{g.c.d.}(q^2-1 \mid q\in\pi(|G|)-\pi_k) & \text{for every } p\in\pi_k \text{ and any } k\geq 1 \\ \text{g.c.d.}(q^2-1 \mid q\in\pi(|G|)), & \text{otherwise.} \end{cases}$$

Evidently, the above congruence improves G. Amit and D. Chillag's congruence given in [1] for finite groups of odd order and also improves A. Vera lópez and M.C. Larrea's congruence given in [7] . Examples at the end show that our results improve the aforementioned.

REFERENCES

[1] G. AMIT and D. CHILLAG, Characters values, conjugacy classes and a problem of Feit. Houston J. Math. **12**, 1-9 (1986).

[2] J.S. CLOWES, On groups of odd order. J. London Math. Soc. **27**, 507-510 (1952).

[3] R. KOCHENDÖRFFER, Group Theory. McGraw-Hill, 1970.

[4] A. MANN, Conjugacy classes in finite groups. Isr. J. Math. **31**, 78-84 (1978).

[5] A. VERA LOPEZ, On the number of Hall π-subgroups in a finite group. Arch. Math. **46**, 102-107 (1986).

[6] A. VERA LOPEZ, Conjugacy classes in finite groups. Proc. Royal Soc. Edinburgh **105A**, 259-264 (1987).

[7] A. VERA LOPEZ and M.C. LARREA, Conjugacy classes in finite groups II. To appear in Houston J. Math..

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