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ORTHOGONALITY IN NORMED LINEAR SPACES: A SURVEY
Part II: RELATIONS BETWEEN MAIN ORTHOGONALITIES

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(DEDICATED TO THE MEMORY OF JOSE M. GARCIA LAFUENTE)

The first part of this survey [2] dealt with the properties (symmetry, homogeneity, additivity, ...) of several concepts of orthogonality in real normed linear spaces.

We continue the survey summarizing known results and open problems on the relations between any two different concepts of orthogonality.

With this aim, we start with a classification of all the concepts of orthogonality we know, which is more suitable for our present purpose than the other one considered in [2], which was less systematic.

DIFFERENT CONCEPTS OF ORTHOGONALITY

According to its greater frecuency in literature we only consider the case in which the normed linear space E is over the real numbers.

When the norm of E is induced by an inner product the orthogonality of two points x and y of E is equivalent to each one of the next (main or secondary) propositions.

In the more general context of normed linear spaces, any one of such propositions is a **DEFINITION** of orthogonality between x and y:

(R) ROBERTS (1934): $\|x-\lambda y\|=\|x+\lambda y\|$, for every $\lambda\in\mathbb{R}$.

(B) BIRKHOFF (1935): $\|x\|\leq\|x+\lambda y\|$, for every $\lambda\in\mathbb{R}$.

(C) CARLSSON (1962): $\sum_{k=1}^{m} a_k \|b_k x + c_k y\|^2 = 0$, where $a_k, b_k, c_k \in \mathbb{R}$ are such that

$$\sum_{k=1}^{m} a_k b_k c_k \neq 0 \ , \ \sum_{k=1}^{m} a_k b_k^2 = \sum_{k=1}^{m} a_k c_k^2 = 0$$

Obviously C–orthogonality is not a singular concept, but a family of them. Before and after Carlsson's paper [12] the following members of such family have been considered separately:

(I) Isosceles (1945): $\|x-y\| = \|x+y\|$

(P) Pythagorean (1945): $\|x-y\|^2 = \|x\|^2 + \|y\|^2$

both introduced by James [19].

(aI) a–Isosceles (1988): $\|x-ay\| = \|x+ay\|$

(aP) a–Pythagorean (1988): $\|x-ay\|^2 = \|x\|^2 + a^2\|y\|^2$

both for some fixed $a \neq 0$. They appear in [3] starting from a hint of [10].

(ab) (1978): $\|ax+by\|^2 + \|x+y\|^2 = \|ax+y\|^2 + \|x+by\|^2$

for some fixed $a,b \in (0,1)$. Considered by Kapoor and Prasad [22].

(a) (1983): $(1+a^2)\|x+y\|^2 = \|ax+y\|^2 + \|x+ay\|^2$

for some fixed $a \neq 1$. Considered by Diminnie, Freese and Andalafte [16].

(U) UNITARY–CARLSSON: Either $\|x\|\|y\| = 0$ or $\|x\|^{-1}x \perp_C \|y\|^{-1}y$

Apparently there is no general study of this family of concepts. However, as for Carlsson's one, some particular members of this family have been considered separately:

(UI) U–Isosceles (1957): Either $\|x\|\|y\| = 0$ or $\|x\|^{-1}x \perp_I \|y\|^{-1}y$

introduced by Singer [25, 26].

(UP) U–Pythagorean (1986): Either $\|x\|\|y\| = 0$ or $\|x\|^{-1}x \perp_P \|y\|^{-1}y$

considered by Diminnie, Andalafte and Freese [15] and Bosznay [11].

(D) DIMINNIE (1983): $\sup\{f(x)g(y) - f(y)g(x) : f,g \in S'\}$, where S' denotes the unit sphere of the topological dual of E, [14].

(A) AREA (1984): Either $\|x\|\|y\| = 0$ or they are linearly independent and such that x, y, –x, –y divide the unit ball of their own plane (identified to $\mathbb{R}^2$) into four equal areas, [1].

As we have pointed out above, all these propositions mean the same (orthogonality between x and y) in inner product spaces, but, in general, they do not mean the same in normed linear spaces.

In this respect, there are many known results of the following type or of a similar one: "if a a given orthogonality is equivalent to (or implies) another one, then E is an inner product space". But there are also many unsolved problems concerning relations (equivalence or implication) between any two given orthogonalities. As was to be expected, results are, in general, easier in case of equivalence than in case of implication.

In what follows we shall give an ordered account of known results and open problems on this topic.

Since Carlsson and Unitary–Carlsson are not simple orthogonality relations, but families of them, we consider firstly relations between any two of the main concepts (Roberts, Birkhoff, Carlsson, Unitary–Carlsson, Diminnie and Area) and then, in a third part of this survey, we shall deal with relations of particular Carlsson and U–Carlsson between themselves and with other orthogonalities, provided that such relations are not covered by a general result.

EQUIVALENCE STATEMENTS FOR MAIN ORTHOGONALITIES

ROBERTS AND OTHERS (B, C, U, D, A): It is known that every orthogonality, excepting Roberts, is existent (i.e. such that $\forall x,y \in E$, $\exists \alpha \in \mathbb{R}$: $x \perp \alpha x+y$) [21, 12, 14, 1]. However, R–orthogonality is existing only in inner product spaces [19]. Therefore:

R–orthogonality is equivalent to any other orthogonality if and only if E is an inner product space.

CARLSSON AND OTHERS (B, U, D, A): It is obvious that B, D, and A orthogonalities are homogeneous (i.e. such that $\lambda \in \mathbb{R}$ and $x \perp y$ imply $x \perp \lambda y$) and that U–orthogonality is positively homogeneous (the above property with $\lambda \geq 0$). However, C–orthogonality is positively homogeneous only in inner product spaces [12]. Therefore:

C–orthogonality is equivalent to any other orthogonality if and only if E is an inner product space.

BIRKHOFF AND OTHERS (U, D, A):

(D, A): It is obvious that D and A orthogonalities are symmetric (i.e. such that $x \perp y$ implies $y \perp x$). However, B–orthogonality is symmetric only in inner product spaces of dimension ≥ 3 [9, 20, 21], or in 2–dimensional spaces that are endowed with a Radon–norm [13]. (Radon–norms were called mixed–norms in [2], but in the light of Gruber's paper [18], we think this name is better). Therefore:

Let dim E $\geq$ 3. B–orthogonality is equivalent to D or A orthogonality if and only if E is an inner product space.

The case dim E=2 requires new arguments:

(D): It is not difficult to see that two linearly independent points x and y are D–orthogonal if and only if $\|x\|^{-1}$ and $\|y\|^{-1}$ determine a parallelogram whose area is a quarter of the minimum area parallelogram circumscribed about the unit sphere of their own plane (identified to $\mathbb{R}^2$).

From the above geometrical interpretation of D–orthogonality it follows that:

Let dim E=2. B–orthogonality is equivalent to D–orthogonality if and only if E is endowed with a Radon–norm.

(A): Invoking arguments of [7, 23] it is stated in [1] that:

Let dim E=2. B–orthogonality is equivalent to A–orthogonality if and only if E is an inner product space.

In short, we have seen that

B≈D iff B is symmetric; B≈A iff E is an inner product space.

(U): We do not know any general answer for the equivalence between B and U orthogonalities.

In constrast, there are a lot of answers for the relation between B and particular U–orthogonalities which will be considered in the third part of this survey.

DIMINNIE AND OTHERS (U, A):

(A): It is easy to see that A–orthogonality is unique (i.e. such that if $x \neq 0$, then $x \perp_A \alpha x+y$ for only one $\alpha \in \mathbb{R}$) and it is known that if D–orthogonality is unique then it agrees with B–orthogonality [14]. Therefore (see the above item):

D–orthogonality is equivalent to A–orthogonality if and only if E is an inner product space.

(U): We do not know any general answer for the equivalence between D and U orthogonalities. However, some partial answers will be given in Part III of this survey.

AREA AND THE OTHER ONE (U): It is known that A, UI and UP are equivalent in the space $\mathbb{R}^2$ endowed with a norm whose spheres are, for example, regular octogons [3]. Therefore it is not true that equivalence between A and U characterizes inner product spaces.

The above counter–example is given in a 2–dimensional space and it is relative to some symmetric U–orthogonalities. Thus, at least, the problems for $\dim E \geq 3$ and for non–symmetric U–orthogonalities remain open.

IMPLICATIONS STATEMENTS FOR MAIN ORTHOGONALITIES

OTHER (B, C, U, D, A) IMPLIES ROBERTS: The same as in the equivalence case, we have that:

If an existing orthogonality (any of the other ones) implies R–orthogonality then is existent and, therefore, E is an inner product space.

ROBERTS IMPLIES OTHER (B, C, U, D, A): There are spaces in which R–orthogonality is strongly–non–existent (i.e. such that $x \perp_R y$ implies $\|x\|\|y\|=0$) [19]. For these non inner product spaces it is obvious that R–orthogonality implies any other orthogonality.

Furthermore, it is easy to see that, in every case, R–orthogonality implies B, I, UI and A orthogonalities, among others.

OTHER (B, U, D, A) IMPLIES CARLSSON: We know that B, D, and A are homogeneous orthogonalities and that U is positively homogeneous. On the other hand, it is stated in [12], by a very involved proof, that C–orthogonality is (positively) homogeneous only in inner product spaces.

However, it is not obvious that C–orthogonality is (positively) homogeneous when it is implied by a (positively) homogeneous orthogonality, even though such orthogonality is, as the present ones, existent too.

Following an indirect way, we think that either B, U, D or A imply C–orthogonality only in inner product spaces, because we conjecture that E is an inner product space when the weaker homogeneity condition that for any $x \in E$ there exists some $y \in E \setminus \{0\}$ such that $x \perp_C \lambda y$, for every $\lambda \in \mathbb{R}$, holds.

CARLSSON IMPLIES OTHER (B, U, D, A):

(A): Since A–orthogonality is homogeneous and unique, the only possibility for C implies A is that C–orthogonality be homogeneous. Therefore:

C–orthogonality implies A–orthogonality if and only if E is an inner product space.

(B, U, D): The last arguments would be valid to obtain analogous conclusions if the orthogonalities under consideration were be unique. But B–orthogonality is unique to the left (right) only in rotund (smooth) spaces [21], D–orthogonality is unique either in inner product spaces of dimension ≥ 3 or in 2–dimensional spaces that are endowed with a Radon–norm [14, 5] and some U–orthogonalities are unique (for example, UI) but other ones are not unique (for example, UP).

From these remarks some consequences can be obtained:

C–orthogonality implies B–orthogonality in a rotund or smooth space E if and only if E is an inner product space.

C–orthogonality implies UI–orthogonality if and only if E is an inner product space.

But we do not know more general results, whereas the second one properly belongs to the third part of this survey.

OTHER (U, D, A) IMPLIES BIRKHOFF OR CONVERSELY:

(D, A): It follows from the geometrical interpretation of D–orthogonality mentioned in the equivalence case, that if D implies B or conversely, then both are equivalent.

On the other hand, it is proved in [1] that the same is true for B and A.

Therefore, what we have already said for the case of "equivalence" remains valid for "implication" (i.e., B ≈ D iff B is symmetric, B ≈ A iff E is an inner product space).

(U): As we have said in the equivalence case we do not know any general answer for the implication between B and U orthogonalities, but there are a lot of partial answers to be considered in Part III of this survey.

OTHER (U, A) IMPLIES DIMINNIE OR CONVERSELY:

(U): All the more reason than in equivalence statements, we do not know any general answer to this question.

(A): If D implies A then D is unique and, therefore, equivalent to B–orthogonality [5]. Thus, B implies A and we have that:

D–orthogonality implies A–orthogonality if and only if E is an inner product space.

Conversely, if A implies D and D is unique, also follows the above conclusion. However, we do not know any more general answer.

OTHER (U) IMPLIES AREA OR CONVERSELY:

We refer to that said for the case of the equivalence.

Equivalence Relations between Main Orthogonalities

⇔	ROBERTS	BIRKHOFF	CARLSSON	U–CARLSSON	DIMINNIE	AREA
ROBERTS		i.p.s.	i.p.s.	i.p.s.	i.p.s.	i.p.s.
BIRKHOFF			i.p.s.	(*)	dim E $\geq$ 3: i.p.s. dim E = 2: Radon–norm	i.p.s.
CARLSSON			To be studied in Part III	i.p.s.	i.p.s.	i.p.s.
U–CARLSSON				To be studied in Part III	(*)	Several counter-examples and open problems
DIMINNIE						i.p.s.
AREA						

i.p.s. = inner product space. (*) = Open problem with partial answers in Part III.

Implication Relations between Main Orthogonalities

⇒	ROBERTS	BIRKHOFF	CARLSSON	U–CARLSSON	DIMINNIE	AREA
ROBERTS		always	sometimes	sometimes	sometimes	always
BIRKHOFF	i.p.s.		(*) (i.p.s. ?)	(*)	dim E $\geq$ 3: i.p.s. dim E = 2: Radon–norm	i.p.s.
CARLSSON	i.p.s.	(*) (i.p.s. ?)	To be studied in Part III	(*) (i.p.s. ?)	(*) (i.p.s. ?)	i.p.s.
U–CARLSSON	i.p.s.	(*)	(*) (i.p.s. ?)	To be studied in Part III	(*)	Several counter-examples and open problems
DIMINNIE	i.p.s.	dim E $\geq$ 3: i.p.s. dim E = 2: Radon–norm	(*) (i.p.s. ?)	(*)		i.p.s.
AREA	i.p.s.	i.p.s.	(*) (i.p.s. ?)	Several counter-examples and open problems	(i.p.s. ?)	

i.p.s. = inner product space. (*) = Open problem with partial answers in Part III.

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