EXTRACTA MATHEMATICAE 4, n.2, 108-110 (1989)

THE CONJUGACY VECTOR OF A p-GROUP OF MAXIMAL CLASS (I) *

by

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A. M. S. CLASSIFICATION 20D99

If G is an arbitrary finite group, we define the **conjugacy vector** of G, Δ_G, in the following way:

$$\Delta_G = (|C_G(x_1)|, \dots, |C_G(x_r)|),$$

where $r = r(G)$ is the number of conjugacy classes of G and $x_1, \dots, x_r$ is a complete system of representatives of the different conjugacy classes of G, ordered so that $|C_G(x_1)| \geq \dots \geq |C_G(x_r)|$. In order to simplify our notation, we write $\Delta_G = ([a_1]^{n_1}, \dots, [a_r]^{n_r})$ instead of $\Delta_G = (a_1, \overset{n_1}{\dots}, a_1, \dots, a_r, \overset{n_r}{\dots}, a_r)$.

In this paper we are concerned with determining the conjugacy vector of a p-group of maximal class and order less or equal than p^7. Thus, we complete the information given in [3], where the invariants $r(G)$ and σ_G of such a group are found, but nothing is said about its conjugacy vector. In subsequent papers, we will determine Δ_G for a p-group of maximal class of order p^8 or p^9.

If $|G| = p^{2n+e}$ with $e \in \{0,1\}$ and $r(G) = n(p^2-1) + p^e + k(p^2-1)(p-1)$, we will write $r(G) = f_k(|G|)$. In [2], the vector Δ_G is given for any p-group G of maximal class which satisfies $r(G) = f_0(|G|)$.

First, we give a result which is fundamental for our purpose:

THEOREM 1. Let G be a nilpotent group and $1 = Z_0 < Z_1 < \cdots < Z_t = G$ the upper central series of G. Then,

$$|C_G(g)| = \prod_{i=0}^{t-1} r_{G/Z_i}\left(\overline{g}\, Z_{i+1}/Z_i\right).$$

COROLLARY. Let G be a p-group of maximal class of order p^m. Then, we can write $|C_G(g)| = p^{\lambda_g}$ with

$$\lambda_g = \left|\left\{\, i \mid 0 \leq i \leq m-1 \text{ and } \overline{g} = gY_{i+1} \in Z\left(N_{G/Y_{i+1}}(\overline{g}\, Y_i/Y_{i+1})\right) \right\}\right|.$$

* Supported by DGICYT grant PS88-0076 and by the University of the Basque Country.

The different conjugacy vectors of the p-groups of maximal class of order less or equal than p^7 are listed in the following theorems:

THEOREM 2. Let G be a p-group of maximal class of order p^m with $m \leq 5$. Then, one of the following cases holds:

1) $m = 3$, $r(G) = f_0(|G|)$ and $\Delta_G = ([p^3]^p, [p^2]^{p^2-1})$.
2) $m = 4$, $r(G) = f_0(|G|)$ and $\Delta_G = ([p^4]^p, [p^3]^{p^2-1}, [p^2]^{p^2-p})$.
3) $m = 5$, $[Y_1, Y_1] = Y_4$, $r(G) = f_0(|G|)$ and

$$\Delta_G = ([p^5]^p, [p^4]^{p-1}, [p^3]^{p^2-1}, [p^2]^{p^2-p}).$$

4) $m = 5$, $[Y_1, Y_1] = 1$, $r(G) = f_1(|G|)$ and

$$\Delta_G = ([p^5]^p, [p^4]^{p^3-1}, [p^2]^{p^2-p}).$$

THEOREM 3. Let G be a p-group of maximal class of order p^6. Then, one of the following cases holds:

1) $[Y_1, Y_1] = Y_4$, $[Y_1, Y_4] = Y_5$, $r(G) = f_0(|G|)$ and

$$\Delta_G = ([p^6]^p, [p^5]^{p-1}, [p^4]^{p-1}, [p^3]^{2p^2-p-1}, [p^2]^{(p-1)^2}).$$

2) $[Y_1, Y_1] = Y_5$, $[Y_1, Y_4] = Y_5$, $r(G) = f_1(|G|)$ and

$$\Delta_G = ([p^6]^p, [p^5]^{p-1}, [p^4]^{p^3-1}, [p^3]^{p^2-p}, [p^2]^{(p-1)^2}).$$

3) $[Y_1, Y_1] = Y_4$, $[Y_1, Y_4] = 1$, $r(G) = f_0(|G|)$ and

$$\Delta_G = ([p^6]^p, [p^5]^{p-1}, [p^4]^{p^2-1}, [p^3]^{p^2-p}, [p^2]^{p^2-p}).$$

4) $[Y_1, Y_1] = Y_5$, $[Y_1, Y_4] = 1$, $r(G) = f_1(|G|)$ and

$$\Delta_G = ([p^6]^p, [p^5]^{p^2-1}, [p^4]^{p^3-p}, [p^2]^{p^2-p}).$$

5) $[Y_1, Y_1] = 1$, $r(G) = f_{p+1}(|G|)$ and $\Delta_G = ([p^6]^p, [p^5]^{p^4-1}, [p^2]^{p^2-p})$.

THEOREM 4. Let G be a p-group of maximal class of order p^7. Then, one of the following cases holds:

A) G is metabelian (i.e., $[Y_2, Y_2] = 1$)

1) $[Y_1, Y_1] = Y_4$, $[Y_1, Y_4] = Y_6$, $r(G) = f_1(|G|)$ and

$$\Delta_G = ([p^7]^p, [p^6]^{p-1}, [p^5]^{p^3-1}, [p^3]^{p^2-p}, [p^2]^{p^2-p}).$$

2) $[Y_1, Y_1] = Y_4$, $[Y_1, Y_4] = 1$, $r(G) = f_2(|G|)$ and

$$\Delta_G = ([p^7]^p, [p^6]^{p^2-1}, [p^5]^{p^3-p}, [p^4]^{p^3-p^2}, [p^2]^{p^2-p}).$$

3) $[Y_1, Y_1] = Y_5$, $r(G) = f_2(|G|)$ and

$$\Delta_G = ([p^7]^p, [p^6]^{p^2-1}, [p^5]^{p^3-p}, [p^4]^{p^3-p^2}, [p^2]^{p^2-p}).$$

4) $[Y_1, Y_1] = Y_6$, $r(G) = f_{p+2}(|G|)$ and

$$\Delta_G = ([p^7]^p, [p^6]^{p^3-1}, [p^5]^{p^4-p^2}, [p^2]^{p^2-p}).$$

5) $[Y_1, Y_1] = 1$, $r(G) = f_{p^2+p+2}(|G|)$ and $\Delta_G = ([p^7]^p, [p^6]^{p^5-1}, [p^2]^{p^2-p})$.

B) G is not metabelian (i.e., $[Y_2, Y_2] = Y_6$)

1) $[Y_1, Y_1] = Y_4$, $[Y_1, Y_4] = Y_6$, $r(G) = f_0(|G|)$ and

$$\Delta_G = ([p^7]^p, [p^6]^{p-1}, [p^5]^{p-1}, [p^4]^{p^2-1}, [p^3]^{p^2-p}, [p^2]^{p^2-p}).$$

2) $[Y_1, Y_1] = Y_4$, $[Y_1, Y_4] = 1$, $r(G) = f_1(|G|)$ and

$$\Delta_G = ([p^7]^p, [p^6]^{p^2-1}, [p^4]^{p^3-1}, [p^2]^{p^2-p}).$$

3) $[Y_1, Y_1] = Y_5$, $r(G) = f_1(|G|)$ and

$$\Delta_G = ([p^7]^p, [p^6]^{p-1}, [p^5]^{p^2-1}, [p^4]^{p^3-p}, [p^2]^{p^2-p}).$$

4) $[Y_1, Y_1] = Y_6$, $r(G) = f_{p+1}(|G|)$ and

$$\Delta_G = ([p^7]^p, [p^6]^{p-1}, [p^5]^{p^4-1}, [p^2]^{p^2-p}).$$

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