EXTRACTA MATHEMATICAE 4, n.2, 99-101 (1989)

Some embeddings for φ_d.

Jesús M.F. Castillo.
Departamento de Matemáticas. Universidad de Extremadura. Avda de Elvas s/n. 06071 Badajoz. España.

AMS (1980) Class. 46A05.

Let K be the scalar field R or C. The sum space $\bigoplus_{N} K$ endowed with the strongest locally convex topology is usually denoted φ.

A theorem of Saxon |5| asserts that if E is a Hausdorff locally convex space not carrying the weak topology, then φ is a subspace of any product E^I for I uncountable.

The first part of this note deals with the possibility of an extension of Saxon's result to nonlocally convex spaces E. This is not completely straightforward since Saxon's proof was based on the locally convex structure of E. Our approach is based on the locally convex structure of φ.

We first note that φ can be written as

$$\varphi = \varinjlim D_{\sigma}(\lambda)$$

where λ represents any of the spaces l_p, $1 \leq p$, ($p=1$ gives the inductive topology) or c_0 (wich gives the so-called box-topology, see |3|), and D_σ is the diagonal operator multiplication by $\sigma \in l_\infty^+ = \{x \in l_\infty : x_n > 0 , n \in N\}$.

Now, subfactorization of suitable D_σ through suitable sequence spaces μ yields an embedding $\varphi \hookrightarrow \mu^I$. Technically, our approach works for μ a pseudo-convex (see |3|) topological vector space, $\varphi \subsetneq \mu \subset l_\infty$, normal and satisfying a regularity condition. It covers the following choices of μ : l_p, $0 < p < 1$, and intersection spaces such as $\bigcap_{p>0} l_p$; nonlocally convex power series sequence spaces $\Lambda_\Phi^p(I)$, $0 < p < 1$ (see |4|) and nonlocally convex Orlicz sequence spaces l_ϱ (see |6|). And, obviously, any topological linear space containing subspaces as above.

The second part shows that there is not an analogue of Saxon's result for uncountable sum spaces.

Let I be a set of cardinality d. We call the sum space $\bigoplus_I K$, φ_d. For d uncountable the inductive topology differs from the box-topology (see |3|).

We have:

Proposition. Let E be an infinite dimensional locally convex space. If φ_d (endowed with the inductive or the box-topology) is a subspace of some product E^J, then E has a basis of zero-neighborhoods, B, such that dim $E_U \geqslant d$ for any $U \in B$.

Since Schwartz spaces have separable associated Banach spaces, it follows:

Corollary. If E is a Schwartz space, then φ_d (endowed with the inductive or box-topology) is not a subspace of any product of copies of E.

And then:

Corollary. Let d be an uncountable cardinal. φ_d (endowed with the inductive or the box-topology) is not a Schwartz space.

In |3 ,p.202| it is given a different proof (only applicable to the inductive topology, but which can be slightly modified to cover the box-topology) of this last corollary.

Finally, it is worthwhile to mention that from |2| it follows that φ_d is not a subspace of any product H^J, H a Hilbert space.

A more detailled study of this and other embeddings involving sum spaces will appear elsewhere.

References.

|1| Jesús M.F.Castillo. A note on some nonlocally convex spaces whose large products (do not) contain φ_d. To appear in Mathematica Japonica.

|2| Jesús M.F.Castillo. Sums and products of Hilbert spaces Proceedings of the AMS 105,2,(1989) 362-366.

|3| H.Jarchow. Locally Convex Spaces. B.G.Teubner Stuttgart 1981.

|4| J.Prada Blanco. Local Convexity in sequence spaces. Math Nachr. 98 (1980) 21-26.

|5| S.Saxon. Nuclear and Product spaces, Baire-like spaces and the strongest locally convex topology. Math Ann 197 (1972) 87-196.

|6| N.J.Kalton. Orlicz sequence spaces without local convexity. Math Proc Cambridge Philos. Soc. (1977) nº2 253-277.

To appear in in Mathematica Japonica