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"Integrability theorems for a generalized Laplace transform"

Jorge J. Betancor[1] and Javier A. Barrios[2]

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The integral transformation defined by

$$L_{\nu,n}\{f\}(y) = F(y) = \int_0^\infty \lambda_\nu^{(n)}(xy)f(x)dx$$

was introduced by E. Kratzel [3]. Here $\lambda_\nu^{(n)}$ denotes the function

$$\lambda_\nu^{(n)}(z) = \frac{(2\pi)^{\frac{n-1}{2}}\sqrt{n}\,(z/n)^{n\nu}}{\Gamma(\nu+1-(1/n))}\int_1^\infty (t^n-1)^{\nu-(1/n)}e^{-zt}dt\ ,\ \nu > \frac{1}{n}-1 \text{ and } z>0.$$

The main properties of $\lambda_\nu^{(n)}(z)$ and $L_{\nu,n}$ were proved in [3],[4] and [5].

Note that $L_{\nu,n}$ reduces to Laplace transformation when n=1 and to Meijer transformation ([3]) for n=2.

The aim of this note is to establish integrability theorems for $L_{\nu,n}$ transforms. Our results extend the one due to P. Heywood [1] and [2] on laplace transforms that can be obtained when n=1.

Theorem 1: "Let f(x) be a real function such that $\lambda_\nu^{(n)}(xy)f(x)\in L(0,\infty)$ for every y>0 and let $\gamma<\alpha=\min\{1,1+n\nu\}$.

a) If there exist $k\in\mathbb{R}$ and $\varepsilon>0$ such that $f(x) > kx^{\varepsilon-\gamma}$ for every x>0, then $y^{-\gamma}F(y)\in L(1,\infty)$ if, and only if, $x^{\gamma-1}f(x)\in L(0,1)$.

b) If there exist $k\in\mathbb{R}$ and $\varepsilon\in(0,\alpha-\gamma)$ such that $f(x) > kx^{-\varepsilon-\gamma}$ for all x>0, then $y^{-\gamma}F(y)\in L(0,1)$ if, and only if, $x^{\gamma-1}f(x)\in L(1,\infty)$."

Sketch of the proof: To prove **a)** we firstly consider a positive function f(x). Then F(y)>0 for every y>0 and

$$\int_0^1 y^{-\gamma}F(y)dy = \int_0^\infty f(x)I(x)dx$$

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[1]Dpto. de Análisis Matemático, Universidad de La Laguna, La Laguna, Tenerife

[2]Dpto. de Economía Aplicada, Universidad de La Laguna, La Laguna, Tenerife

where $I(x) = \int_0^1 \lambda_\nu^{(n)}(xy)y^{-\gamma}dy$. I(x) is a continuous and bounded function on $(0,\infty)$ because $\gamma<1$. Therefore I(x)f(x) is absolutely integrable on (0,T), for T>0.

On the other hand

$$\frac{I(x)}{x^{\gamma-1}} \longrightarrow (2\pi)^{\frac{n-1}{2}} n^{-(1/2)-n\nu} \frac{\Gamma(n\nu+1-\gamma)\Gamma((1-\gamma)/n)}{\Gamma(\nu+1-(1/n))} \quad \text{as } x\longrightarrow\infty$$

Hence, $y^{-\gamma}F(y)\in L(0,1)$ if, and only if, $x^{\gamma-1}f(x)\in L(1,\infty)$.

Part (a) is established by considering the function $f(x)-kx^{\varepsilon-\gamma}$.

Statement in (b) can be proved in a similar way.

The following lemma is required in the proof of Theorem 2.

Lemma 1: "Let $0<\alpha<1$ and $\nu>1$. If

$$g(x) = \begin{cases} -x^{\alpha-1} & , 0 < x \le 1 \\ x^{-\alpha-1} & , x > 1 \end{cases}$$

then

$$|G(y)| \le \frac{y^\alpha}{\alpha}(2\pi)^{\frac{n-1}{2}} n^{(1/2)-(n+1)\nu} \frac{\Gamma(n(1+\nu)-\alpha)\Gamma(1-(\alpha/n))}{\Gamma(2-((\alpha+1)/n)+\nu)} + 2\frac{y^n \lambda_{\nu-1}^{(n)}(0)}{\alpha n^{n-1}(n-\alpha)}$$

where $G(y) = \int_0^\infty \lambda_\nu^{(n)}(xy)g(x)dx$."

Theorem 2: "Let $f(x)\in L(0,\infty)$, $\nu>1$ and $1<\gamma<2$. If $\int_0^\infty f(x)dx=0$,and there exist $k\in\mathbb{R}$, $\varepsilon\in(0,2-\gamma)$ and $x_0>0$ such that $f(x) > kx^{-\gamma-\varepsilon}$ for every $x > x_0$, then a necessary and sufficient conditions for $y^{-\gamma}F(y)\in L(0,1)$ is $x^{\gamma-1}f(x)\in L(1,\infty)$.

Moreover $\frac{F(y)}{y} \in L(0,1)$ if, and only if, $(\log x)f(x)\in L(1,\infty)$."

Sketch of the proof: Let $1<\gamma<2$. Assume firstly that $f(x)>0$ for large values of x. We define the functions

$$f_1(x) = \max\{f(x),0\} \quad \text{and} \quad f_2(x) = \max\{-f(x),0\}$$

Hence $F(y) = F_2(y) - F_1(y)$, where

$$F_i(y) = \int_0^\infty [\lambda_\nu^{(n)}(0)-\lambda_\nu^{(n)}(xy)]f_i(x)dx \quad , i = 1,2$$

Since $y^{-\gamma}F_2(y)\in L(0,1)$ then $y^{-\gamma}F(y)\in L(0,1)$ if, and only if, $y^{-\gamma}F_2(y)\in L(0,1)$. Therefore $y^{-\gamma}F(y)\in L(0,1)$ if, and only if, $x^{\gamma-1}f(x)\in L(1,\infty)$.

When $\gamma=1$ we can establish the corresponding result by making use of the auxiliar function

$$J(x) = \int_0^x (\lambda_\nu^{(n)}(0)-\lambda_\nu^{(n)}(u))/u \, du$$

To complete the proof of this theorem it is sufficient to consider the function f(x)-kg(x) where g(x) is defined as in lemma 1 with $\alpha=\gamma+\varepsilon-1$.

References:

[1] **P. Heywood,** "Integrability theorems for power series and Laplace transform", J. of the London Math. Soc. , 30, (1955), 302-310.

[2] **P. Heywood,** "Integrability theorems for power series and Laplace transform (II)", J. of the London Math. Soc. ,32, (1957), 22-27.

[3] **E. Kratzel,** "Eine verallgemeinerung der Laplace- und Meijer-Transformation", Wiss. Z. Univ. Jena Math. Naturw. Reihe, Heft 5, (1965), 369-381.

[4] **E. Kratzel,** "Die faltung der L-transformation" Wiss. Z. Univ. Jena Math. Naturw. Reihe, Heft 5, (1965), 383-390.

[5] **E. Kratzel,** "Differentiationssätze der L-Transformation und Differentialgleichungen nach dem operator $\frac{d}{dt}\left(t^{\frac{1}{n}-\nu}\left(t^{1-\frac{1}{n}}\frac{d}{dt}\right)^{n-1} t^{\nu+1-\frac{2}{n}}\right)$ " Math. Nachr. , 8 (1967), 105-114.

[6] **K. Soni and R.P. Soni,** "Integrability theorems for a class of integral transforms I", J. of Math. Anal. and Appl. , 43 (2) (1973), 397-418.

[7] **K. Soni and R.P. Soni,** "Integrability theorems for a class of integral transforms II", J. of Math. Anal. and Appl. , 44 (3) (1973), 553-580.