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UTUMI ENDOMORPHISM RINGS OF DUAL MODULES

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Several left-right properties of the endomorphism ring S of a left R-module $_RM$ which are symmetrically -or almost symmetrically- represented on $_RM$ and its dual module $M^*_R = Hom_R(M,R)$ (among them the Utumi conditions for S) have been investigated in [7], where the machinery used to deal with this problem is the standard Morita context associated to $_RM$ (see [8]). In order to ensure that this context behaves properly, it is assumed therein that the modules involved and the pairings defined by the context are faithful, i.e., that the context is nondegenerate, in which case $_RM$ is said to be a nondegenerate module. For these modules, S is canonically a subring of B = $End(M^*_R)$ and the results obtained in [7] have led Khuri to raise the general question of how S sits in B.

In this paper, we pay attention to an aspect of this question, more specifically, to the relation between the Utumi conditions for S and B. We work in the rather more general setting obtained by considering two left R-modules $_RM$ and $_RU$, their endomorphism rings $S = End(_RM)$ and $T = End(_RU)$ and the U-dual module of M, $M^*_T = Hom_R(M,U)$. The first thing to do is to analyze whether the Utumi properties for S are symmetrically represented in $_RM$ and M^*_T and to make explicit the conditions on which the existence of such a symmetry depends; these conditions will replace the nondegeneracy condition in this broader situation. The relevant conditions are: M self-faithful (see definition below), M U-torsionless (i.e., cogenerated by U) and U M-torsionless. Of course, nondegenerate modules satisfy these properties (for U = R) but the converse is not true.

A module M is called self-faithful [2] when $Hom_R(M,X)g = 0$ with $g \in Hom_R(X,M)$ implies g = 0, for every R-module X which is (isomorphic to) a submodule of a quotient module of a direct sum of copies of M (i.e., X is a M-subgenerated module). This notion plays a fundamental role in the characterization of the modules whose endomorphism rings have a specified properties (se , e.g., [2], [3], [4], [5] and [6]). We will denote by r and l the usual annihilator mappings between left and right ideals of S and we set $X^{\perp} = \{f \in M^* \mid f(X) = 0\} \leq M^*_T$ for $X \leq {}_RM$ and ${}^{\perp}L = \{x \in M \mid f(x) = 0$ for each $f \in L\} \leq {}_RM$ for $L \leq M^*_T$. We will use the symbol $X \leq^e Y$ to indicate that X is an essential submodule of Y.

In [9, Theorem 3], Utumi has shown that if S is a left and right nonsingular ring, then its maximal left and right rings of quotients coincide if and only if S satisfies the conditions K_l: "For each left ideal I of S, $r(I) = 0$ implies I essential in ${}_SS$" and K_r: "For each right ideal J of S, $l(J) = 0$ implies J essential in S_S". The left nonsingular rings which satisfy K_l are called left Utumi rings and right Utumi rings are defined symmetrically. When R is left and right Utumi, then we say that R is an Utumi ring. We get the following result which extends [7, Theorem 7].

Theorem 1. Let ${}_RM$ be self-faithful and U-torsionless and U M-torsionless. Then the following statements hold:

i) S satisfies K_l if and only if, for each $X \leq {}_RM$, $X^{\perp} = 0$ implies $X \leq^e M$.
ii) S satisfies K_r if and only if, for each $L \leq M^*_T$, ${}^{\perp}L = 0$ implies $L \leq^e M^*_T$.

This result is, even for U = R, more general than [7, Theorem 7], for we can give an example of a self-faithful torsionless and faithful module which is not nondegenerate but has an Utumi endomorphism ring:

Example 2. Let A be a left primitive ring with a faithful simple module ${}_AV$ such that $\mathrm{Hom}_A(V,A) = 0$ (in particular, we can take A to be a non-artinian simple ring) and $D = \mathrm{End}({}_AV)$. Let $R = \begin{pmatrix} A & V \\ 0 & D \end{pmatrix}$, ${}_RP = Re_{22} = \begin{pmatrix} 0 & V \\ 0 & D \end{pmatrix}$, ${}_RX = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}$ and ${}_RM = {}_RX \oplus {}_RP$. Then ${}_RM$ has all the properties stated above (in particular, the endomorphism ring $S = \mathrm{End}({}_RM)$ is isomorphic to the 2x2 upper triangular matrix ring over D, which is, of course, an Utumi ring).

If ${}_RM$ is an U-torsionless module, the canonical homomorphism $S \longrightarrow B$ is injective. We are now ready to study how the Utumi conditions on S relate with the corresponding ones on B. In this sense, using Theorem 1 we are able to show that, under our standard assumptions, the property of being right Utumi goes up from S to B. To go in the opposite direction we need to make an additional assumption on ${}_RM$. Recall that ${}_RU$ is said to be balanced [1] when the canonical homomorphism from R to the biendomorphism ring $\mathrm{End}(U_T)$ of ${}_RU$ is surjective.

Theorem 3. Let ${}_RM$ and ${}_RU$ be modules such that M is M-distinguished and U-torsionless and U is M-torsionless. Then S right Utumi implies B right

Utumi. If, moreover, ${}_RU$ is balanced, then B left Utumi implies S left Utumi.

The converses of the two assertions of Theorem 3 fail, but in the two-sided case we can get a more satisfactory result:

Theorem 4. Let ${}_RM$ and ${}_RU$ be R-modules such that ${}_RM$ is M-distinguished and U-torsionless and ${}_RU$ is M-torsionless and balanced. Then the following conditions are equivalent:

i) B is an Utumi (resp. left Utumi) ring.

ii) S is an Utumi (resp. left Utumi) ring and the canonical homomorphism $\phi: {}_RM \longrightarrow {}_RM^{**}$ has essential image.

Moreover, if these equivalent conditions hold, then the maximal (left) rings of quotients of S and B coincide.

The proofs of the foregoing results will appear in [5], where other properties of endomorphism rings of dual modules are also investigated.

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