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# A SYSTEM OF AXIOMS FOR MEASURES OF NONCOMPACTNESS

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**1. INTRODUCTION.** Several authors have defined systems of axioms for measures of noncompactness (mnc): Sadovskii [8] gives a system of axioms on a locally convex space; Banas and Goebel [2] , Banas [1] and Weis [9] on a Banach space; and Duc [4] on a Fréchet space.

In this paper we give a system of axioms on a complete metric space, inspired in Banas [1], but simpler; our system is more complex than Sadovskii, Weis and Duc systems, but we can derive from it the metric properties of the usual mnc. Moreover our axioms are independent.

We give the definition of measure and we characterize its kernel, or class of sets in which the measure takes value equal to 0 (the mnc are measures whose kernel is the class of relatively compact sets). Each measure has a module and its kernel is kernel of a measure of module one, the canonical measure. For mnc we give a Generalized Cantor Theorem.

**2. NOTATION.** Let (E,d) be a complete metric space. We use the following notation: a,b,.. are elements of E; A,B,..are elements of $P_b(E)$, the class of nonempty bounded subsets of E; $\varepsilon > 0$ is a real number. Moreover

$B(A,\varepsilon) := \{x\in E : d(x,A) < \varepsilon\}$

$D'(A,B) := \sup \{d(A,b) : b\in B\}$

$D(A,B) := \max \{D'(A,B),D'(B,A)\}$

D is the **Hausdorff metric**. Given $\emptyset \neq \mathcal{W} \subset P_b(E)$, we define:

$D(A,\mathcal{W}) := \inf \{D(A,W) : W\in\mathcal{W}\}$

**3. DEFINITION. THE SYSTEM OF AXIOMS.** The map

$$\mu: P_b(E) \longrightarrow \mathbb{R}$$

is called a **premeasure** if

(1) $\mu(A)\geq 0$

(2) there exists C such that $\mu(C)=0$

(3) there exists $r(\mu)\geq 0$ such that $\mu(B(A,\varepsilon))\leq\mu(A)+r(\mu)\varepsilon$

The map $\mu$ is called a **measure** if it is a premeasure and

(4) if $A \subset B$, then $\mu(A) \le \mu(B)$

(5) $\mu(A \cup B) \le \max \{\mu(A), \mu(B)\}$

$\mu$ is called a **measure of noncompactness** (mnc) if it is measure and

(6) $\mu(A) = 0$ if and only if A is relatively compact

The **kernel** of $\mu$ is $Ker(\mu) := \{A \in P_b(E) : \mu(A)=0\}$. Each $r(\mu)$ that verifies (3) is called a **modular bounded** of $\mu$ (If $\mu$ is a measure, then the modular bounds of $\mu$ have minimum, which is called the **module** and it is denoted $m(\mu)$).

**OBSERVATION.** We can give examples showing that the axioms of measure are independent.

**4. PROPERTIES** The following results show that our measures satisfy the usual properties of a mnc.

**PROPOSITION.** If $\mu$ is a measure, then

(1) $\mu(A \cup B) = \max \{\mu(A), \mu(B)\}$

(2) if $A \cap B \ne \emptyset$, then $\mu(A \cap B) \le \min \{\mu(A), \mu(B)\}$

(3) $\mu(A) = \mu(\bar{A})$ ($\bar{A}$ is the clousure of A)

(4) $|\mu(A) - \mu(B)| \le m(\mu)\, D(A,B)$

(5) $\mu$ is uniformly continuous on $P_b(E)$ with respect to D

(6) if $N \in Ker(\mu)$, then $\mu(A \cup N) = \mu(A)$

**GENERALIZED CANTOR THEOREM.** If $\mu$ is a mnc and if $(C_n)$ is a decreasing sequence of nonempty bounded closed subsets of E and $\lim \mu(C_n)=0$, then $\bigcap_{n=1}^{\infty} C_n$ is nonempty and compact.

**5. PROPOSITION. THE KERNEL OF A MEASURE.** $\mathcal{N} \subset P_b(E)$ is kernel of a measure if and only if

(1) $\mathcal{N} \ne \emptyset$

(2) if $\emptyset \ne A \subset B$ and $B \in \mathcal{N}$, then $A \in \mathcal{N}$

(3) if $A, B \in \mathcal{N}$, then $A \cup B \in \mathcal{N}$

(4) $\mathcal{N}$ is closed in $P_a(E)$ respect to D

If $\mathcal{N}$ verify (1)-(4) then

$$\mu_{\mathcal{N}}(A) := D(\mathcal{N}, A)$$

defines a measure with kernel $\mathcal{N}$, **with** modular bound $r(\mu_{\mathcal{N}}) = 1$.

**6. the CANONICAL MEASURE.** Two measures are equivalents if they have the same kernel. Each measure $\mu$ is equivalent to a **canonical measure** $\mu_c$ defined by $\mu_c = \mu_{Ker(\mu)}$. We have $\mu \le m(\mu)\mu_c$, and $m(\mu_c) = 1$ when $\mu_c \neq 0$.

**7. EXAMPLES.**

1. [6], [7]. The **Kuratowski measure** of A, k(A), is the infimum of the $\varepsilon>0$ such that A admits a finite cover by sets of diameter less than $\varepsilon$. k is a mnc and $m(k) \le 2$.

2. (see [2]). The **Hausdorff measure** of A, h(A), is the infimum of the $\varepsilon>0$ such that A can be covered by a finite number of balls of radius less than $\varepsilon$. h is the canonical mnc; consequently, $m(h) = 1$.

3. [5]. The **Istratescu measure** of A, i(A), is the infimum of the $\varepsilon>0$ such that if $B \subset A$ verifies "$x,y \in B$, $x \neq y \Rightarrow d(x,y)>\varepsilon$", then B is finite. i is a mnc and $m(i) \le 2$.

4. [3]. The **Danes measure** of A, d(A), is defined analogously as h, but taking the centers of the balls inside of A. d is not a measure, but it is a premeasure of noncompactness and $m(d) \le 1$.

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