

"FINITE GROUPS WITH EXACTLY THIRTEEN CONJUGACY CLASSES"

by

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In the following G is a finite group, $r(G)$ is the number of conjugacy classes and Δ_G is the conjugacy vector of G .

The classification of all finite groups with $r(G) \leq 9$ was carried out in a serie of papers by W. Burnside and G. A. Miller ($r(G)=5$, cf. [3], 1910), D. I. Sigley ($r(G)=6$, [8], 1935), J. Poland ($r(G)=7$ [7], 1966), L. F. Kosvintsev ($r(G)=8$, [5], 1974), and V. A. Odincov and A. I. Starostin ($r(G)=9$, [6], 1976). In 1978, A. G. Aleksandrov and E. Komissarcik (cf. [1]) found all simple finite groups with $r(G) \leq 12$. In [9] and [10], A. Vera López and J. Vera López classify all finite groups with $r(G) \leq 12$. Ja. G. Berkovic (Cf. [2]) obtains these groups using the character theory's machinery. However, his list is incomplete and contains some mistakes. E. Komissarcik proves (750 hours of CPU time) that there are no nonabelian finite simple groups with 13 conjugacy classes (Cf. [4]). In this work all the finite groups with 13 conjugacy classes are classified.

In order to simplify our notation we write $\Delta_G = (a_1^{n_1}, \dots, a_t^{n_t})$ instead of $\Delta_G = (a_1, \dots, a_1, \dots, a_t, \dots, a_t)$.

We shall follow the notation introduced in [9] and [10].

In addition we consider the following groups:

$$(C_2 \times Q_8) \times_{\lambda} \Sigma_3 = (\langle e \rangle \times \langle e_1, e_2 \rangle) \times_{\lambda} \langle a, b \rangle \quad \text{with } e^a = e, \quad e_1^a = e_2, \quad e_2^a = e_2 e_1, \\ e^b = e e_1^2, \quad e_1^b = e_1^{-1}, \quad e_2^b = e_1 e_2.$$

$$2^4 \Gamma_2 b \times_{\lambda} \Sigma_3 = ((C_4 \times C_2) \times_{\lambda} C_2) \times_{\lambda} \Sigma_3 = \langle x_1, x_2, e \rangle \times_{\lambda} \langle a, b \rangle \quad \text{with } e^a = e, \\ x_2^a = x_1^2 x_2, \quad x_1^a = x_1, \quad x_2^a = e, \quad e^b = x_1^{-1} x_2 e, \quad x_1^b = x_1^{-1}, \quad x_2^b = x_2, \quad e^c = x_1^{-1} x_2 e.$$

$$2^4 \Gamma_2 b \cdot DC_3 = \langle x_1, x_2, e \rangle \cdot \langle a, b \rangle \quad \text{with the above relations}$$

$$C_{13} \times_f C_{12} = \langle a \rangle \times_f \langle b \rangle \quad \text{with } a^b = a^2$$

$$R_1 \times_{\lambda} C_2 = ((C_9 \times_{\lambda} C_3) \times_{\lambda} C_3) \times_{\lambda} C_2 = \langle a_2, a_1, d \rangle \times_{\lambda} \langle b \rangle \quad \text{with } a_2^a = a_2 a_1, \quad a_1^a = a_1, \quad a_1^b = a_1 a_2, \quad a_2^b = a_2, \quad d^b = d^{-1}.$$

$$R_2 \times_{\lambda} C_2 = ((C_9 \times_{\lambda} C_3) \cdot C_9) \times_{\lambda} C_2 = (\langle a_2, a_1 \rangle \cdot \langle d \rangle) \times_{\lambda} \langle b \rangle \quad \text{with } a_2^a = a_2^4, \\ d^a = a_2^3, \quad a_2^d = a_2 a_1, \quad a_1^d = a_1, \quad a_1^b = a_1 a_2^3, \quad a_2^b = a_2^{-1}, \quad d^b = d^{-1}.$$

$$R_3 \times_{\lambda} C_2 = ((C_9 \times_{\lambda} C_3) \cdot C_9) \times_{\lambda} C_2 = (\langle a_2, a_1 \rangle \cdot \langle d \rangle) \times_{\lambda} \langle b \rangle \quad \text{with } a_2^a = a_2^4, \\ d^a = a_2^{-3}, \quad a_2^d = a_2 a_1, \quad a_1^d = a_1, \quad a_1^b = a_1 a_2^3, \quad a_2^b = a_2^{-1}, \quad d^b = d^{-1}.$$

$$R_4 \times_{\lambda} C_2 = ((C_3 \times C_9) \times_{\lambda} C_3) \times_{\lambda} C_2 = \langle a_1, a_2, d \rangle \times_{\lambda} \langle b \rangle \text{ with } a_1^d = a_1 a_2^{-3}, \\ a_2^d = a_2 a_1, \quad a_1^b = a_1 a_2, \quad a_2^b = a_2^{-1}, \quad d^b = d^{-1}.$$

$$(Q_8 Q_8)_{C_2} \times_{\lambda} \Sigma_3 = (\langle a_1, b_1 \rangle \cdot \langle a_2, b_2 \rangle) \times_{\lambda} \langle a, b \rangle \text{ with } a_i^a = b_i a_i, \quad b_i^a = a_i^{-1}, \\ a_i^b = b_i, \quad b_i^b = a_i, \quad i=1,2.$$

$$(Q_8 Q_8)_{C_2} \cdot DC_3 = (\langle a_1, b_1 \rangle \cdot \langle a_2, b_2 \rangle) \cdot \langle a, b \rangle \text{ with the above relations} \\ \text{and } b^2 = a_1^2 = a_2^2.$$

$$Q_1 \times_{\lambda} C_8 = (C_3^2 \times_{\lambda} C_3) \times_{\lambda} C_8 = \langle d_1, e, d_2 \rangle \times_{\lambda} \langle a \rangle \text{ with } d_1^{d_2} = d_1 e, \quad e^{d_2} = e, \\ e^a = e^{-1}, \quad d_1^a = d_2, \quad d_2^a = d_1 d_2^2.$$

$$\text{Hol}((Q_8 D_8)_{C_2}, C_5) = (\langle x_1, x_2 \rangle \cdot \langle x_3, x_4 \rangle) \times_{\lambda} \langle a \rangle \text{ with } x_1^a = x_2 x_3 x_4, \\ x_2^a = x_1^{-1}, \quad x_3^a = x_1 x_4, \quad x_4^a = x_1^2 x_3 x_4.$$

$$C_5^2 \times_{\lambda} (C_3 \times_{\lambda} C_8) = \langle x_1, x_2 \rangle \times_{\lambda} \langle a, b \rangle \text{ with } a^b = a^{-1}, \quad x_1^a = x_2, \quad x_2^a = x_1^{-1} x_2^{-1}, \\ x_1^b = x_1 x_2^{-1}, \quad x_2^b = x_1^{-2} x_2^{-1}.$$

Relations for the rest of the groups appearing in the table can be found in [10].

THEOREM: Let G be a group with 13 conjugacy classes. Then G is isomorphic to one group of the following table:

G	Δ_G
C_{13}	(13^{13})
$C_5 \times_{\lambda} D_8$	$(40^2, 20^9, 4^2)$
$C_5 \times_{\lambda} D_8$	$(40^2, 20^9, 4^2)$
$C_5 \times_{\lambda} Q_8$	$(40^2, 20^9, 4^2)$
$C_{23} \times_{\lambda} C_2$	$(46, 23^{11}, 2)$
$2^6 \Gamma_{22} a_1$	$(64^2, 32, 16^5, 8^5)$
$2^6 \Gamma_{22} a_2$	$(64^2, 32, 16^5, 8^5)$
$2^6 \Gamma_{23} a_1$	$(64^2, 32, 16^5, 8^5)$
$2^6 \Gamma_{23} a_2$	$(64^2, 32, 16^5, 8^5)$
$2^6 \Gamma_{23} a_3$	$(64^2, 32, 16^5, 8^5)$
$2^6 \Gamma_{23} a_4$	$(64^2, 32, 16^5, 8^5)$
$C_{31} \times_{\lambda} C_3$	$(93, 31^{10}, 3^2)$
$(C_2 \times Q_8) \times_{\lambda} \Sigma_3$	$(96^2, 48, 16^2, 12^4, 8^4)$
$2^4 \Gamma_{2b} \times_{\lambda} \Sigma_3$	$(96^2, 48, 16^2, 12^4, 8^4)$
$2^4 \Gamma_{2b} \cdot DC_3$	$(96^2, 48, 16^2, 12^4, 8^4)$
$C_5^2 \times_{\lambda} C_4$	$(100, 50^2, 25^3, 20, 10^2, 4^2)$
$C_2^2 \times_{\lambda} (C_{15} \times_{\lambda} C_2)$	$(120, 60^2, 40, 20^2, 15^5, 4^2)$
$C_{37} \times_{\lambda} C_4$	$(148, 37^9, 4^3)$
$C_5^2 \times_{\lambda} \Sigma_3$	$(150, 50^4, 25^2, 10^5, 3)$
$C_{13} \times_{\lambda} C_{12}$	$(156, 13, 12^{11})$
$\text{Hol}((Q_8 D_8)_{C_2}, C_5)$	$(160^2, 16^3, 10^8)$
$R_1 \times_{\lambda} C_2$	$(162, 54^2, 81, 27^3, 9^3, 6^3)$

$R_2 \times_{\lambda} C_2$	(162, 54^2 , 81, 27^3 , 9^3 , 6^3)
$R_3 \times_{\lambda} C_2$	(162, 54^2 , 81, 27^3 , 9^3 , 6^3)
$R_4 \times_{\lambda} C_2$	(162, 54^2 , 81, 27^3 , 9^3 , 6^3)
$((Q_8 Q_8)_{C_2}) \times_{\lambda} \Sigma_3$	(192^2 , 32^3 , 16^3 , 8^3 , 6^2)
$((Q_8 Q_8)_{C_2}) \cdot DC_3$	(192^2 , 32^3 , 16^3 , 8^3 , 6^2)
$C_{41} \times_f C_5$	(205, 41^8 , 5^4)
$Q_1 \times_{\lambda} C_8$	(216, 108, 24^3 , 12^3 , 9, 8^4)
$C_{23} \times_f C_{11}$	(253, 23^2 , 11^{10})
$C_{43} \times_f C_6$	(258, 43^7 , 6^5)
$C_{43} \times_f C_7$	(301, 43^6 , 7^6)
$C_{31} \times_f C_{10}$	(310, 31^3 , 10^9)
$C_3^3 \times_{\lambda} A_4$	(324, 81^2 , 54, 27, 12, 9^6 , 6)
$C_{41} \times_f C_8$	(328, 41^5 , 8^7)
$C_{37} \times_f C_9$	(333, 37^4 , 9^8)
$C_5^2 \times_{\lambda} M_{16}$	(400, 50, 40, 25, 16^3 , 10, 8^5)
$C_2^4 \times_{\lambda} (C_3^2 \times_f C_4)$	(576, 96, 64, 36, 16^2 , 12, 9, 8^5)
$C_5^2 \times_f (C_3 \times_{\lambda} C_8)$	(600, 25, 24^3 , 12^4 , 8^4)
$SL(2,9)$	(720^2 , 18^4 , 10^4 , 8^3)
$C_9^3 \times_{\lambda} (C_{13} \times_f C_3)$	(1053, 81^2 , 13^4 , 9^6)
$PGL(2,11)$	(1320, 24, 20, 12^5 , 11, 10^4)
$P\Gamma L(2,9)$	(1440, 48, 40, 32, 18, 16^2 , 10^2 , 8^3 , 6)

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