EXTRACTA MATHEMATICAE 3, n.3, 92-94 (1988)

ON THE LENGTH OF THE ABSOLUTE SAMUEL STRATUM

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1980 Mathematics Subject Classification: 14B05
Key words: Samuel Stratum, infinitely near point, maximal contact.

Throughout this note x will be a point of a scheme X of finite type over the field of complex numbers, $\mathcal{O}$ will be the local ring of X at x and M will be its maximal ideal. The Hilbert-Samuel function of X at x is defined by:

$$H_{X,x}(n) = \text{length}(M^n/M^{n+1})$$

The flatness strata ([1]) of the sheaves of jets J^n_X are said to be the (scheme-theoretic) Samuel strata of X. The Samuel stratum of X passing through x is denoted by $S_x(X)$ and $J_{X,x}$ will be its ideal in $\mathcal{O}$.

We define the tangent cone to X at x as $C_x(X) = \text{Spec}(Gr_M\mathcal{O})$ and the Samuel stratum of $C_x(X)$ passing through the vertex is said to be the strict tangent space to X at x. It is denoted by $T_x(X)$ and it is a vector subspace because the characteristic of the residue field $\mathbb{C}(x)$ is zero.

A monoidal transformation $X' \longrightarrow X$ is said to be permissible if the center is regular and X is normally flat along it.

If $X_i \longrightarrow \dots \longrightarrow X_1 \longrightarrow X$ is a sequence of permissible monoidal transformations, for each point x' of X_i above x we have $H_{X_i,x'} \leq H_{X,x}$ (counted with appropriate transcendence degrees). If the equality holds, then x' is said to be an infinitely near H-point of X at x (From now on, H will be the function $H_{X,x}$).

Our results are based on the following theorem:

Theorem: Let $X' \longrightarrow X$ be a permissible monoidal transformation and let $x' \in X'$ be an infinitely near H-point of X at x. If $g \in J_{X,x}$, then g/t belongs to $J_{X',x'}$ where t=0 is a local equation for the exceptional divisor at x'.

Length of the Samuel stratum

If $J_{X,x}$ is the ideal of $\mathcal{O}$ corresponding to the Samuel stratum $S_x(X)$, it seems natural to consider the length $s_{X,x}$ of $\mathcal{O}/J_{X,x}$ as a significant invariant in the study of singularities. Our aim is to study the effect of permissible monoidal transformations on this invariant.

Let $X' \longrightarrow X$ be a permissible monoidal transformation and let x be the generic point of its center. By [3] we know that, for any point x' of X' above x, the Hilbert-Samuel function does not increase and that if $H_{X',x'} = H_{X,x}$ then x' is a point of the projectivization $\mathbb{P}(T_xX)$ of the strict tangent space to X at x.

We prove, when $s_{X,x}$ is finite, that the singularity of X' at the generic point of $\mathbb{P}(T_xX)$ is always better than the singularity of X at x:

Theorem: Let x be the generic point of the center of a permissible monoidal transformation $X' \longrightarrow X$ and let y be the generic point of $\mathbb{P}(T_xX)$. If $s_{X,x}$ is finite and y is an infinitely near H-fold point of X at x, then

$$s_{X',y} < s_{X,x}$$

Corollary: If $s_{X,x}$ is finite and the dimension of T_xX is 1, then there exists a sequence of permissible monoidal transformations $X_s \rightarrow \dots \rightarrow X_1 \rightarrow X$, with $s \leq s_{X,x}$, such that X_s has no infinitely near H-fold point of X at x.

Maximal contact

If X is a closed subscheme of a regular scheme Z of finite type over the complex numbers, we say that a subscheme W of Z has maximal contact with X at x if W is regular at x and for any infinitely near H-fold point x' of X at x we have

$$x' \in W \qquad \text{and} \qquad T_{x'}(X) \subseteq T_{x'}(W)$$

(Naturally in these conditions X and W must be replaced by their respective strict transforms).

Theorem: Let W be a subscheme of Z regular at x. If W contains the Samuel stratum $S_x(X)$ then W has maximal contact with X at x.

Examples:

1) Let $f(x_1,\ldots,x_n)=0$ be an hypersurface and let x be a point of multiplicity m. If you find m-1 derivations $D_1,\ldots,D_{m-1}$ such that the multiplicity of $g=D_1(\ldots D_{m-1}(f))$ at x is 1 (it is always possible to find such derivations when the characteristic of the base field is zero: take generic directions!), then the hypersurface g=o has maximal contact with the given hypersurface f=0 at x.

If X is the hypersurface defined by

$$f = z^m + \sum_{i=1}^{m} A_i(t_1,\ldots,t_n)z^{m-i} = 0 \qquad , \qquad A_i \in (t_1,\ldots,t_n)^{i+1}$$

then the multiplicity of X at the origin is m and $\partial^{m-1}f/\partial z^{m-1} = (m-1)!\,(mz + A_1)$. Hence, the hypersurface $z + A_1(t_1,\ldots,t_n)/m = 0$ has maximal contact with X at the origin.

2) Let X be the plane curve defined by $z^3 = t^5$. Then the ideal of the Samuel stratum of X at the origin is (z,t^3), but the curve $z=t^2$ has maximal contact with X at the origin. Hence, the condition $S_x(X) \subseteq W$ is not necessary for W to have maximal contact with X at x.

References

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(To appear in Journal of Algebra)