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INVERSION FORMULAS FOR DIRICHLET SERIES

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Let χ_1 and χ_2 be characters modulo q_1 and q_2, respectively, where q_1 and q_2 are positive integers and let

(1.1) $$f(n) = (d_k N^a \chi_1 * d_\ell N^b \chi_2)(n)\ , \quad k,\ell \in Z^+, \quad o \le b \le a,$$

being $N(n) = n$ for any positive integer n and $d_k(n)$ the number of representations of n as a product of k factors.

Theorem

Let $f(n)$ be the arithmetical function defined in (1.1), let $\delta(\chi)$ the characteristic function such that $\delta(\chi)=1$ if χ is the principal character modulo q and $\delta(\chi)=0$ otherwise and let r be a positive integer such that $2r + 1 - 2ka - 2\ell b - k - \ell > 0$.

(I) Let $b < a$. Then we have , as $x \to \infty$,

$$\sum_{n\le x} f(n) \log^r(x/n) = \delta(\chi_1)\ r!\ x^{1+a}\ P_{k-1}(\log x) +$$

(1.2) $$+ \delta(\chi_2)\ r!\ x^{1+b}\ P_{\ell-1}(\log x) + r!\ P_r(\log x) +$$

$$+ O(x^{1/2 - (r-ka-\ell b-1/2)/(k+\ell)})$$

where

i) $$P_{k-1}(\log x) = x^{-1-a} \operatorname*{Res}_{\omega=1+a} (F(\omega)\, x^\omega\, \omega^{-r-1}) =$$

$$= \sum_{\alpha=o}^{k-1} \sum_{\substack{n=-k \\ n+m+\beta=-\alpha-1}}^{-1} \sum_{m=o}^{k-1} \sum_{\beta=o}^{k-1} \frac{(-1)^\beta (r+\beta)!\, a_n(q_1)\, L^{\ell^{(m)}}(1+a-b_1\chi_2)}{r!\beta!m!\alpha!\ (1+a)^{r+1+\beta}} \log^\alpha x$$

being $L^{k}(\omega-a,\chi_1) = \sum_{n=-k}^{\infty} a_n(q_1)(\omega-a-1)^n$

and $$F(\omega) = \sum_{n=1}^{\infty} f(n)\, n^{-\omega}$$

for $\mathrm{Re}\,\omega > 1 + a$.

ii) $$P_{\ell-1}(\log x) = x^{-1-b} \operatorname*{Res}_{\omega=1+b} (F(\omega)\, x^{\omega}\, \omega^{-r-1}) =$$

$$\sum_{\alpha=0}^{\ell-1} \sum_{\substack{m=-\ell \\ n+m+\beta=-\alpha-1}}^{-1} \sum_{n=0}^{\ell-1} \sum_{\beta=0}^{\ell-1} \frac{(-1)^{\beta}(r+\beta)!\, a_m(q_2) L^{k^{(n)}}(1+b-a,\chi_1)}{r!\,\beta!\,n!\,\alpha!\,(1+b)^{r+1+\beta}} \log^{\alpha} x$$

being $L^{\ell}(\omega-b,\chi_2) = \sum_{m=-\ell}^{\infty} a_m(q_2)(\omega-1-b)^m$.

iii) $$P_r(\log x) = \operatorname*{Res}_{w=0} (F(\omega)x^{\omega}\omega^{-r-1}) = \sum_{m=0}^{r} \frac{F^{(m)}(0)}{m!(r-m)!} \log^{r-m} x .$$

(II) Let $b=a$. Then we have, as $x \to \infty$,

$$\sum_{n\le x} f(n) \log^{r}(x/n) = \alpha(\chi_1,\chi_2)\, r!\, x^{1+a} P_{\gamma(\chi_1,\chi_2)}(\log x) + r!\, P_r(\log x) + O(x^{1/2+a-(r+1/2)/(k+\ell)}) \tag{1.3}$$

where

i) $\alpha(\chi_1,\chi_2) = 1$ and $\gamma(\chi_1,\chi_2) = k - 1$ if χ_1 principal and χ_2 nonprincipal ; $\alpha(\chi_1,\chi_2) = 1$ and $\gamma(\chi_1,\chi_2) = \ell - 1$ if χ_1 nonprincipal and χ_2 principal ; $\alpha(\chi_1,\chi_2) = 1$ and $\gamma(\chi_1,\chi_2) = k+\ell-1$ if χ_1 and χ_2 are principals ; and $\alpha(\chi_1,\chi_2) = 0$ otherwise.

ii) $P_r(\log x)$ and $P_{k-1}(\log x)$ are the polynomials above mentioned

iii) $P_{\ell-1}(\log x) = x^{-1-a} \operatorname*{Res}_{\omega=1+a} (F(\omega)\, x^{\omega}\, \omega^{-r-1}) =$

$$= \sum_{\alpha=0}^{\ell-1} \sum_{n=0}^{\ell-1} \sum_{m=-\ell}^{-1} \sum_{\beta=0}^{\ell-1} \frac{(-1)^{\beta}(r+\beta)!\; a_m(q_2) L^{k^{(n)}}(1,\chi_1)}{r!\beta!n!\alpha!\,(1+a)^{r+1+\beta}} \log^{\alpha} x$$
$n+m+\beta = \alpha-1$

iv) $P_{k+\ell-1}(\log x) = x^{-1-a} \operatorname*{Res}_{\omega=1+a} (F(\omega)\, x^{\omega}\, \omega^{-r-1}) =$

$$= \sum_{\alpha=0}^{k+\ell-1} \sum_{\beta=0}^{k+\ell-1} \sum_{n=-k}^{\ell-1} \sum_{m=-\ell}^{k-1} \frac{(-1)^{\beta}(r+\beta)!\; a_n(q_1)\; a_m(q_2)}{r!\beta!\alpha!\,(1+a)^{r+1+\beta}} \log^{\alpha} x .$$
$\beta+n+m = -\alpha-1$

For $q_1=1$ $\chi_1\equiv1$, $\chi_2(n)=\chi_d(n) =(\frac{d}{n})$ the Kronecker Symbol , $k=\ell=1$ and $a=b=0$, $F(w)$ is the Dedekind zeta function of K , $\zeta_K(w)$, where K is an imaginary quadratic field of discriminant d. For $r=1$ from (1.3) formula we deduce , as a particular case, a result of Ayoub and Chowla [2]

$$(1.5)\quad \sum_{n\le x} f(n)\log(x/n) = L_d(1)x+\zeta_K(0)\log x+\zeta_K'(0)+O(x^{-1/4})$$

where

$$L_d(s) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s}$$

References

[1] Apostol, T. M. .- Dirichlet L-series and character powers sums. J. Number Theory, 2, 223-234, (1970).

[2] Ayoub, R ., and Chowla, S. .- On a theorem of Müller and Carlitz. J. Number Theory, 2, 342-344, (1970).

[3] Copson , E.T. .- Theory of Functions of a Complex Variable . Oxford, (1935)

[4] Landau, E. .- Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale. Chelsea Publ. Co., New York, 1949.

[5] Redmond, D. .- Corrections and additions to "A generalization of a theorem of Ayoub and Chowla". Proc. Amer. Math. Soc., 90, 2, 345-346 (1984).