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SEMI-FREDHOLM OPERATORS AND SEMIGROUPS ASSOCIATED WITH SOME CLASSICAL OPERATOR IDEALS II

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In [3] (see also [4]) we studied semigroups SU_+ associated with operator ideals U defined in terms of sequences as Co : compact, WCo : weakly compact, Ro : Rosenthal, CC : completely continuous and WCC : weakly completely continuous operators.

In this note we complete some results in [3] and introduce another semigroups SU_- for the above operator ideals, and some others defined in terms of w^*-convergence of sequences in dual spaces. We prove that the semigroups SU_- are stable by addition of operators in the corresponding U, and other basic properties. From this we derive characterizations for lower semi-Fredholm operators SF_- in terms of the semigroups SU_-, and for certain coincomparability classes of Banach spaces [1] by means of SF_-.

For $U \in \{ Co, WCo, Ro, CC, WCC \}$, we denote
$U^d := \{ K \in L / K' \in U \}$ and $SU_- = \{ T \in L / T' \in SU_+ \}$

The definition of SU_+ can be found in this journal [4].

Moreover, for the operator ideals Gr: Grothendieck [5], Lm: limited [2], and Cd: condensed, we define
$T \in L(X,Y)$ belongs to SGr_- (resp. SLm_- , SCd_-) if every w^*-convergent sequence (f_n) in Y' such that $(T'f_n)$ is w-convergent (resp. convergent, w-Cauchy), has a w-convergent (resp. convergent, w-Cauchy) subsequence.

<u>Theorem</u>. Suppose $U \in \{ Co, WCo, CC, Ro, WCC, Gr, Cd \}$; denote $V := U$ for Gr and Cd, $V := U^d = \{ K \in L \,/\, K' \in U \}$ in the remaining cases, and let $T \in L(X,Y)$. Then we have

(1) SU_- is a semigroup which contains SF_-.

(2) $T + K \in SU_-$ for every $T \in SU_-$ and $K \in V$.

(3) If $T \in SU_-$ then $Y/\overline{R(T)} \in Sp(V) = \{ X \,/\, I_X \in V \}$.

(4) If $R(T)$ is closed and $Y/R(T) \in Sp(V)$ then $T \in SU_-$.

(5) $T \in SF_-$ if and only if $T \in SU_-$ and $q'_N T \in SF_-$ for every $Y/N \in Sp(V)$.

(6) $SF_-(Z,Y) = SU_-(Z,Y)$ for any Banach space Z if and only if $Y \in Sp(V)^c = \{$ X without infinite-dim. quotients in $Sp(V)$ $\}$.

(7) $SLm_- = SF_-$.

<u>Observations</u>.

1. For U = Co, WCo, CC, Ro, or WCC, the coincomparability classes $Sp(U^d)^c$ are respectively the finite dimensional spaces; and the spaces without infinite dimensional quotients in the classes of reflexive, Dunford-Pettis without copies of l_1, without copies of l_1 nor c_o quotients, and weakly sequentially complete dual spaces.

2. In [6] upper semi-Fredholm operators in Hilbert space are characterized as those $T \in L(H)$ for which given a weakly null sequence (x_n) such that (Tx_n) is norm null, we have that (x_n) is norm null (Wolf condition).

We showed in [3] that the above condition characterize $SF_+(X,Y)$ for X,Y Banach exactly when X has no copies of l_1.

Noting that Hilbert spaces are reflexive; hence all operators are conjugate $(T = T'')$ and the topologies weak and weak* are the same, the above proposition shows that the right extension of

Wolf condition to general Banach spaces, characterizing conjugate operators in SF_+, is the following assertion, which is obviously equivalent to the definition of SLm_-:

If (f_n) is weak* null and $(T'f_n)$ is norm null, then (f_n) is norm null.

3. In [4] we considered the class WSC^i of all Banach spaces without infinite dimensional WSC subspaces. Because c_o is not weakly sequentially complete, it is clear that $Swc_o \subset WSC^i$, where Swc_o is the class of all somewhat-c_o Banach spaces for which every infinite dimensional subspace has a copy of c_o. The equality is equivalent with the James tricothomy problem:

$WSC^i = Swc_o$ if and only if every Banach space contains c_o, l_1 or an infinite-dimensional reflexive subspace.

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