EXTRACTA MATHEMATICAE 2, n.3, 93-95 (1987)

FINITELY SELF-COGENERATED QUASI-INJECTIVE MODULES AND THEIR ENDOMORPHISM RINGS

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A.M.S. Classification: 16A65, 16A52

An useful tool in the study of the endomorphism ring S of a quasi-injective left R-module U is the duality defined by ${}_RU_S$ which has been recently investigated in several papers:[4], [8]. The closed submodules of ${}_RU$ and S_S (i.e., the annihilators in U and S of subsets of S and U respectively) have a good behaviour in relation with this duality for they are precisely the U-reflexive submodules of U and the right ideals Z of S such that S/Z is U-reflexive (and, in particular, the finitely generated right ideals of S are closed). This allows us by using the duality defined by ${}_RU_S$, to characterize the quasi-injective modules such that their endomorphism rings are right coherent or right semihereditary, since these rings are defined by properties of closed submodules of finitely generated free right S-modules. Moreover, we give necessary and sufficient conditions on ${}_RU$ for every maximal right ideal of S to be closed, that is, for every simple right S-module to be cogenerated by U_S. This class of modules is interesting because of its connections with the polyform modules in the sense of [9], and also because every simple right S-module has, in this case, a projective cover by results in [6] and hence S is semiperfect (which, in turn, is well-known to be equivalent to ${}_RU$ being finite-dimensional).

In what follows, R will be an associative ring with 1, R-mod (resp., mod-R) the category of left (resp., right) R-modules. For a given module ${}_RU$, a module ${}_RX$ is said to be (finitely) U-generated (resp., U-cogenerated) if X is a quotient (resp., a submodule) of a (finite) direct sum (resp., product) of copies of U. X is called U-injective if for each submodule L of X, the induced morphism $\mathrm{Hom}_R(X,U) \longrightarrow \mathrm{Hom}_R(L,U)$ is an epimorphism and, in particular, U is quasi-injective when U is U-injective. If $S=\mathrm{End}({}_RU)$, then ${}_RU_S$ is a bimodule and the U-dual functors $\mathrm{Hom}_R(-,U)$:R-mod $\longrightarrow$ mod-S and $\mathrm{Hom}_S(-,U)$:mod-S $\longrightarrow$ R-mod will be denoted, as usual, by ( )*. The evaluation maps yield natural transformations $\psi: \mathbf{1}_{R\text{-mod}} \longrightarrow (\ )^{**}$ and $\psi: \mathbf{1}_{\text{mod-}S} \longrightarrow (\ )^{**}$. A module X (in R-mod or mod-S) is called U-reflexive when ψ_X is an isomorphism. E(X) will denote the injective hull of X.

In [9] a module M is called polyform when for each submodule N of M and every homomorphism $f:N \longrightarrow M$, Ker f is an essentially closed submodule of N. On the other hand, a lattice L with least element 0 has the finite intersection property when every subset of L with intersection 0 has a finite subset whose intersection is also 0. We have the following result (in which, as well as in all the results that follow, U is a quasi-injective left R-module and $S=End(_RU)$).

Theorem 1. The following conditions are equivalent.

i) Each simple right S-module is U-reflexive (U-cogenerated).

ii) For every monomorphism $u:U \longrightarrow U^I$ there exists a finite subset F of I such that if $p_F:U^I \longrightarrow U^F$ denotes the canonical projection, then $p_F o u$ is a monomorphism.

iii) The lattice of closed submodules of $_RU$ has the finite intersection property.

iv) $_RU$ has an essential finite-dimensional polyform submodule.

We call a quasi-injective module U with the properties of Theorem 1 finitely self-cogenerated. If U is finitely self-cogenerated, then S is semiperfect, but the converse does not hold, for there exist (commutative) self-injective local rings which are not cogenerator rings (the example of Levy in [1]) and hence are not finitely self-cogenerated.

Recall that a module M is called monoform if every nonzero homomorphism from a submodule N of M to M is a monomorphism. It follows from Theorem 1 and [9, Prop. 3.3] that a quasi-injective module U is finitely self-cogenerated if and only if U has an essential submodule of the form $\overset{n}{\underset{1}{\oplus}} M_i$, such that every nonzero homomorphism $f:M_i \longrightarrow E(M_j)$ is a monomorphism. These essential submodules are therefore finite direct sums of monoform modules; and U is of this form precisely when S is semisimple, as showed in the next result.

Corollary 2. i) S is semisimple if and only if there exists a decomposition $U = \overset{n}{\underset{1}{\oplus}} M_i$ such that every nonzero homomorphism $f:M_i \longrightarrow E(M_j)$ is a monomorphism.

ii) S is simple artinian if and only if U is a finite direct sum of copies of a monoform module.

Semihereditary endomorphism rings of injective modules have been studied in several papers: [3], [5], [7]. We give a more general result.

Theorem 3. The following conditions are equivalent.

i) S is right semihereditary.

ii) For each finitely U-generated and finitely U-cogenerated module X, X** is U-injective.

iii) For each finitely U-generated submodule X of U^n there is a direct summand Y of U^n such that $X \subset Y$ and $(Y/X)^* = 0$.

[3, Satz 4] states that a ring R is left hereditary if and only if $End(_RQ)$ is right semihereditary for every injective left R-module Q. This follows easily from Theorem 3, and also [5, Prop. 2.1] and [7, Coroll. 6.9] can be deduced from this theorem.

Finally, we study coherent endomorphism rings.

Theorem 4. S is right coherent if and only if for every finitely U-cogenerated and finitely U-generated left R-module X there exists an exact sequence $0 \longrightarrow X \longrightarrow U^k \longrightarrow Y \longrightarrow 0$, with Im ψ_Y finitely U-cogenerated.

When $_RU$ is assumed to be a quasi-injective self-cogenerator (i.e., $_RU$ cogenerates all nonzero quotients of each U^n), then Theorems 3 and 4 specialize to the following results.

Corollary 5. S is right semihereditary if and only if every finitely U-generated and finitely U-cogenerated module is U-injective.

Corollary 6. S is right coherent if and only if for every homomorphism $f: U^k \longrightarrow U^r$ there exists an exact sequence $U^k \xrightarrow{f} U^r \longrightarrow U^n$.

The proofs of the foregoing results will appear in [2].

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