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LOCAL CONVEXITY IS A THREE-SPACE PROPERTY FOR NON-ARCHIMEDEAN FRECHET SPACES

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The three-space problem we consider is the following: If X is an F-space (that is, a complete metrizable topological vector space) with a closed locally convex subspace M such that X/M is locally convex, must X be locally convex?. The negative answer to this question was obtained independently at the end of the seventies by N.J. Kalton, M. Ribe and J.W. Roberts.

In order to solve the above question, N.J. Kalton and N.T. Peck [2] had introduced the concept of K-space whose definition for a Banach space is as follows: A Banach space X is said to be a K-space if whenever Y is a quasi-Banach space and L is a one-dimensional subspace of Y such that Y/L and X are linearly isomorphic, then L is complemented in Y.

In this note we are going to study K-spaces and the above three-space problem within the context of F-spaces over a complete non-trivially non-archimedean valued field $\mathbb{K}$. Proofs of results included in this note will appear in our papers [4] and [5].

First we give conditions under which a Banach space over $\mathbb{K}$ is a K-space. For that we recall that a Banach space over $\mathbb{K}$ has a basis if it possess a Schauder basis of an arbitrary cardinality.

Theorem 1: *Every Banach space over $\mathbb{K}$ with a basis is a K-space.*

This theorem is in sharp contrast with the corresponding results in case of real or complex ground fields. In fact N.J. Kalton [1] and other authors showed that l^1 is not a K-space.

A Banach space over $\mathbb{K}$ is said to be *non-archimedean* if the norm verifies the ***strong triangle inequality*** (that is, $\|x+y\| \le \max\{\|x\|, \|y\|\}$). Also $\mathbb{K}$ is said to be ***spherically complete*** if it verifies the Nachbin's property of binary intersection for balls.

Theorem 2: *Every non-archimedean Banach space over a spherically complete field $\mathbb{K}$ is a K-space.*

This theorem can become false if the ground field $\mathbb{K}$ is not spherically complete; in fact the non-archimedean Banach space l^∞/c_0 over any not spherically complete $\mathbb{K}$ is not a K-space.

One can deduce that local convexity is not a three-space property in the category of real or complex complete locally bounded spaces (see [3], p.95) as a consequence of the fact that l^1 is not a K-space. In our case the situation is very different for spherically complete ground fields.

Theorem 3: *Local convexity (a la Monna) is a three-space property in the category of complete locally bounded spaces over a spherically complete ground field.*

As a result of our theorem 1, every non-archimedean Banach space of countable type (that is, with a countable subset whose linear span is dense) is a K-space. In [4] we have found an example of a Banach space of countable type with separating dual which is not a K-space. In particular *this space cannot have a basis.* So we derive a strong difference with respect to the well-known fact that every non-archimedean Banach space of countable type has a basis.

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