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SEMISIMPLE REPRESENTATIONS OF QUIVERS

research announcement

Lieven LE BRUYN
University of Antwerp,UIA-NFWO

A quiver Q consists of a finite set $Q_0 = \{1, ..., n\}$ of vertices, a set Q_1 of arrows and two maps $t, h : Q_1 \to Q_0$ assigning to an arrow ϕ its tail $t\phi$ and its head $h\phi$, respectively. We do not exclude loops nor multiple arrows.

A representation V of Q is a family $\{V(i) : i \in Q_0\}$ of finite dimensional $\mathbb{C}$-vectorspaces together with a famnily of linear maps $\{V(\phi) : V(t\phi) \to V(h\phi) \mid \phi \in Q_1\}$. The vector $dim(V) = (dim(V(1)), ..., dim(V(n)))$ is called the dimension type of V. A morphism $f : V \to W$ between two representations of Q is a family of linear maps $\{f_i : V(i) \to W(i) \mid i \in Q_0\}$ such that $W(\phi) \circ f(t\phi) = f(h\phi) \circ V(\phi)$ for all $\phi \in Q_1$.

The representation space $R(Q, \alpha)$ of Q of dimension type $\alpha = (\alpha(1), ..., \alpha(n)) \in \mathbb{N}^n$ is the set of representations

$$R(Q, \alpha) = \{V : V(i) = \mathbb{C}^{\alpha(i)}; 1 \le i \le n\}$$

Because $V \in R(Q, \alpha)$ is determined by the maps $V(\phi)$

$$R(Q, \alpha) = \bigoplus_{\phi \in Q_1} M_\phi(\mathbb{C})$$

where $M_\phi(\mathbb{C})$ is the set of $\alpha(h\phi)$ by $\alpha(t\phi)$ matrices over $\mathbb{C}$. We consider $R(Q, \alpha)$ as an affine variety with coordinate ring $\mathbb{C}[Q, \alpha]$. The linear reductive group $GL(\alpha) = \prod_{i=1}^n GL_{\alpha(i)}(\mathbb{C})$ acts linearly (and regularly) on $R(Q, \alpha)$ by for $g = (g_1, ..., g_n) \in GL(\alpha)$. The $GL(\alpha)$-orbits in $W(\phi) \circ f(t\phi) = f(h\phi) \circ V(\phi)$ for all $R(Q, \alpha)$ are the isomorphism classes of representations.

We want to study the quotient variety

$$Z(Q, \alpha) = R(Q, \alpha)/GL(\alpha)$$

i.e. the affine variety parametrizing the closed $GL(\alpha)$-orbits or equivalently the isoclasses of semi-simple representations of dimension type α.In view of Mumford's theory [Mu] , the coordinate ring of $Z(Q, \alpha)$ coincides with the ring of polynomial invariants $\mathbb{C}[Q, \alpha]^{GL(\alpha)}$.Note that $\mathbb{C}[Q, \alpha] = \mathbb{C}[X_\phi : \phi \in Q_1]$ where X_ϕ is an $\alpha(h\phi)$ by $\alpha(t\phi)$ matrix of indeterminates $(x_{ij}(\phi))_{i,j}$.We can give a precise description of this invariant ring :

Theorem 1 (Le Bruyn,Procesi)
The ring of polynomial invariants is generated by elements of the form $Tr(X_{\phi_1}.X_{\phi_2}...X_{\phi_k})$ where $(\phi_1, \phi_2, ..., \phi_k)$ is an oriented cycle in Q and $k \le \sum_{i=1}^n \alpha(i)$.

The main point of the proof is to note that $Z(Q, \alpha)$ is an irreducible component of the variety parametrizing semi-simple $\sum \alpha(i)$- representations of the path algebra of the opposite quiver Q^* which reduces the problem modulo [AS] to [Pr].

If $\xi \in Z(Q, \alpha)$, then ξ determines the isoclass of a semi- simple representation $V = e_1.V_1 \oplus ... \oplus e_k.V_k$ where the V_i are distinct simple representations with dimension vectors β_i and occuring with multiplicity e_i.We say that ξ is of type $\tau = (e_1, \beta_1; ...; e_k, \beta_k)$. In order to describe all possible types we need to know the dimension types of simple representations of Q. For all $i, j \in Q_0$ let us denote $r_{ij} = \sharp\{\phi \in Q_1 : t\phi = i, h\phi = j\}$. The Ringel bilinear form of Q , R , on $\mathbb{Z}^n$ is then determined by the n by n matrix $(\delta_{ij} - r_{ij})_{i,j}$. Let $\alpha_i = (\delta_{ij})_j$ be the standard basevectors for $\mathbb{Z}^n$, then we get

Theorem 2 (Le Bruyn,Ringel) $\alpha \in \mathbb{N}^n$ is the dimension type of a simple representation of the quiver Q iff either $supp(\alpha)$ is an oriented cycle and all $\alpha(i)$ are 0 or 1

or $supp(\alpha)$ is a 'club' i.e. any two vertices in it belong to an oriented cycle in $supp(\alpha)$ and $R(\alpha, \alpha_i) \leq 0$, $R(\alpha_i, \alpha) \leq 0$ for all $i \in supp(\alpha)$.

The proof goes by induction on $\sum \alpha(i)$ and a shrinking process stating that for a quiver Q having verices i and j such that there is only one directed arrow between them, the dimension types of simple representations of Q with $\alpha(i) = \alpha(j)$ are those of the simples of Q' where we have identified i with j.

Let $\tau = (e_1, \beta_1; ...; e_k, \beta_k)$ be the type of a semi-simple α-representation of Q, then we denote by $Z(\tau)$ the set of all points $\xi \in Z(Q, \alpha)$ of type τ. The next result is an easy application of the Luna slice lemma [Lu].

Theorem 3

$\{Z(\tau) \mid \tau \text{an admissible type}\}$ is a finite stratification of $Z(Q, \alpha)$ into locally closed irreducible smooth subvarieties.

Of course, $Z(\tau')$ lies in the closure of $Z(\tau)$ if representations of type τ' are degenerations of those of type τ.Now, let $\xi \in Z(Q, \alpha)$ be a point of type $\tau = (e_1, \beta_1; ...; e_k, \beta_k)$, then we construct a new quiver Q_{tau} in the following way : $Q_{\tau 0} = \{1, ..., k\}$, in i there are $1 - R(\beta_i, \beta_i)$ loops and there are $-R(\beta_i, \beta_j)$ oriented arrows from i to j if $i \neq j$. onsider the dimension type $\alpha_\tau = (e_1, ..., e_k)$, then we get

Theorem 4

There is an analytic isomorphism between a neighborhood of ξ in $Z(Q, \alpha)$ and a neighborhood of the origin in $Z(Q_\tau, \alpha_\tau)$.

Again,this follows from the Luna slice theorem,see [LP] for some special cases.It follows from 4 that there is a generic type $\tau_{gen} = (e_1, \gamma_1; ...; e_k, \gamma_k)$ i.e. such that $Z(\tau_{gen})$ is an open subvariety. By counting the loops in the quiver associated to this τ_{gen} one can show that the Krull dimension of the quotient variety $Z(Q, \alpha)$ is equal to $\sum_{i=1}^{k}(1 - R(\gamma_i, \gamma_i)$. Moreover, $Z(\tau_{gen})$ is precisely the open set of regularity of $Z(\tau)$ except for low- dimensional anomalities.

References

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