EXTRACTA MATHEMATICAE 1, n.2, 73-75 (1986)

ON MEASURABILITY OF VECTOR-VALUED FUNCTIONS WITH RESPECT TO OPERATOR-VALUED MEASURES

Soledad Rodríguez Salazar *

Dpto.Teoría de Funciones.Facultad de Matemáticas.
Universidad Complutense.28040,Madrid.

A.M.S. class. (1980): 28B05; 28C15; 46G10.

Introduction.One of the most important facts in the integration process is the kind of measurability which is used,because it shows the power and generality of the studied integration.Many theorems had been given to characterize the measurable functions by their range,specially the Pettis Theorem for strongly measurable functions used in Bochner integral (2).In many cases these theorems give properties of the measure.Important examples can be seen in the Diestel and Uhl survey (2).Also Chi (1), Gilliam (5), Rodríguez-Salinas (13) and De María (8) have succesively given theorems of this type.

In this paper we give conditions to characterize measurable functions with a view to the bilinear integration.The completeness of the bilinear measure had been usefull and we have ensured it by a Caratheodory type theorem for measures verifying Price Axiom (9,p.20).The semivariation definition follows the Kluvanek- Knowles work and the Dobrakov papers (3,4).

1.Notation. Let X, Y be Hausdorff l.c.s.,and let Y be complete ,$L_c(X,Y)$ is the linear continuous operator space from X in Y , T is any set, $\mathcal{P}$ is a σ-algebra in T and $m : \mathcal{P} \longrightarrow L_c(X,Y)$ is a countable additive measure for the strong operators topology in $L_c(X,Y)$. We denote by $(p)_X$ and $(q)_Y$ the continuous seminorms in X and Y. We use functions $f : T \longrightarrow X$.

*This research was partially supported by CAICYT nº0338/84

We call 0-simple functions those of the form $f = \sum_{i=1}^{n} x_i \chi_{E_i}$.
The integral of f is defined by $\int_E f\, dm = \sum_{i=1}^{n} m(E \cap E_i) x_i$.

We call simple functions those functions which are the uniform limit of a net of 0-simple functions. A function f is simple if and only if $f(T)$ is a precompact subset of X and $p \circ (f-x)$ is $\mathcal{P}$-measurable for every $p \in (p)_X$ and every $x \in X$ ($\mathcal{P}$-measurable means measurable in the sense of inverse images). We define the semivariation of m associated to p and q by:

$$m_{q,p}(E) = \sup\{q(\int_E f\, dm) : f \text{ 0-simple}, p_E(f) \leq 1\}.$$

<u>2.DEFINITION</u>. A function f is m-measurable if for any $q \in (q)_Y$ there is a $p \in (p)_X$ such that for any $\varepsilon > 0$ there is a $K_\varepsilon \in \mathcal{P}$ which verify $m_{q,p}(T \setminus K_\varepsilon) < \varepsilon$ and $f\chi_{K_\varepsilon}$ is simple.

<u>3.PROPOSITION</u>. f is m-measurable <==> f is the almost uniform limit of a net of simple functions.
<u>4.DEFINITION</u>. A function f is $\bar{m}$-measurable if it is the uniform limit of a net of m-measurable functions.

With the assumption of the Weak Price Axiom for the measure: "For any $E \in \mathcal{P}$, $m(E)=0$ or $m(E)$ is bijective". we obtain the following main result:
<u>5.THEOREM-DEFINITION</u>. Let $\mathcal{P}^*$ be the family of subsets, E, of T such that there are $A, B \in \mathcal{P}, p \in (p)_X$ and $q \in (q)_Y$ which verify $A \subset E \subset B$ and $m_{q,p}(B \setminus A) = 0$. Then:
i) $\mathcal{P}^*$ is a σ-algebra.
ii) The following mapping m^* is a countable additive measure:

$$m^* : \mathcal{P}^* \longrightarrow L_c(X,Y),$$

$m^*(E) = m(A)$, where A is the associated set to E in the construction of $\mathcal{P}^*$, $A \subset E$. We call $\mathcal{P}^*$ the completion of $\mathcal{P}$. A measure m is said to be complete if $\mathcal{P} = \mathcal{P}^*$.

With the assumption of the completeness of the measure m, we obtain the following range-characterization for measurable functions:

<u>6.THEOREM</u>. Suppose $m_{q,p}$ continuous for every $q \in (q)_Y$ and every $p \in (p)_X$. Then f is $\bar{m}$-measurable $\Longleftrightarrow$

i) f is essentially-ω-precompact.

ii) $p_\varphi(f-x)$ is P-measurable for every $p \in (p)_X$ and every $x \in X$.

<u>7.COROLLARY</u>. The almost everywhere limit of a $\bar{m}$-measurable functions sequence is a $\bar{m}$-measurable function.

In order to study a theory of integration related to this measurability,as well as to compare it with other integration theories in Banach spaces we refer to (10),(11),(12).

<u>References</u>

(1) G.Y.CHI,"On the Radon Nikodym Theorem in locally convex spaces",Lect.Notes in Math.541,Springer N.Y.1976,199-209

(2) J.DIESTEL-J.UHL,"Vector measures",Math.Surveys,Amer.Math. Soc.,Providence,R.I.,1977.

(3),(4) I.DOBRAKOV,"On integration in Banach Spaces,I,II", Czech.Math.J.20,1970,pp 511-536,680-695.

(5) D.GILLIAM,"On integration and the Radon Nikodym theorem in quasicomplete l.c.t.s.",J.Reine Angew Math.292,1977 pp.125-137.

(6) P.R.HALMOS,"Measure Theory",Van Nostrand,Princeton 1950.

(7) I.KLUVANEK-G.KNOWLES,"Vector Measures and Contol Systems" North Holland Math. Studies 20,1976.

(8) J.L.de MARIA,"A characterization theorem for l.c.s. valued functions",Illinois J. Math.28,4,1984,592-596.

(9) G.B.PRICE,"The theory of integration",Trans.Amer.Math. Soc.47,1940,pp 1-50.

(10) S.RODRIGUEZ SALAZAR,"Integracion general en e.l.c." Univ.Complutense,Madrid,Tesis Doctoral,1985.

(11) S.RODRIGUEZ SALAZAR-M.SOLER,"Una integral tipo Dobrakov en e.l.c."Actas VII Congreso GMEL,Coimbra 1985.

(12) S.RODRIGUEZ SALAZAR,"Teorema del Valor Medio para una integral bilineal en e.l.c.",Actas XI Jornadas Hispano Lusas de Matematicas,Badajoz,1986.

(13) B.RODRIGUEZ-SALINAS,"Integracion de funciones con valores en un e.l.c.".Rev.Real Acad.Ciencias,Madrid,1979.