1. INTRODUCTION

Let $A$ be an $mxn$ complex matrix. The singular values
(1) $\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_{\min(m,n)}(A)$
of $A$ are the common eigenvalues of the positive semidefinite matrices $(AA^*)^{1/2}$ and $(A^*A)^{1/2}$.

Since $AA^*$ is $m$-square and $A^*A$ is $n$-square, the
eigenvalues of $(AA^*)^{1/2}$ and $(A^*A)^{1/2}$ do not coincide in full. However, it is well known that the nonzero eigenvalues (including multiplicities) of these two matrices always coincide.

It is convenient to define $\sigma_t(C)$, $C \in \mathbb{C}^{mxn}$, to be zero for $t \geq \min(m,n)$. This frees us from constantly indicating the variation of the indices in the theorems.

We consider the following theorem (see [1] and [5]):

Let $A$ be an $mxn$ complex matrix with singular values $\sigma_1(A) \geq \sigma_2(A) \geq \ldots$. Let $B$ be a $(m-s) \times (n-r)$ submatrix of $A$ with singular values $\sigma_1(B) \geq \sigma_2(B) \geq \ldots$. Then $\sigma_k(A) \geq \sigma_k(B) \geq \sigma_{k+s+r}(A)$, $k = 1, 2, \ldots$.

This paper deals essentially with the question of finding conditions under which this interlacing theorem for singular values survives a generalization (cf. [3]), in their definition; namely, the introduction of two arbitrary norms in the Courant-Fischer minimax characterization of singular values (see [4], pag. 321).

Let $A$ be an $mxn$ complex matrix and let $p: \mathbb{C}^m \rightarrow \mathbb{R}$, $q: \mathbb{C}^n \rightarrow \mathbb{R}$ be vector norms. The generalized singular va-
stes of $A$ are the non-negative numbers defined as

$$
\sigma_k(A) = \min_{G_{n-k+1} \in \mathbb{C}^n} \max_{x \in G_{n-k+1} \setminus \{0\}} \frac{p(Ax)}{q(x)}
$$

where $G_{n-k+1}$ is the set of all subspaces of $\mathbb{C}^n$ of dimension $n-k+1$, $k = 1, \ldots, n$.

Also, we agree that $\sigma_i(C)$, $C \in \mathbb{C}^{m \times n}$, is zero if $i \geq \min(m,n)$. If $p$, $q$ are the Euclidean norms, then $\sigma_k(A) = \sigma_k(A)$ (see [4], pag. 321).

2.-RESULTS

We show that the generalized singular values (briefly p,q-s.v.) verify the well-known inequalities of Ky Fan for the sum of matrices (cf. [2]): Let $A$, $B$ be $m \times n$ complex matrices, then the p,q-s.v. of $A$, $B$ and $A+B$ verify

$$
\sigma_{k+j-1}(A+B) \leq \sigma_k(A) + \sigma_j(B).
$$

If $u = (u_1, \ldots, u_n)^T \in \mathbb{C}^n$ and $v = (v_1, \ldots, v_n)^T \in \mathbb{C}^n$, write $(u,v)$ for the vector $(u_1, \ldots, u_n, v_1, \ldots, v_n)^T \in \mathbb{C}^{n+1}$. The main result is the following theorem:

**Theorem.** Let $A$ be an $m \times n$ complex matrix and let $B$ be a $(m-s) \times (n-r)$ submatrix of $A$ obtained by eliminating the $s$ last rows and $r$ last columns of $A$. Let $\tau(x) := p(x,0)$ be the norm induced by $p$ in $\mathbb{C}^{m-s}$, and let $\chi(z) := q(z,0)$ be the norm induced by $q$ in $\mathbb{C}^{n-r}$. Then

$$
\sigma_k(B) \leq \sigma_{k+s+r}(A),
$$

$\sigma_k(B)$ being the $\tau, \chi$-s.v. of $B$.

Suppose that the conditions

$$
\tau(x) \leq \tau(x,y) \text{ for all } x, y \in \mathbb{C}^{m-s}, \quad y \in \mathbb{C}^s
$$

$$
\chi(z) \leq \chi(z,w) \text{ for all } z, w \in \mathbb{C}^{n-r}, \quad w \in \mathbb{C}^r
$$

are satisfied. Then

$$
\sigma_k(A) \geq \sigma_k(B).
$$

The Hölder norms verify the hypothesis of the above result. We say that a norm $\mu: \mathbb{C}^k \rightarrow \mathbb{R}$ is symmetric when $\mu(x_1, \ldots, x_k)$ is unchanged if $(x_1, \ldots, x_k)$ is replaced by $(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$, $\sigma$ being any permutation of $(1,2, \ldots, k)$. Note that if $p$ and $q$ are symmetric norms,
this Theorem holds for any submatrix $B$ of $A$, not necessarily that of the upper left corner.

REFERENCES


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