Multiscale finite element method combined with local preconditioning for solving Euler equations

Método de elementos finitos em multiescala combinado com pré-condicionamento local para resolver equações de Euler

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ABSTRACT
In this work we present a nonlinear multiscale finite element method combined with local preconditioning for solving compressible Euler equations in conservative variables. The formulations are based on the strategy of separating scales, in which it is the core of the variational multiscale (finite element) methodology. The subgrid scale space is defined using bubble functions that vanish on the boundary of the elements, allowing to use a local Schur complement to define the resolved scale problem. The resulting numerical procedure allows the fine scales to depend on time. The formulation proposed added artificial viscosity isotropically in all scales of the discretization. Due to the fact that, density-based schemes suffer with undesirable effects of low speed flow including low convergence rate and loss of accuracy, local preconditioning is applied to the set of equations in the continuous case. We evaluate the multiscale formulation with local preconditioning in the low Mach number comparing with the non-preconditioned case.

The experiments show that density-based schemes combined with local preconditioning yields good results.

Keywords: multiscale formulation, local preconditioning, Euler equations, conservative variables.

RESUMO
Neste trabalho apresentamos um método de elemento finito multiescala não linear combinado com pré-condicionamento local para resolver equações de Euler compressíveis em variáveis conservadoras. As formulações são baseadas na estratégia de separação de escalas, na qual é o núcleo da metodologia de multiescala variacional (elemento finito). O espaço de escala de subgrade é definido usando funções de bolha que desaparecem no limite dos elementos, permitindo usar um complemento de Schur local para definir o problema de escala resolvido. O procedimento numérico resultante permite que as escalas finas dependam do tempo. A formulação proposta adicionou viscosidade artificial isotropicamente em todas as escalas da discretização. Devido ao fato de que esquemas baseados em densidade sofrem com efeitos indesejáveis de fluxo de baixa velocidade, incluindo baixa taxa de convergência e perda de precisão, pré-condicionamento local é aplicado ao conjunto de equações no caso contínuo. Avaliamos a formulação multiescala com pré-condicionamento local no número Mach baixo em comparação com o caso não pré-condicionado. Os experimentos mostram que esquemas baseados em densidade combinados com pré-condicionamento local produzem bons resultados.

Palavras-chave: formulação em escala múltipla, pré-condicionamento local, equações de Euler, variáveis conservadoras.

1 INTRODUCTION

There are many challenges in developing numerical methods for solving problems from low to high speed compressible flows. As an example, flows at a low speed show an incompressible behavior, because the density variation is almost negligible. Numerical methods addressed for solving low speed are usually pressure-based, since the flow is approaching to the incompressibility. On the other hand, in transonic and supersonic regimes the numerical methods generally are density-based. It is known that density-based strategy to solve compressible flow suffers severe deficiencies when applied to very low Mach number problems, degrading convergence speeds, and impacting the efficiency and accuracy of the numerical formulations (LI; XIANG, 2013). In the low Mach number limit the system of Euler equations becomes stiff due to large disparity in the timescales (BASSI et al., 2009).

With some adjustments, numerical methods can handle the full spectrum of speeds, as well as situations where the density does not change. Local preconditioning or mass matrix preconditioning schemes have been proposed as a way to address this drawback using density-based method for low-Mach number flow, whose goal is to get an uniformization of the eigenvalues, smoothing the discrepancy of the time scales (COLIN; DENIAU; BOUSSUGE, 2011; GINARD; VÁZQUEZ; HOUZEAUX, 2016). Local preconditioning is applied to the set of continuous differential equations premultiplying the time derivative by a suitable preconditioning matrix. However, the original problem and the preconditioned one have different time evolution but the same steady-state solution. The application of these methodologies to unsteady problems requires the use of the "dual-time-stepping" technique (LOPEZ et al., 2012), in which the physical time derivative terms are treated as source and/or reactive terms.

In this work we use the Van Leer-Lee-Roe (VLR) preconditioner, proposed in (LEER; LEE; ROE, 1991) for Euler steady flow. The VLR preconditioner is symmetric and optimal, in the sense that it equalizes the eigenvalues of the problem for all Mach number regimes. We apply local preconditioning in the Euler equation and after that the continuous equations are discretized by a nonlinear multiscale viscosity method proposed in (BENTO et al., 2016).

2 GOVERNING EQUATIONS

We consider the two-dimensional compressible Euler equations for an ideal gas. The equations may be written in conservative variables without source terms as a system of conservation laws,

$$\frac{\partial}{\partial t}\mathrm{U}+\mathrm{F}(U)\cdot\nabla\mathrm{U}=0, \text{ in } \Omega\times(0,\mathrm{t}_f] \qquad (1)$$

Where:

t_f is a positive real number, representing the final time and Ω is a domain in $\mathbb{R}^2$,with boundary Γ,$U \subset \mathbb{R}^4$is the vector of conservative variables, and $F(U) \subset \mathbb{R}^{4x2}$,is the Euler flux vector.

Here:

$$U=\underbrace{\begin{bmatrix}U_1\\U_2\\U_3\\U_4\end{bmatrix}}_{\text{conservative variables}}=\underbrace{\rho\begin{bmatrix}1\\u\\v\\E\end{bmatrix}}_{\text{primitive variables}}, \qquad (2)$$

Where:

ρ is the fluid density, $\mathbf{u} = [u \quad v]^T$ is the velocity vector, ρE is the total energy, and E is the total specific energy. Others important physical quantities are the pressure p and the Mach number, $M = \|\mathbf{u}\|_2/c$, where $c = \sqrt{\gamma \frac{p}{\rho}}$ is the speed of sound, with $\gamma = \frac{c_p}{c_v}(\gamma>1)$ being the ratio of specific heats, and c_p and c_v are the coefficients of specific heat at constant pressure and volume, respectively. The system of equations (1) is closed by the equation of state for pressure

$$p=(\gamma-1)(\rho E-\frac{\rho}{2}\|u\|_2^2). \qquad (3)$$

Alternatively, Eq. (1) can be rewritten as in the quasi-linear form:

$$\frac{\partial}{\partial t}U+A_x\frac{\partial}{\partial x}U+A_y\frac{\partial}{\partial y}U=0, \text{ in } \Omega\times(0,t_f] \qquad (4)$$

Where:

$A_x = \frac{\partial F_x}{\partial U}$ and $A_y = \frac{\partial F_y}{\partial U}$ are the Jacobian matrices. Associated with Eq. (4) we have an appropriate set of boundary and initial conditions.

We assume the following boundary and initial conditions,

$$BU=Z, \text{on } \Gamma\times(0,t_f],$$
$$U(x,t)=U_0,$$

Where:

B denotes a general boundary operator, and **Z** and $\mathbf{U}_0$ are given functions.

3 MULTISCALE FINITE ELEMENT DISCRETIZATION

To define the finite element method, we consider a triangular partition T_h of the domain Ω into *nel* elements,

Where:

$$\Omega = \cup_{e=1}^{nel} \quad \Omega_e \text{with } \Omega_i \cap \Omega_j = \emptyset, \text{ for } i, j = 1, 2, \ldots, nel \text{ and } i \neq j.$$

In order to define the multiscale finite element methods, we introduce the function space V_{Zhb}, which is written as the direct sum,

$$V_{zh} \oplus V_b \tag{5}$$

Where the subspaces V_{zh} and V_{Zb} are given by:

$$V_{zh} = \{U_h \in [H^1(\Omega)]^4; U_h|_{\Omega_e} \in [\mathrm{P}_1(\Omega_e)]^4, BU_h = Z on \Gamma_D\} \tag{6}$$

$$V_b = \{U_b \in [H_0^1(\Omega)]^4; U_b|_{\Omega_e} \in [span(\psi_b)]^4, \forall \Omega_e \in T\} \tag{7}$$

With $\mathcal{P}_1(\Omega_e)$ representing the set of first order polynomials in Ω_e, $H^1(\Omega)$ denotes the Sobolev space of square-integrable functions whose first derivatives are also square-integrable, $H_0{}^1(\Omega)$ is a space of function in $H^1(\Omega)$ that vanish at the boundary of Ω, and$\psi_b$is a bubble function. The space V_{zh} represents the resolved (coarse) scale space whereas V_b stands for the subgrid (fine) scale space. The space defined in (5) with $\boldsymbol{Z} = \mathbf{0}$ on Γ_D is written as $V_{0hb} = V_{0h} \oplus V_b$.

The nonlinear multiscale method used here can be found in (BENTO et al., 2016), which is referred to as NMV (Nonlinear Multiscale Viscosity) method. Those method adds artificial viscosity isotropically in all scales of the discretization, where the amount of artificial viscosity is given by the YZβ shock-capturing viscosity parameter, as described in (TEZDUYAR; SENGA, 2006). The NMV method for the Euler equation consists of finding $\mathbf{U}_{hb} = \mathbf{U}_h + \mathbf{U}_b \in V_{Zhb}$ with $\mathbf{U}_h \in V_{Zh}$, $\mathbf{U}_b \in V_b$ such that:

$$\int_{\Omega} W_{hb} \cdot (\frac{\partial}{\partial t} U_{hb} + A_x^h \frac{\partial}{\partial x} U_{hb} + A_y^h \frac{\partial}{\partial y} U_{hb}) d\Omega + \sum_{e=1}^{nel} \int_{\Omega_e} \delta_h (U_h)(\frac{\partial}{\partial x} W_{hb} \cdot \frac{\partial}{\partial x} U_{hb} + \frac{\partial}{\partial y} W_{hb} \cdot \frac{\partial}{\partial y} U_{hb}) d\Omega = 0, \quad \text{for all } W_{0hb} \in \tag{8}$$

Where:

$\mathbf{W}_{hb} = \mathbf{W}_h + \mathbf{W}_b \in V_{0hb}$ with $\mathbf{W}_h \in V_{0h}$, $\mathbf{W}_b \in V_b$ and the amount of artificial viscosity, $\delta_h(\mathbf{U}_h)$, is calculated on the element-level by using the **YZβ** shock-capturing viscosity parameter (TEZDUYAR; SENGA, 2006),

$$\delta_h(U_h) = \|Y^{-1}R(U_h)\|(\sum_{i=1}^{2} \|Y^{-1}\frac{\partial U_h}{\partial x_i}\|_2^2)^{\frac{\beta}{2}-1}\|Y^{-1}U_h\|^{1-\beta}h^{\beta}, \tag{9}$$

Where:

$$R(U_h) = \frac{\partial}{\partial t}U_h + A_x^h\frac{\partial}{\partial x}U_h + A_y^h\frac{\partial}{\partial y}U_h \tag{10}$$

Is the residue of the problem on Ω_e , **Y** is a diagonal matrix constructed from the reference values of the components of **U**, given by:

$$Y = diag((U_1)_{ref}, (U_2)_{ref}, (U_3)_{ref}, (U_4)_{ref}), \tag{11}$$

h is the local length scale defined as follow$h = \sum_{a=1}^{3} |j \cdot \nabla N_a|$, **j** is a unit vector defined as **j** = $\nabla\rho/\|\nabla\rho\|_2$ and N_a is the interpolation function associated with node a. It is important to note that, the local length h is defined automatically taking into account the directions of high gradients and spatial discretization domain.

4 LOCAL PRECONDITIONING FOR THE EULER EQUATION

A system of differential algebraic equations (DAE) is *stiff* due to the large disparity in their timescales (KNOLL; KEYES, 2004). In the same way, the system of the compressible Euler equations is also stiff if it covers a wide range of timescales (BASSI et al., 2009). In the context of conservation laws, precisely, compressible Euler equations, the *stiffness* is measured through of the disparities related to the characteristic propagation speeds of the system, that are given by the eigenvalues of the Euler flux Jacobian (LOPEZ et al., 2012; GINARD; VÁZQUEZ; HOUZEAUX, 2016). Stiffness causes convergence problems regardless of the discretization method utilized, and it is measured (for one and two dimensions) by the so called condition number (GINARD; VÁZQUEZ; HOUZEAUX, 2016),

$$\kappa = \begin{cases} \frac{M+1}{M}, if M < 1/2; \\ \frac{M+1}{1-M}, if\ 1/2 \leqslant M \leqslant 1; \\ \frac{M+1}{M-1}, if M > 1. \end{cases}$$

When M → 0 or M → 1, the condition number κ → ∞ and the problem (4) becomes stiff. A strategy to reduce the disparity between the eigenvalues of the problem (4) and consequently decrease the condition number is the use of local preconditioning or preconditioning mass matrix schemes.

Local preconditioning or preconditioning mass matrix scheme consists of premultiplying the time derivatives by a properly matrix in order to uniform the eigenvalues, smoothing the discrepancy of the different time scales. It is applied to the set of continuous equations before any discretization is done. Denoting by ***P*** the (nonsingular) preconditioning matrix, then the system of equations (4) after the preconditioning process reads

$$P^{-1}\frac{\partial}{\partial t}\text{U}+\text{A}_x\frac{\partial}{\partial x}\text{U}+\text{A}_y\frac{\partial}{\partial y}\text{U}=0 \Rightarrow \frac{\partial}{\partial t}\text{U}+\text{PA}_x\frac{\partial}{\partial x}\text{U}+\text{PA}_y\frac{\partial}{\partial y}\text{U}=0, \text{ in } \Omega\times(0,\text{t}_f] \quad (12)$$

Even the solution evolves in time differently from that of the original problem, the time derivatives go to zero and (4) and (12) will share the same steady-state solution. We described below a local preconditioner technique: the VLR.

4.1 VAN LEER-LEE-ROE PRECONDITIONER

The Van Leer-Lee-Roe's (VLR) preconditioner for the Euler equations was introduced in (LEER; LEE; ROE, 1991) using the symmetrizing variables with the streamline coordinates. The resulting preconditioning matrix satisfies some properties as *optimality*, *accuracy*, *continuity at the sonic point*, *preservation of the decoupled entropy equation*, *positivity*, and *symmetrizability.* The VLR preconditioner is considered optimal because it equalizes the eigenvalues of the system for all Mach numbers (COLIN;

DENIAU; BOUSSUGE, 2011). An explicit expression for the VLR preconditioner in conservative variables (GINARD; VÁZQUEZ; HOUZEAUX, 2016) is

$$P_{VLR} = \begin{bmatrix} a_1 & a_2 u & a_2 v & a_3 \\ a_4 u & a_5 uu + \tau & a_5 uv & a_6 u \\ a_4 v & a_5 uv & a_5 vv + \tau & a_6 v \\ a_7 & a_8 u & a_8 v & a_9 \end{bmatrix} \tag{13}$$

All coefficients definitions of the Eq. (13) can be found in (GINARD; VÁZQUEZ; HOUZEAUX, 2016).

5 NUMERICAL RESULTS

The flow over an airfoil is an interesting problem to examine the numerical instability coming from Mach numbers variations, that occurs in the Euler equations. This section shows the results of a flow passing through a NACA 0012 airfoil at an angle of attack of 0º and inflow Mach number from 0.01 up to 0.5.

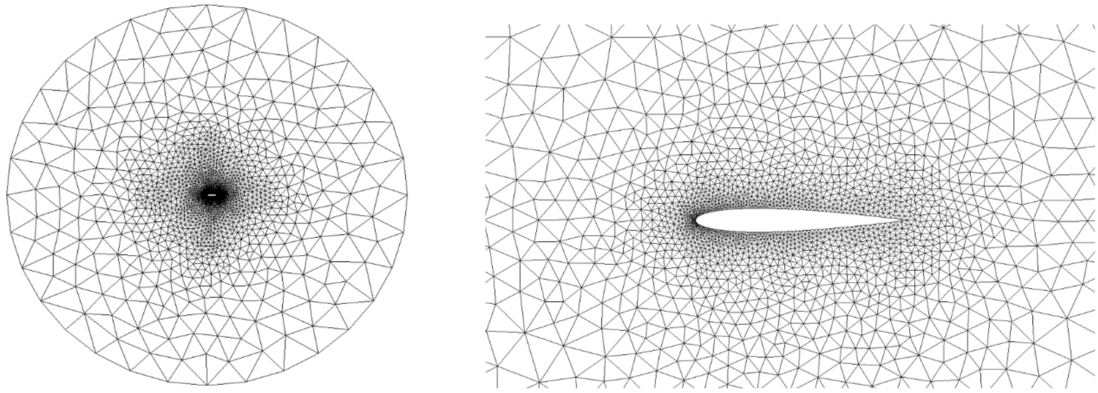
An unstructured triangular mesh of 5,606 elements and 2,886 nodes was used for the simulation, in the computational domain given by a circle centered at the (0,0) with radius 15 (Fig. 1). A distance is taken ahead the leading edge of the airfoil to the inflow and outflow boundaries in order to avoid numerical instabilities of reflecting waves (GINARD; VÁZQUEZ; HOUZEAUX, 2016). The inflow data is set up by:

$$inflow \begin{cases} \rho = 1.0 \\ u = 1.0 \\ v = 0.0' \\ T = 1.0 \end{cases} \tag{14}$$

Where T is the temperature. As in (GINARD; VÁZQUEZ; HOUZEAUX, 2016), the coefficients c_v and c_p are set to obtain the desired inflow Mach numbers. The numerical solution is advanced in time by the predictor-corrector algorithm adapted for the multiscale framework in (BENTO et al., 2016) for the Euler equations. A restarted version of the GMRES solver is used to find the solution of the linearized system in each nonlinear and time iterations. The GMRES parameters are: 30 vectors to the restart process and tolerance equals 10^{-5}. The time-step size is 10^{-3} and the simulation runs until $t_f = 20.0$ (20,000 steps), and 3 fixed nonlinear iterations. For the reference values used in Eq. (11), we consider the inflow data given by Eq. (14). In this example, we evaluate the

VLR preconditioner comparing it with the non-preconditioned (NP) case. The tests are carried out with the intention of analyzing accuracy issues, specially in the incompressibility limit. Due to flow at a low speed to demonstrate an incompressible behavior, i.e., density variation is almost negligible, we use the pressure contour to analyze this experiments.

Figure 1: Unstructured triangular mesh of 5,606 elements and 2,886 nodes.



(a) NACA 0012 mesh (b) Detail of the NACA 0012 airfoil

Source: Authors.

Figures 2-5 show the pressure contours for different inflow Mach numbers. The solutions are in good agreement with the solutions found in (GINARD; VÁZQUEZ; HOUZEAUX, 2016), where it is used a first order forward finite difference scheme, with the time step satisfying CFL condition. As happened in the work of (GINARD; VÁZQUEZ; HOUZEAUX, 2016) for the non-preconditioned case, our multiscale methodology does not work in the low Mach number limit, i.e., when the Mach number approaches to zero. We can see in Fig. 2-4, when $M \leq 0.3$, the flow at a low speed demonstrates an incompressible behavior, and methods based on conservative variables suffer with undesirable effects (LI; XIANG, 2013). The numerical solutions in the low Mach number limit are completely oscillatory, e.g. Fig. 2(a). On the other hand, the NMV method local preconditioned is able to solve problems with an incompressible behavior, as shown in Fig. 2(b), 3(b), and 4(b). It is worth pointing out that the non-preconditioned NMV becomes more stable as the Mach number increases and solutions obtained from $M = 0.3$ are comparable with the preconditioned case.

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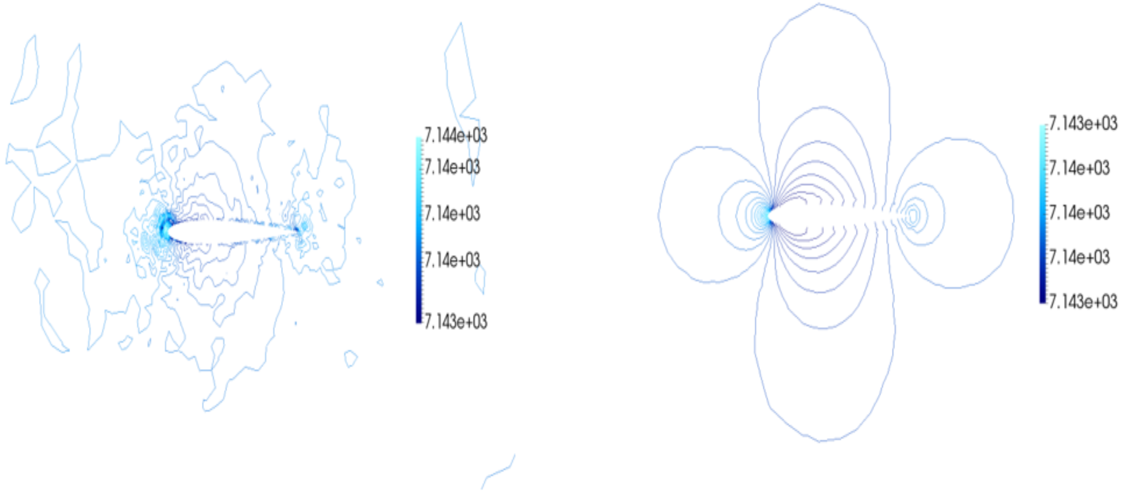
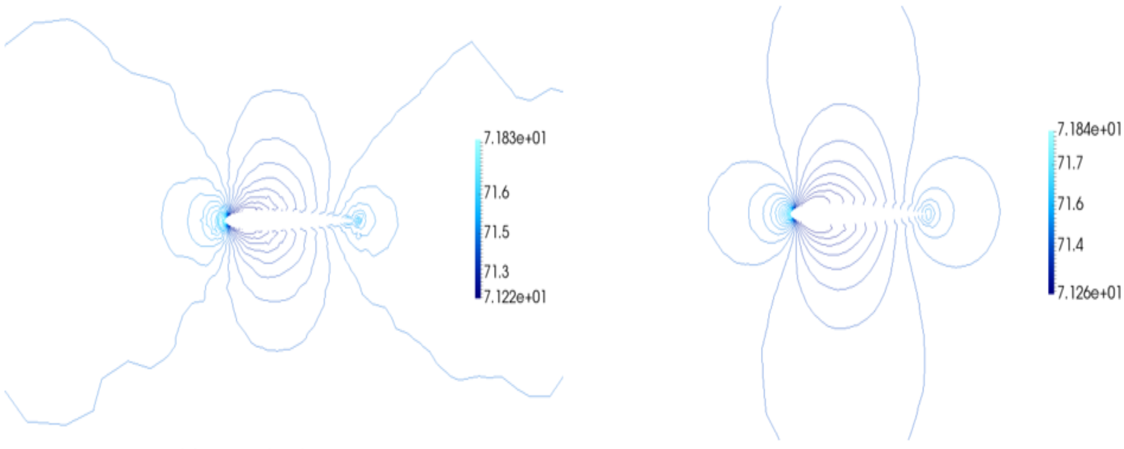
Figure 2: NACA 0012: Pressure contours for $M = 0.01$ at the inflow.



(a) NMV(NP) (b) NMV(VLR)

Source: Authors.

Figure 3: NACA 0012: Pressure contours for $M = 0.1$ at the inflow.



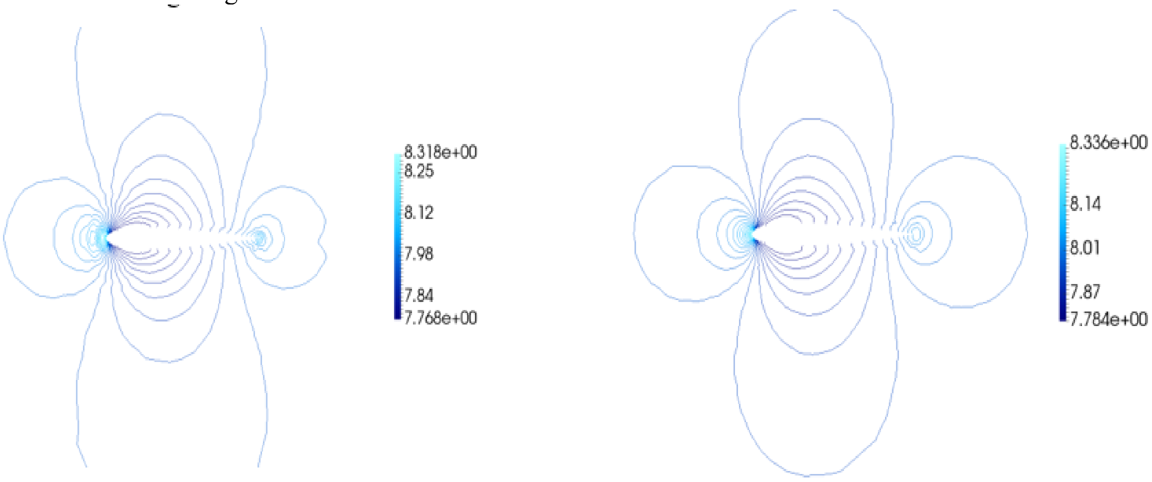
(a) NMV(NP) (b) NMV(VLR)

Source: Authors.

6 CONCLUSIONS

The NMV method combined with VLR preconditioner was applied in the NACA 0012 airfoil problem for the incompressible flow limit. We simulate the flow over the NACA 0012 airfoil under various regimes of inflow Mach numbers: 0.01; 0.1; 0.3; 0.5. The solutions obtained with the NMV without local preconditioning are completely oscillatory in the low Mach number limit (e.g. for $M = 0.01$), showing that methods based on conservative variables fail in this case. On the other hand, the NMV method combined with local preconditioning presents good results, as shown in the numerical results section.

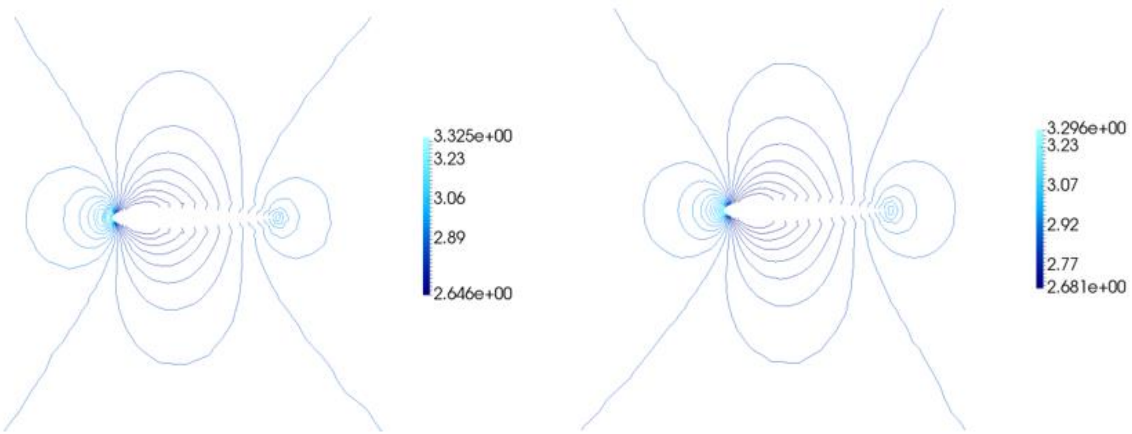
Figure 4: NACA 0012: Pressure contours for $M = 0.3$ at the inflow.



(a) NMV(NP) (b) NMV(VLR)

Source: Authors.

Figure 5: NACA 0012: Pressure contours for $M = 0.5$ at the inflow.



(a) NMV(NP) (b) NMV(VLR)

Source: Authors.

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