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**ENDOGENOUS FINANCIAL CONSTRAINTS:
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FLUCTUATIONS**

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ENDOGENOUS FINANCIAL CONSTRAINTS: PERSISTENCE AND INTEREST RATE FLUCTUATIONS

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Resumen

Este trabajo analiza la dinámica de las firmas cuando los empresarios tienen una capacidad limitada de cumplir los contratos financieros. El contrato restringido óptimo es caracterizado bajo esta imperfección en la presencia de fluctuaciones de productividad y de tasa de interés. Se muestra que el contrato óptimo implica que las fluctuaciones de productividad y tasa de interés despliegan efectos amplificados sobre las dinámicas de las firmas, más allá de lo predicho en el caso de un perfecto cumplimiento de contratos. Más aun, la persistencia de estas fluctuaciones es mayor cuando los problemas de cumplimiento de contratos son más severos. Estos resultados pueden estar relacionados con el hecho de que países con un mayor grado de cumplimiento de contratos poseen un desarrollo financiero más profundo y un mejor desempeño económico.

Abstract

This article analyzes firm dynamics when the entrepreneurs have limited capacity to comply with their financial contracts. We characterize the optimal constrained contract under this imperfection in the presence of productivity and interest rate fluctuations. We show that under the optimal contract, productivity and interest rate fluctuations have amplified effects on the firms' dynamics, beyond what would be predicted in the case of perfect enforceability. Moreover, the persistence of these fluctuations is higher when the compliance problems are more severe. These findings can be related to the fact that countries with better contract enforceability display deeper financial development and better economic performance.

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1 Introduction

The impact of financial constraints on the macroeconomic dynamics has been studied deeply in the last twenty years.¹ There have also been many studies that emphasize financial constraints faced by consumers so as to improve the features of consumption based asset pricing.² At the microeconomic level, the implications of financial constraints over investment have been largely analyzed empirically. In this context, the financial constraints manifests as an external premium over internal financing that is modelled exogenously. This premium constrains investment decisions and may imply a higher sensitivity of investment to cash flow at the firm level.³ More recently, the effect of this external premium on cross-section asset pricing has been considered by Gomes et al. (2003).

These models tend to treat the external financial premium at the microeconomic level as exogenous, which is quite arbitrary. One exception, however, is the work of Albuquerque and Hopenhayn (2004) which proposes a model where the financing constraints are endogenously derived from limited enforceability problems. In this way, borrowing constraints and firm size dynamics are jointly determined.⁴ In this paper, the authors analyze the implications of productivity fluctuations in the firm dynamics using the optimal contract. This is important because the financial arrangements that are considered in most of the literature of propagation mechanism with borrowing constraints are never intertemporally optimal, meaning that contracts provisions are not contingent on all public information. One exception is the work of Cooley, Marimon

¹The most influential works in this area are Bernanke and Gertler (1989), Scheinkman and Weiss (1986), Carlstrom and Fuerst (1997) and Kiyotaki and Moore (1997).

²Examples of this kind are Mankiw (1986), Constatanides and Duffie (1996), Alvarez and Jermann (2000) and Lustig (2001).

³The cash flow sensitivity of firm investment decisions and its link to financial constraints has been studied by Fazzari et al. (1988), Hoshi et al. (1991), Himmelberg and Gilchrist (1995, 1998) and Gomes (2001).

⁴In other work Clementi and Hopenhayn (2002) characterize the optimal contract when there is asymmetric information between entrepreneur and lender in a dynamic setting as well.

and Quadrini (2004) which embeds the model of Albuquerque and Hopenhayn (2004) in a general equilibrium framework.

Following the work of Albuquerque and Hopenhayn (2004), this article analyzes the firm dynamics when the entrepreneurs have limited enforcement to the financial contracts. The optimal contract is characterized and the persistence of the productivity and interest rate movements is analyzed. One interesting result we obtain is that the persistence of the shocks increases in economies with lower enforceability.

In the economic environment, entrepreneurs have to borrow from a lender to finance the initial investment required to start the firm because they do not have funds. Entrepreneurs use their firm cash flows each period to repay back they initial debt. However, they also need to borrow to finance the capital advancement required each period to run the firm. Therefore, the cash flows of each period depend on the capital advanced to the entrepreneur. Additionally, the entrepreneur has limited commitment to the contract which is given by an outside option that she can gets if she breaches the contract and the firm disappears. In this context, a bigger outside option implies a lower level of contract enforceability.

Initially, the value of the firm entitled by the optimal contract to the entrepreneur is low and consequently she has a high incentive to default the contract. This implies that capital advancement prescribed by the optimal contract is below its efficient level. This, in turn, translates into young firms being smaller and constrained. Hence, the marginal value of capital advancement is above the opportunistic cost given by the interest rate. Over time, the entrepreneur will pay the initial debt acquired until a point where the value of the firm entitled to her by the optimal contract is big enough such that she does not have incentive to default on the contract.

The persistence of the productivity and interest rate is due to the fact that during the phase where the entrepreneur is financially constrained, all excess of cash flow goes to pay the initial debt acquired when starting the firm. Comparing two paths of productivity, we can see that the one with higher productivity for at least one period will pay back the initial debt faster and the firm will reach the unconstrained level sooner. Similarly, a path with higher interest rate will pay back the initial debt slowly and the firm will reach the unconstrained level later. Moreover, this persistence of productivity and interest fluctuations is bigger and lasts longer in an environment where the enforceability of contract is more imperfect.

These findings posit an analytic framework to understand why economies with low levels of systematic enforceability have less developed financial markets and economic performance. In this model, low enforceability will imply an inefficiency in terms of the size and external finance for young firms. This link among institutional factors (e.g., enforceability of contracts), financial markets development and economic performance across countries has been vastly analyzed empirically. La Porta et al. (1997) using cross-country regressions show that countries with better investor protection have bigger and broader equity and debt markets. Similarly, Knack and Keefer (1995) analyzing a cross-country sample concluded that institutions that protect property rights are crucial for economic growth and higher investment rates. Levine (1999) provided cross-country evidence supporting the fact that countries with more developed legal and regulatory system have better developed financial intermediaries, and consequently grow faster.

The result that economies with lower enforceability of contracts display less external finance and investment rates can be consistent with the pattern shown in figure 1. This figure relates the Investment-GDP ratio to an index of enforceability of contracts for a cross-section of countries. This index is constructed by La Porta et al. (1998) and measures whether the country's laws are efficiently and impartially enforced and

whether governments tend to change the nature of contracts ex post. Higher values of this index indicate greater efficiency in enforcing contracts. The Investment-GDP ratio is computed from the Penn World Table 6.1 as the average for the period 1980-2000. As it can be observed from the figure, there is a positive relationship between the aggregate investment rates and the degree of contract enforceability.

The other element implied by the financial constraints is the premium of the external funds over the interest rate. This model emphasizes that this spread is bigger for younger firms. Moreover, this spread will be higher and lasts longer for firms in economies with lower contract enforceability. This implication is supported by figure 2 which shows the negative association between the lending spread and the level of contract enforceability in a cross-section of countries. We use the same index of enforceability displayed in figure 1. The lending spread is calculated from the International Monetary Fund Financial Statistics as the difference between the lending rate and deposit rate in each country for the period 1980-2000. Hence, countries with better contract enforceability in average have lower lending spread.

The remainder of this article is organized as follows. The model is explained in Section 2. In Section 3, we characterize the main properties of the optimal contract. In particular, we describe the optimal capital advancement and repayment policies, and the evolution of the value entitled to the entrepreneur over time as implied by the contract. The implications for firm dynamics and persistence of shocks are described in section 4. Section 5 states final remarks. Two appendixes are in the end of the article. Appendix A contains the proofs of the main results and appendix B describes shortly the parameters considered for the numerical simulations.

2 Model

The model is built on the work of Albuquerque and Hopenhayn (2004) to include interest rate fluctuations and analyze the link between persistence and contract enforceability. Time is discrete and the horizon is infinite. At $t = 0$ an entrepreneur has an investment opportunity of starting a new firm which requires an initial investment of I_0 . After this initial investment, the firm has a stream of revenues $R(k, s)$ in each period where k is the capital input and $s \in \mathcal{S} \subset \mathbb{R}$ is a productivity shock. The entrepreneur has limited liability and starts with zero wealth requiring a lender to finance the initial investment and advancement of capital every period. Entrepreneur and lender are both risk neutral and discount the flows between the beginning of period and the end of it with same interest rate $r > 0$.

The productivity shock s follows a Markov process with a cumulative transition probability function given by $\Pr(s_{t+1} = s' | s_t = s) = F(s' | s)$. Also, the interest rate $r \in \Lambda \subset \mathbb{R}$ follows a Markov process with cumulative transition of probability given by $\Pr(r_{t+1} = r' | r_t = r) = G(r' | r)$.

Timing of events is as follows. First, the productivity s and interest rate r are realized, then the capital input is purchased, sales takes place, and revenues $R(k, s)$ are collected.⁵ The entrepreneur has lack of commitment to the contracts, but she is the only one able to run the firm. This is the root of the endogenous financial constraints. In contrast, the lender has full commitment to the contract and access to a perfect capital markets.

If the entrepreneur defaults on the contract she can get an outside option $O(k, s, r)$.

⁵Albuquerque and Hopenhayn (2004) also include the possibility of liquidation just after the realization of productivity. Here is not included this possibility to focus the analysis on the persistence of interest rate and productivity shocks and its connection with the level enforceability. However, when the liquidation value is low the decision of liquidation is never optimal in the contract.

This outside option depends on the capital input k , the productivity s and interest rate r . Additionally, the outside option will depend on a cost of defaulting. This cost of defaulting is connected to the level of enforceability guaranteed in the economy. The lower the level of enforceability of contracts the cheaper the cost of defaulting. Hence, we consider economies with a more imperfect level of financial contract enforceability as economies with a cost of defaulting lower and consequently a higher outside option for the entrepreneur.

In this context, a long term contract specifies a contingent capital advancements k_t from lender to the firm that take place at the beginning of the period and cash flow distribution consisting of a dividend d_t and a payments to the lender $R(k_t, s_t) - d_t$ which take place at the end of the period. Since the entrepreneur has no additional funds the limited liability implies that $d_t \geq 0$. A history at t is $h_t = \{k_n, d_n, s_{n+1}, r_{n+1}\}_{n=0}^{t-1}$. \mathcal{H} is the set of all possible histories.

Definition 1 *A feasible contract is a mapping $\mathcal{C} : \mathcal{H} \rightarrow \mathbb{R}_+^2$ such that $\forall h_t \in \mathcal{H}$, $\mathcal{C}(h_t) = (k_t, d_t)$.*

At time zero, a competitive financial intermediary or lender offers a long term contract to the entrepreneur. If the entrepreneur accepts the contract, the lender pays for the initial investment I_0 and makes the advancement of capital as described by the contract as long as the entrepreneur meets the payments stipulated in the contract and there is no default. If the entrepreneur accomplishes the terms of the contract, the firms remains active. Otherwise, the firm is terminated.

2.1 Contract with Perfect Enforceability

In the case of perfect enforceability, the entrepreneur can credibly commit to long term contracts without any additional conditions. The presence of many competitive lenders and the fact that lender and entrepreneur discount the cash flow at the same rate guarantee that a long term contract will maximize the total expected discounted profits.

Let denote the unconstrained profit function conditional in the current level of productivity and interest as:⁶

$$\pi(s, r) = \max_{k \geq 0} \{R(k, s) - (1 + r)k\} \quad (1)$$

To guarantee that $\pi(s, r)$ exists we consider the following assumption:

Assumption 1

1. $R(k, s)$ is increasing in both arguments
2. $R(k, s)$ is continuous in k
3. For each $s \in \mathcal{S}$, $R(k, s) - (1 + r)k$ is quasiconcave in k and has a maximum
4. The function $R(k, s) - (1 + r)k$ is bounded

To obtain the total surplus of the match in this unconstrained optimal setting we use the following recursive equation:

$$\tilde{W}(s, r) = \frac{1}{1 + r} \{ \pi(s, r) + \mathbb{E}[\tilde{W}(s', r') | (s, r)] \} \quad (2)$$

⁶From the specification below is clear that since the capital is advanced at the beginning of the period and the revenues are collected at the end of the period, the opportunity cost of the capital k in the project is $(1 + r)$.

With perfect capital markets Modigliani and Miller (1958) showed that the financial structures of the firms is irrelevant for their value. As it is expected, this model with perfect enforceability obtains the same conclusion. When the Modigliani-Miller theorem applies, the effects of productivity and interest fluctuations on the firm dynamics are independent of its financial structure. As we will see below this irrelevance conclusion breaks down when there is limited contract enforceability.

2.2 Contracts with Limited Enforceability

When the entrepreneur has the choice to default, one key object is the outside option which will determinate the severity of the financial constraint. In principle, we just assume a generic form for the outside option $O(k, s, r)$. In this way, the model can nest alternative form for the outside investment opportunities and cost of defaulting.

To simplify the characterization of the optimal contract we consider the following assumption:

Assumption 2

1. $O(0, s, r) = 0$.
2. O is a continuous function
3. O is non-decreasing in k and s .

Assumption 2.1 helps to simplify the derivation of the optimal constrained contract.

A long term contract specifies a history dependent contingent advances of capital and a dividend distribution. The contract implicitly defines a value V_t for the entrepreneur and long term debt B_t for the lender. The total value of the firm is $W_t = V_t + B_t$. B_t is labelled as the long term debt and k_t as the short term debt. Let $V_{t+1}(s', r')$ denote

the continuation entrepreneur value at the beginning of period $t + 1$ after the history $h_{t+1} = (h_t, k_t, d_t, s', r')$. The enforcement constraint then can be written as:

$$O(k_t, s_t, r_t) \leq \frac{1}{1+r} (d_t + \mathbb{E}_t[V_{t+1}(s', r') | (s_t, r_t)]) \quad (3)$$

A feasible contract is enforceable if the triplet $(k_t, d_t, V_{t+1}(s', r'))$ satisfies (3) after any history h_t . Now we can see that the recursive formulation is more apparent because the continuation contract after any history is also enforceable. In other words, let $\Omega(s, r) \subseteq \mathbb{R}^2$ be the set of values for the long term debt B and entrepreneur value V such that there exists an enforceable contract with initial values $B_0 = B$ and $V_0 = V$ when the initial productivity and interest rate are s and r , respectively. The set of optimal contracts gives (B_t, V_t) in the frontier of $\Omega(s_t, r_t)$ for all t . To close what point of the frontier we are going to be looking at, we assume the existence of many competitive lenders. This condition gives all bargaining power to the entrepreneur and the lender will break even. We summarize this discussion with the following definition:

Definition 2 *An equilibrium contract $\mathcal{C}(\cdot)$ is feasible, enforceable, and gives the highest possible value to the borrower consistent with the lender breaking even if $V_0 = \sup \{V : (B, V) \in \Omega(s_0, r_0), B \geq I_0\}$ when the initial productivity and interest rate are s_0 and r_0 , respectively.*

In other words, the optimal contract that we will describe in the next section it will be the equilibrium contract that prevails in a decentralized economy with many competitive lenders.

3 Characterization of the Optimal Contract

The purpose of this section is to give an analytical solution to the optimal contract. This characterization will allow us to describe the role of the endogenous financial constraints

in the propagation of interest rate and productivity fluctuations in the firm dynamics. It can be inferred from last section it is easier to use a recursive formulation to describe the optimal contract.

To obtain the recursive specification note that the values $V_{t+1}(s', r')$ for future contingencies provide a summary for the future contract and together with (k_t, d_t, s_t, r_t) are sufficient to verify the enforcement constraint (3). Also, V_t is a state variable that summarizes the history of firm value that the contract gives to the entrepreneur. This is the part of the total value of the firm that is entitled to the entrepreneur by the contract.⁷ In every period, given initial values $V_t = V$, $s_t = s$, $r_t = r$, the contract specifies a pair (k, d) and a continuation values $V(s', r')$. In turn, the continuation values $V(s', r')$ will determine investment and dividends in the future. The entrepreneur value has to be the result of the dividend and continuation share deducted from the optimal contract:

$$V = \frac{1}{1+r} (d + \mathbb{E}[V(s', r')|(s, r)]) \quad (4)$$

This is also called the ‘*promise-keeping constraint*’.

By Assumption 2.1 we can conclude that $V(s', r')$ can be supported by an enforceable contract if and only if $V(s', r') \geq 0$. This is a domain restriction that simplifies the determination of the optimal contract. Also, the limited liability condition prevents negative dividends and therefore, we can write the enforcement constraint as:

$$O(k, s, r) \leq V \quad (5)$$

Moreover, we can rewrite the limited liability condition ($d \geq 0$) as:

⁷We also refer V as the entrepreneur value or (inside) equity value that the contract implicitly defines.

$$\frac{1}{1+r} \mathbb{E}[V(s', r') | (s, r)] \leq V \quad (6)$$

Since the total value of the firm is the sum of the long term debt (B_t) and entrepreneur value (V_t), maximizing the long term value given a level of entrepreneur value is equivalent to maximizing the total value of the firm given that level of entrepreneur value. Hence, the optimal debt contract maximize the total value of the firm W given V , s and r . In other words, the total value of the firm in the optimal contract is a function $W(V, s, r)$ which represents the total present value of the firm starting with entrepreneur value V , productivity s , and interest rate r . Therefore, the dynamic programming problem can be written as:

$$W(V, s, r) = \max_{k, V(s', r') \geq 0} \frac{[R(k, s) - (1+r)k + \mathbb{E}[W(V(s', r'), s', r') | (s, r)]]}{1+r} \quad (7)$$

s.t. (5) and (6)

From above, we can see that the static decision of k can separated of the intertemporal decision of $V(s', r')$ in (7). Hence, we can write the following static maximization problem:

$$\Pi(V, s, r) = \max_{k \geq 0} \{R(k, s) - (1+r)k\} \quad (8)$$

s.t. (5)

Denote $\tilde{k}(s, r) = \inf\{k : R(k, s) - (1+r)k = \pi(s, r)\}$ and $V^u(s, r) = O(\tilde{k}(s, r), s, r)$. This last term is the smallest continuation entrepreneur value that is compatible with static profit maximization. Thus, if $V < V^u(s, r)$ the enforcement constraint (5) is binding and $\Pi(V, s, r) < \pi(s, r)$. In the other case when $V \geq V^u(s, r)$, the enforcement constraint is not binding and $\Pi(V, s, r) = \pi(s, r)$. We make the following assumption

regarding $V^u(s, r)$:

Assumption 3 *The function V^u is bounded.*

Now the problem in (7) can be written as:

$$W(V, s, r) = \max_{V(s', r') \geq 0} \frac{[\Pi(V, s, r) + \mathbb{E}[W(V(s', r'), s', r')|(s, r)]]}{1 + r} \quad (9)$$

s.t. (6)

Result 1 *Under some conditions on the first and second derivatives of the functions $R(\cdot)$ and $O(\cdot)$, $\Pi(\cdot)$ have the following properties:*

1. Π is continuous, uniformly bounded
2. Π strictly increasing in V for $V < V^u(s, r)$
3. Π is concave in V , and strictly concave if $V < V^u(s, r)$

Proof: See appendix A

As we concluded above when $V \geq V^u(s, r)$, the static problem of choosing k is equivalent to the unconstrained efficient. However, this may not ensure that the total present value of the firm $W(V, s, r) = \tilde{W}(s, r)$ because the entrepreneur value V must also guarantee that the enforcement constraint will not bind in the future. To see this, suppose that $V^u(s, r) < V < \frac{1}{1+r}\mathbb{E}[V^u(s', r')|(s, r)]$ and let $V(s', r')$ be the solution to (9) starting from (V, s, r) . Thus, it must be the case that $V^u(s', r') > V(s', r')$ for a subset of the space state of non-zero measure. Hence, the unconstrained profit maximization cannot be guaranteed in some states in the next period.

Let $V^n(s, r)$ denote the minimum level of current initial value for the entrepreneur that is needed to guarantee that the enforcement constraint will not bind for at least n periods, including the current one, when the state is (s, r) . A recursive formulation for this can be expressed as:

$$V^n(s, r) = \max\{V^u(s, r), \frac{1}{1+r}\mathbb{E}[V^{n-1}(s', r')|(s, r)]\} \quad (10)$$

where $V^0(s, r) = 0$.

Define $\tilde{V}(s, r) = \lim_{n \rightarrow \infty} V^n(s, r)$. Since $V^n(s, r)$ is an increasing sequence and uniformly bounded, the limit exists. Moreover, using the Lebesgue's dominated convergence theorem, it follows that $\forall (s, r)$:

$$\tilde{V}(s, r) = \max\{V^u(s, r), \frac{1}{1+r}\mathbb{E}[\tilde{V}(s', r')|(s, r)]\} \quad (11)$$

Result 2

1. $W(V, s, r)$ is weakly increasing in V
2. $\forall V \geq \tilde{V}(s, r)$, $W(V, s, r) = \tilde{W}(V, s, r)$
3. $\forall V < \tilde{V}(s, r)$, $W(V, s, r) < \tilde{W}(s, r)$

Proof: See Appendix A

Result 3 W is strictly concave in V if $W(V, s, r) < \tilde{W}(s, r)$.⁸

Proof: See Appendix A

Result 4 Suppose $W(V, s, r)$ is concave in V for all (s, r) . Then if $V_1 < \tilde{V}(s, r)$ it follows that $W(V_2, s, r) < W(V_1, s, r)$ for all $V_2 < V_1$

⁸Weak concavity of W in V is obtained directly from the fact that Π is weakly concave in V using dynamic programming arguments.

Proof: See Appendix A

Result 5 If $V_t(1 + r_t) < \mathbb{E}_t[\tilde{V}(s_{t+1}, r_{t+1})|(s_t, r_t)]$ then the optimal contract requires that $V_t(1 + r_t) = \mathbb{E}_t[V_{t+1}(s_{t+1}, r_{t+1})|(s_t, r_t)]$ so that no dividends are distributed.⁹

Proof: See Appendix A

The last four results establish some basic properties of the optimal constrained value function. This will be the prevailing total value of the firm when there exists many competitive lenders. A numerical simulation confirms these properties of W .¹⁰ The shape of W for this numerical example is showed in figure 3 and 4. These figures highlight that the value function W is increasing and concave in V as long as V is below \tilde{V} . For values of V above \tilde{V} the value function is horizontal because the firm is no longer constrained. Also, we can see that W is decreasing in the interest rate (r) and increasing in the productivity level (s).

The monotonicity of W on V shows that the Modigliani and Miller Theorem fails with limited contract enforceability because highlights a tradeoff between the value of the firm (W) and the financial structure (V). The concavity of W on V implies that the magnitude of this tradeoff is less significative as the entrepreneur value on the firm (V) increases. Finally, when the value entitled to the entrepreneur is high enough ($V \geq \tilde{V}(s, r)$) this tradeoff disappears.

4 Firm Dynamics

As opposed to a model that takes the premium of the external financing as exogenous, this model offer a crucial link between borrowing constraints and firms dynamics. The

⁹The limited liability condition is a lower bound in the dividends. Hence, we can have a lower bound bigger than zero and the firm would distribute dividends even when it was constrained.

¹⁰The explanations of the parameters used in the numerical simulation is described in Appendix B.

premium over external financing will depend on the entire history of shocks hitting the firm. In this context, this section addresses three main issues. First, it analyzes how on average the firms grows over time as they are constrained. Second, it states conditions that imply persistence of the productivity and interest fluctuations over time. Third, it considers how the persistence can increase as the enforceability problems become more severe.

4.1 Age Effects

The following result states that on average the firm increases over time when it is constrained. Intuitively, when the firm is constrained the limited liability will be binding and the entrepreneur value V will increase in average to the rate of the interest rate. Since the total value of the firm W is increasing in the entrepreneur value V , the total value of the firm W will also expand in average over time.

Result 6 *Conditional on the state (s, r) $V_t(s, r)$ in the optimal contract increases over time.*

Proof. Assuming the firm is still constrained, the first order condition for the continuation value $V_{t+1}(s_{t+1}, r_{t+1})$ at the state (s_{t+1}, r_{t+1}) is given by:

$$W_1(V_{t+1}(s_{t+1}, r_{t+1}), s_{t+1}, r_{t+1}) = \lambda \quad (12)$$

where λ is the lagrange multiplier on (6). The envelope theorem gives us the following relationship:

$$W_1(V_t(s_t, r_t), s_t, r_t) = \frac{1}{1+r_t} \Pi_1(V_t(s_t, r_t), s_t, r_t) + \lambda \quad (13)$$

Combing (12) and (13) we get:

$$W_1(V_t(s_t, r_t), s_t, r_t) = \frac{1}{1+r_t} \Pi_1(V_t(s_t, r_t), s_t, r_t) + W_1(V_{t+1}(s_{t+1}, r_{t+1}), s_{t+1}, r_{t+1}) \quad (14)$$

If we consider $s_t = s_{t+1} = s$ and $r_t = r_{t+1} = r$ then $V_t(s, r) \leq V_{t+1}(s, r)$ given $\Pi_1 \geq 0$ and concavity of W in V .

4.2 Persistence

A key issue in macroeconomics is understanding what lies behind the propagation of economics fluctuations. Along this line is the question of what creates the persistence of the shocks. Several models have been developed to capture some mechanism of persistence. However, few models address this question in an optimal contract framework and therefore it is not clear whether the borrowing constraints itself or the lack of more financial instruments to diversify idiosyncratic risks drive the result. This distinction is very important since each time there are more financial instruments that allows the agents to hedge better. If the reason of the macroeconomic propagation mechanism of financial frictions is only due to a lack of financial instruments, one would expect that this propagation will loss relevance as new financial instruments are available. However, this model offers an approach of borrowing constraints in a optimal contract context (i.e. with all contingent financial instruments available) where one can see the extent of this constraints alone as a propagation mechanism. The focus in this section is to give conditions that characterize persistence of productivity and interest rate shocks.

Result 7 *If W is strictly concave in V for $V < \tilde{V}(s, r)$ then $W_{12} > 0$ and $W_{13} < 0$ at an interior solution so that any optimal continuation entrepreneur value $V(s', r')$ must be non-decreasing in s' and non-increasing in r' .*

Proof: See Appendix A

Assumption 4

1. $\forall s', F(s'|s)$ is non-increasing in s
2. $\forall r', G(r'|r)$ is non-increasing in r

This assumption states a first order stochastic dominance property in the conditional distribution of the shocks.

Assumption 5

1. If $\Pi_1(V(s, r), s, r)$ is non-decreasing in s then $\Pi_1\left(\frac{\mathbb{E}[V(s, r)|(s_0, r_0)]}{1 + r_0}, s_0, r_0\right)$ is non-decreasing in s_0
2. If $\frac{\Pi_1(V(s, r), s, r)}{(1+r)}$ is non-increasing in r then $\frac{\Pi_1\left(\frac{\mathbb{E}[V(s, r)|(s_0, r_0)]}{1+r_0}, s_0, r_0\right)}{1 + r_0}$ is non-increasing in r_0

We define positive persistence of the productivity shocks as the fact that firm size and entrepreneur value increase with past productivity shocks after controlling for the current one. Formally, let assume two paths that start from the same value V_0 : $\{(s_0, r_0), (s_1, r_1), (s_2, r_2), \dots, (s_t, r_t)\}$ and $\{(s_0, r_0), (s'_1, r_1), (s'_2, r_2), \dots, (s'_t, r_t)\}$ such that $s_i \geq s'_i$ $i = 1, \dots, t$. Denoting as $\{V_j\}_{j=1}^t$ and $\{V'_j\}_{j=1}^t$ the optimal value to the entrepreneur implied by these two paths, the positive persistence of productivity shocks implies that $V_j \geq V'_j \forall j = 1, \dots, t$.

Similarly, we can state the positive persistence of interest rate shocks. Having two paths $\{(s_0, r_0), (s_1, r_1), (s_2, r_2), \dots, (s_t, r_t)\}$ and $\{(s_0, r_0), (s_1, r'_1), (s_2, r'_2), \dots, (s_t, r'_t)\}$ such that $r_i \leq r'_i$ $i = 1, \dots, t$, the positive persistence of interest rate shocks implies that $V_j \geq V'_j$ where $\{V_j\}_{j=1}^t$ and $\{V'_j\}_{j=1}^t$ are the optimal entrepreneur value deduced from

these two paths.

Result 8 *Suppose that Π , F and G satisfy assumptions 4 and 5 then the optimal contract displays positive persistence.*

Proof: See Appendix A

The persistence comes from the fact that the entrepreneur is constrained. We know that in some moment in the future she will reach the unconstrained region where she will not have incentive to default. The entrepreneur will get faster to the unconstrained region if she has a better history of shocks because that allows to pay back the initial debt sooner and increases faster her value on the firm.

4.3 Level of Enforceability and Persistence

Several studies have remarked that the volatility of economic performance in developing countries is bigger than in those more developed. Other empirical studies have emphasized the higher propagation of shocks in developing economies are due to a more severe financial frictions. Lower of enforceability is one institutional factor that makes that the firms of the developing countries face more severe financing constraints. Hence, we can use this model to ask whether lower enforceability constraints imply higher persistence of shocks.

We do not have general results, but using the numerical simulation we analyze this question comparing the persistence of one period shock in productivity and interest rate for two economies with different levels of enforceability. The persistence of these two type of shocks are displayed in figure 5 and 6.

The responses to these shocks are computed comparing the profile of the variables with a base path. The base path is the one obtained with a constant productivity level of $s = 1$ and interest rate of $r = 3\%$. Hence, the response of the entrepreneur value is the percentage deviation of the path deducted with a reduction of 7.7% in productivity at period $t = 1$ with respect to the base path.

The outside option is denoted by $O(k, s, r) = \lambda k$. As noted earlier a lower outside option indicates a higher level of contract enforceability. For that reason, we denote $\lambda = 2.8$ as an economy with better contract enforceability than with $\lambda = 3.2$.

It is worth noting that these figures also display the variable called premium. This is the external finance premium that has been computed as the excess of marginal value of the capital advancement. Formally, the premium at t is computed as $R_1(k_t, s_t) - (1 + r_t)$.

Figure 5 shows the responses to a shock in the firm productivity. After a productivity reduction of one period, the firm dynamics in both economies display persistence reflected in that the entrepreneur value stays at a lower level than the base path for many periods. More interestingly, the speed of going back to the base path is faster in the economy with better contract enforceability. The response in the capital advancement resembles that of the entrepreneur value showing again that the deviations die out faster in an economy with better enforceability. The premium responses also stress the persistence. After an instantaneous reduction in the premium at the moment of the reduction in productivity, the persistence implies that the firm will be constrained a longer numbers of period and consequently it will have a higher premium than the base path. Moreover, the premium will return to the base path slower in the economy with lower contract enforceability.

The effects on the same variables under an interest rate shock are displayed in fig-

ure 6. The entrepreneur value stays a lower level even after the interest reduction has occurred too many periods ago. This is the manifestation of the persistence of this kind of fluctuations. The effect of the interest rate reduction disappear a little faster over time in the economy with better enforceability. Similarly, the reduction in the capital advancement is persistent but in the economy with lower enforceability this effect stays longer. Initially, the premium reduces with the drop in the interest rate. However, since the firm will be constrained longer, the premium has to be above the base path afterwards. Again, this persistence in the premium is higher for the economy with worse contract enforceability.

Hence, a better enforceability of financial contract –understood as lower outside option to the entrepreneur– implies that young firms will be constrained for a shorter period of time and they will reach the unconstrained region sooner. As a consequence, the propagation mechanism of financial constraints will be less severe under better enforceability of contracts. Also, less imperfect enforceability will lead to less inefficiency in the size and external finance of young firms.

5 Final Remarks

This article presents a model where the propagation of shocks under borrowing constraints operates in an optimal contract context. One interesting element has been the inclusion of the fluctuations of the interest rate. Hence, there is a pass-through of interest rate movements to the external financing premium that is modelled endogenously.

We have proved that in the optimal constrained contract the effect of productivity and interest rate movements affect the size and capital input advancement beyond what a model with perfect enforceability of the financial contracts would predict. Under some conditions the amplifications can be very powerful when the enforceability problems are

severe.

This model also offers a rationale to understand why economies with lower enforceability can be exposed to higher amplification and persistence of the productivity and interest rate fluctuations. This element is very important because traditionally macroeconomics models with financial frictions characterize developing and developed countries in the same degree of extension. The endogenous financial constraints obtained from problems of lack of commitment of contracts provided one way to distinguish the level of borrowing constraints depending on the degree of systematic enforceability in an economy. Recently the work of Cooley, Marimon and Quadrini (2004) has emphasized the persistence of productivity shock and volatility of the output increase in economies with lower enforceability of contracts. Also, Bergoing et al. (2002) suggest banking deregulation and bankruptcy laws in Chile that made that inefficient firms be replaced for more efficient firms as key factor to understand the better economic performance of Chile than Mexico after the debt crises of early 1980. This conclusions also can be expressed as the higher level of enforceability of contract in Chile than Mexico could be a force behind the weaker economic growth in Mexico than Chile during the eighties and nineties. The severity of enforceability problems in developing countries has also been stressed for several works to explain crises and high vulnerability to external shocks.¹¹

One interesting extension to consider along these lines is to analyze empirically the microeconomic implications in economies with different level of enforceability. For instance, the stationary cross-sections distribution of profits and size of firms can have a different pattern in two identical industries with a dissimilar levels of enforceability. This could be analyzed comparing the cross-sectional distribution of firms in the same

¹¹Schneider and Tornell (2004) have enforceability problems as one ingredient to explain boom-bust episodes in middle income countries. Caballero and Krishnamurthy (2001) also use enforceability problems to explain the relevance of collateral to secure the loans in a developing economy which in turn has significative macroeconomic amplification of external shocks.

industry for countries with high enforceability vis-a-vis countries with low enforceability. Braun (2003) is recent example that provides empirical evidence about industrial performance in different countries as a way to shed light on the relevance of the financial markets imperfections across countries.

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6 Appendix A: Properties of the profit function and Proofs of Results

This appendix contains proofs of the main results stated in the article. It also describes further features of the objects that characterize the optimal contract.

A.1. Definitions of derivatives of Π . If R and O are twice continuously differentiable and $V < V^u(s, r)$, applying the envelope theorem and the first order condition in (8) we have the following expressions:

- $\Pi_1 = (R_1(k, s) - (1 + r))/O_1(k, s, r).$
- $\Pi_2 = R_2(k, s) - (R_1(k, s) - (1 + r))O_2(k, s, r)/O_1(k, s, r)$
- $\Pi_3 = -(R_1(k, s) - (1 + r))O_3(k, s, r)/O_1(k, s, r) - k$
- $\Pi_{11} = \frac{R_{11}(k, s)O_1(k, s, r) - O_{11}(k, s, r)(R_1(k, s) - (1 + r))}{O_1(k, s, r)^3}$
- $\Pi_{12} = -O_2(k, s, r)\frac{R_{11}(k, s)O_1(k, s, r) - O_{11}(k, s, r)(R_1(k, s) - (1 + r))}{O_1(k, s, r)^3}$
 $+ \frac{R_{12}(k, s)O_1(k, s, r) - O_{12}(k, s, r)(R_1(k, s) - (1 + r))}{O_1(k, s, r)^2}$
- $\Pi_{12} = -O_2(k, s, r)\Pi_{11}(V, s, r)$
 $+ \frac{R_{12}(k, s)O_1(k, s, r) - O_{12}(k, s, r)(R_1(k, s) - (1 + r))}{O_1(k, s, r)^2}$
- $\Pi_{13} = -O_3(k, s, r)\frac{R_{11}(k, s)O_1(k, s, r) - O_{11}(k, s, r)(R_1(k, s) - (1 + r))}{O_1(k, s, r)^3}$
 $- \frac{O_1(k, s, r) - O_{13}(k, s, r)(R_1(k, s) - (1 + r))}{O_1(k, s, r)^2}$

$$\Pi_{13} = -O_3(k, s, r)\Pi_{11}(V, s, r)$$

$$\frac{O_1(k, s, r) - O_{13}(k, s, r)(R_1(k, s) - (1 + r))}{O_1(k, s, r)^2}$$

A.2. Sign of Derivatives. If $V < V^u(s, r)$

- $\Pi_1 > 0$ just by the assumptions
- $\Pi_2 > 0$ if $R_2 > (R_1 - (1 + r))O_2/O_1$
- $\Pi_3 < 0$ if $k > -(R_1 - (1 + r))O_3/O_1$
- $\Pi_{11} \leq 0$ if $R_{11}O_1 \leq O_{11}(R_1 - (1 + r))$
- $\Pi_{12} \geq 0$ if $R_{12}O_1 - O_{12}(R_1 - (1 + r)) \geq 0$
- $\Pi_{13} \leq 0$ if $-O_3\Pi_{11}O_1^2 \leq O_1 - O_{13}(R_1 - (1 + r))$

Proof of Result 1.

The first property is a direct application of the maximum theorem using the conditions stated in assumptions 1 and 2. The continuous differentiability of Π comes from the continuous differentiability of R and O . The second property is a direct consequence from the fact that the lagrange multiplier in the constraint will be strict positive when $V < V^u(s, r)$ and by the envelope theorem Π_1 is equal to that lagrange multiplier. Using the fact that $R_{11}O_1 < O_{11}(R_1 - (1 + r))$ from **A.2.** we get that $\Pi_{11} < 0$ when $V < V^u(s, r)$. Also, when $V \geq V^u(s, r)$ we know that $\Pi(V, s, r) = \pi(s, r)$ and $\Pi_1 = 0$ which implies that $\Pi_{11} = 0$.

Proof of Result 2.

First, the property that W is weakly increasing in V is a direct application of the dynamic programming arguments using the fact that Π is weakly increasing in V .

The second property can be proved using the fact that the dynamic programming equation preserve this property. Let assume that $W(\hat{V}, s', r') = \tilde{W}(s, r)$ for $\hat{V} \geq \tilde{V}(s', r')$. If $V \geq \tilde{V}(s, r)$ by the definition of \tilde{V} we have that $V \geq \mathbb{E}[\tilde{V}(s', r')|(s, r)]/(1+r)$ and therefore, $\tilde{V}(s', r')$ can be implemented as continuation value. By assumption we obtain $W(\tilde{V}(s', r'), s', r') = \tilde{W}(s', r')$. Since $V \geq \tilde{V}(s, r)$ we know that $V \geq V^u(s, r)$ and we can conclude $\Pi(V, s, r) = \pi(s, r)$. Hence, we can write the following:

$$\begin{aligned} W(V, s, r) &= \frac{1}{1+r} \{ \Pi(V, s, r) + \mathbb{E}[W(V(s', r'), s', r')|(s, r)] \} \\ &= \frac{1}{1+r} \{ \pi(s, r) + \mathbb{E}[\tilde{W}(s', r')|(s, r)] \} \\ &= \tilde{W}(s, r) \end{aligned}$$

The last equality comes directly from the unconstrained dynamic programming equation (2).

The third property can be shown using an induction argument. The idea is to prove that $V < V^n(s, r)$ implies $W(V, s, r) < \tilde{W}(s, r)$ and this holds for all $n \in \mathbb{N}$. For $n = 1$ we can see that $V < V^1(s, r) = V^u(s, r)$ implies that $\Pi(V, s, r) < \pi(s, r)$ and directly we can obtain $W(V, s, r) < \tilde{W}(s, r)$. Now, let assume that the statement is true for any $n \in \mathbb{N}$ and then show that the same statement is also true for $n + 1$. If $V < V^{n+1}(s, r)$ we have two cases: either $V < V^u(s, r)$ or $V^u(s, r) \leq V < \mathbb{E}[V^n(s', r')|(s, r)]/(1+r)$. In the first case we know that $\Pi(V, s, r) < \pi(s, r)$ and we obtain the statement as before. In the second case, we know that some subset $\Theta \subseteq \mathcal{S} \times \Lambda$ with strictly positive measure satisfies $V^n(s', r') > V(s', r') \forall (s', r') \in \Theta$, where $V(s', r')$ is the optimal con-

tinuation value for the next period. Hence, $W(V(s', r'), s', r') < \tilde{W}(s', r')$ for $(s', r') \in \Theta$ which implies $\mathbb{E}[W(V(s', r'), s', r')|(s, r)] < \mathbb{E}[\tilde{W}(s', r')|(s, r)]$. This, in turn, makes that $W(V, s, r) < \tilde{W}(s, r)$ concluding then the induction proof.

Proof of Result 3.

If $V < \tilde{V}(s, r)$ it should be that there exists $n \in \mathbb{N}$ such that $V < V^n(s, r)$. We will show by induction on n that this implies that W is strictly concave in a neighborhood of V . For $n = 1$ we have $V < V^u(s, r)$ and Π will be strictly concave in a neighborhood of V which implies W will be also strictly concave applying standard dynamic programming arguments. Let assume that the statement is true for n (i.e., $V < V^n(s, r) \Rightarrow W$ strictly concave in V) and prove that it is true for $n + 1$. By assumption if $V < V^{n+1}(s, r)$, it is possible that either $V < V^u(s, r)$ or $V^u(s, r) \leq V < \mathbb{E}[V^n(s', r')|(s, r)]/(1 + r)$. In the first case, we get the strictly concavity using the same arguments as when $n = 1$. In the second case, we know there exists a set $\Theta \subset \mathcal{S} \times \Lambda$ with strictly positive measure such that $V(s', r') < V^n(s', r') \forall (s', r') \in \Theta$, where $V(s', r')$ is the optimal continuation value. Since $W(V(s', r'), s', r')$ is strictly concave function in a neighborhood of $V(s', r')$ $\forall (s', r') \in \Theta$ then

$$W(V, s, r) = \frac{1}{1 + r} \max\{\Pi(V, s, r) + \mathbb{E}[W(V(s', r'), s', r')|(s, r)]\}$$

will be strictly concave in a neighborhood of V . This completes the induction argument.

Proof of Result 4.

We will show this result by contradiction. Suppose that $V_2 < V_1$ and $W(V_2, s, r) = W(V_1, s, r)$ for some $V_2 < V_1 < \tilde{V}(s, r)$. Then by concavity:

$$W(V_1, s, r) \geq aW(V_2, s, r) + (1 - a)\tilde{W}(s, r)$$

where $a = (\tilde{V}(s, r) - V_1)/(\tilde{V}(s, r) - V_2) \in (0, 1)$. By assumption this leads to $W(V_1, s, r) \geq \tilde{W}(s, r)$ which is a contradiction with the result 2.

Proof of Result 5.

Since $V_t(1 + r_t) < \mathbb{E}_t[\tilde{V}(s_{t+1}, r_{t+1})|(s_t, r_t)]$ we obtain that $\mathbb{E}_t[V_{t+1}(s_{t+1}, r_{t+1})|(s_t, r_t)] < \mathbb{E}_t[\tilde{V}(s_{t+1}, r_{t+1})|(s_t, r_t)]$, where $V_{t+1}(s_{t+1}, r_{t+1})$ is the optimal continuation value. There exists then a set $\Theta \subset \mathcal{S} \times \Delta$ with strict positive measure such that $V_{t+1}(s', r') < \tilde{V}(s', r') \forall (s', r') \in \Theta$. Using the result 2 we can conclude that $\mathbb{E}_t[W(V_{t+1}(s_{t+1}, r_{t+1}), s_{t+1}, r_{t+1})|(s_t, r_t)] < \mathbb{E}_t[\tilde{W}(s_{t+1}, r_{t+1})|(s_t, r_t)]$. This implies that the constraint (6) is binding in the constrained dynamic programming problem (9) so that $V_t(1 + r_t) = \mathbb{E}_t[V_{t+1}(s_{t+1}, r_{t+1})|(s_t, r_t)]$ and therefore $d_t = 0$.

Proof of Result 7.

Consider the problem:

$$\begin{aligned} g(V, s, r) &= \max \mathbb{E}[W(V(s', r'), s', r')|(s, r)] \\ \text{s.t. } &\mathbb{E}[V(s', r')|(s, r)] \leq (1 + r)V \end{aligned}$$

Consider $s_2 > s_1$ and assuming $W_{12} \geq 0$ and using the optimality condition for the continuation of the entrepreneur value, we get $W_1(V(s_1, r'), s_1, r') = W_1(V(s_2, r'), s_2, r') < W_1(V(s_1, r'), s_2, r')$ which in turn implies $V(s_2, r') \geq V(s_1, r')$ by concavity of W in V . Also, the stochastic dominance assumed implies that any optimal continuation value $V(s', r')$ satisfies:

$$\mathbb{E}[V(s', r')|(s_2, r)] \geq \mathbb{E}[V(s', r')|(s_1, r)]$$

Now let $V^i(s', r')$ the optimal continuation value from (V, s_i, r) . Since

$$V(1 + r) = \mathbb{E}[V^1(s', r')|(s_1, r)] = \mathbb{E}[V^2(s', r')|(s_2, r)]$$

we can obtain that $V^1(s', r') \geq V^2(s', r')$ for $(s', r') \in \Theta \subseteq \mathcal{S} \times \Delta$ with Θ having strictly positive measure. Moreover, the optimality condition for the continuation value and the strictly concavity of W in V imply that $V^1(s', r') \geq V^2(s', r') \forall (s', r') \in \mathcal{S} \times \Delta$. Having this, we can prove that if $W(V, s, r)$ is strictly concave in V for $V < \tilde{V}(s, r)$, $\Pi_{12} \geq 0$ and Π is concave in V , then $W_{12} \geq 0$. We will show that the Bellman equation maps the set of functions with positive cross-partial derivatives into itself. Suppose that we start with a function $W_{12} \geq 0$. Using the envelope theorem, it follows that $g_1(V, s_i, r) = (1 + r)W_1(V^i(s', r'), s', r')$. If $s_2 > s_1$ from above we have $V^1(s', r') \geq V^2(s', r')$ which implies $W_1(V^1(s', r'), s', r') \leq W_1(V^2(s', r'), s', r')$. We then conclude $g_1(V, s_1, r) \leq g_1(V, s_2, r)$ implying that $g_{12} \geq 0$. By definition we have $W = \Pi + g$, then if $\Pi_{12} \geq 0$ and $g_{12} \geq 0$ we can see that $W_{12} \geq 0$.

Analogously, let consider $r_1 < r_2$. Following a similar argument as above, we can obtain $V(s', r_1) \geq V(s', r_2)$ if we assume that W is concave in V and $W_{13} \leq 0$. This conclusion together with stochastic dominance will imply that for any optimal continuation value $V(s', r')$:

$$\mathbb{E}[V(s', r')|(s, r_1)] \geq \mathbb{E}[V(s', r')|(s, r_2)]$$

Let $y^i(s', r')$ be the optimal continuation value from (V, s, r_i) we can write:

$$(1 + r_1)V = \mathbb{E}[y^1(s', r')|(s, r_1)] < (1 + r_2)V = \mathbb{E}[y^2(s', r')|(s, r_2)] \leq \mathbb{E}[y^2(s', r')|(s, r_1)]$$

which implies $y^1(s', r') < y^2(s', r')$. We will show that Bellman equation maps the set of function with $W_{13} \leq 0$ into itself since W is strictly concave in V for $V < \tilde{V}(s', r')$, $\Pi_{13} \leq 0$ and Π concave in V . Suppose that we start with a function W with $W_{13} \leq 0$. Using the envelope condition:

$$\begin{aligned}
g_1(V, s, r_1) &= \frac{W_1(y^1(s', r'), s', r')}{1 + r_1} \\
&\geq \frac{W_1(y^2(s', r'), s', r')}{1 + r_1} \\
&> \frac{W_1(y^2(s', r'), s', r')}{1 + r_2} \\
&= g_1(V, s, r_2)
\end{aligned}$$

which states that $g_{13} \leq 0$. This is a sufficient condition to obtain $W_{13} \leq 0$. Taking derivatives we can see that

$$W_{13} = \frac{\Pi_{13} + g_{13}}{1 + r} - \frac{\Pi_1 + g_1}{(1 + r)^2} \leq 0$$

Proof of Result 8.

This proof is long and is made in two steps. The first step is to prove that in the optimal contract $\Pi_1(V(s, r), s, r)/(1 + r)$ is weakly increasing in s and decreasing in r using assumption 5. Let assume that W satisfies the following:

If $W_1(V(s, r), s, r)$ is constant in s and r then

$$\frac{\Pi_1\left(\frac{\mathbb{E}[V(s, r)|(s_0, r_0)]}{1 + r_0}, s_0, r_0\right)}{1 + r_0} \text{ is non-decreasing in } s_0 \text{ and non-increasing in } r_0 \tag{15}$$

Defining TW as the function of the left hand side of (9), we can prove that TW

also satisfies (15). This guarantees that the statement in (15) is true using standard dynamic programming arguments that state that the set of continuous functions that satisfies this property maps itself. Hence, assuming that $TW_1(V(s, r), s, r)$ is constant in s and r , first order and envelope conditions give us:

$$TW_1(V(s_0, r), s_0, r) = \frac{\Pi_1(V(s_0, r), s_0, r)}{1+r} + W_1(V_0(s', r'), s', r') \quad (16)$$

$$TW_1(V(s_1, r), s_1, r) = \frac{\Pi_1(V(s_1, r), s_1, r)}{1+r} + W_1(V_1(s', r'), s', r')$$

where $s_0 < s_1$ and $V_i(s', r')$ is the optimal continuation value from $(V(s_i, r), s_i, r)$. We will see that $\Pi(V(s_0, r), s_0, r)/(1+r) \leq \Pi(V(s_1, r), s_1, r)/(1+r)$ by contradiction. If that is not true using (16) we can get that $W_1(V_0(s', r'), s', r') < W_1(V_1(s', r'), s', r')$. By concavity of W in V we obtain that $V_0(s', r') > V_1(s', r')$ which in turn implies:

$$\begin{aligned} \Pi_1(V(s_1, r), s_1, r) &= \Pi_1\left(\frac{\mathbb{E}[V_1(s', r')|(s_1, r)]}{1+r}, s_1, r\right) \\ &\geq \Pi_1\left(\frac{\mathbb{E}[V_0(s', r')|(s_1, r)]}{1+r}, s_1, r\right) \end{aligned} \quad (17)$$

where the second inequality above comes from the concavity of Π in V . Now since $W_1(V_0(s', r'), s', r')$ is constant in s' and r' we know by the assumption over W that:

$$\Pi_1\left(\frac{\mathbb{E}[V_0(s', r')|(s_1, r)]}{1+r}, s_1, r\right) \geq \Pi_1\left(\frac{\mathbb{E}[V_0(s', r')|(s_0, r)]}{1+r}, s_0, r\right) = \Pi_1(V(s_0, r), s_0, r) \quad (18)$$

Combining (17) and (18) we establish that $\Pi_1(V(s_1, r), s_1, r) > \Pi_1(V(s_0, r), s_0, r)$ which closes the contradiction argument. Using this condition that states $\Pi(V(s, r), s, r)$ is non-decreasing in s and the assumption 5 we are able to show that TW also holds the first part of property (15).

Similarly, we can prove the second part of property in (15). Suppose $r_0 < r_1$ and

denote $y_i(s', r')$ the optimal continuation value from $(V(s, r_i), s, r_i)$. The optimality condition is described as in (16):

$$TW_1(V(s, r_0), s, r_0) = \frac{\Pi_1(V(s, r_0), s, r_0)}{1 + r_0} + W_1(y_0(s', r'), s', r')$$

$$TW_1(V(s, r_1), s, r_1) = \frac{\Pi_1(V(s, r_1), s, r_1)}{1 + r_1} + W_1(y_1(s', r'), s', r')$$
(19)

Again, by contradiction we can prove that

$$\frac{\Pi_1(V(s, r_1), s, r_1)}{1 + r_1} < \frac{\Pi_1(V(s, r_0), s, r_0)}{1 + r_0}$$

If that is not the case, we get that $W_1(y_0(s', r'), s', r') > W_1(y_1(s', r'), s', r')$ which in turn implies $y_0(s', r') < y_1(s', r')$ by concavity of W in V . This last conclusion implies

$$\begin{aligned} \Pi_1(V(s, r_0), s, r_0)/(1 + r_0) &= \Pi_1\left(\frac{\mathbb{E}[y_0(s', r')|(s, r_0)]}{1 + r_0}, s, r_0\right)/(1 + r_0) \\ &\geq \Pi_1\left(\frac{\mathbb{E}[y_1(s', r')|(s, r_0)]}{1 + r_0}, s, r_0\right)/(1 + r_0) \end{aligned}$$
(20)

Since $W_1(y_1(s', r'), s', r')$ is constant and the property (15) is true for W we obtain:

$$\begin{aligned} \Pi_1\left(\frac{\mathbb{E}[y_1(s', r')|(s, r_0)]}{1 + r_0}, s, r_0\right)/(1 + r_0) &\geq \Pi_1\left(\frac{\mathbb{E}[y_1(s', r')|(s, r_1)]}{1 + r_1}, s, r_1\right)/(1 + r_1) \\ &= \Pi_1(V(s, r_1), s, r_1)/(1 + r_1) \end{aligned}$$
(21)

As before, combining (20) and (21) the contradiction appears and we can conclude that in the optimal contract $\Pi_1(V(s, r), s, r)/(1 + r)$ is weakly decreasing in r . This conclusion and assumption 5 finally imply the second part of property in (15).

As a summary of this first step we conclude that W satisfies property in (15) and in

the optimal contract $\Pi(V(s, r), s, r)/(1+r)$ is weakly increasing in s and decreasing in r .

The second step of the proof uses the following claim. Take states described by (V_1, s_1, r) , (V_0, s_0, r) , (V_1, s, r_1) and (V_0, s, r_0) . Let $V_i(s', r')$ and $y_i(s', r')$ denote the continuation value from (V_i, s_i, r) and (V_i, s, r_i) , respectively. The claim states:

1. If $s_1 > s_0$ and $W_1(V_1, s_1, r) \leq W_1(V_0, s_0, r)$ then

$$V_1(s', r') \geq V_0(s', r') \text{ and } W_1(V_1(s', r'), s', r') \leq W_1(V_0(s', r'), s', r') \quad (22)$$
2. If $r_1 > r_0$ and $W_1(V_0, s, r_0) \leq W_1(V_1, s, r_1)$ then

$$y_1(s', r') \leq y_0(s', r') \text{ and } W_1(y_1(s', r'), s', r') \geq W_1(y_0(s', r'), s', r')$$

Let \hat{V}_1 satisfy $W_1(\hat{V}_1, s_1, r) = W_1(V_0, s_0, r) \geq W_1(V_1, s_1, r)$. By concavity of W in V we can see that $\hat{V}_1 \leq V_1$. Let $\hat{V}_1(s', r')$ be the optimal continuation value from (\hat{V}_1, s_1, r) . From monotonicity of the policy function we know

$$\hat{V}_1(s', r') \leq V_1(s', r') \quad (23)$$

Using similar arguments as in the first step we will prove that $\Pi_1(\hat{V}_1, s_1, r) \geq \Pi_1(V_0, s_0, r)$. The optimality conditions for the continuation values establish:

$$W_1(\hat{V}_1, s_1, r) = \frac{\Pi_1(\hat{V}_1, s_1, r)}{1+r} + W_1(\hat{V}_1(s', r'), s', r') \quad (24)$$

$$W_1(V_0, s_0, r) = \frac{\Pi_1(V_0, s_0, r)}{1+r} + W_1(V_0(s', r'), s', r')$$

If $\Pi_1(\hat{V}_1, s_1, r) < \Pi_1(V_0, s_0, r)$ from (24) we get that $W_1(\hat{V}_1(s', r'), s', r') > W_1(V_0(s', r'), s', r')$. The concavity of W implies that $\hat{V}_1(s', r') < V_0(s', r')$ which induces that

$$\begin{aligned}
\Pi_1(\hat{V}_1, s_1, r)/(1+r) &= \Pi_1\left(\frac{\mathbb{E}[\hat{V}_1(s', r')|(s_1, r)]}{1+r}, s_1, r\right)/(1+r) \\
&> \Pi_1\left(\frac{\mathbb{E}[V_0(s', r')|(s_1, r)]}{1+r}, s_1, r\right)/(1+r)
\end{aligned} \tag{25}$$

Also, the fact that $W_1(V_0(s', r'), s', r')$ is constant implies that

$$\begin{aligned}
\Pi_1\left(\frac{\mathbb{E}[V_0(s', r')|(s_1, r)]}{1+r}, s_1, r\right)/(1+r) &\geq \Pi_1\left(\frac{\mathbb{E}[V_0(s', r')|(s_0, r)]}{1+r}, s_0, r\right)/(1+r) \\
&= \Pi_1(V_0, s_0, r)/(1+r)
\end{aligned} \tag{26}$$

Putting (25) and (26) together we obtain the contradiction. Therefore, we can conclude that $\Pi_1(\hat{V}_1, s_1, r) \geq \Pi_1(V_0, s_0, r)$ and $W_1(\hat{V}_1(s', r'), s', r') \leq W_1(V_0(s', r'), s', r')$. This last inequality implies that $\hat{V}_1(s', r') \geq V_0(s', r')$. Combining this with (23) we obtain $V_1(s', r') \geq V_0(s', r')$ and therefore

$$W_1(V_1(s', r'), s', r') \leq W_1(V_0(s', r'), s', r')$$

This ends the proof of the first part of the claim in (22).

Analogously, to prove the second part of the claim in (22) we first define \hat{V}_0 such that $W_1(V_1, s, r_1) = W_1(\hat{V}_0, s, r_0) \geq W_1(V_0, s, r_0)$. Let $\hat{y}_0(s', r')$ be the continuation value from (\hat{V}_0, s, r_1) . By concavity of W we get that $\hat{V}_0 \leq V_0$. Hence, the monotonicity of the optimal continuation values implies that

$$\hat{y}_0(s', r') \leq y_0(s', r') \tag{27}$$

As before we can show that $\Pi_1(V_1, s, r_1)/(1+r_1) \leq \Pi_1(\hat{V}_0, s, r_0)/(1+r_0)$ by contradiction.

If that is not true [$\Pi_1(V_1, s, r_1)/(1 + r_1) > \Pi_1(\hat{V}_0, s, r_0)/(1 + r_0)$] we conclude that

$$W_1(y_1(s', r'), s', r') > W_1(\hat{y}_0(s', r'), s', r') \quad (28)$$

which in turn implies $y_1(s', r') \leq \hat{y}_0(s', r')$ by concavity of W . Using the concavity of Π we infer that:

$$\begin{aligned} \Pi_1(\hat{V}_0, s, r_0)/(1 + r_0) &= \Pi_1\left(\frac{\mathbb{E}[\hat{y}_0(s', r')|(s, r_0)]}{1 + r_0}, s, r_0\right)/(1 + r_0) \\ &\geq \Pi_1\left(\frac{\mathbb{E}[y_1(s', r')|(s, r_0)]}{1 + r_0}, s, r_0\right)/(1 + r_0) \end{aligned} \quad (29)$$

By the conclusion in the first step we then can write:

$$\begin{aligned} \Pi_1(V_1, s, r_1)/(1 + r_1) &= \Pi_1\left(\frac{\mathbb{E}[y_1(s', r')|(s, r_1)]}{1 + r_1}, s, r_1\right)/(1 + r_1) \\ &\leq \Pi_1\left(\frac{\mathbb{E}[y_1(s', r')|(s, r_0)]}{1 + r_1}, s, r_0\right)/(1 + r_0) \end{aligned} \quad (30)$$

because $W_1(y_1(s', r'), s', r')$ is constant. Combining (29) and (30) we get $\frac{\Pi_1(\hat{V}_0, s, r_0)}{1 + r_0} > \frac{\Pi_1(V_1, s, r_1)}{1 + r_1}$ which states the contradiction. Hence,

$$W_1(\hat{y}_0(s', r'), s', r') \leq W_1(y_1(s', r'), s', r')$$

Using concavity of W and (27), we conclude that

$$y_1(s', r') \leq y_0(s', r') \text{ and } W_1(y_1(s', r'), s', r') \geq W_1(y_1(s', r'), s', r').$$

Having these two steps we can easily prove the Result 8. Taking two histories starting from the same entrepreneur value V_0 :

$$\begin{aligned} & \{(s_0, r_0), (s_1, r_1), \dots, (s_{T-1}, r_{T-1}), (s_T, r_T)\} \\ & \{(s_0, r_0), (\hat{s}_1, r_1), \dots, (\hat{s}_{T-1}, r_{T-1}), (\hat{s}_T, r_T)\} \end{aligned}$$

where $s_t \geq \hat{s}_t \forall t = 1, \dots, T$ and $s_1 > \hat{s}_1$. Denote V_t and \hat{V}_t as the entrepreneur values at t for the first and second history, respectively. Using the optimality condition of the contract we have $W_1(V_1, s_1, r_1) = W_1(\hat{V}_1, \hat{s}_1, r_1)$ which implies $V_2(s', r') \geq \hat{V}_2(s', r')$ applying the claim in (22). Also, using implications obtained in Result 7 we conclude that $V_2 > \hat{V}_2$. Moreover, the optimality condition for the continuation value and the fact that $W_{12} \geq 0$ imply $W_1(V_2, s_2, r_2) \leq W_1(\hat{V}_2, \hat{s}_2, r_2)$. Applying the same logic recursively, the claim in (22) and the result 7 we conclude that $V_t \geq \hat{V}_t$ and $W_1(V_t, s_t, r_t) \leq W_1(\hat{V}_t, \hat{s}_t, r_t) \forall t \leq T$.

Making use of similar steps we can use to prove the persistence in the interest rate fluctuations. Assuming two histories:

$$\begin{aligned} & \{(s_0, r_0), (s_1, r_1), \dots, (s_{T-1}, r_{T-1}), (s_T, r_T)\} \\ & \{(s_0, r_0), (s_1, \hat{r}_1), \dots, (s_{T-1}, \hat{r}_{T-1}), (s_T, \hat{r}_T)\} \end{aligned}$$

such that $r_t \leq \hat{r}_t \forall t = 1, \dots, T$ and $r_1 < \hat{r}_1$. Applying the claim in (22) and the result 7 regarding the fact that $W_{13} \leq 0$ we can show recursively that $V_t \geq \hat{V}_t$ and $W_1(V_t, s_t, r_t) \leq W_1(V_t, s_t, \hat{r}_t)$, where V_t and \hat{V}_t are the optimal paths of the entrepreneur value for the first and second history, respectively.

7 Appendix B: Description of Parameters for the Simulation

To highlight the main analytical conclusions from the optimal contract we consider a very simple specification of the revenue function and outside option:

- $R(s, k) = sk^\alpha$
- $O(k, s, r) = \lambda k$
- s, r are *iid*

For the baseline simulation α is set at 0.8, λ is 3.2. The comparison for the case of better enforceability is analyzed changing λ to 2.8. We assume that productivity (s) and interest rate (r) are independent and identically distributed over time and independent each other. The mean and variance of s are 1 and 0.01, respectively. These same moments for the interest rate are fixed at 0.04 and 0.001. In the implementation of the simulation we consider a grid of five points for s and r to match these moments.

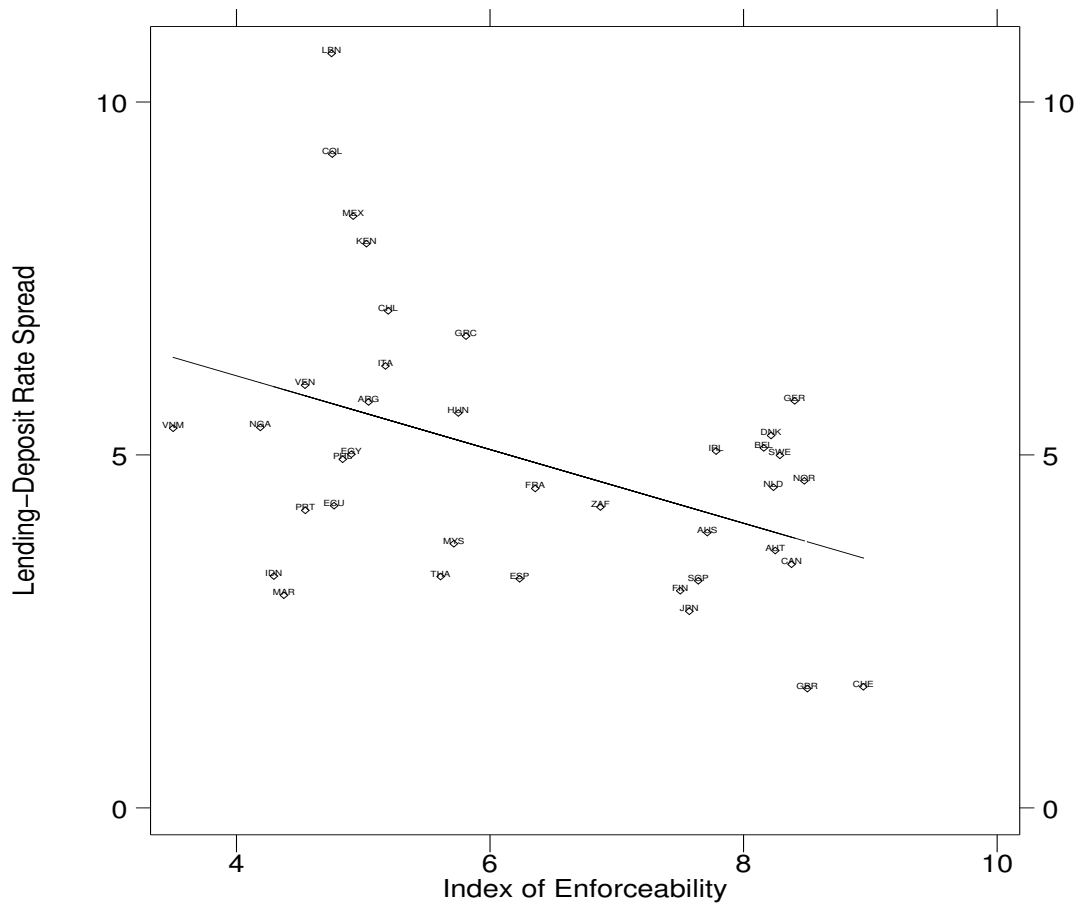


Figure 2: Enforceability of Contracts and Interest Rate Spread

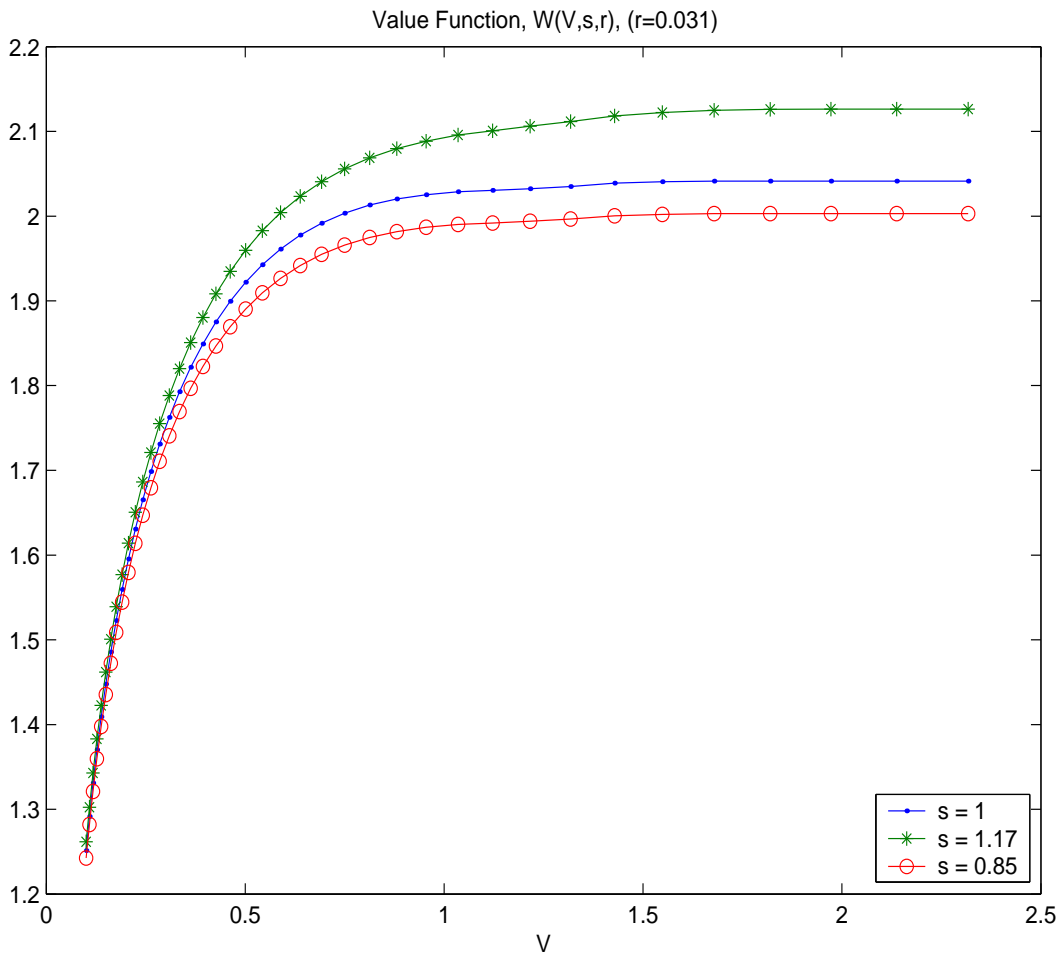


Figure 3: Value Function for different level of current productivity

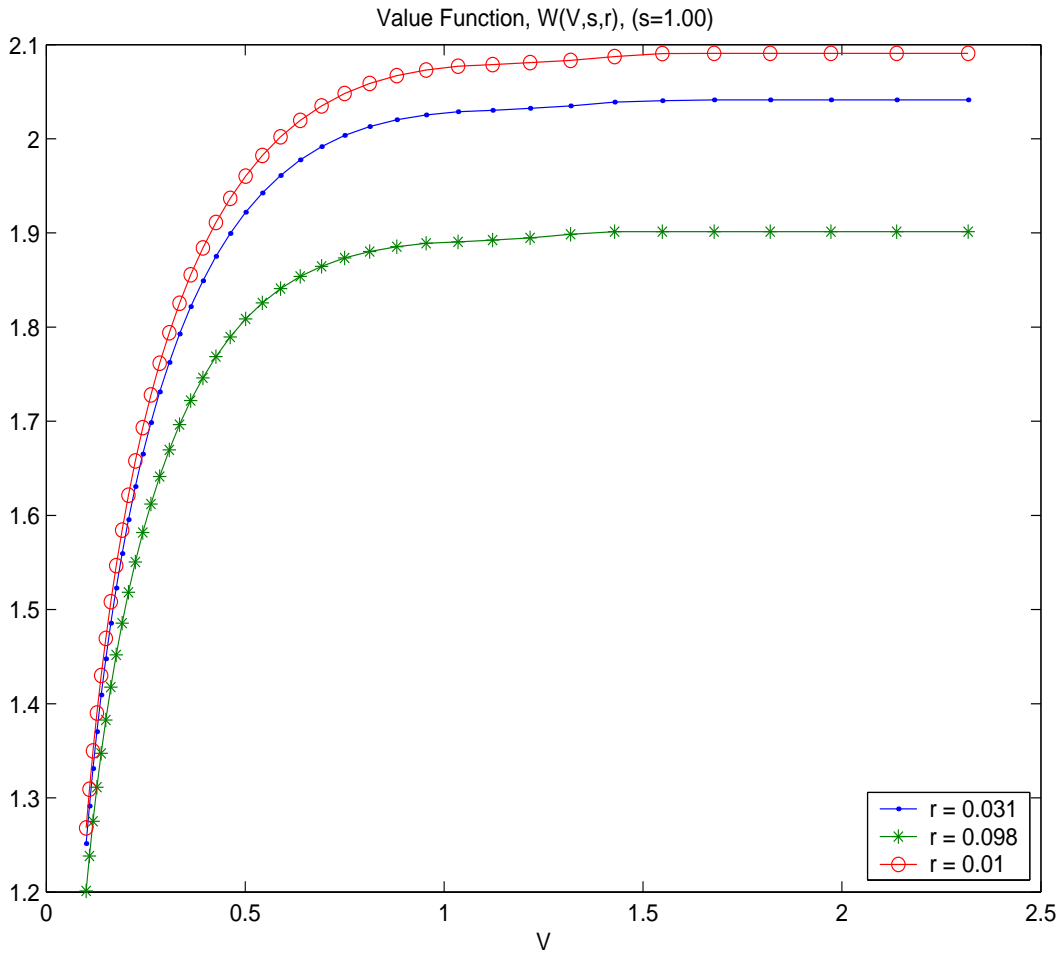


Figure 4: Value Function for different level of current interest rates

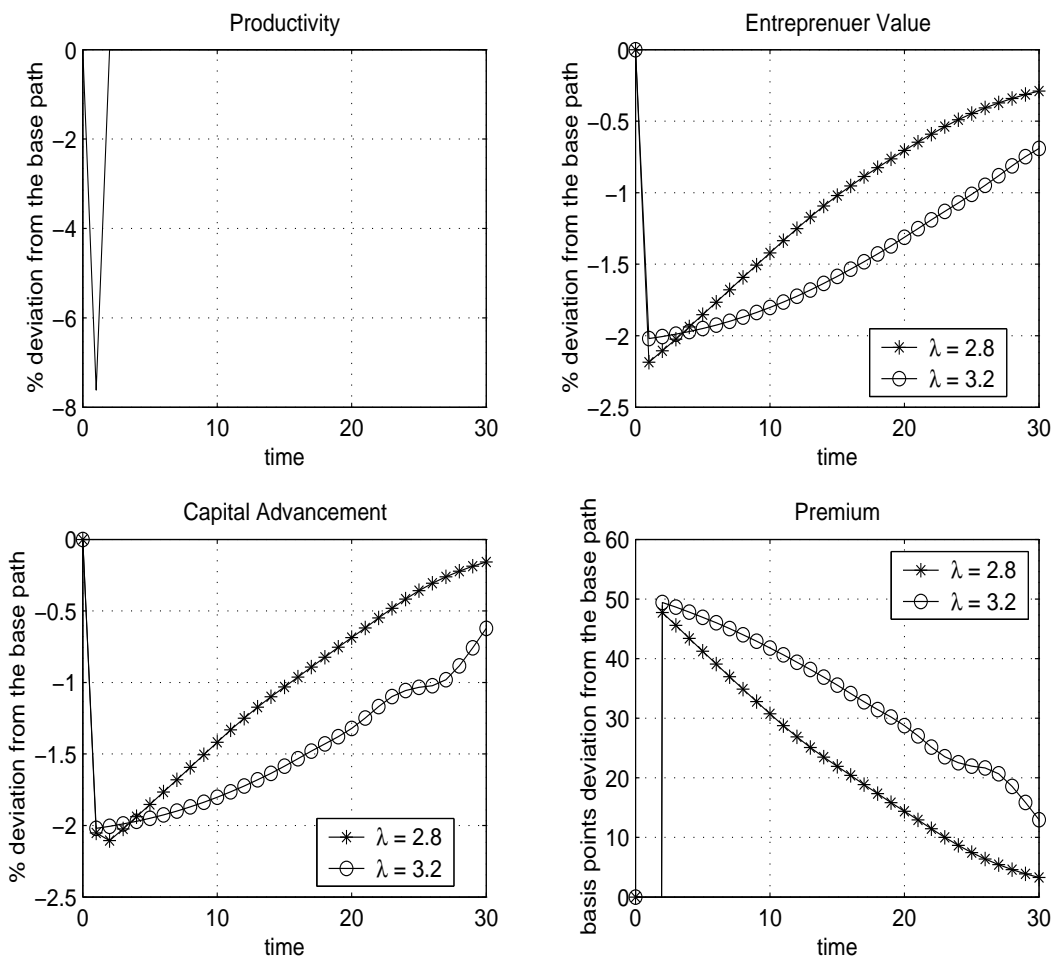


Figure 5: Impulse Responses to a Reduction in Productivity

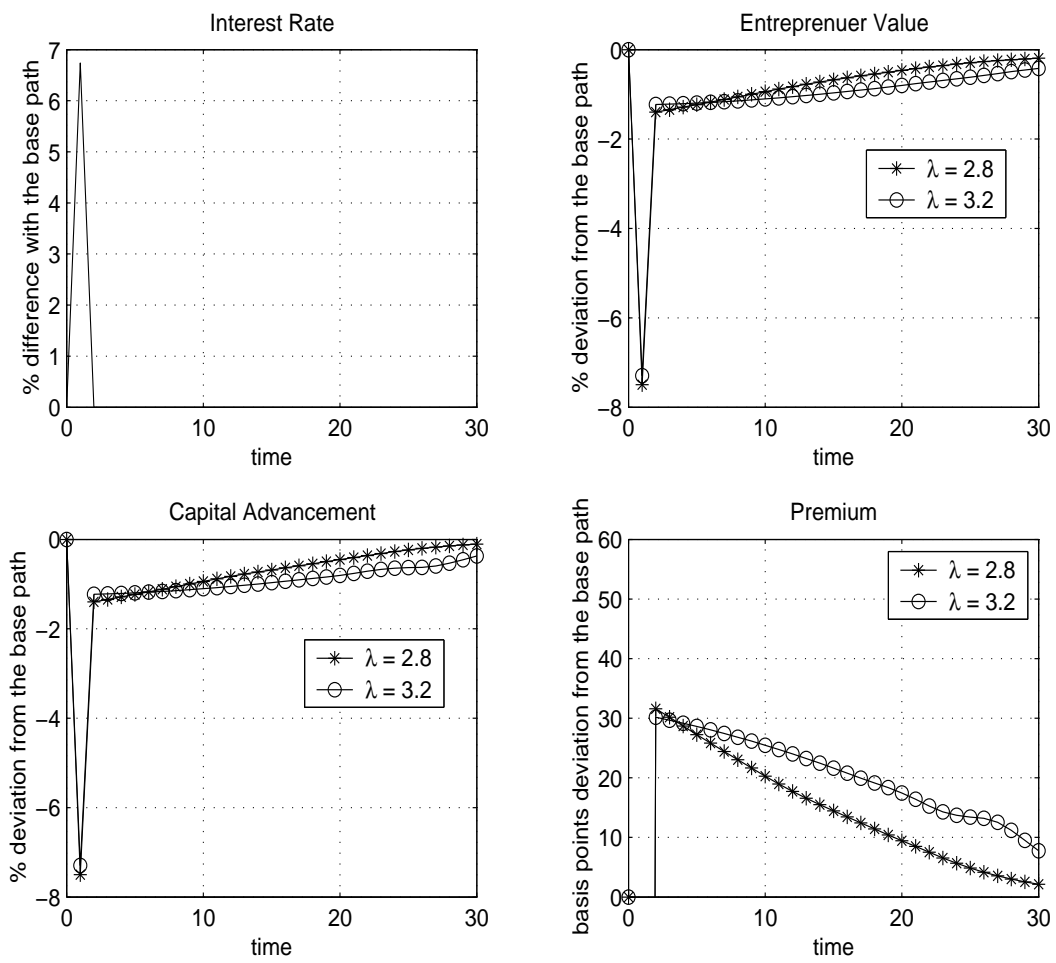


Figure 6: Impulse Responses to an Increase in the Interest Rate

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