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Sensitivity Analysis and Lipschitzian Properties in Linear Optimization

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María Josefa Cánovas Cánovas
y Francisco Javier Toledo Melero.

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la meua segona mare,
allà on estigues.

Als meus pares,
per tots els llibres comprats.

*"La mente que se abre a una nueva idea
jamás regresa a su tamaño original."
– Albert Einstein*

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Abstract

The objective of the present thesis is studying the stability of linear optimization problems through Lipschitzian constants. In other words, we pretend to quantify the rate of variation, around a given solution, of the optimal value and the feasible set with respect to the variation of the parameters of the model. Therefore, our research may be situated in the fields of Optimization, Linear Programming and Variational Analysis.

Specifically, Chapter 1 deals with the computation of formulae which provide measures of rates of variation (decrease and increase) of the optimal value associated with a finite linear problem under canonical perturbations of the data. It is worth mentioning that canonical perturbations are those that affect the vector of coefficients of the objective function and the independent terms (the right-hand-side) of the constraints. Formally, we want to compute (or estimate) the so-called *calmness modulus* as well as the *calmness from below and above moduli*. Chapter 2 focus on the computation of the *Lipschitz modulus* of the same optimal value function in the previous parametric setting, i.e., finite linear optimization problems under canonical perturbations. These two first chapters can be included in the paradigmatic topic of Sensitivity Analysis as they are concerned with quantifying the stability of the optimal value of optimization problems.

Chapter 3 aims to study the *Lipschitz lower semicontinuity* (Lipschitz-lsc, in brief) of the feasible set mapping. Roughly speaking, this property measures the rate of local contraction (in a neighborhood of a nominal solution fixed in advance) of the feasible set under perturbations of the data's problem. In this chapter there is a notable jump with respect to the previous ones in terms of the parametric context: we now work with linear optimization problems where there is not necessarily a finite number of constraints but the index set is arbitrary; in particular, it may be finite or infinite. In this last case, we deal with the Linear Semi-Infinite Programming. About the type of perturbations, besides perturbations of right-hand-side of the constraints, we also analyse the feasible set mapping under left-hand-side perturbations. Additionally, the study of Lipschitz-lsc has led to study, at the same time, other Lipschitzian property of lower semicontinuity which we have called *Lipschitz lower semicontinuity-star* (Lipschitz-lsc*, in brief).

In addition to these three mentioned chapters, the manuscript contains an unnumbered section, *Introduction* (and its Spanish version), as a preamble, where we present the work and expose the objectives that we pretend to cover in this research as well as we comment how these are integrated in the literature. Then, we have included the preliminary Chapter 0 in order to detail the parametric framework, the notation, the used tools and the previous results needed on to achieve the proposed objectives. To end the work, we have created another unnumbered section, *Conclusions and future work* (and its Spanish version), where we summarize and remark the main contributions of our research and give some brief comments on future research lines.

Resumen

El objetivo de la presente tesis es el estudio de la estabilidad de problemas de optimización lineal a través de constantes de tipo Lipschitz. En otros términos, se pretende cuantificar la tasa de variación, alrededor de una solución dada, del valor óptimo del problema y del conjunto de soluciones factibles con respecto a la variación de los parámetros del modelo. Por tanto, la presente investigación se engloba en las áreas de Optimización, Programación Lineal y Análisis Variacional.

De forma más explícita, el Capítulo 1 versa sobre la obtención de fórmulas capaces de proporcionar medidas de tasas de variación (de crecimiento y decrecimiento) del valor óptimo de un problema de programación lineal finita bajo perturbaciones canónicas de sus datos. Cabe mencionar que las perturbaciones canónicas son aquellas que afectan al vector de coeficientes de la función objetivo y los términos independientes (el miembro derecho) de las restricciones. Formalmente, se trata de calcular (o estimar) el llamado *módulo de calmness*, así como los *módulos de calmness por arriba y por abajo*. El Capítulo 2 se centra en el cálculo del *módulo de Lipschitz* para la misma función valor optimo en el mismo contexto paramétrico que la anterior, el de los problemas de optimización lineal finitos sujetos a perturbaciones canónicas. Estos dos primeros capítulos pueden ubicarse en el

tema paradigmático del análisis de sensibilidad, en tanto que se ocupan de cuantificar la estabilidad del valor óptimo de problemas de optimización.

Por su parte, el Capítulo 3 pretende estudiar la propiedad de *Lipschitz lower semicontinuity* (Lipschitz-lsc, por brevedad) para la multifunción conjunto factible. En términos informales esta propiedad cuantifica la tasa por la que se encoge localmente (en torno a una solución nominal prefijada de antemano) el conjunto factible con respecto a perturbaciones de los datos de los problemas. En este capítulo se produce un salto notable con respecto a los anteriores en cuanto al contexto paramétrico en el que se desarrolla la investigación: se trabaja con problemas de optimización lineal donde no necesariamente hay un número finito de restricciones, sino que el conjunto de índices del problema es arbitrario pudiendo ser en particular finito o infinito. Éste último caso se corresponde con la optimización lineal semi-infinita. Con respecto al tipo de perturbaciones, además de perturbaciones del miembro derecho de las restricciones, se analiza la multifunción conjunto factible en función de perturbaciones del miembro izquierdo de las mismas. Adicionalmente, el estudio de Lipschitz-lsc ha dado lugar a estudiar a su vez otro tipo de propiedad tipo Lipschitz de semicontinuidad inferior, a la que hemos denominado *Lipschitz lower semicontinuity-star* (Lipschitz-lsc*, por brevedad).

Además de estos tres capítulos mencionados, el manuscrito incluye una primera sección no numerada, *Introduction* (y su versión en castellano), a modo de preámbulo donde se presenta el trabajo con más detalle, se exponen los objetivos que pretende cubrir esta investigación y cómo se integran estos en la literatura. A continuación, se ha creado un Capítulo 0 preliminar para detallar el contexto paramétrico, la notación y las herramientas utilizadas y los resultados previos que han sido necesarios para conseguir los objetivos propuestos. Para finalizar el trabajo, se ha creado otra sección no numerada, *Conclusions and future work* (con su correspondiente versión en castellano), donde se resumen y remarcan los resultados principales obtenidos y se dan algunas pinceladas de las líneas de investigación a seguir en un futuro.

Introduction

The aim of the present dissertation is to quantify the stability of linear optimization problems through the use of Lipschitzian properties. In other words, our goal is to measure the rate of variation of the elements of a fixed problem, called nominal one, via the computation or estimation of Lipschitz type moduli. For this issue, one can study the behavior of certain associated mappings of the problem which are the feasible set and optimal set multifunctions and the optimal value function. We denote them by $\mathcal{F}$, $\mathcal{F}^{op}$ and ϑ, respectively (see Sections 0.1 and 0.2 for formal definitions). As a first stage, we focus on *calmness* (Chapter 1) and *Lipschitz* (Chapter 2) properties of the optimal value function restricted to its domain (where it has finite values), ϑ^R, in the framework of Linear Programming (LP, for short), i.e., when the index set of the constraints, T, is finite. In the second part of our research, we step forward considering T arbitrary and focus on the *Lipschitz lower semicontinuity* (Lipschitz-lsc, in brief) of $\mathcal{F}$ (Chapter 3). In particular, when T is infinite, we deal with Linear Semi-Infinite Programming (LSIP, for short). These Lipschitzian properties constitute important concepts in the field of Variational Analysis; with respect to this point, the reader is addressed to the monographs [18, 37, 43, 51] and references therein.

Taking into account the previous results in the literature, where the computation of calmness and Lipschitz moduli of multifunctions $\mathcal{F}$ and $\mathcal{F}^{op}$ have been already addressed, the study of calmness and Lipschitz continuity of ϑ turns out in a natural way. Specifically, exact formulae for the calmness moduli of $\mathcal{F}$ and $\mathcal{F}^{op}$ are established, respectively, in [13] and [9] (see [33] and [39] in relation to the calmness of $\mathcal{F}$ and $\mathcal{F}^{op}$ in nonlinear contexts). In relation to Lipschitz modulus, we refer to [8], [14], [41] and [42].

The present research could also be integrated in the widely explored field of Sensitivity Analysis, where, from different approaches, one tries to answer the natural question of *what* happens with the optimal value *if* one modifies the nominal problem's data. Specifically, our focus is on a *local aspect* of sensitivity analysis in contrast to the classical theory of parametric linear optimization, which usually concerns the behavior of ϑ^R and $\mathcal{F}^{op}$ on the domain of $\mathcal{F}^{op}$ or some of its subsets. Along this work different types of perturbations of the nominal problem are considered and, in each of these perturbation frameworks, our goal is to compute (or at least estimate) the corresponding Lipschitzian moduli. This process is addressed not only to know how much a set of solutions (feasible or optimal) varies with respect to the variation of the parameters of the model; but also we want to know how them vary. For example, in which direction (or set of directions) the variation (expansion/contraction) of these sets takes place, or in which direction the parameters must be perturbed to obtain the maximum variation.

The theory of parametric linear optimization goes back to the early time of LP (see, e.g., [21] and [52]). A systematic development of LP with canonical perturbations started in the 1970s. One research direction was focussed on the behavior of ϑ^R. Specifically, the continuity of ϑ^R was demonstrated through different approaches: via parametric analysis (see [44]), from a parametric approach using Berge's theory (see [3, 6]), and by a primal-dual approach (see [34] and [57]). A second direction of development of sensitivity analysis in LP starting in the late 1960s was the analysis of semicontinuity and Lipschitz semicontinuity properties, which was based on approaches of variational analysis like Berge's theory or Hoffman's error bounds (see [3, 17, 19, 40, 47, 49, 55, 57]). Along this work the continuity in the Painlevé-Kuratowski sense of multifunctions $\mathcal{F}$ and $\mathcal{F}^{op}$ restricted to their domains plays a crucial role; see Section 0.4 for details and specific references on these results.

In the 1990s and continuing until today, both directions became of great interest; see the survey [56] on different approaches to sensitivity, and the monograph [20]. See also [22] for the study of regions in which ϑ is affine and [1, 24, 31, 32] for an approach to the sensitivity analysis from an optimal partition perspective, related to support set invariancy. For extensions to LSIP, the reader is addressed to [27], [29] and [30]. In the context of conic linear systems (which includes our framework as a particular case), the pioneer works [45] and [46] provide a quantitative approach to the stability of optimization problems by using as an ingredient the *distance to the infeasibility*.

As we have advanced, Chapter 1 is focussed on the calmness of ϑ^R. This property is approached through the *calmness from below* and *calmness from above* which, roughly speaking, measure the local rate of decrease and increase, respectively, with respect to the nominal problem. In this chapter we estimate (in some cases compute) the calmness from above and below moduli under different kind of perturbations. Section 1.1 deals with right-hand-side (RHS, in brief) perturbations of the constraints. In Section 1.2 we perturb exclusively the coefficients of the objective function (c-perturbations, for simplicity). After that we tackle the setting of the so-called *canoni-*

cal perturbations, i.e., RHS perturbations together with c-perturbations (see Section 1.3).

To the authors' knowledge, the contributions of this work about the computation (or estimation) of calmness moduli, which are contained in Theorems between 28 to 46 and Corollaries 32 and 44, are new. As immediate antecedents of this part of the work, we refer to [45] (see also [46]) and [56]. In order to better integrate the current work in the literature, a comparative analysis between Theorem 46 and a certain consequence of [45, Theorem 1.1(5)] is developed in the final section; the details are gathered in Theorem 47 and Corollary 48. Specifically, from [45, Theorem 1.1(5)] one immediately derives an upper bound for the calmness modulus of ϑ, provided that the nominal problem belongs to the interior of the domain of $\mathcal{F}^{op}$, in terms of the distances to primal and dual infeasibility. In the same case, our Theorem 46 provides an exact formula for the calmness modulus of ϑ, which constitutes a refinement of Corollary 48 as far as the referred upper bound might be far from the exact value of such modulus (see Remark 49). On the other hand, [56, Theorem 18], translated into our notation, provides a particular constant k_1 involved in the calmness of ϑ from below (0.12) in the context of RHS perturbations.

Chapter 2 deals with the Lipschitz continuity of ϑ^R in the same framework of LP, which for arbitrary functions is known to be a stronger property than calmness. Roughly speaking, the Lipschitz modulus provides a local measure of the greatest rate of variation of the optimal value with respect to data perturbation. While calmness property compares the nominal optimal value with the optimal value of a perturbed problem, Lipschitz property involves the optimal values of two different perturbed problems around the nominal one. This fact entails notable differences between both properties and the computation of their moduli. We take advantage of the background on calmness but the new contributions of Chapter 2 are not direct consequences of the ones of Chapter 1, as we shall emphasize in the corresponding proofs. In particular, a key strategy (inspired by [42, Section 2]) based on computing the aimed Lipschitz modulus through a uniform calmness constant is appealed to.

The immediate antecedent of Chapter 2 can be traced out from [12] which, instead of ϑ^R, deals with the optimal value function, ϑ, defined on the whole space (and, so, taking values in the extended real line). As a counterpart, the local study is made around a problem which is in the interior of the domain of ϑ. This interiority condition characterizes the Lipschitz continuity of ϑ at such a problem (this fact is held in the more general setting of LSIP; see [28, Lemma 10.2]) and it is equivalent to the well-known Slater constraint qualification together with the boundedness (and nonemptiness) of the nominal optimal set. Specifically, [12, Theorem 4.3] provides a formula for a particular Lipschitz constant of ϑ in terms of the so-called *distance to ill-posedness* (see also [45] and [46], developed in the context of conic linear problems). Let us point out that the new results of this chapter constitute an improvement of [12, Theorem 4.3] in different directions: first, here we do not require any interiority assumption; moreover, the Lipschitz modulus provides –roughly speaking– the tightest Lipschitz constant; and, finally, we also tackle the case of partial perturbations (RHS or c-perturbations).

The structure of Chapter 2 is parallel to Chapter 1. In Section 2.1 we consider the case of RHS perturbations where a formula for the exact Lipschitz modulus of ϑ^R at a nominal problem is obtained in Theorem 51. Section 2.2 is developed in the context of c-perturbations, and mainly consists of Theorem 54, where lower and upper estimates (exact value when the nominal optimal set is nonempty and bounded) for the aimed modulus are provided. The last section deals with canonical perturbations. In it, Theorem 56 provides a lower bound of the Lipschitz modulus, while Theorem 58 provides an upper one based on a certain uniform calmness constant which is established in Lemma 57. The last theorem also provides the exact Lipschitz modulus under the boundedness (and nonemptiness) of the nominal optimal set.

For its part, Lipschitz-lsc quantifies the lower semicontinuity behavior of a set-valued mapping in Lipschitzian terms (see [37]). In our framework, it allows us to measure the rate of contraction of the feasible set mapping under data perturbations. This property has been studied by many authors and may be harnessed to characterize other Lipschitzian properties as the Aubin property or calmness; see, for example, [23], [37] and [39]. Lipschitz-lsc itself has been addressed by A. Uderzo in [53] and [54]. Specifically, [53, Theorem 4.1] provides a sufficient condition for the Lipschitz-lsc of the variational system associated with a general parameterized problem, while [54, Proposition 1] is focussed on the solution map in the same setting. On parametric constrained optimization problems, conditions for the Lipschitz-lsc of the feasible set and solution mappings are given in [53, Proposition 6.1 and Corollaries 6.1 and 6.2] and [54, Theorem 2].

The third chapter claims to be a new research line and constitutes a remarkable change with respect to the previous framework. As mentioned in the beginning, the parametric context now deals with linear optimization problems where there is not necessarily a finite number of constraints, in other words, the index set T is arbitrary and, in particular, it may be finite (LP) or infinte (LSIP). Additionally, we focus on the Lipschitz-lsc of $\mathcal{F}$ so, we change the functional thinking used in Chapters 1 and 2 and deal with multifunction arguments. About perturbations, in addition to RHS perturbations, we also consider perturbations of the left-hand-side (LHS, in brief) of the constraints. The main goal is to clarify the existent relationship between Aubin property and Lipschitz-lsc. This approach has lead to study at the same time the intermediate property between Aubin and Lipschitz-lsc, which we have called *Lipschitz lower semicontinuity-star* (Lipschitz-lsc*, in brief); see Section 0.2 for definitions. The starting background for our purpose is gathered in Proposition 60, where equivalences between Lipschitzian properties and other well-known statements are given. We note that the equivalence between Lipschitz-lsc and Aubin property fails where there are not RHS perturbations involved, hence in Section 3.1 we focus on the case of Lipschitz-lsc* when only LHS perturbations are allowed. Corollary 69 (for T arbitrary) and Theorem 73 (for T finite) establish the main difference between Lipschitz-lsc and Lipschitz-lsc* with respect to Aubin property. Section 3.2 is devoted to compute the Lipschitz-lsc and Lipschitz-lsc* moduli under simultaneous LHS and RHS perturbations. The direct antecedent is [7, Theorem 1] where authors give an exact expression of the Lipschitz modulus of $\mathcal{F}$ via the computation of the equivalent modulus for $\mathcal{F}^{-1}$, i.e., the modulus of the metric regularity (see [37, Lemma 1.12]). As we will show in Theorem 78, all these moduli turn out to be equal.

Chapters 1, 2 and 3 contain the original contributions of the present document. Most of the results about calmness have been included in a paper which has been already published (see [25]). About Lipschitz part of the optimal value, another paper has been written and submitted for possible publication (see [26]). The contents of Chapter 3 have been mainly developed during my research stay in the Weierstrass Institute for Applied Analysis and Stochastics (WIAS) in Berlin, from February 1 to May 4 of

2018, under the supervision of Dr. René Henrion. We intend it to be part of a new publication in the future. Besides these three mentioned chapters, the present document includes a prelude part, Chapter 0, when we introduce the model we are dealing with, the notation, the main goals of our research, as well as the formal definitions, key tools and preliminary results used later on. In order to complete our dissertation, we finish with the unnumbered section "Conclusions and future work". All the new results about the calmness and Lipschitz moduli of the optimal value function are gathered in Tables 3.1 and 3.2, respectively. Moreover, we stand out the existing relationships between these two properties. On the other hand, we also remark the main results of Chapter 3 and bring forward new future research lines. This research has been partially supported by grant BES-2015-073220 associated with project MTM2014-59179-C2-2-P from MINECO, Spain, and FEDER "Una manera de hacer Europa", European Union.

Introducción

La presente tesis tiene por objeto cuantificar la estabilidad de problemas de optimización lineal a través del uso de propiedades tipo Lipschitz. En otras palabras, pretendemos medir la tasa de variación de los elementos de un problema fijo dado, que llamamos problema nominal, calculando o estimando módulos tipo Lipschitz. En concreto, estudiamos el comportamiento de las multifunciones conjunto factible y conjunto óptimo y la función valor óptimo asociadas al problema nominal, a las que denotamos por $\mathcal{F}$, $\mathcal{F}^{op}$ y ϑ, respectivamente (véanse las Secciones 0.1 y 0.2 para las definiciones formales). La primera parte de nuestra investigación se centra en la propiedad de "calmness" (Capítulo 1) y la continuidad Lipschitz (Capítulo 2) de la función valor óptimo restringida a su dominio (donde tiene valor finito), denotada por ϑ^R, en el contexto de la Programación Lineal (PL, para abreviar), es decir, cuando el conjunto de índices T es finito. En la segunda parte, generalizamos el contexto considerando T arbitrario. Aquí nos centramos en la *semicontinuidad inferior Lipschitz* (Lipschitz-lsc por brevedad; del inglés *Lipschitz lower semicontinuity*) de $\mathcal{F}$ (Capítulo 3). En el caso particular donde T es infinito estamos en el contexto de la Programación Lineal Semi-Infinita (PLSI, para abreviar). Dentro del campo del Análisis Variacional, las propiedades de tipo Lipschitz tienen especial importancia; con respecto a este punto, remitimos al lector a las monografías [18, 37, 43, 51] y a las referencias que se incluyen en ellas.

Teniendo en cuenta los resultados previos existentes en la literatura, cabe mencionar que el cálculo de los módulos de calmness y Lipschitz de las multifunciones $\mathcal{F}$ y $\mathcal{F}^{op}$ ya ha sido abordado. Así pues, el estudio de calmness y la continuidad Lipschitz de ϑ surge de una manera natural. Concretamente, en [13] y [9] se dan las fórmulas exactas de los módulos de calmness de $\mathcal{F}$ y $\mathcal{F}^{op}$, respectivamente (véase también [33] y [39] para la propiedad de calmness de $\mathcal{F}$ y $\mathcal{F}^{op}$ en contextos no lineales). En cuanto al módulo de Lipschitz, véanse las referencias [8], [14], [41] y [42].

Esta línea de investigación puede integrarse también en el amplio campo del análisis de sensibilidad, donde, desde diferentes enfoques, se pretende responder a la pregunta natural de *qué* ocurre con el valor óptimo *si* se modifican los datos del problema nominal. Específicamente, nos centramos en un *aspecto local* del análisis de sensibilidad con respecto a la teoría clásica de la optimización lineal paramétrica, la cual normalmente se ocupa del comportamiento de ϑ^R y $\mathcal{F}^{op}$ en el dominio de $\mathcal{F}^{op}$ o en algunos subconjuntos de él. A lo largo del presente trabajo consideramos distintos tipos de perturbaciones del problema nominal, y en cada uno de dichos contextos de perturbaciones pretendemos calcular (o al menos estimar) los módulos de tipo Lipschitz. Este proceso está dirigido no solo a saber cuánto varía un conjunto de soluciones (factibles u óptimas) con respecto a la variación de los parámetros del modelo, sino que también queremos saber cómo varía. Por ejemplo, en qué dirección (o conjunto de direcciones) varían (se expanden o se contraen) estos conjuntos de soluciones, o en qué dirección debemos perturbar los parámetros para obtener la máxima variación.

La teoría sobre optimización lineal paramétrica se remonta a los primeros tiempos de la PL (véanse, por ejemplo, [21] y [52]). En los años 70 empezó un desarrollo sistemático de la PL con *perturbaciones canónicas*, donde una línea de investigación se centró en el comportamiento de ϑ^R. En concreto, la continuidad de ϑ^R fue demostrada a través de distintos enfoques: vía el análisis paramétrico (véase [44]), desde un enfoque paramétrico usando la teoría de Berge (véanse [3, 6]) y a través de un planteamiento primal-dual (véanse [34] y [57]). Una segunda línea de desarrollo se centró en el estudio de propiedades de semicontinuidad y semicontinuidad tipo Lipschitz. Dicho estudio se basó en los enfoques del análisis variacional como la teoría de Berge o las cotas de error de Hoffman (véanse [3, 17, 19, 40, 47, 49, 55, 57]). En el presente trabajo la continuidad en el sentido de Painlevé-Kuratowski de las multifunciones $\mathcal{F}$ y $\mathcal{F}^{op}$ restringidas a sus dominios juega un papel importante; remitimos a la Sección 0.4 para más detalles y referencias sobre estos resultados.

Desde los años 90 hasta hoy, ambas líneas de investigación han tomado gran interés; véase [56] sobre diferentes enfoques de la sensibilidad, y la monografía [20]. Véase también [22] para el estudio de regiones en las que ϑ es afín, y [1, 24, 31, 32] para un planteamiento del análisis de sensibilidad desde una perspectiva de partición óptima, relacionada con el conjunto soporte invariante. Para extensiones a la PLSI referenciamos a [27], [29] y [30]. En el contexto de los sistemas cónicos lineales (que incluyen nuestro contexto como caso particular), los trabajos pioneros de J. Renegar, [45] y [46], proporcionan un enfoque cuantitativo de la estabilidad de problemas de optimización usando la *distancia a la infactibilidad*.

Como ya hemos avanzado, el Capítulo 1 se centra en la propiedad de calmness de ϑ^R. El estudio de esta propiedad se realiza mediante el uso las propiedades de calmness por debajo y calmness por arriba, las cuales, en términos informales, miden la tasa local de decrecimiento y crecimiento, respectivamente, con respecto al problema nominal. En este capítulo calculamos (en algunos casos estimamos) los módulos de calmness por debajo y calmness por arriba bajo distintos tipos de perturbaciones. En la Sección 1.1 trabajamos con perturbaciones del lado derecho de las restricciones, (perturbaciones de b). En la Sección 1.2 perturbamos exclusivamente los coeficientes de la función objetivo (perturbaciones de c). Tras estudiar estos dos casos, completamos los resultados obtenidos abordando el caso de las conocidas *perturbaciones canónicas*, es decir, perturbaciones simultáneas de b y c (véase la Sección 1.3).

Hasta nuestro conocimiento, las contribuciones de este trabajo sobre el cálculo (o estimación) de los módulos de calmness, contenidas entre los Teoremas 28 y 46 y los Corolarios 32 y 44, son nuevas. Como antecedentes inmediatos de esta parte del trabajo, debemos hacer referencia a [45] (véanse también [46]) y [56]. Con el objetivo de integrar mejor nuestro resultados en la literatura, en la sección final del capítulo hemos incluido un análisis comparativo entre el Teorema 46 y una cierta consecuencia de [45, Teorema 1.1(5)]. Para ser más precisos, a partir de [45, Teorema 1.1(5)] se puede obtener de forma inmediata una cota superior para el módulo de calmness de ϑ en términos de las distancias a la infactibilidad primal y dual, siempre y cuando el problema nominal esté en el interior del dominio de $\mathcal{F}^{op}$. En este caso, en el Teorema 46 damos una fórmula exacta para el módulo de calmness de ϑ que resulta ser un refinamiento del Corolario 48 en tanto que la cota superior citada anteriormente podría estar alejada del valor exacto de dicho módulo (véase la Observación 49). Por otro lado, [56, Teorema 18], trasladado a nuestra notación, proporciona una constante particular k_1 para la propiedad de calmness por debajo (0.12) bajo perturbaciones de b.

En el Capítulo 2 se trata la continuidad Lipschitz de ϑ^R en el mismo contexto que en el Capítulo 1, sabiendo que, para funciones arbitrarias es una propiedad más fuerte que calmness. En términos informales, el módulo de Lipschitz proporciona una medida local de la mayor tasa de variación del valor óptimo con respecto a perturbaciones del problema nominal. Mientras calmness compara el valor óptimo del problema nominal con el valor óptimo de un problema perturbado, Lipschitz compara los valores óptimos de dos problemas perturbados (distintos) alrededor del nominal. Este hecho pone de manifiesto la existencia de diferencias notables entre ambas propiedades y, por ende, en el cálculo de sus módulos. Aunque aprovechamos los antecedentes sobre calmness, las contribuciones del Capítulo 2 no son consecuencia directa de las del Capítulo 1, como así enfatizamos en las pruebas. En este punto, adelantamos ya el uso que haremos de una estrategia clave (inspirada por [42, Sección 2]) basada en calcular el módulo de Lipschitz a través de una constante uniforme de calmness.

Como antecedente inmediato en la literatura al Capítulo 2 citamos [12] donde, en lugar de ϑ^R, se trabaja con la función valor óptimo, ϑ, definida en todo el espacio (por lo que puede tomar valores en la recta real extendida). Como contrapartida, el estudio local se realiza alrededor de un problema situado en el interior del dominio de ϑ. Esta condición de interioridad permite caracterizar la continuidad Lipschitz de ϑ en tal problema (este hecho se verifica en el contexto más general de la PLSI; véase [28, Lema 10.2]) y es equivalente a la bien conocida condición de Slater ("Slater constraint qualification", SCQ de forma abreviada) junto con la acotación del conjunto óptimo nominal (no vacío). Concretamente, [12, Teorema 4.3] da una fórmula para una constante de Lipschitz particular de ϑ en términos de la *distancia al mal planteamiento* (véanse también [45] y [46] en el contexto de problemas cónicos lineales). Cabe señalar que los resultados de este segundo capítulo constituyen una mejora de [12, Teorema 4.3] en diferentes aspectos: (i) no se requiere la hipótesis de interioridad, (ii) el módulo de Lipschitz proporciona –informalmente hablando– la constante de Lipschitz más ajustada y (iii) también estudiamos el caso de perturbaciones parciales (perturbaciones de b y c de forma separada).

La estructura del Capítulo 2 es paralela a la del Capítulo 1. En la Sección 2.1 consideramos el caso de perturbaciones de b, donde en el Teorema 51 se obtiene la fórmula exacta del módulo de Lipschitz de ϑ^R en un problema nominal. La Sección 2.2 se desarrolla en el contexto de perturbaciones de c, y su resultado principal es el Teorema 54, donde se dan cotas inferiores y superiores para el correspondiente módulo de Lipschitz. El valor exacto se consigue cuando el conjunto óptimo nominal es no vacío y acotado. La última sección aborda el contexto de las perturbaciones canónicas. El Teorema 56 da una cota inferior del módulo de Lipschitz y, por otro lado, el Teorema 58 da una cota superior basándose en una cierta constante uniforme de calmness obtenida previamente en el Lema 57. Además, en el Teorema 58 probamos que dicha cota superior se alcanza cuando el conjunto óptimo nominal es no vacío y acotado.

Por otra parte, Lipschitz-lsc cuantifica el comportamiento semicontinuo inferiormente de una multifunción en términos de la propiedad de Lipschitz (véase [37]). En nuestro contexto, dicho comportamiento permite medir la tasa de contracción de la multifunción conjunto factible bajo perturbaciones de los datos. Esta propiedad ha sido estudiada por otros autores con anterioridad y puede usarse como una vía para caracterizar otras propiedades de tipo Lipschitz como la propiedad de Aubin o la de calmness; véanse, por ejemplo, [23], [37] y [39]. Lipschitz-lsc ha sido estudiada en particular por A. Uderzo en [53] y [54]. En concreto, [53, Teorema 4.1] proporciona una condición suficiente para Lipschitz-lsc del sistema variacional asociado a un problema paramétrico general, mientras que [54, Proposición 1] se centra en la multifunción solución en el mismo contexto. Para problemas de optimización paramétrica con restricciones, en [53, Proposición 6.1 y Corolarios 6.1 y 6.2] y [54, Theorem 2] se dan condiciones para que se verifique Lipschitz-lsc de las multifunciones conjunto factible y conjunto óptimo.

El tercer capítulo pretende ser una nueva línea de investigación y, en parte, constituye un cambio considerable con respecto al contexto anterior. Como se ha mencionado al inicio, el contexto paramétrico ahora trata con problemas de optimización lineales donde no necesariamente hay un número finito de restricciones, en otras palabras, T es arbitrario y, en particular, puede ser finito (PL) o infinito (PLSI). En esta parte nos centramos en Lipschitz-lsc de $\mathcal{F}$ por lo que cambiamos el enfoque funcional usado en los Capítulos 1 y 2 y usamos argumentos relativos a las multifunciones. Sobre las perturbaciones, además de perturbaciones de b como en los capítulos anteriores, también consideramos perturbaciones del lado izquierdo de las restricciones (perturbaciones de a). El primer objetivo ahora es clarificar la relación que existe entre la propiedad de Aubin y la de Lipschitz-lsc. Este enfoque ha dado lugar a estudiar al mismo tiempo una propiedad intermedia entre Aubin y Lipschitz-lsc, a la que hemos llamado *semicontinuidad Lipschitz-estrella* (Lipschitz-lsc*, por abreviar); véase Sección 0.2 para las definiciones. El punto de partida para abordar estas cuestiones y los antecedentes se resumen en la Proposición 60, donde se dan equivalencias entre propiedades tipo Lipschitz y otros resultados bien conocidos. De esta forma, observamos que la equivalencia entre Lipschitz-lsc y la propiedad de Aubin no se tiene cuando no intervienen perturbaciones de b. Así pues, en la Sección 3.1 nos centramos en estudiar qué ocurre con Lipschitz-lsc* en relación con la propiedad de Aubin cuando solo se perturba el parámetro a. En el Corolario 69 (para T arbitrario) y en el Teorema 73 (para T finito) se demuestran las principales diferencias entre ambos tipos de Lipschitz-lsc (con respecto a la propiedad de Aubin). La Sección 3.2 está destinada al cómputo de los módulos de Lipschitz-lsc y Lipschitz-lsc* bajo perturbaciones simultáneas de a y b. El antecedente directo a esta parte del trabajo es [7, Teorema 1] donde los autores proporcionan una expresión exacta del módulo de Lipschitz de $\mathcal{F}$ en términos del módulo de regularidad métrica de $\mathcal{F}^{-1}$ (véase [37, Lema 1.12]). Como veremos en el Teorema 78, todos estos módulos resultan ser iguales.

Los Capítulos 1, 2 y 3 contienen las contribuciones originales de la tesis. La mayor parte de los resultados sobre calmness del valor óptimo han sido previamente incluidos en un artículo que ya está publicado (véase [25]). Sobre la parte de la continuidad Lipschitz del valor óptimo, se ha escrito también otro artículo el cual ya ha sido enviado para su posible publicación (véase [26]). Por su parte, los contenidos que forman el Capítulo 3 han sido mayoritariamente desarrollados durante mi estancia de investigación en Berlín en el "Weierstrass Institute for Applied Analysis and Stochastics" (WIAS), del 1 de febrero al 4 de mayo de 2018, bajo la supervisión del Dr. René Henrion. Nuestra intención es que este trabajo forme parte de una nueva publicación en un futuro cercano. Además de estos tres capítulos mencionados, el presente documento incluye el Capítulo 0, donde se presenta el modelo que consideramos, la notación, los objetivos principales de nuestra investigación con más detalle, así como las definiciones formales, las herramientas clave y los resultados preliminares necesarios. Para completar el trabajo, terminamos con una sección no numerada llamada Conclusiones y trabajo futuro. Los resultados más relevantes obtenidos sobre la propiedad de calmness y la continuidad Lipschitz están recogidos en las Tablas 3.3 y 3.4, respectivamente. Además, hacemos hincapié en las relaciones existentes entre estas dos propiedades y sus respectivos módulos. Por otro lado, también resumimos los principales resultados obtenidos en el Capítulo 3 y damos unas pinceladas sobre futuras líneas de investigación. Esta investigación ha sido realizada con el soporte de la beca BES-2015-073220 asociada al proyecto MTM2014-59179-C2-2-P del MINECO y FEDER "Una manera de hacer Europa".

Chapter 0

Preliminaries

0.1 Notation and definitions

We consider the general parameterized linear optimization problem

$$\begin{array}{rll} \pi: & \text{Min} & c'x \\ & \text{s.t.} & a_t'x \leq b_t, \;\; t \in T \;, \end{array} \tag{0.1}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, $c \in \mathbb{R}^n$, $a \equiv (a_t)_{t \in T} \in (\mathbb{R}^n)^T$ and $b \equiv (b_t)_{t \in T} \in \mathbb{R}^T$ are the parameters to be perturbed around nominal ones. All elements in $\mathbb{R}^n$ are regarded as column-vectors and y' denotes the transpose of $y \in \mathbb{R}^n$. The linear inequality system is denoted by

$$\sigma := \left\{ a_t'x \leq b_t, \;\; t \in T \right\} \;. \tag{0.2}$$

In this general setting the index set T is arbitrary (without specific topological structure) and the functions $t \mapsto a_t$ and $t \mapsto b_t$ have no particular property.

Chapters 1 and 2 are focused on the stability of problem (0.1) when T is finite and parameter a is fixed, say $\overline{a} \equiv (\overline{a}_t)_{t \in T}$. For the sake of simplicity in the notation, we identify the parameters to be perturbed (c, b) with the associated optimization problem π, i.e., $\pi \equiv (c, b)$. This is the context of the so-called *canonical perturbations*, where the right-hand-side (RHS) of the constraints and the objective function coefficients are allowed to be perturbed simultaneously.

In Chapter 3, we study the stability of system (0.2) under perturbations of a or b, i.e., under left-hand-side (LHS) or RHS perturbations, allowing the possibility of perturbing both simultaneously. Again, for the sake of simplicity, we identify system σ with the parameters to be perturbed around nominal ones, that is, $\sigma \equiv (a, b)$. Also for simplicity, we denote by Θ the parameter space $(\mathbb{R}^n \times \mathbb{R})^T$. In this part of the work, T is considered arbitrary in general, although some results concern the particular case when T is finite.

Associated with the previous parameterized problem, and taking into account our framework, we consider the *feasible set mapping* $\mathcal{F} : \Theta \rightrightarrows \mathbb{R}^n$, given by

$$\mathcal{F}(\sigma) := \left\{x \in \mathbb{R}^n \mid a_t' x \leq b_t, \;\; t \in T\right\} \ ,$$

the *optimal value function* $\vartheta : \mathbb{R}^n \times \mathbb{R}^T \to [-\infty, +\infty]$, given by

$$\vartheta(\pi) := \inf\{c' x \mid x \in \mathcal{F}(\overline{a}, b)\} \ ,$$

(with the convention $\vartheta(\pi) := +\infty$ when $\mathcal{F}(\overline{a}, b) = \emptyset$), and the *optimal set mapping* $\mathcal{F}^{op} : \mathbb{R}^n \times \mathbb{R}^T \rightrightarrows \mathbb{R}^n$, given by

$$\mathcal{F}^{op}(\pi) := \left\{x \in \mathcal{F}(\overline{a}, b) \mid c' x = \vartheta(\pi)\right\} \ .$$

The *set of active indices* at $x \in \mathcal{F}(\sigma)$, for $\sigma \in \Theta$, is denoted by $T_\sigma(x)$; i.e.,

$$T_\sigma(x) := \left\{t \in T \mid a_t' x = b_t\right\} \ . \tag{0.3}$$

If the index set T of system (0.2) is infinite, the problem (0.1) is a LSIP problem. In contrast with ordinary LP, it may happen that there is no optimal solution although the optimal value is finite, as we show in the following example.

Example 1 Consider the problem

$$\begin{array}{ll} \text{Inf} & x_1 \\ \text{s.t.} & -tx_1 - \frac{1}{t}x_2 \leq -2\,, \;\; t \in \,]0, +\infty[\;\; . \end{array}$$

The boundary of the feasible set is formed by points $(\frac{1}{t}, t)$, with $t > 0$, hence the optimal value is 0 while the optimal set is empty.

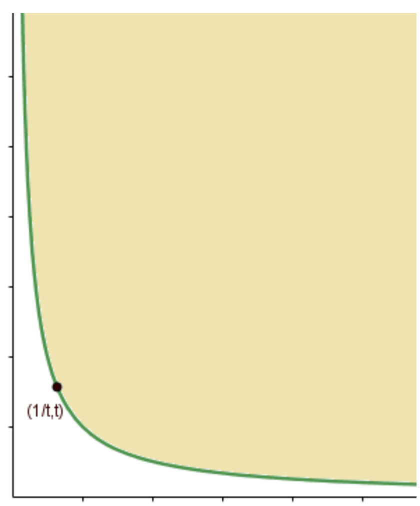

Figure 0.1

When we fix a or b separately, we consider particular feasible set mappings

$$\mathcal{F}_{\overline{a}} : \mathbb{R}^T \rightrightarrows \mathbb{R}^n \ ,$$

and

$$\mathcal{F}_{\overline{b}} : (\mathbb{R}^n)^T \rightrightarrows \mathbb{R}^n \ ,$$

defined as

$$\mathcal{F}_{\overline{a}}(b) := \mathcal{F}(\overline{a}, b) \text{ and } \mathcal{F}_{\overline{b}}(a) := \mathcal{F}(a, \overline{b}) \ ,$$

respectively.

Given $X \subset \mathbb{R}^p$, $p \in \mathbb{N}$, we denote by conv$X$, cone$X$, aff$X$, and span$X$, the *convex hull*, the *conical convex hull,* the *affine hull,* and the *linear hull* of X, respectively. Moreover, $X^{\perp}$ denotes the orthogonal complement of span$X$ (with respect to the usual inner product), and, provided that X is convex, extr$X$ stands for the set of extreme points of X. Recall that $x \in$ extr$X$ means that it is impossible to express x as a convex combination of two points of $X \backslash \{x\}$. It is assumed that cone$X$ always contains the zero-vector 0_p, in particular cone $\emptyset = \{0_p\}$.

From the topological side, if X is a subset of any topological space, int$X$, cl$X$ and bd$X$ stand, respectively, for the interior, the closure, and the boundary of X.

Along this work the space of variables $\mathbb{R}^n$ is equipped with an arbitrary norm, $\|\cdot\|$, with *dual norm* is, as usual, denoted by $\|\cdot\|_*$ and defined as

$$\|u\|_* = \max_{\|x\| \leq 1} \left| u'x \right| \ .$$

In Chapters 1 and 2, the parameter spaces $\mathbb{R}^T$, $\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^T$ (associated with the contexts of RHS perturbations, c-perturbations and canonical perturbations) are endowed, respectively, with the norms

$$\|b\|_\infty := \max_{t \in T} |b_t| \, , \ \|c\|_* \ \text{ and } \ \|\pi\| := \max \left\{ \|c\|_* \, , \|b\|_\infty \right\} \ . \tag{0.4}$$

Note that the choice of the dual norm $\|\cdot\|_*$ for measuring the perturbations of c comes from the fact that it is seen as the functional $x \mapsto c'x$. Moreover, the use of supremum (maximum indeed) norm for both b and π is a usual choice for measuring errors, and it is followed, for instance, in previous works with the same parametric context, as [9] and [13].

In Chapter 3, since we change the context of the previous two chapters, we consider another way of measuring perturbations (the same as in [7]). Specifically, in Section 3.2 the parameter space Θ is endowed with the *extended distance* $d : \Theta \times \Theta \to [0, +\infty]$ given by

$$d(\sigma^1, \sigma^2) := \sup_{t \in T} \left\{ \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t^2 \\ b_t^2 \end{pmatrix} \right\| \right\} ,$$

where now $\| \cdot \|$ is the norm in $\mathbb{R}^{n+1}$ defined as

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| := \max \{ \|u\|_*, |v| \} \text{ for all } \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^{n+1} . \tag{0.5}$$

We also define the distance between a system and a subset $\widetilde{\Theta}$ of Θ as

$$d(\sigma, \widetilde{\Theta}) := \inf \left\{ d(\sigma, \widetilde{\sigma}), \widetilde{\sigma} \in \widetilde{\Theta} \right\} \text{ for a given } \sigma \in \Theta ,$$

where $d(\sigma, \emptyset) := +\infty$.

0.2 About Lipschitz type properties

Let $\mathcal{G} : Y \rightrightarrows X$ be a multifunction between metric spaces Y and X, with both distances denoted by d (see [51]), and let $(\overline{y}, \overline{x}) \in \text{gph}\mathcal{G}$ where $\text{gph}\mathcal{G}$ denotes the graph of $\mathcal{G}$. It is said that $\mathcal{G}$ verifies the *Aubin property* (cf. [51]) at $(\overline{y}, \overline{x})$ if there exist a constant $\kappa \geq 0$ and neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that

$$d\left(x, \mathcal{G}(\widetilde{y})\right) \leq \kappa d\left(y, \widetilde{y}\right) \text{ for all } x \in \mathcal{G}(y) \cap U \text{ and all } y, \widetilde{y} \in V . \tag{0.6}$$

This property is also called in the literature *Aubin continuity*, *pseudo-Lipschitz* (cf. [37]) or *Lipschitz-like* (cf. [43]). The infimum of those constants κ for which (0.6) holds (for some associated neighborhoods) is called the *Lipschitz modulus* of $\mathcal{G}$ at $(\overline{y}, \overline{x})$ and it is denoted by $\mathrm{lip}\mathcal{G}(\overline{y}, \overline{x})$. By convention, $\mathrm{lip}\mathcal{G}(\overline{y}, \overline{x}) = +\infty$ when $\mathcal{G}$ does not satisfy the Aubin property at $(\overline{y}, \overline{x})$.

It is well known that the Aubin property of a multifunction is equivalent to the *metric regularity* of its inverse mapping [37, Remark 1.11 and Lemma 1.12]. Recall that $\mathcal{G}^{-1} : X \rightrightarrows Y$ is given by $y \in \mathcal{G}^{-1}(x) \Leftrightarrow x \in \mathcal{G}(y)$. Formally, $\mathcal{G}^{-1}$ is said to be *metrically regular* at $(\overline{x}, \overline{y})$ if there exist some neighborhoods U of $\overline{x}$ and V of $\overline{y}$, and a constant $\kappa \geq 0$ such that

$$d\left(x, \mathcal{G}(y)\right) \leq \kappa d\left(y, \mathcal{G}^{-1}(x)\right) \text{ for all } x \in U \text{ and } y \in V \ .$$

It is also known that $\mathrm{lip}\mathcal{G}(\overline{y}, \overline{x}) = \mathrm{reg}\mathcal{G}^{-1}(\overline{x}|\overline{y})$, where $\mathrm{reg}\mathcal{G}^{-1}(\overline{x}|\overline{y})$ denotes the *modulus of metric regularity* (or *regularity modulus*) of $\mathcal{G}^{-1}$ at $\overline{x}$ for $\overline{y} \in \mathcal{G}^{-1}(\overline{x})$.

We can find other (weaker) definitions by fixing the variables involved in (0.6). In this sense, if we fix $\widetilde{y} = \overline{y}$ we obtain *calmness* property while if we fix $y = \overline{y}$ we obtain a different property without an specific name up to our knowledge. Let us call it from now on *Lipschitz lower semicontinuity** (Lipschitz-lsc*, in brief). Once we have fixed $y = \overline{y}$, if we additionally fix $x = \overline{x}$ we get the *Lipschitz lower semicontinuity* (cf. [37]; Lipschitz-lsc, in brief). Formally, $\mathcal{G}$ is said to be *calm* at $(\overline{y}, \overline{x})$ if there exist a constant $\kappa \geq 0$ and neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that

$$d\left(x, \mathcal{G}(\overline{y})\right) \leq \kappa d\left(y, \overline{y}\right) \text{ for all } x \in \mathcal{G}(y) \cap U \text{ and } y \in V \ . \tag{0.7}$$

We say that $\mathcal{G}$ is *Lipschitz-lsc** at $(\overline{y}, \overline{x})$ if there exist a constant $\kappa \geq 0$ and neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that

$$d\,(x, \mathcal{G}(\widetilde{y})) \leq \kappa d\,(\overline{y}, \widetilde{y}) \;\text{ for all } x \in \mathcal{G}(\overline{y}) \cap U \text{ and } \widetilde{y} \in V \;. \tag{0.8}$$

Finally, $\mathcal{G}$ is said to be *Lipschitz-lsc* at $(\overline{y}, \overline{x})$ if there exist a constant $\kappa \geq 0$ and a neighborhood V of $\overline{y}$ such that

$$d\,(\overline{x}, \mathcal{G}(y)) \leq \kappa d\,(\overline{y}, y) \;\text{ for all } y \in V \;. \tag{0.9}$$

The respective moduli are analogously defined like the Lipschitz modulus and they are denoted by $\text{clm}\mathcal{G}(\overline{y}, \overline{x})$, $\text{liplsc*}\mathcal{G}(\overline{y}, \overline{x})$ and $\text{liplsc}\mathcal{G}(\overline{y}, \overline{x})$, respectively. It is easy to see that

$$\text{clm}\mathcal{G}(\overline{y}, \overline{x}) \leq \text{lip}\mathcal{G}(\overline{y}, \overline{x}) \;,$$

and

$$\text{liplsc}\mathcal{G}(\overline{y}, \overline{x}) \leq \text{liplsc}^*\mathcal{G}(\overline{y}, \overline{x}) \leq \text{lip}\mathcal{G}(\overline{y}, \overline{x}) \;.$$

In general, the latter inequality may be strict as we show in the next example. Moreover, we also show in it that there is no a general ordering relation between $\text{clm}\mathcal{G}(\overline{y}, \overline{x})$ and $\text{liplsc}\mathcal{G}(\overline{y}, \overline{x})$ or $\text{liplsc}^*\mathcal{G}(\overline{y}, \overline{x})$.

Example 2 Consider the mapping $\mathcal{G}(y) :=]-\infty, f(y)]$, where function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$f(y) := \begin{cases} 2y + y^2 \sin \frac{1}{y} & \textit{if} \quad y > 0 \\ 0 & \textit{if} \quad y = 0 \\ \alpha y & \textit{if} \quad y < 0 \;, \end{cases}$$

with $0 \leq \alpha \leq 3$. Since

$$f'(y) := \begin{cases} 2 + 2y \sin \frac{1}{y} - \cos \frac{1}{y} & \textit{if} \quad y > 0 \\ \alpha & \textit{if} \quad y < 0 \;, \end{cases}$$

we have that $f'_+(0) = 2$ and $f'_-(0) = \alpha$, where $f'_+(0)$ and $f'_-(0)$ denote the *right and left derivatives* of function f at point 0. Nevertheless, $\lim_{y\to 0^+} f'(y)$ does not exist since, clearly, $\liminf_{y\to 0^+} f'(y) = 1$ and $\limsup_{y\to 0^+} f'(y) = 3$. It can be showed that $\text{lip}\mathcal{G}(0,0) = 3$ (by using the Mean Value Theorem) while $\text{clm}\mathcal{G}(0,0) = 2$ and $\text{liplsc}\mathcal{G}(0,0) = \text{liplsc}^*\mathcal{G}(0,0) = \alpha$.

From the definitions it is obvious that Aubin property implies calmness and Lipschitz-lsc*, and the latter implies Lipschitz-lsc, but the converse implications are not held in general. We can find in the literature some specific counterexamples: Example 3.2 in [53] shows that calmness does not imply Aubin property and Examples 1 and 2 in [54] show that there is no relationship between calmness and Lipschitz-lsc. Moreover, Lipschitz-lsc does not imply Lipschitz-lsc* in general and then neither Aubin property as we show in the next example. This example is used in [39, p. 711] to show that calmness together with Lipschitz-lsc does not imply Aubin property in general.

Example 3 Consider the multifunction $\mathcal{G} : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as $\mathcal{G}(0) = \mathbb{R}$ and $\mathcal{G}(y) = \{0\}$ when $y \neq 0$. It is easy to see that $\mathcal{G}$ is Lipschitz-lsc at $(0,0)$ but not Lipschitz-lsc* at the same point.

On the other hand, note that Lipschitz-lsc (and, hence, Lipschitz-lsc*) of $\mathcal{G}$ at $(\overline{y}, \overline{x})$ ensures the lower semicontinuity of the multifunction at $\overline{y}$. Recall that $\mathcal{G}$ is *lower semicontinuous in the sense of Berge* (lsc, in brief) at $\overline{y}$ if for all open set U verifying $\mathcal{G}(\overline{y}) \cap U \neq \emptyset$ there exists an open neighborhood V of $\overline{y}$ such that $\mathcal{G}(y) \cap U \neq \emptyset$ for all $y \in V$ (cf. [3, Section 2.2]). However, the converse statement is not true in general. Example 3.1 in [53] is a clear situation where lsc at $\overline{y}$ does not imply Lipschitz-lsc at $(\overline{y}, \overline{x})$.

Going further on all these properties, Aubin continuity at $(\overline{y}, \overline{x})$ can be understood as a quantitative way to measure the local deformation or distortion of $\mathcal{G}(\overline{y})$. In this sense, calmness property controls the rate of shift/expansion while Lipschitz-lsc measures the shift/contraction one. From this point of view, we can ask ourselves what happens with a multifunction which is calm and Lipschitz-lsc at the same time (recall Example 3). On this subject, it is usual to find in the literature that Lipschitz-lsc appears with the name *inner calmness* (see [4, Definition 2.2]). In [23, Theorem 3.8] the authors provide some sufficient assumptions to obtain calmness and Lipschitz-lsc of a class of solution maps. More recently, J. V. Outrata named the double property of being calm and Lipschitz-lsc as *two-sided calmness* in his contribution *On two variants of calmness and their verification in a class of solution maps* to the 11th International Conference on Parametric Optimization and Related Topics XI (Prague, September 19-22, 2017).

An analogous definition but invoking Lipschitz-lsc* instead of Lipschitz-lsc was given earlier by D. Klatte in [35] and was called *pseudo-Lipschitz**. That is the reason why we call property (0.8) *Lipschitz-lsc**.

All previous facts give us new possibilities of research and constitute the starting point for study the relationships between all the previous properties in different contexts of parametric optimization problems. In Chapter 3, we focus on the case of the feasible set mapping $\mathcal{F}$ associated with system (0.2).

Otherwise, main part of our research is focussed on the calmness and Lipschitz of function ϑ at a nominal parameter $\overline{\pi}$ such that $\vartheta(\overline{\pi})$ is finite (see Chapters 1 and 2). It is well known that a function $f : \mathbb{R}^p \to [-\infty, +\infty]$, $p \in \mathbb{N}$, is said to be *Lipschitz continuous* at $\overline{z} \in \mathbb{R}^p$ (with $f(\overline{z})$ finite) if there exist a constant $\kappa \geq 0$ and a neighborbood U of $\overline{z}$ such that

$$|f(z) - f(\widetilde{z})| \leq \kappa \|z - \widetilde{z}\|, \text{ for all } z, \widetilde{z} \in U \ . \tag{0.10}$$

The infimum of those constants κ for which (0.10) holds, for some associated neighborhood, is the Lipschitz modulus of f at $\overline{z}$. Alternatively, it can be expressed as

$$\mathrm{lip} f(\overline{z}) = \limsup_{\substack{z, \widetilde{z} \to \overline{z} \\ z \neq \widetilde{z}}} \frac{|f(z) - f(\widetilde{z})|}{\|z - \widetilde{z}\|} \ ,$$

under the convention $\infty - \infty := 0$.

The Lipschitz modulus of f is related to its calmness modulus, as in the case of multifunctions. In fact, a function f is defined as calm by also fixing $\widetilde{z} = \overline{z}$ in (0.10); the same occurs for its corresponding modulus. Obviously,

$$\mathrm{clm} f(\overline{z}) \leq \mathrm{lip} f(\overline{z}) \ . \tag{0.11}$$

The concept of calmness of f can be also introduced through the simultaneous fulfilment of the so-called calmness from below and calmness from above (see, e.g., [51, Section 8.F]). Let $\overline{z} \in \mathbb{R}^p$ be such that $f(\overline{z})$ is finite; recall that f is *calm at* $\overline{z}$ *from below* if there exist a constant $\kappa_1 \geq 0$ and a neighborhood U_1 of $\overline{z}$ such that

$$f(\overline{z}) - f(z) \leq \kappa_1 \|z - \overline{z}\|, \text{ for all } z \in U_1 \ . \tag{0.12}$$

Respectively, f is *calm at* $\overline{z}$ *from above* if

$$f(z) - f(\overline{z}) \leq \kappa_2 \|z - \overline{z}\|, \text{ for all } z \in U_2 \ , \tag{0.13}$$

for some constant $\kappa_2 \geq 0$ and some neighborhood U_2 of $\overline{z}$.

Along this work, the infimum of those constants κ_1 and κ_2 for which (0.12) and (0.13), respectively, hold (for some associated neighborhoods) are called the *calmness modulus from below* and *above* of f at $\overline{z}$, and they are denoted by $\underline{\mathrm{clm}}f(\overline{z})$ and $\overline{\mathrm{clm}}f(\overline{z})$, respectively; these moduli can alternatively be expressed as

$$\underline{\mathrm{clm}}f(\overline{z}) = \limsup_{z\to\overline{z}} \frac{f(\overline{z}) - f(z)}{\|z-\overline{z}\|} \text{ and } \overline{\mathrm{clm}}f(\overline{z}) = \limsup_{z\to\overline{z}} \frac{f(z) - f(\overline{z})}{\|z-\overline{z}\|}\ . \tag{0.14}$$

Remark 4 Hereinafter, in expressions of this kind we allow the possibility of approaching $\overline{z}$ by constants sequences under the convention $\frac{0}{0} := 0$; so, $\underline{\mathrm{clm}}f(\overline{z})$ and $\overline{\mathrm{clm}}f(\overline{z})$ are always non-negative. Alternatively, in order to ensure its nonnegativity , we could define these moduli as the maximum between 0 and the corresponding '$\limsup$' in (0.14) with $z \to \overline{z}$, $z \neq \overline{z}$. Observe that, for $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(z) := |z|$ and $\overline{z} := 0$, we have $\limsup_{z\to\overline{z}, z\neq\overline{z}} \frac{f(\overline{z})-f(z)}{|z-\overline{z}|} = -1$, while, under our convention, $\limsup_{z\to\overline{z}} \frac{f(\overline{z})-f(z)}{|z-\overline{z}|} = 0$.

Combining both definitions, f is said to be *calm at* $\overline{z}$ if it is calm from below and above at $\overline{z}$, and the *calmness modulus* of f at $\overline{z}$, $\mathrm{clm}f(\overline{z})$, is defined as

$$\mathrm{clm}f(\overline{z}) := \limsup_{z\to\overline{z}} \frac{|f(z) - f(\overline{z})|}{\|z-\overline{z}\|} = \max\left\{\underline{\mathrm{clm}}f(\overline{z}), \overline{\mathrm{clm}}f(\overline{z})\right\}\ .$$

Note that $\underline{\mathrm{clm}}f(\overline{z})$ is nothing else but the *strong slope* of f at $\overline{z}$, while $\overline{\mathrm{clm}}f(\overline{z})$ corresponds to that of $-f$ (see, e.g., [2]). Roughly speaking, they respectively provide measures of maximum rates of decrease and increase of f at $\overline{z}$.

Coming back to the optimal value function ϑ, in the case when T is finite it is well known that $\vartheta(\pi)$ is finite if and only if $\mathcal{F}^{op}(\pi) \neq \emptyset$; i.e., if and only if $\pi \in \mathrm{dom}\mathcal{F}^{op}$ (the domain of $\mathcal{F}^{op}$). The following remark motivates the fact of considering ϑ restricted to $\mathrm{dom}\mathcal{F}^{op}$, denoted by ϑ^R (the notation is inspired by [16], where the feasible set mapping restricted to its domain is analyzed); so,

$$\vartheta^R : \mathrm{dom}\mathcal{F}^{op} \to \left]-\infty, +\infty\right[\ . \tag{0.15}$$

We aim to compute the corresponding calmness and Lipschitz moduli of ϑ^R at $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$ which are given by

$$\mathrm{clm}\vartheta^R(\overline{\pi}) = \limsup_{\pi \underset{\mathrm{dom}\mathcal{F}^{op}}{\longrightarrow} \overline{\pi}} \frac{|\vartheta(\pi) - \vartheta(\overline{\pi})|}{\|\pi - \overline{\pi}\|} \ ,$$

and

$$\mathrm{lip}\vartheta^R(\overline{\pi}) = \limsup_{\pi, \widetilde{\pi} \underset{\mathrm{dom}\mathcal{F}^{op}}{\longrightarrow} \overline{\pi}} \frac{|\vartheta(\pi) - \vartheta(\widetilde{\pi})|}{\|\pi - \widetilde{\pi}\|} \ ,$$

where $\pi \underset{\mathrm{dom}\mathcal{F}^{op}}{\longrightarrow} \overline{\pi}$ means that $\pi \to \overline{\pi}$ and $\pi \in \mathrm{dom}\mathcal{F}^{op}$. The calmness moduli of ϑ^R from below and above are analogously defined. From now on we also appeal to the *Slater constraint qualification* (SCQ, in brief) which is satisfied at $\sigma \in \mathrm{dom}\mathcal{F}$ if there exists $\widehat{x} \in \mathbb{R}^n$ (called a *Slater point*) such that $a_t'\widehat{x} < b_t$ for all $t \in T$.

Remark 5 It is clear from the definitions that if $\overline{\pi} \in \mathrm{intdom}\mathcal{F}^{op}$ (the interior of $\mathrm{dom}\mathcal{F}^{op}$), then

$$\mathrm{clm}\vartheta^R(\overline{\pi}) = \mathrm{clm}\vartheta(\overline{\pi}) \ \text{ and } \ \mathrm{lip}\vartheta^R(\overline{\pi}) = \mathrm{lip}\vartheta(\overline{\pi}) \ .$$

It is well known that condition $\overline{\pi} \in \mathrm{intdom}\mathcal{F}^{op}$ is equivalent to ensure that SCQ holds at $\overline{\sigma}$ and $\mathcal{F}^{op}(\overline{\pi})$ is nonempty and bounded (with T finite). These comments extend even for linear semi-infinite problems, i.e., with infinitely many constraints (see Remark 6 and [28, Theorem 6.1 and Lemma 10.2]). The simultaneous fulfilment of these two conditions is frequently used in the literature since they provide high stability of the problem, and it also happens in our context. Theorems 46 and 58 provide exact formulae for

$\text{clm}\vartheta(\overline{\pi})$ and $\text{lip}\vartheta(\overline{\pi})$ in such a case, respectively. Otherwise, if $\overline{\pi}$ is in the boundary of $\text{dom}\mathcal{F}^{op}$, $\text{clm}\vartheta(\overline{\pi}) = +\infty$ and $\text{lip}\vartheta(\overline{\pi}) = +\infty$ (ϑ is not calm neither Lipschitz continuous at $\overline{\pi}$), while we will show that $\text{clm}\vartheta^R(\overline{\pi})$ and $\text{lip}\vartheta^R(\overline{\pi})$ are always finite. In this way, $\text{clm}\vartheta^R(\overline{\pi})$ and $\text{lip}\vartheta^R(\overline{\pi})$ still represent quantitative measures of the stability of our problem $\overline{\pi}$ when either SCQ fails at $\overline{\sigma}$ or $\mathcal{F}^{op}(\overline{\pi})$ is unbounded. On the other hand, if $\mathcal{F}^{op}$ satisfies the Aubin property at $(\overline{\pi}, \overline{x}) \in \text{gph}\mathcal{F}^{op}$, then ϑ is Lipschitz continuous and consequently ϑ^R does (see [37, Corollary 4.7] and [10, Theorem 16]).

Remark 6 (Strong Slater constraint qualification) Actually [28, Theorem 6.1] refers to the *strong Slater constraint qualification* (SSCQ, in brief.). It is said that $\sigma \in \text{dom}\mathcal{F}$ satisfies the SSCQ if there exists $\widehat{x} \in \mathbb{R}^n$ (called a *strong Slater element*, SS element in brief) and a positive scalar ε such that $a_t'\widehat{x} \leq b_t - \varepsilon$ for all $t \in T$. In the case when T is finite, SCQ and SSCQ are equivalent. That is the reason why in Chapters 1 and 2 we appeal to the SCQ, while in Chapter 3 we will use the SSCQ.

Remark 7 Observe that the concept of calmness for our function ϑ^R does not coincide with the corresponding one to the multifunction $\pi \mapsto \left\{\vartheta^R(\pi)\right\}$. The latter does not entail the continuity of function ϑ^R.

0.3 Minimal KKT subsets of indices

This section and the next one are exclusively to be used in Chapters 1 and 2; recall that in these chapters T is finite and we consider $\overline{a} \equiv (a_t)_{t\in T}$. The dual problem (in the sense of Haar) of (0.1) is given by

$$\begin{aligned} \text{Max} \quad & -b'\lambda \\ \text{s.t.} \quad & \sum_{t\in T} \lambda_t \overline{a}_t = -c, \\ & \lambda = (\lambda_t)_{t\in T} \in \mathbb{R}_+^T \ . \end{aligned} \tag{0.16}$$

Let us denote by $\Lambda : \mathbb{R}^n \rightrightarrows \mathbb{R}^T$ and $\Lambda^{op} : \mathbb{R}^n \times \mathbb{R}^T \rightrightarrows \mathbb{R}^T$ the feasible and optimal set mappings corresponding to the family of (dual) problems (0.16); i.e., $\Lambda(c)$ is the feasible set of (0.16), which does not depend on b, and $\Lambda^{op}(\pi)$ denotes the optimal set of (0.16).

From now on, we use the following notation associated with any $\pi \equiv (c, b) \in \mathrm{dom}\mathcal{F}^{op}$. For $x \in \mathcal{F}^{op}(\pi)$, we denote by $\mathcal{K}_\pi(x)$ the following family of subsets of indices involved in the Karush-Kuhn-Tucker (KKT in brief) conditions:

$$\mathcal{K}_\pi(x) := \{D \subset T_b(x) \mid |D| \leq n, \ -c \in \mathrm{cone}\{\overline{a}_t, \ t \in D\}\} \ ,$$

where $|D|$ is the cardinality of D. To ease the notation in this framework, $T_b(x)$ denotes the set of active indices at $x \in \mathcal{F}_{\overline{a}}(b)$ for $b \in \mathbb{R}^T$, i.e., the particular case of $T_\sigma(x)$ defined in (0.3) when we consider a fixed. The condition '$|D| \leq n$' comes from Carathéodory's Theorem. Moreover, we shall appeal to the family of *minimal KKT subsets of indices*

$$\mathcal{M}_\pi(x) := \{D \in \mathcal{K}_\pi(x) \mid D \text{ is minimal for the inclusion order}\} \ , \tag{0.17}$$

which constitutes a key ingredient in the formula of the calmness modulus of $\mathcal{F}^{op}$ established in [9]. Trivially, $\mathcal{M}_\pi(x) = \{\emptyset\}$ when $c = 0_n$.

The next example presents a very simple illustration of subsets $\mathcal{K}_\pi(x)$ and $\mathcal{M}_\pi(x)$, as well as, advances the fact that, indeed, $\mathcal{M}_\pi(x)$ does not depend on the point (see Remark 9).

Example 8 Consider the nominal problem in $\mathbb{R}^2$ with the Euclidean norm:

$$\begin{array}{lll} \overline{\pi}: & \text{Min} & x_1 \\ & \text{s.t.} & -x_1 + x_2 \leq 0 \ , \quad (t=1) \\ & & -x_1 - x_2 \leq 0 \ , \quad (t=2) \\ & & -x_1 \leq -2 \ . \qquad (t=3) \end{array}$$

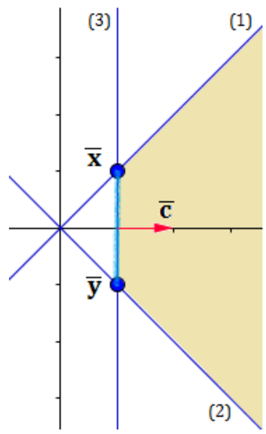

Figure 0.2

It is easy to see that for points $\overline{x} = (2,2)'$ and $\overline{y} = (2,-2)'$ one has $\mathcal{K}_{\overline{\pi}}(\overline{x}) = \{\{3\},\{1,3\}\}$ and $\mathcal{K}_{\bar{\pi}}(\bar{y}) = \{\{3\},\{2,3\}\}$. So, $\mathcal{M}_{\overline{\pi}}(\overline{x}) = \mathcal{M}_{\overline{\pi}}(\overline{y}) = \{\{3\}\}$.

Remark 9 Recall a stardard fact of LP theory: the dual optimal set $\Lambda^{op}(\pi)$ does coincide with the set of KKT multipliers associated with any primal solution $x \in \mathcal{F}^{op}(\pi)$. As a direct consequence, $\mathcal{M}_{\pi}(x)$ does not depend on point x. Formally,

$$\mathcal{M}_{\pi}(x) = \mathcal{M}_{\pi}(y) \, , \ \text{whenever } x, y \in \mathcal{F}^{op}(\pi) \ \ ,$$

and, accordingly, we may remove the optimal point in the notation of the minimal KKT subsets of indices. So, from now on, we just denote

$$\mathcal{M}_\pi := \mathcal{M}_\pi(x), \text{ for any } x \in \mathcal{F}^{op}(\pi) \ ,$$

provided that $\pi \in \mathrm{dom}\mathcal{F}^{op}$.

Remark 10 Observe that,¡ a standard argument of linear algebra (in the line of Carathéodory's Theorem) yields the linear independence of $\{\overline{a}_t, t \in D\}$, whenever $D \in \mathcal{M}_\pi$. Specifically, arguing by contradiction, assume that $\sum_{t\in D} \mu_t \overline{a}_t = 0_n$ for some $(\mu_t)_{t\in D}$; without loss of generality, $\mu_t > 0$, for some $t \in D$. Write $\sum_{t\in D} \lambda_t \overline{a}_t = -c$, for certain $(\lambda_t)_{t\in D}$, and consider $t_0 \in D$ such that $\frac{\lambda_{t_0}}{\mu_{t_0}} = \min\left\{\frac{\lambda_t}{\mu_t} : \mu_t > 0, t \in D\right\}$. Then,

$$-c = \sum_{t\in D\setminus\{t_0\}} \left(\lambda_t - \frac{\lambda_{t_0}}{\mu_{t_0}}\mu_t\right) \overline{a}_t \in \mathrm{cone}\{\overline{a}_t, t \in D \setminus \{t_0\}\} \ ,$$

which contradicts the minimality of D. In this way, for any $D \in \mathcal{M}_\pi$, we define $\lambda^D := \left(\lambda_t^D\right)_{t\in T}$ as the unique element in $\mathbb{R}_+^T$ verifying

$$\sum_{t\in D} \lambda_t^D \overline{a}_t = -c, \text{ and } \lambda_t^D = 0, \text{ whenever } t \in T \setminus D \ . \tag{0.18}$$

Observe that the minimality of D entails $\lambda_t > 0$ for all $t \in D$. In the case $c = 0_n$, we have $\lambda^\emptyset = 0_T$.

Lemma 11 *Let $\pi \in \mathrm{dom}\mathcal{F}^{op}$. We have*

$$\left\{\lambda^D, \ D \in \mathcal{M}_\pi\right\} = \mathrm{extr}\Lambda^{op}(\pi) \ .$$

Proof. Consider the nontrivial case $\mathcal{M}_{\overline{\pi}} \neq \{\emptyset\}$ (otherwise, $c = 0$ and it is clear that $\mathrm{extr}\Lambda^{op}(\overline{b}) = \{0_T\}$). One easily sees (according to the previous remark) that $D \in \mathcal{M}_\pi$ if and only if $D \subset T_b(x)$, for any $x \in \mathcal{F}^{op}(\overline{\pi})$, the set of vectors $\{\overline{a}_t, \ t \in D\}$ is linearly independent, and

$$\sum_{t\in D} \lambda_t \overline{a}_t = -c \ ,$$

for some $\lambda_t > 0$, $t \in D$. The latter condition (with the componets λ_t there) is equivalent to

$$\lambda^D \in \mathrm{extr}\Lambda^{op}(\pi) \ ,$$

for $\lambda_t^D := \lambda_t > 0, t \in D$, $\lambda_s^D := 0$, $s \in T \setminus D$. ■

0.4 On the continuity of $\mathcal{F}_{\overline{a}}$, ϑ, and $\mathcal{F}^{op}$ restricted to their domains

The following proposition gathers well-known results characterizing the consistency and optimality in linear programming which are consequences of the celebrated Farkas Lemma (see also [28, Theorem 4.4]). Statement (ii) provides a reformulation of the family $\mathcal{M}_\pi$ defined in the previous section.

Proposition 12 *Let $\pi \equiv (c, b) \in \mathbb{R}^n \times \mathbb{R}^T$. One has:*

(i) $b \in \mathrm{dom}\mathcal{F}_{\overline{a}}$ if and only if

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \notin \mathrm{cone}\left\{ \begin{pmatrix} \overline{a}_t \\ b_t \end{pmatrix},\ t \in T \right\}.$$

(ii) $\pi \in \mathrm{dom}\mathcal{F}^{op}$ if and only if $b \in \mathrm{dom}\mathcal{F}_{\overline{a}}$ and $-c \in \mathrm{cone}\{\overline{a}_t,\ t \in T\}$.

(iii) $\mathcal{F}^{op}(\pi) \neq \emptyset$ if and only if $\mathcal{F}_{\overline{a}}(b) \neq \emptyset$ and $\mathcal{M}_\pi \neq \emptyset$.

The proof of the next corollary follows standard LP arguments which we include here for completeness.

Corollary 13 *The following statements are true:*

(i) $\mathrm{dom}\mathcal{F}_{\overline{a}}$ is a closed and convex subset in $\mathbb{R}^T$.

(ii) $\mathrm{dom}\mathcal{F}^{op}$ is a closed and convex subset in $\mathbb{R}^n \times \mathbb{R}^T$.

Proof. (i) Let us show that $\mathbb{R}^T \setminus \mathrm{dom}\mathcal{F}_{\overline{a}}$ is open. Take any $\overline{b} \in \mathbb{R}^T \setminus \mathrm{dom}\mathcal{F}_{\overline{a}}$, and write (according to Proposition 12(i)),

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} = \sum_{t\in T} \lambda_t \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix},$$

for some $\lambda_t \geq 0$, $t \in T$. Take $\varepsilon = (2\sum_{t\in T} \lambda_t)^{-1} > 0$. Then, if $\left\|\overline{b} - b\right\|_\infty < \varepsilon$,

$$\begin{aligned} \sum_{t\in T} \lambda_t b_t &= \sum_{t\in T} \lambda_t \overline{b}_t + \sum_{t\in T} \lambda_t \left(b_t - \overline{b}_t\right) \\ &\leq -1 + \varepsilon \sum_{t\in T} \lambda_t = -\frac{1}{2}. \end{aligned}$$

So,

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \mathrm{cone}\left\{ \begin{pmatrix} \overline{a}_t \\ b_t \end{pmatrix},\ t \in T \right\},$$

and then, obviously, $b \in \mathbb{R}^T \setminus \mathrm{dom}\mathcal{F}_{\overline{a}}$. The convexity of $\mathrm{dom}\mathcal{F}_{\overline{a}}$ is trivial.

(ii) It comes straightforward from (i) and Proposition 12(ii). ■

It is well known that the lsc of $\mathcal{G} : Y \rightrightarrows X$ at $\overline{y}$ introduced in Section 0.2 can be characterized in terms of the Painlevé-Kuratowski lower/inner limit as follows: $\mathcal{G}$ is *lsc* at $\overline{y} \in \mathrm{dom}\mathcal{G}$ if and only if

$$\mathcal{G}(\overline{y}) \subset \ \mathrm{Lim\,inf}_r\, \mathcal{G}\left(y^r\right) \ , \tag{0.19}$$

for any $\{y^r\} \subset Y$ converging to $\overline{y}$. Since we are restricting our mappings $\mathcal{F}_{\overline{a}}$ and $\mathcal{F}^{op}$ to their domains, we may confine ourselves to the case $\mathcal{G}\left(y^r\right) \neq \emptyset$ for all r. In such a case, recall that the Painlevé-Kuratowski lower/inner limit, $\mathrm{Lim\,inf}_r\, \mathcal{G}\left(y^r\right)$, is formed by all possible limits of sequences $\{x^r\}$, with $x^r \in \mathcal{G}\left(y^r\right)$, for all r. Recall also that the Painlevé-Kuratowski upper/outer limit, $\mathrm{Lim\,sup}_r\, \mathcal{G}\left(y^r\right)$, consists of all the cluster points (limits of subsequences) of sequences $\{x^r\}$, with $x^r \in \mathcal{G}\left(y^r\right)$, for all r. It is clear that

$$\mathrm{Lim\,inf}_r\, \mathcal{G}\left(y^r\right) \subset \mathrm{Lim\,sup}_r\, \mathcal{G}\left(y^r\right) \ .$$

When these two sets coincide, we say that there exists the limit of $\{\mathcal{G}(y^r)\}_{r\in\mathbb{N}}$ in the Painlevé-Kuratowski sense, and we write

$$\mathrm{Lim}_r\,\mathcal{G}(y^r) = \mathrm{Lim\,inf}_r\,\mathcal{G}(y^r) = \mathrm{Lim\,sup}_r\,\mathcal{G}(y^r) \ .$$

Remark 14 In this work we opt for stating Painlevé-Kuratowski convergence results in a sequential form, following [51, p. 109]. Functional expressions of the type $\mathrm{Lim\,inf}_{y\to\overline{y}}\,\mathcal{G}(y)$ can be found, for instance, in [18, p. 142] or [43, p. 13]. In the latter, the notation Lim inf*,* Lim sup and Lim with capital L is used for multifunctions in order to distinguish this concept from its counterpart for real-valued functions.

The following theorem, which can be traced out from the literature, establishes the Painlevé-Kuratowski continuity of $\mathcal{F}_{\overline{a}}$ restricted to $\mathrm{dom}\mathcal{F}_{\overline{a}}$ (closed from Corollary 13(i)). Indeed, it can be found under different approaches. It comes from [17, Corollary II.3.1] (dealing with the continuity of $\mathcal{F}_{\overline{a}}$ in the Hausdorff metric); see also [3, Theorem 3.4.1] for a proof of this result in terms of the representation of $\mathcal{F}_{\overline{a}}(b)$ as a compact polyhedron plus its recession cone, similar to [42, Lemma 3.3].

Theorem 15 *Let $\{b^r\} \subset \mathrm{dom}\mathcal{F}_{\overline{a}}$ be a sequence converging to $\overline{b}$. Then*

$$\mathcal{F}_{\overline{a}}\left(\overline{b}\right) = \mathrm{Lim}_r\,\mathcal{F}_{\overline{a}}(b^r) \ .$$

Remark 16 The situation is different when dealing with $\mathcal{F}^{op}$. Specifically, one has

$$\mathrm{Lim\,sup}_r\,\mathcal{F}^{op}(\pi^r) \subset \mathcal{F}^{op}(\overline{\pi}) \ , \tag{0.20}$$

for any $\{\pi^r\} \subset \mathrm{dom}\mathcal{F}^{op}$ converging to $\overline{\pi}$, as it follows from the Berge's theory (see [3, Theorem 5.5.1]) or from the upper Lipschitz property for polyhedral multifunctions, which is the case of $\mathcal{F}^{op}$ (see [49]). However $\mathcal{F}^{op}(\overline{\pi})$ may not be included in $\operatorname{Lim\,inf}_r \mathcal{F}^{op}(\pi^r)$. Just consider the counterexample, in $\mathbb{R}$, Min cx s.t. $x \in [-1, 1]$ around $\overline{c} = 0$.

For the computation of our aimed calmness and Lipschitz moduli of the optimal value function, we need to introduce the following sets of extreme points:

$$\mathcal{E}(b) := \operatorname{extr}\left(\mathcal{F}_{\overline{a}}(b) \cap \operatorname{span}\{\overline{a}_t,\ t \in T\}\right),\ b \in \operatorname{dom}\mathcal{F}_{\overline{a}}\ , \tag{0.21}$$
$$\mathcal{E}^{op}(\pi) := \operatorname{extr}\left(\mathcal{F}^{op}(\pi) \cap \operatorname{span}\{\overline{a}_t,\ t \in T\}\right),\ \pi \in \operatorname{dom}\mathcal{F}^{op}\ .$$

In order to motivate the use of mappings $\mathcal{E}$ and $\mathcal{E}^{op}$ from a geometrical point of view, recall (see, e.g., [50, p. 65]) that any nonempty convex set F can be decomposed as the direct sum

$$F = L_F + \left(F \cap L_F^{\perp}\right),$$

where L_F is the lineality space of F and $L_F^{\perp}$ is the orthogonal complement of L_F. In our case, when either $F = \mathcal{F}_{\overline{a}}(b)$ for $b \in \operatorname{dom}\mathcal{F}_{\overline{a}}$ or $F = \mathcal{F}^{op}(\pi)$ for $\pi \in \operatorname{dom}\mathcal{F}^{op}$, one has that $L_F^{\perp} = \operatorname{span}\{\overline{a}_t, t \in T\}$. In other words, $\{\overline{a}_t,\ t \in T\}^{\perp}$ consists of those 'directions' $d \in \mathbb{R}^n$ such that $x+\mu d \in \mathcal{F}_{\overline{a}}(b)$ (resp. $x+\mu d \in \mathcal{F}^{op}(\pi)$) for all $x \in \mathcal{F}_{\overline{a}}(b)$ (resp. $x \in \mathcal{F}^{op}(\pi)$) and all $\mu \in \mathbb{R}$. It is easy to see that, for $\pi \in \operatorname{dom}\mathcal{F}^{op}$, we have $\operatorname{extr}\mathcal{F}^{op}(\pi) = \emptyset$, equivalently $\operatorname{extr}\mathcal{F}_{\overline{a}}(b) = \emptyset$, if and only if $\{\overline{a}_t,\ t \in T\}^{\perp} \neq \{0_n\}$. In such a case, a way to ensure the existence of extreme points is intersecting $\mathcal{F}_{\overline{a}}(b)$, and $\mathcal{F}^{op}(\pi)$, with $\left(\{\overline{a}_t,\ t \in T\}^{\perp}\right)^{\perp} = \operatorname{span}\{\overline{a}_t,\ t \in T\}$, as we show in the following example. This construction is inspired by the definition of multifunction F_0 considered in [42, p. 142].

Example 17 Consider the nominal problem in $\mathbb{R}^2$ with the Euclidean norm:

$$\begin{aligned} \overline{\pi}: \quad \text{Min} \quad & x_1 \\ \text{s.t.} \quad & x_1 \leq 2 \ , \\ & -x_1 \leq 2 \ . \end{aligned}$$

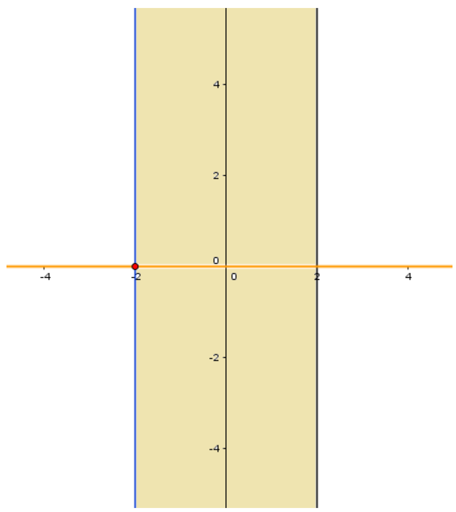

Figure 0.3

Then $\mathcal{F}^{op}(\overline{\pi}) = \left\{(-2,y)' : y \in \mathbb{R}\right\}$ so, extr$\mathcal{F}^{op}(\overline{\pi}) = \emptyset$, while span$\{\overline{a}_t, t \in T\} = \left\{(x,0)' : x \in \mathbb{R}\right\}$ and $\mathcal{E}^{op}(\overline{\pi}) = \left\{(-2,0)'\right\}$.

In fact, in the case when span $\{\overline{a}_t,\ t \in T\} \subsetneqq \mathbb{R}^n$, we can take a basis of $\{\overline{a}_t,\ t \in T\}^{\perp}$, $\{u_1, ..., u_p\}$, and form the matrix Q whose rows are u_i', $i = 1, ..., p$; then, in order to apply the results of [42] we consider the following convenient representation of $\mathcal{E}(b)$ and $\mathcal{E}^{op}(\pi)$, for $\pi = (c,b) \in \text{dom}\mathcal{F}^{op}$: take any $D \in \mathcal{M}_{\pi}$, and write

$$\mathcal{E}(b) = \text{extr}\left\{x \in \mathbb{R}^n \mid \overline{a}_t' x \leq b_t,\ t \in T;\ Qx = 0\right\} \ , \tag{0.22}$$

and

$$\mathcal{E}^{op}(\pi) = \text{extr}\left\{x \in \mathbb{R}^n \mid \overline{a}_t' x \leq b_t,\ t \in T \setminus D;\ \overline{a}_t' x = b_t,\ t \in D\ ;\ Qx = 0\right\}\ . \tag{0.23}$$

(In the case when $\text{span}\{\overline{a}_t,\ t \in T\} = \mathbb{R}^n$, we just omit equation '$Qx = 0$'.) Then, as a consequence of [42, Lemma 3.3] we derive the following lemma (recall that $\mathcal{E}(b)$ and $\mathcal{E}^{op}(\pi)$ are always nonempty, whenever $\pi \in \text{dom}\mathcal{F}^{op}$). In it, and throughout Chapters 1 and 2, π^r is identified with $(c^r, b^r) \in \mathbb{R}^n \times \mathbb{R}^T$ for all $r \in \mathbb{N}$ and the nominal problem $\overline{\pi}$ is with parameter $(\overline{c}, \overline{b})$.

Lemma 18 *Let $\{\pi^r\} \subset \text{dom}\mathcal{F}^{op}$ converge to $\overline{\pi}$. We have:*

(i) $\{\mathcal{E}(b^r)\}_{r\in\mathbb{N}}$ *is uniformly bounded and*

$$\emptyset \neq \text{Lim}_r\, \mathcal{E}(b^r) = \mathcal{E}\left(\overline{b}\right)\ .$$

(ii) $\{\mathcal{E}^{op}(\pi^r)\}_{r\in\mathbb{N}}$ *is uniformly bounded and*

$$\emptyset \neq \text{Lim}\sup_r \mathcal{E}^{op}(\pi^r) \subset \mathcal{E}^{op}(\overline{\pi})\ .$$

Proof. (i) According to (0.22), $\mathcal{E}(b)$ is nothing else but $\text{extr}F_0(b)$ in [42, Lemma 3.3] (here we omit d therein since we have no equations). So, the current statement is a direct consequence of [42, Lemma 3.3] where the Lipschitz contituity of $\mathcal{E}$ in the Hausdorff metric is established.

(ii) The uniform boundedness of the sequence $\{\mathcal{E}^{op}(\pi^r)\}$ is a direct consequence of the uniform boundedness of $\{\mathcal{E}(b^r)\}$ due to $\mathcal{E}^{op}(\pi^r) \subset \mathcal{E}(b^r)$. Hence, $\text{Lim}\sup_r \mathcal{E}^{op}(\pi^r)$ turns out to be nonempty. Now, we consider any element of $\text{Lim}\sup_r \mathcal{E}^{op}(\pi^r)$, say $z = \lim_k x^{r_k}$ with $x^{r_k} \in \mathcal{E}^{op}(\pi^{r_k})$. For each k, by (0.23) x^{r_k} verifies

$$\overline{a}_t' x^{r_k} \leq b_t^{r_k},\ t \in T \setminus D^k;\ \overline{a}_t' x^{r_k} = b_t^{r_k},\ t \in D^k;\ Qx^{r_k} = 0\ , \tag{0.24}$$

for $D^k \in \mathcal{M}_{\pi^{r_k}}$. The extremality of each x^{r_k} entails that

$$\operatorname{span}\Big(\{u_1, ..., u_p\} \cup \{\overline{a}_t, t \in T_{b^{r_k}}(x^{r_k})\}\Big) = \mathbb{R}^n \ . \tag{0.25}$$

From the finiteness of T we can assume that $\{D^k\}_{k\in\mathbb{N}}$ is a constant sequence (by taking a subsequence if necessary), say $D^k = D$ for all k. A new refinement allows to assume that $T_{b^{r_k}}(x^{r_k})$ is also constant. By taking limits in (0.24) over k we obtain

$$\overline{a}_t' z \leq \overline{b}_t, \ t \in T \setminus D; \ \overline{a}_t' z = \overline{b}_t, \ t \in D; \ Qz = 0 \ .$$

Since $-c^{r_k} \in \operatorname{cone}\{\overline{a}_t, \ t \in D\}$ for all k, it follows that $-\overline{c} \in \operatorname{cone}\{\overline{a}_t, \ t \in D\}$ (although the minimality of D in relation to $-c^r$ does not entail the minimality of D for $-\overline{c}$). Then, $z \in \mathcal{F}^{op}(\overline{\pi})$ and $Qz = 0$, which implies that $z \in \mathcal{E}^{op}(\overline{\pi})$; the extremality of z comes from (0.25). ■

The next result is well known in the literature. The reader is addressed to [3, Theorem 4.5.2] for a proof based on the Berge's theory, or to [34, Satz 2.7] and [57, Theorem 14] for a primal-dual approach to the continuity of ϑ^R (see also [44] for a parametric analysis). Indeed, one can find stronger versions: ϑ^R is Lipschitz continuous on bounded subsets of $\operatorname{dom}\mathcal{F}^{op}$; see [49, p. 214] in the context of canonically perturbed convex quadratic problems (see also [57, p. 25]). On the other hand, [40] proved the continuity of the optimal value function for a (generally non-convex) quadratic program under canonical perturbations. Nevertheless, for completeness, we include here a direct proof based on the previous lemma.

Theorem 19 *ϑ^R is continuous on* $\operatorname{dom}\mathcal{F}^{op}$*.*

Proof. Let $\{\pi^r\} \subset \operatorname{dom}\mathcal{F}^{op}$ be convergent to $\overline{\pi}$ (belonging to $\operatorname{dom}\mathcal{F}^{op}$ because Corollary 13(ii)) and let us see that

$$\lim_r \vartheta(\pi^r) = \vartheta(\overline{\pi}) \ .$$

Take any $\overline{x} \in \mathcal{F}^{op}(\overline{\pi})$ and, appealing to Theorem 15, take any sequence $\{x^r\}$ converging to $\overline{x}$, with $x^r \in \mathcal{F}_{\overline{a}}(b^r)$ for all r. Then,

$$\vartheta(\overline{\pi}) = \overline{c}'\overline{x} = \lim_r (c^r)' x^r \geq \limsup_r \vartheta(\pi^r) \ .$$

Now, reasoning by contradiction, assume that $\liminf_r \vartheta(\pi^r) < \vartheta(\overline{\pi})$. Write $\liminf_r \vartheta(\pi^r)$ as $\lim_k \vartheta(\pi^{r_k})$ for an appropriate subsequence. By Lemma 18 (ii), without loss of generality (taking a subsequence if necessary), we can assume the existence of $x^k \in \mathcal{E}^{op}(\pi^{r_k})$, for all k, such that $\{x^k\}$ converges to some $\overline{x} \in \mathcal{E}^{op}(\overline{\pi})$. Therefore, we attain the contradiction

$$\overline{c}'\overline{x} = \lim_k (c^{r_k})' x^k = \lim_k \vartheta(\pi^{r_k}) < \vartheta(\overline{\pi}) \ .$$

■

Finally, recall that the restriction of $\mathcal{F}^{op}$ to its domain is not continuous (in the Painlevé-Kuratowski sense) as shown in Remark 16. However, it is if we only perturb b, as the following theorem asserts. In fact, it is a well-known result of stability theory in LP. Specifically, it can be derived from the fact that $\mathcal{F}^{op}(\overline{c}, \cdot)$ is Lipschitzian on $\mathrm{dom}\mathcal{F}_{\overline{a}}$, provided that

$$-\overline{c} \in \mathrm{cone}\{\overline{a}_t, \ t \in T\} \ ; \tag{0.26}$$

see, e.g. [36, p. 232] or [19, Chapter IX (Section 7)].

Theorem 20 *Let $\overline{c} \in \mathbb{R}^n$ verify (0.26). For any $\{b^r\}_{r\in\mathbb{N}} \subset \mathrm{dom}\mathcal{F}_{\overline{a}}$ converging to $\overline{b}$ we have*

$$\mathcal{F}^{op}(\overline{\pi}) = \mathrm{Lim}_r\, \mathcal{F}^{op}(\overline{c}, b^r) \ .$$

Remark 21 In general, the boundedness of a Painlevé-Kuratowski limit of sets does not imply the uniform boundedness of those sets. For instance, $\mathrm{Lim}_r\{1, r\} = \{1\}$. Nevertheless, in the previous theorem the boundedness of

$\mathcal{F}^{op}(\overline{\pi})$ does imply the uniform boundedness of $\{\mathcal{F}^{op}(\overline{c}, b^r)\}_{r\in\mathbb{N}}$. This follows from the convexity of each $\mathcal{F}^{op}(\overline{c}, b^r)$ or, alternatively, from [28, Corollary 6.2.1] together with Theorem 19.

Chapter 1

Calmness of the optimal value function

The main goal of this chapter consists in computing (or estimating) $\mathrm{clm}\vartheta^R(\overline{\pi})$ (recall $\overline{\pi} \equiv (\overline{c}, \overline{b})$) via the computation of the corresponding calmness moduli from below and above. At this moment we point out the fact that the present work establishes point-based formulae for the aimed calmness moduli (sometimes estimations), i.e., formulae which only involve the nominal data $\overline{\pi}$ (not appealing to parameters in a neighborhood). In relation to this point, our main contributions are gathered in Theorems 38, 39, and 42 (the last one under the boundedness of $\mathcal{F}^{op}(\overline{\pi})$); see also Theorem 46 (stated for $\overline{\pi}$ in the interior of $\mathrm{dom}\mathcal{F}^{op}$). Our first step will be developed in the context of RHS perturbations, in which case the corresponding optimal value function is specially tractable; in fact, an explicit formula for computing the optimal values around $\overline{\pi}$ is provided (Corollary 26) and it is used as a starting point for deriving the results about the calmness modulus of the optimal value under RHS perturbations (Theorem 28 and Corollary 32). The second step will be studying the calmness moduli under perturbations of the coefficients in the objective function (c-perturbations). The results in this section will allow us to provide in Corollary 44 an alternative upper bound for $\mathrm{clm}\vartheta^R(\overline{\pi})$ (assuming that $\mathcal{F}^{op}(\overline{\pi})$ is bounded) in terms of the calmness moduli previously considered (i.e, under RHS perturbations and under perturbations of the objective function in a separate way). For the interest of the reader, it is worth announcing at this moment that in the part *Conclusions and future work* at the end of this document we include a table gathering the new results about the calmness moduli. In order to better integrate the current work in the literature, we conclude the chapter with a comparative analysis between Theorem 46 and a certain consequence of [45, Theorem 1.1(5)].

To start with, we establish the followings lemmas.

Lemma 22 *There exists $\overline{\delta} > 0$ such that if $\pi, \overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$ satisfy $\|\pi - \overline{\pi}\| < \overline{\delta}$, with $\pi \equiv (c, b)$ and $\overline{\pi} \equiv \left(\overline{c}, \overline{b}\right)$, then*

$$\mathcal{F}^{op}(\pi) \subset \mathcal{F}^{op}(\overline{c}, b) \ .$$

Proof. Reasoning by contradiction, assume the existence of a sequence of problems $\{\pi^r \equiv (c^r, b^r)\} \subset \mathrm{dom}\mathcal{F}^{op}$ converging to $\overline{\pi}$ and a sequence of points $\{x^r\} \subset \mathbb{R}^n$ such that $x^r \in \mathcal{F}^{op}(\pi^r) \setminus \mathcal{F}^{op}(\overline{c}, b^r)$ for all r. We have that

$$-c^r \in \mathrm{cone}\{\overline{a}_t,\ t \in T_{b^r}(x^r)\},\ \text{for all } r\ .$$

We may assume (by taking a subsequence if necessary) that $T_{b^r}(x^r) = D$ for all r (not depending on r). Then

$$-\overline{c} \in \mathrm{cone}\{\overline{a}_t,\ t \in D\}\ ,$$

(by the closedness of a finitely generated cone in $\mathbb{R}^n$) which yields the contradiction $x^r \in \mathcal{F}^{op}(\overline{c}, b^r)$ for each r. ■

Lemma 23 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$ and $\{\pi^r \equiv (c^r, b^r)\}_r$ be a sequence converging to $\overline{\pi}$, with $\{b^r\}_r \subset \mathrm{dom}\mathcal{F}_{\overline{a}}$ for all $r \in \mathbb{N}$. If $x \mapsto (c^r)'x$ is bounded from below on $\mathcal{F}^{op}(\overline{c}, b^r)$ for all r, then*

$$\pi^r \in \mathrm{dom}\mathcal{F}^{op} \tag{1.1}$$

for r large enough.

Proof. Assume, reasoning by contradiction, that $\vartheta(\pi^r) = -\infty$ for all r (replacing, if necessary, the sequence with an appropriate subsequence). From [42, Lemma 4.1], we can write

$$\mathcal{F}\left(b^r\right) = \mathrm{conv}\,\mathcal{E}\left(b^r\right) + \left\{d \in \mathbb{R}^n : \overline{a}_t' d \leq 0,\ t \in T\right\}\ ,$$

(recall that $\mathcal{E}\left(b^r\right) := \mathrm{extr}\left(\mathcal{F}\left(b^r\right) \cap \mathrm{span}\left\{\overline{a}_t,\ t \in T\right\}\right.$ for all r) where the last term is the recession cone of $\mathcal{F}\left(b^r\right)$, which does not depend on r (only on the fact that $b^r \in \mathrm{dom}\mathcal{F}_{\overline{a}}$). Let us write this (polyhedral) recession cone as $\mathrm{cone}\left\{d_1, ..., d_p\right\}$ for a certain $p \in \mathbb{N}$. On the other hand, Lemma 18 (i) ensures that $\left\{\mathcal{E}\left(b^r\right)\right\}_{r\in\mathbb{N}}$ is a sequence of uniformly bounded nonempty compact sets. Because of the compactness of $\mathrm{conv}\mathcal{E}\left(b^r\right)$, assumption $\vartheta\left(\pi^r\right) = -\infty$ implies (again considering an appropriate subsequence, if necessary) that $\left(c^r\right)' d_k < 0$ for all r and some fixed $k \in \{1, ..., p\}$. Letting $r \to \infty$ we obtain $\overline{c}' d_k \leq 0$, which entails that d_k is not only a recession direction of $\mathcal{F}\left(b^r\right)$, but also of $\mathcal{F}^{op}\left(\overline{c}, b^r\right)$, for all r. This, together with $\left(c^r\right)' d_k < 0$ ensures that, for each $r \in \mathbb{N}$, $x \mapsto \left(c^r\right)' x$ is not bounded from below on $\mathcal{F}^{op}\left(\overline{c}, b^r\right)$, contradicting the hypothesis of the statement. $\blacksquare$

Remark 24 When $\mathcal{F}^{op}(\overline{\pi}) \neq \emptyset$ is bounded, one immediately has (1.1) for r large enough as a consequence of [28, Lemma 10.2].

1.1 Calmness modulus under RHS perturbations

Along this section we deal with linear optimization problems with a fixed c, say $\overline{c}$, which is assumed to verify (0.26). So, the only parameter to be considered here is $b \in \mathrm{dom}\mathcal{F}_{\overline{a}}$ (equivalently $(\overline{c}, b) \in \mathrm{dom}\mathcal{F}^{op}$). Hence, we consider the particular optimal value function $\vartheta_{\overline{c}}^{R} : \mathrm{dom}\mathcal{F}_{\overline{a}} \to \left]-\infty, +\infty\right[$ defined as

$$\vartheta_{\overline{c}}^{R}(b) := \vartheta(\overline{c}, b), \text{ for all } b \in \mathrm{dom}\mathcal{F}_{\overline{a}} .$$

We aim to to compute/estimate the calmness modulus of $\vartheta_{\overline{c}}^{R}$ at $\overline{b} \in \mathrm{dom}\mathcal{F}_{\overline{a}}$, which is given by

$$\mathrm{clm}\vartheta_{\overline{c}}^{R}(\overline{b}) = \limsup_{b \underset{\mathrm{dom}\mathcal{F}_{\overline{a}}}{\to} \overline{b}} \frac{|\vartheta(\overline{c}, b) - \vartheta(\overline{\pi})|}{\left\|b - \overline{b}\right\|_{\infty}} ,$$

and the corresponding calmness moduli from below and above (which are analogously defined; see Section 0.2). Observe that we use indistinctly the notation $\vartheta_{\overline{c}}^{R}(b)$ and $\vartheta(\overline{c}, b)$ whenever $b \in \mathrm{dom}\mathcal{F}_{\overline{a}}$. In fact, for clarity, we usually write $\vartheta_{\overline{c}}^{R}$ when talking about the function itself and write $\vartheta(\overline{c}, b)$ for the image of $\vartheta_{\overline{c}}^{R}$ at $b \in \mathrm{dom}\mathcal{F}_{\overline{a}}$.

To start with, we have the well-known expression for $\vartheta_{\overline{c}}^{R}$ as a piecewise linear function (see, e.g., [5, p. 214]) given by

$$\vartheta(\overline{c}, b) = \max_{\lambda \in \mathrm{extr}\Lambda(\overline{c})} -b'\lambda, \text{ for all } b \in \mathrm{dom}\mathcal{F}_{\overline{a}} .$$

The following results are devoted to refine the previous expression in a neighborhood of $\overline{b}$ by appealing to the family of minimal KKT subsets of indices, $\mathcal{M}_{\overline{\pi}}$; specifically, to replace $\mathrm{ext}\Lambda(\overline{c})$ with a smaller set written in terms of $\mathcal{M}_{\overline{\pi}}$.

The following result is standard (the finiteness of $\mathrm{extr}\Lambda(\overline{c})$ is a key fact).

Lemma 25 *Let $\overline{b} \in \mathrm{dom}\mathcal{F}_{\overline{a}}$. There exists a neighborhood $U_{\overline{b}} \subset \mathbb{R}^T$ of $\overline{b}$ such that*

$$\mathrm{extr}\Lambda^{op}(\overline{c}, b) \subset \mathrm{extr}\Lambda^{op}(\overline{\pi}), \textit{ whenever } b \in \mathrm{dom}\mathcal{F}_{\overline{a}} \cap U_{\overline{b}}.$$

As a consequence of the previous lemma together with Lemma 11, and taking into account the obvious fact that

$$\vartheta(\overline{c}, b) = \max_{\lambda \in \mathrm{extr}\Lambda^{op}(\overline{c},b)} -b'\lambda, \text{ for all } b \in \mathrm{dom}\mathcal{F}_{\overline{a}}$$

we derive the following corollary.

Corollary 26 *Let $\overline{b} \in \mathrm{dom}\mathcal{F}_{\overline{a}}$ and let $U_{\overline{b}}$ be as in Lemma 25. Then*

$$\vartheta(\overline{c}, b) = \max_{D \in \mathcal{M}_{\overline{\pi}}} -b'\lambda^D, \textit{ for all } b \in \mathrm{dom}\mathcal{F}_{\overline{a}} \cap U_{\overline{b}}.$$

Now, applying the previous corollary, and using the fact that $\vartheta(\overline{\pi}) = -\overline{b}'\lambda^D$ for all $D \in \mathcal{M}_{\overline{\pi}}$ into account, we deduce

$$\vartheta(\overline{c}, b) - \vartheta(\overline{\pi}) = \left(\max_{D \in \mathcal{M}_{\overline{\pi}}} -b'\lambda^D \right) - \vartheta(\overline{\pi}) = \max_{D \in \mathcal{M}_{\overline{\pi}}} -\left(b - \overline{b}\right)' \lambda^D, \tag{1.2}$$

while

$$\vartheta(\overline{\pi}) - \vartheta(\overline{c}, b) = \vartheta(\overline{\pi}) - \max_{D \in \mathcal{M}_{\overline{\pi}}} -b'\lambda^D = \min_{D \in \mathcal{M}_{\overline{\pi}}} (b - \overline{b})'\lambda^D, \tag{1.3}$$

for all $b \in \mathrm{dom}\mathcal{F}_{\overline{a}} \cap U_{\overline{b}}$, where $U_{\overline{b}}$ is as in Lemma 25. Consequently, if we denote

$$k^- := \min_{D \in \mathcal{M}_{\overline{\pi}}} \left\| \lambda^D \right\|_1 \text{ and } k^+ = \max_{D \in \mathcal{M}_{\overline{\pi}}} \left\| \lambda^D \right\|_1, \tag{1.4}$$

(where, as usual, $\left\| \lambda^D \right\|_1 = \sum_{t \in D} \lambda_t^D$) we deduce the following result saying that k^- and k^+ are, respectively, a calmness constant from below and above for our optimal value function $\vartheta_{\overline{c}}^R$.

Corollary 27 *Let* $\overline{b} \in \text{dom}\mathcal{F}_{\overline{a}}$ *and let* $U_{\overline{b}}$ *be as in Lemma 25. Then, for* $b \in \text{dom}\mathcal{F}_{\overline{a}} \cap U_{\overline{b}}$, *one has*

(i) $\vartheta(\overline{\pi}) - \vartheta(\overline{c}, b) \leq k^- \left\|b - \overline{b}\right\|_\infty$.

(ii) $\vartheta(\overline{c}, b) - \vartheta(\overline{\pi}) \leq k^+ \left\|b - \overline{b}\right\|_\infty$.

So, k^- and k^+ are, respectively, upper bounds on the calmness moduli from below and above of $\vartheta_{\overline{c}}^R$ at $\overline{b}$, given by

$$\underline{\text{clm}}\vartheta_{\overline{c}}^R(\overline{b}) = \limsup_{b \underset{\text{dom}\mathcal{F}_{\overline{a}}}{\rightarrow} \overline{b}} \frac{\vartheta(\overline{\pi}) - \vartheta(\overline{c}, b)}{\left\|b - \overline{b}\right\|_\infty} \text{ and } \overline{\text{clm}}\vartheta_{\overline{c}}^R(\overline{b}) = \limsup_{b \underset{\text{dom}\mathcal{F}_{\overline{a}}}{\rightarrow} \overline{b}} \frac{\vartheta(\overline{c}, b) - \vartheta(\overline{\pi})}{\left\|b - \overline{b}\right\|_\infty} .$$

The following theorem shows that k^- is always attained as the calmness modulus from below of $\vartheta_{\overline{c}}^R$. The counterpart for k^+ is no longer true, as Example 30 shows.

Theorem 28 *Let* $\overline{b} \in \text{dom}\mathcal{F}_{\overline{a}}$. *One has:*

$$\underline{\text{clm}}\vartheta_{\overline{c}}^R(\overline{b}) = k^- \text{ and } \overline{\text{clm}}\vartheta_{\overline{c}}^R(\overline{b}) \leq k^+ .$$

Consequently,

$$k^- \leq \text{clm}\vartheta_{\overline{c}}^R(\overline{b}) \leq k^+ .$$

Proof. As commented above, it is clear that $\underline{\text{clm}}\vartheta_{\overline{c}}^R(\overline{b}) \leq k^-$ and $\overline{\text{clm}}\vartheta_{\overline{c}}^R(\overline{b}) \leq k^+$. So, we only need to prove that $\underline{\text{clm}}\vartheta_{\overline{c}}^R(\overline{b}) \geq k^-$. Consider the sequence

$$b^r := \overline{b} + \frac{1}{r}1_T, \text{ for all } r ,$$

where $1_T \in \mathbb{R}^T$ represents the vector having all its coordinates equal to 1.

Clearly, $\{b^r\} \subset \mathrm{dom}\mathcal{F}_{\overline{a}}$. Then, appealing to (1.3) we have

$$\begin{aligned}\underline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) &\geq \limsup_r \frac{\vartheta\left(\overline{c},\overline{b}\right)-\vartheta\left(\overline{c},b^r\right)}{\left\|b^r-\overline{b}\right\|_\infty} \\ &= \limsup_r \frac{\min_{D\in\mathcal{M}_{\overline{\pi}}}\left(b^r-\overline{b}\right)'\lambda^D}{\left\|b^r-\overline{b}\right\|_\infty} \\ &= \lim_r \frac{\frac{1}{r}\min_{D\in\mathcal{M}_{\overline{\pi}}}\left\|\lambda^D\right\|_1}{\frac{1}{r}} = k^- \ .\end{aligned}$$

■

The following proposition is intended to provide an alternative approach for determining $\left\|\lambda^D\right\|_1$, with $D \in \mathcal{M}_{\overline{\pi}}$. In it, u_D represents the projection of 0_n on aff $\{\overline{a}_t,\ t \in D\}$ in the Euclidean norm, provided that $D \in \mathcal{M}_{\overline{\pi}}$ (observe that $0_n \notin$ aff $\{\overline{a}_t,\ t \in D\}$ as a consequence of the linear independence of $\{\overline{a}_t,\ t \in D\}$).

Proposition 29 *For each $D \in \mathcal{M}_{\overline{\pi}}$, one has*

$$\left\|\lambda^D\right\|_1 = \frac{-\overline{c}'u_D}{\left\|u_D\right\|_2^2} \ . \tag{1.5}$$

Proof. Take any $D \in \mathcal{M}_{\overline{\pi}}$. Since $-\left\|\lambda^D\right\|_1^{-1}\overline{c} \in$ aff $\{\overline{a}_t,\ t \in D\}$, the definition of u_D yields

$$\left(-\left\|\lambda^D\right\|_1^{-1}\overline{c} - u_D\right)' u_D = 0 \ ,$$

which entails the aimed equality (1.5). ■

The following example shows that the calmness modulus from above of $\vartheta_{\overline{c}}^R$ can take any positive value less than or equal to k^+.

Example 30 Consider the problem in $\mathbb{R}$ given by

$$\begin{array}{llll}\overline{\pi}: & \mathrm{Min} & x_1 & \\ & \mathrm{s.t.} & -x_1 \leq 0\ , & (t=1) \\ & & -2x_1 \leq 0\ , & (t=2) \\ & & \theta x_1 \leq 0\ . & (t=3)\end{array}$$

where $\theta > 0$. Trivially, $\mathcal{M}_{\overline{\pi}} := \{\{1\},\{2\}\}$, $\lambda^{\{1\}} = 1 = k^{+}$, $\lambda^{\{2\}} = \frac{1}{2} = k^{-}$. Let us check that

$$\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) = \min\left\{1, \frac{1}{\theta}\right\} .$$

Observe that $\overline{b} = 0_3$. According to Corollary 26, in some neighborhood $U_{\overline{b}}$ of $\overline{b}$ we have

$$\vartheta\left(\overline{c}, b\right) = \max\{-b_1, -\tfrac{1}{2}b_2\}, \text{ for } b = (b_1, b_2, b_3)' \in \mathrm{dom}\mathcal{F}_{\overline{a}} \cap U_{\overline{b}} .$$

Moreover, it is immediate that

$$b \in \mathrm{dom}\mathcal{F}_{\overline{a}} \Leftrightarrow \max\{-b_1, -\tfrac{1}{2}b_2\} \leq \tfrac{1}{\theta}b_3 .$$

So,

$$\frac{\vartheta\left(\overline{c}, b\right) - \vartheta\left(\overline{\pi}\right)}{\left\|b - \overline{b}\right\|_{\infty}} = \frac{\max\{-b_1, -\frac{1}{2}b_2\}}{\|b\|_{\infty}} \leq \frac{\frac{1}{\theta}b_3}{\|b\|_{\infty}} \leq \tfrac{1}{\theta}, \text{ for all } b \in \mathrm{dom}\mathcal{F}_{\overline{a}} \cap U_{\overline{b}} .$$

Consequently, one always have

$$\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) \leq \frac{1}{\theta} .$$

Now, we distinguish two cases:

Case 1: $\theta > 1$. Just consider the sequence

$$b^{r} = \left(-\frac{1}{\theta r}, -\frac{1}{\theta r}, \frac{1}{r}\right)', \ r = 1, 2...$$

It is clear that $b^r \in \mathrm{dom}\mathcal{F}_{\overline{a}}$ for all r. Moreover, for r large enough (to ensure $b^r \in U_{\overline{b}}$) one has

$$\frac{\vartheta\left(\overline{c}, b^{r}\right) - \vartheta\left(\overline{\pi}\right)}{\left\|b^{r} - \overline{b}\right\|_{\infty}} = \frac{\frac{1}{\theta r}}{\frac{1}{r}} = \frac{1}{\theta} ,$$

So, $\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) \geq \frac{1}{\theta}$.

Case 2: $0 < \theta \leq 1$, yielding $\frac{1}{\theta} \geq 1 = k^{+} \geq \overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b})$. Consider the sequence

$$b^{r} = \left(-\frac{1}{r}, -\frac{1}{r}, \frac{1}{r}\right)', \ r = 1, 2...$$

One has $b^r \in \mathrm{dom}\mathcal{F}_{\overline{a}}$ for all r and, for r large enough,

$$\frac{\vartheta\left(\overline{c}, b^r\right) - \vartheta\left(\overline{\pi}\right)}{\left\|b^r - \overline{b}\right\|_\infty} = \frac{\frac{1}{r}}{\frac{1}{r}} = 1 \ .$$

Inspired by the previous example, the following proposition provides a sufficient condition for having the equality $\overline{\mathrm{clm}}\vartheta_{\overline{c}}^R(\overline{b}) = k^+$.

Proposition 31 *Let $\overline{b} \in \mathrm{dom}\mathcal{F}_{\overline{a}}$. Assume that there exists some $\overline{D} \in \mathcal{M}_{\overline{\pi}}$, with $\left\|\lambda^{\overline{D}}\right\|_1 = k^+$, and some $\varepsilon > 0$ such that $b^\varepsilon \in \mathrm{dom}\mathcal{F}_{\overline{a}}$, with*

$$b_t^\varepsilon := \begin{cases} \overline{b}_t - \varepsilon & \text{if } t \in \overline{D}, \\ \overline{b}_t + \varepsilon & \text{if } t \in T \backslash \overline{D} \ . \end{cases} \tag{1.6}$$

Then,

$$\overline{\mathrm{clm}}\vartheta_{\overline{c}}^R(\overline{b}) = \mathrm{clm}\vartheta_{\overline{c}}^R(\overline{b}) = k^+ \ .$$

Proof. Take $\overline{D} \in \mathcal{M}_{\overline{\pi}}$ and $\varepsilon > 0$ verifying the assumptions of the current proposition. First, observe that for any $0 < \widetilde{\varepsilon} < \varepsilon$, the associated $b^{\widetilde{\varepsilon}}$ defined as in (1.6) (replacing ε with $\widetilde{\varepsilon}$) also verifies that $b^{\widetilde{\varepsilon}} \in \mathrm{dom}\mathcal{F}_{\overline{a}}$, as far as $\mathrm{dom}\mathcal{F}_{\overline{a}}$ is a convex set. Specifically, observe that $b^{\widetilde{\varepsilon}} = \left(1 - \frac{\widetilde{\varepsilon}}{\varepsilon}\right)\overline{b} + \frac{\widetilde{\varepsilon}}{\varepsilon} b^\varepsilon$. Let $U_{\overline{b}}$ be as in Lemma 25, and consider the sequence $\{b^{\frac{1}{r}}\}$, where $b^{\frac{1}{r}}$ comes again from replacing ε with $\frac{1}{r}$ in (1.6). Let r_0 be large enough to guarantee $\frac{1}{r_0} < \varepsilon$ (so, $b^{\frac{1}{r}} \in \mathrm{dom}\mathcal{F}_{\overline{a}}$, $r \geq r_0$) and $b^{\frac{1}{r}} \in U_{\overline{b}}$, for all $r \geq r_0$. Then

$$\begin{aligned} \mathrm{clm}\vartheta_{\overline{c}}^R(\overline{b}) &\geq \limsup_r \frac{\vartheta\left(\overline{c}, b^{\frac{1}{r}}\right) - \vartheta\left(\overline{\pi}\right)}{\left\|b^{\frac{1}{r}} - \overline{b}\right\|_\infty} \\ &= \limsup_r \frac{\max_{D \in \mathcal{M}_{\overline{\pi}}} - \left(b^{\frac{1}{r}} - \overline{b}\right)' \lambda^D}{\frac{1}{r}} = \left\|\lambda^{\overline{D}}\right\|_1 \ , \end{aligned}$$

where the last equality comes from the fact that

$$\left| -\left(b^{\frac{1}{r}} - \overline{b}\right)' \lambda^D \right| \leq \left\| b^{\frac{1}{r}} - \overline{b} \right\|_\infty \left\| \lambda^D \right\|_1 = \frac{1}{r} \left\| \lambda^D \right\|_1 ,$$

for all $D \in \mathcal{M}_{\overline{\pi}}$, and $-\left(b^{\frac{1}{r}} - \overline{b}\right)' \lambda^{\overline{D}} = \frac{1}{r} \left\| \lambda^{\overline{D}} \right\|_1$. ■

As a consequence of the previous proposition we have the following corollary under SCQ. It is well known that SCQ at $\overline{b} \in \mathrm{dom}\mathcal{F}_{\overline{a}}$ is equivalent to $\overline{b} \in \mathrm{intdom}\mathcal{F}_{\overline{a}}$.

Corollary 32 *Let $\overline{b} \in \mathrm{dom}\mathcal{F}_{\overline{a}}$ and assume that SCQ holds at $\overline{b}$. Then*

$$\overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) = \mathrm{clm}\vartheta^R_{\overline{c}}(\overline{b}) = k^+ .$$

Remark 33 In the case when the SCQ fails, we may have either $\mathrm{clm}\vartheta^R_{\overline{c}}(\overline{b}) < k^+$ or $\mathrm{clm}\vartheta^R_{\overline{c}}(\overline{b}) = k^+$. For the strict inequality just consider $\theta = 2$ in Example 30, in which case $\mathrm{clm}\vartheta^R_{\overline{c}}(\overline{b}) = \max\left\{\underline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}), \overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b})\right\} = \left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2}$, while $k^+ = 1$. The following example illustrates the possibility of having $\mathrm{clm}\vartheta^R_{\overline{c}}(\overline{b}) = k^+$ together with the fulfilment of the SCQ.

Example 34 Consider the parametrized problem in $\mathbb{R}^2$

$$\begin{array}{lll} (\overline{c}, b): & \mathrm{Min} & x_2 \\ & \mathrm{s.t.} & -\frac{1}{2}x_1 - \frac{3}{2}x_2 \leq b_1 , \quad (t=1) \\ & & x_1 - 3x_2 \leq b_2 , \qquad (t=2) \\ & & x_2 \leq b_3 . \qquad\qquad (t=3) \end{array}$$

Take $\overline{b} = 0_3$ and observe that $\mathcal{M}_{\overline{\pi}} = \{\{1,2\}\}$ and $-\overline{c} = \frac{1}{3}\overline{a}_1 + \frac{1}{6}\overline{a}_2$. So, $k^- = k^+ = \frac{1}{2}$ and, then $\mathrm{clm}\vartheta^R_{\overline{c}}\left(\overline{b}\right) = \frac{1}{2}$ by applying Theorem 28. Figure 1.1 illustrates the elements of Proposition 29 (observe that $u_{\{1,2\}} = (-1,-1)'$).

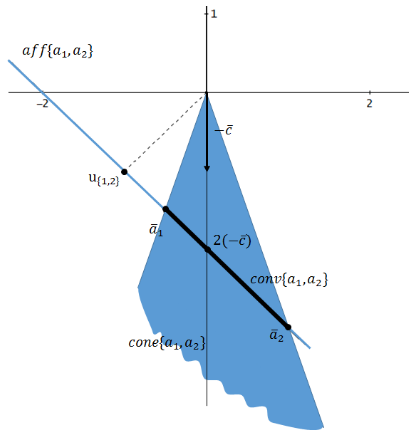

Figure 1.1

1.2 Calmness modulus under c-perturbations

Let us focus now on the computation of the calmness moduli from below and above under perturbations only of the objective function's coefficients. For this issue, if parameter b is considered fixed, say $\overline{b}$, we define a new optimal value function only depending on c,

$$\vartheta^R_{\overline{b}} : -\mathrm{cone}\{\overline{a}_t, t \in T\} \rightarrow]-\infty, +\infty[\ ,$$

as $\vartheta^R_{\overline{b}}(c) := \vartheta^R(c, \overline{b})$ provided that $\overline{b} \in \mathrm{dom}\mathcal{F}_{\overline{a}}$.

Theorem 35 *Let* $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$*. Then*

$$\overline{\mathrm{clm}}\vartheta^R_{\overline{b}}(\overline{c}) = d(0_n, \mathcal{F}^{op}(\overline{\pi})) \ .$$

Proof. Take $\overline{x} \in \mathcal{F}^{op}(\overline{\pi})$ with $\|\overline{x}\| = d(0_n, \mathcal{F}^{op}(\overline{\pi}))$. Consider $(c, \overline{b}) \in \mathrm{dom}\mathcal{F}^{op}$ such that $\|(c, \overline{b}) - \overline{\pi}\| = \|c - \overline{c}\|_* < \overline{\delta}$ for $\overline{\delta} > 0$ as in Lemma 22. Pick an arbitrary $\widehat{x} \in \mathcal{F}^{op}(c, \overline{b}) \subset \mathcal{F}^{op}(\overline{\pi})$. Clearly, $\vartheta(c, \overline{b}) = c'\widehat{x} \leq c'\overline{x}$ (because $\overline{x} \in \mathcal{F}(\overline{b})$). Then,

$$\vartheta(c, \overline{b}) - \vartheta(\overline{\pi}) = c'\widehat{x} - \overline{c}'\overline{x} \leq c'\overline{x} - \overline{c}'\overline{x} = (c - \overline{c})'\overline{x} \leq \|c - \overline{c}\|_* \|\overline{x}\| \ .$$

Therefore,

$$\overline{\mathrm{clm}}\vartheta_{\overline{b}}^{R}(\overline{c}) \leq d(0_n, \mathcal{F}^{op}(\overline{\pi})) \ .$$

Now we show the case "$\geq$". If $0_n \in \mathcal{F}^{op}(\overline{\pi})$, then

$$0 \leq \overline{\mathrm{clm}}\vartheta_{\overline{b}}^{R}(\overline{c}) \leq d(0_n, \mathcal{F}^{op}(\overline{\pi})) = 0 \ ,$$

and, trivially, $\overline{\mathrm{clm}}\vartheta_{\overline{b}}^{R}(\overline{c}) = 0$. Hence, let us assume $0_n \notin \mathcal{F}^{op}(\overline{\pi})$. Let $\overline{x} \in \mathcal{F}^{op}(\overline{\pi})$ such that $\|\overline{x}\| = d(0_n, \mathcal{F}^{op}(\overline{\pi}))$. According to [11, Lemma 9], there exists $u \in \mathbb{R}^n$ with $\|u\|_* = 1$ such that $u'x \geq u'\overline{x} = \|\overline{x}\|$, for all $x \in \mathcal{F}^{op}(\overline{\pi})$. Define the perturbation

$$c^r := \overline{c} + \frac{1}{r}u \ \text{ for each } r \in \mathbb{N} \ .$$

For all $x \in \mathcal{F}^{op}(\overline{\pi})$, we have

$$(c^r)'x = \overline{c}'x + \frac{1}{r}u'x \geq \overline{c}'\overline{x} + \frac{1}{r}u'\overline{x} = (c^r)'\overline{x} \ . \tag{1.7}$$

This implies that $x \mapsto (c^r)'x$ is bounded from below on $\mathcal{F}^{op}(\overline{\pi})$ and, by Lemma 23, $(c^r, \overline{b}) \in \mathrm{dom}\mathcal{F}^{op}$ for r large enough (say for all r). Lemma 22 entails $\mathcal{F}^{op}(c^r, \overline{b}) \subset \mathcal{F}^{op}(\overline{\pi})$ for r large enough, and indeed (1.7) yields $\overline{x} \in \mathcal{F}^{op}(c^r, \overline{b})$. Then, we have

$$\begin{aligned} \overline{\mathrm{clm}}\vartheta_{\overline{b}}^{R}(\overline{c}) &\geq \limsup_r \frac{\vartheta(c^r, \overline{b}) - \vartheta(\overline{\pi})}{\|c^r - \overline{c}\|_*} \\ &= \limsup_r \frac{(c^r - \overline{c})' \, \overline{x}}{\frac{1}{r}\|u\|_*} = u'\overline{x} = \|\overline{x}\| = d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right) \ . \end{aligned}$$

■

Along this work $e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$ represents the Hausdorff excess of $\mathcal{E}^{op}\left(\overline{\pi}\right)$ over $\{0_n\}$, which may alternatively be written as

$$\max\left\{\|x\| \mid x \in \mathcal{E}^{op}\left(\overline{\pi}\right)\right\} \ ,$$

i.e., the maximum norm in a finite set.

Theorem 36 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$. Then*

$$\underline{\mathrm{clm}}\vartheta_{\overline{b}}^{R}(\overline{c}) \leq e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right) \ .$$

Equality holds when $\mathcal{F}^{op}(\overline{\pi})$ is bounded in which case $\mathcal{E}^{op}(\overline{\pi})$ can be replaced by $\mathcal{F}^{op}(\overline{\pi})$. Indeed, in such a case one has

$$\mathrm{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) = \underline{\mathrm{clm}}\vartheta_{\overline{b}}^{R}(\overline{c}) = e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right) \ .$$

Proof. Write

$$\underline{\mathrm{clm}}\vartheta_{\overline{b}}^{R}(\overline{c}) = \limsup_r \frac{\vartheta(\overline{\pi}) - \vartheta(c^r, \overline{b})}{\|\overline{c} - c^r\|_*} \ ,$$

for an appropriate sequence $\{(c^r, \overline{b})\}_r \subset \mathrm{dom}\mathcal{F}^{op}$ converging to $\overline{\pi}$. According to Lemma 18 (ii), there exists a certain $\overline{x} \in \mathrm{Lim\,sup}_r\, \mathcal{E}^{op}(c^r, \overline{b})$ and associated $x^k \in \mathcal{F}^{op}(c^{r_k}, \overline{b})$, for $r_1 < r_2 < \ldots < r_k < \ldots$, such that $x^k \to \overline{x} \in \mathcal{E}^{op}(\overline{\pi})$. Then, for all $k \in \mathbb{N}$ we have

$$\vartheta(\overline{\pi}) - \vartheta(c^{r_k}, \overline{b}) \leq \overline{c}' x^k - (c^{r_k})' x^k \leq \|\overline{c} - c^{r_k}\|_* \|x^k\| \ ,$$

which implies

$$\begin{aligned} \underline{\mathrm{clm}}\vartheta_{\overline{b}}^{R}(\overline{c}) &= \limsup_k \frac{\vartheta(\overline{\pi}) - \vartheta(c^{r_k}, \overline{b})}{\|\overline{c} - c^{r_k}\|_*} \\ &\leq \limsup_k \|x^k\| = \|\overline{x}\| \leq e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right) \ . \end{aligned}$$

For the second part, note that the boundedness of $\mathcal{F}^{op}(\overline{\pi})$ entails that $\mathrm{span}\{\overline{a}_t, t \in T\} = \mathbb{R}^n$, hence $\mathcal{E}^{op}(\overline{\pi}) = \mathrm{extr}\mathcal{F}^{op}(\overline{\pi})$ and $e\left(\mathrm{extr}\mathcal{F}^{op}(\overline{\pi}), 0_n\right) = e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right)$ (this last follows a standard argument by using the convexity of the norm). So, we only have to prove that $\underline{\mathrm{clm}}\vartheta_{\overline{b}}^{R}(\overline{c}) \geq e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right)$. Let $\overline{x} \in \mathcal{F}^{op}(\overline{\pi})$ with $\|\overline{x}\| = e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right)$. Take $u \in \mathbb{R}^n$, with $\|u\|_* = 1$, be such that $u'\overline{x} = \|\overline{x}\|$. Define the perturbation

$$c^r := \overline{c} - \frac{1}{r}u \ \text{ for each } r \in \mathbb{N} \ .$$

Since $\mathcal{F}^{op}(\overline{\pi})$ is bounded, from Remark 24 $(c^r, \overline{b}) \in \text{dom}\mathcal{F}^{op}$ for r large enough. Then, since both problems $(c^r, \overline{b})$ and $\overline{\pi}$ have the same feasible set, we have

$$\vartheta(c^r, \overline{b}) \leq (c^r)'\overline{x} = \overline{c}'\overline{x} - \frac{1}{r}u'\overline{x} = \vartheta(\overline{\pi}) - \|c^r - \overline{c}\|_* \|\overline{x}\| \ .$$

Therefore,

$$\underline{\text{clm}}\vartheta_{\overline{b}}^R(\overline{c}) \geq \limsup_r \frac{\vartheta(\overline{\pi}) - \vartheta(c^r, \overline{b})}{\|\overline{c} - c^r\|_*} \geq \|\overline{x}\| = e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right) \ .$$

■

Inequality of Theorem 36 may be strict, as we show in the following example:

Example 37 Consider the nominal problem, in $\mathbb{R}^2$ with the Euclidean norm,

$$\begin{array}{lll} \overline{\pi}: & \text{Min} & x_2 \\ & \text{s.t.} & x_1 \leq -1 \ , \\ & & -x_2 \leq 1 \ . \end{array}$$

Clearly $\mathcal{E}^{op}(\overline{\pi}) = \left\{(-1,-1)'\right\}$ hence, $e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right) = \sqrt{2}$. Let us see that $\underline{\text{clm}}\vartheta_{\overline{b}}^R(\overline{c}) = 1$. Take $c = (\varepsilon_1, 1 + \varepsilon_2)'$ with $\sqrt{\varepsilon_1^2 + \varepsilon_2^2} < 1$ and observe that $\left(c, \overline{b}\right) \in \text{dom}\mathcal{F}^{op}$ if and only if $\varepsilon_1 \leq 0$, in which case $\vartheta\left(c, \overline{b}\right) = -\varepsilon_1 - 1 - \varepsilon_2$. Then

$$\vartheta\left(\overline{\pi}\right) - \vartheta\left(c, \overline{b}\right) = \varepsilon_1 + \varepsilon_2 \leq \varepsilon_2 \leq \|c - \overline{c}\|_* \ .$$

Accordingly, $\underline{\text{clm}}\vartheta_{\overline{b}}^R(\overline{c}) \leq 1$, and equality happens by taking $\varepsilon_1 = 0$.

1.3 Calmness modulus under canonical perturbations

This section is devoted to compute/estimate the calmness moduli from below and above of the optimal value function restricted to $\mathrm{dom}\mathcal{F}^{op}$, ϑ^R defined in (0.15). Recall that they are respectively given by

$$\underline{\mathrm{clm}}\vartheta^R(\overline{\pi}) = \limsup_{\pi \underset{\mathrm{dom}\mathcal{F}^{op}}{\to} \overline{\pi}} \frac{\vartheta(\overline{\pi}) - \vartheta(\pi)}{\|\pi - \overline{\pi}\|} \text{ and } \overline{\mathrm{clm}}\vartheta^R(\overline{\pi}) = \limsup_{\pi \underset{\mathrm{dom}\mathcal{F}^{op}}{\to} \overline{\pi}} \frac{\vartheta(\pi) - \vartheta(\overline{\pi})}{\|\pi - \overline{\pi}\|} ,$$

and, roughly speaking, provide a measure of the maximum rate of decrease and increase, respectively, under perturbations of the data (regarding solvable problems only).

Theorem 38 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$. Then*

$$\overline{\mathrm{clm}}\vartheta^R(\overline{\pi}) = \overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + d\left(0_n, \mathcal{F}^{op}\left(\overline{\pi}\right)\right) .$$

Proof. Take $\overline{x} \in \mathcal{F}^{op}(\overline{\pi})$ with $\|\overline{x}\| = d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right)$. Fix arbitrarily $\varepsilon > 0$ and let $\pi \equiv (c, b) \in \mathrm{dom}\mathcal{F}^{op}$ be such that $\|\pi - \overline{\pi}\| < \delta$ for a certain $\delta > 0$ satisfying the following:

$$\begin{array}{rcl} \delta & \leq & \overline{\delta} \text{ (the one from Lemma 22)}, \\ \left\|b - \overline{b}\right\|_\infty & < & \delta \Rightarrow \left\{ \begin{array}{l} d\left(\overline{x}, \mathcal{F}^{op}\left(\overline{c}, b\right)\right) < \varepsilon \text{ (by Theorem 20)}, \\ b \in U_{\overline{b}} \text{ (that of Lemma 25)}, \\ \vartheta\left(\overline{c}, b\right) - \vartheta\left(\overline{\pi}\right) \leq \left(\overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + \varepsilon\right) \left\|b - \overline{b}\right\|_\infty . \end{array} \right. \end{array}$$

Now pick an arbitrary $\widehat{x} \in \mathcal{F}^{op}\left(\pi\right) \subset \mathcal{F}^{op}\left(\overline{c}, b\right)$ (because $\delta \leq \overline{\delta}$) and $\widetilde{x} \in \mathcal{F}^{op}\left(\overline{c}, b\right)$ with $\|\widetilde{x} - \overline{x}\| < \varepsilon$. Clearly $\vartheta\left(\pi\right) = c'\widehat{x} \leq c'\widetilde{x}$ and $\vartheta\left(\overline{c}, b\right) = \overline{c}'\widehat{x} =$

$\overline{c}'\widetilde{x}$. Then we have

$$\begin{aligned}\vartheta(\pi)-\vartheta(\overline{\pi}) &= c'\widehat{x}-\overline{c}'\widehat{x}+\vartheta(\overline{c},b)-\vartheta(\overline{\pi})\\ &\leq c'\widetilde{x}-\overline{c}'\widetilde{x}+\left(\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\bar{b})+\varepsilon\right)\left\|b-\bar{b}\right\|_{\infty}\\ &\leq \|c-\overline{c}\|_{*}\left(\|\overline{x}\|+\varepsilon\right)+\left(\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\bar{b})+\varepsilon\right)\left\|b-\bar{b}\right\|_{\infty}\\ &\leq \left(\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\bar{b})+\|\overline{x}\|+2\varepsilon\right)\|\pi-\overline{\pi}\|\ .\end{aligned}$$

Since $\varepsilon>0$ has been arbitrarily chosen, we get $\overline{\mathrm{clm}}\vartheta^{R}(\overline{\pi})\leq\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\bar{b})+d\left(0_n,\mathcal{F}^{op}(\overline{\pi})\right)$.

Next we show that the previous inequality holds as an equality. The case $0_n\in\mathcal{F}^{op}(\overline{\pi})$ is trivial, since in such a case we have $\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\bar{b})\leq\overline{\mathrm{clm}}\vartheta^{R}(\overline{\pi})\leq\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\bar{b})+0$; i.e., $\overline{\mathrm{clm}}\vartheta^{R}(\overline{\pi})=\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\bar{b})$. Hence, let us assume $0_n\notin\mathcal{F}^{op}(\overline{\pi})$. Let us consider a sequence $\{b^r\}_{r\in\mathbb{N}}\subset\mathrm{dom}\mathcal{F}_{\overline{a}}$ such that

$$\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\bar{b})=\lim_r\frac{\vartheta(\overline{c},b^r)-\vartheta(\overline{\pi})}{\left\|b^r-\bar{b}\right\|_{\infty}}\ .$$

Because of Theorem 20 we may assume $0_n\notin\mathcal{F}^{op}(\overline{c},b^r)$ for all r. The same theorem ensures the existence of $x^r\in\mathcal{F}^{op}(\overline{c},b^r)$ with $\|x^r\|=d\left(0_n,\mathcal{F}^{op}(\overline{c},b^r)\right)$, for all r, and $\|x^r\|\rightarrow\|\overline{x}\|$ (we do not need to guarantee $x^r\rightarrow\overline{x}$). More in detail, replacing $\{b^r\}$ with an appropriate subsequence (we do not relabel for simplicity) we could choose $x^r\in B\left(0_n,\|\overline{x}\|+\frac{1}{r}\right)$, i.e., the open ball centered at 0_n with radius $\|\overline{x}\|+\frac{1}{r}$, which is an open set containing $\overline{x}$; again considering an appropriate subsequence we may assume that $\{x^r\}$ converges to certain $z\in\mathrm{cl}B\left(0_n,\|\overline{x}\|\right)$, and if $\|z\|<\|\overline{x}\|$ we attain a contradiction with Theorem 20. Now, for each r, we appeal to [11, Lemma 9] to ensure the existence of $u^r\in\mathbb{R}^n$ with $\|u^r\|_*=1$ such that

$$\left(u^r\right)'x\geq\left(u^r\right)'x^r=\|x^r\|\ \text{ for all } x\in\mathcal{F}^{op}\left(\overline{c},b^r\right)\ . \tag{1.8}$$

Let us define $c^r := \overline{c} + \left\| b^r - \overline{b} \right\|_\infty u^r$, which entails $\|c^r - \overline{c}\|_* = \left\| b^r - \overline{b} \right\|_\infty$. First we note that, for all $x \in \mathcal{F}^{op}(\overline{c}, b^r)$, we have

$$(c^r)' x = \overline{c}' x + \left\| b^r - \overline{b} \right\|_\infty (u^r)' x \geq \vartheta(\overline{c}, b^r) + \|c^r - \overline{c}\|_* \|x^r\| \ . \tag{1.9}$$

Then, from Lemma 23, we can ensure the existence of $r_0 \in \mathbb{N}$ such that

$$\pi^r \equiv (c^r, b^r) \in \mathrm{dom}\mathcal{F}^{op}, \text{ for } r \geq r_0 \ .$$

It yields $\mathcal{F}^{op}(\pi^r) \subset \mathcal{F}^{op}(\overline{c}, b^r)$ for $r \geq r_0$ large enough (apply Lemma 22). Then, by repeating inequality (1.9) with any $x \in \mathcal{F}^{op}(\pi^r)$, we deduce $\vartheta(\pi^r) \geq \vartheta(\overline{c}, b^r) + \|c^r - \overline{c}\|_* \|x^r\|$ and therefore, recalling $\|c^r - \overline{c}\|_* = \left\| b^r - \overline{b} \right\|_\infty$, we have

$$\begin{aligned} \overline{\mathrm{clm}}\vartheta^R(\overline{\pi}) &\geq \limsup_r \frac{\vartheta(\pi^r) - \vartheta(\overline{\pi})}{\|\pi^r - \overline{\pi}\|} \\ &= \limsup_r \frac{\vartheta(\pi^r) - \vartheta(\overline{c}, b^r)}{\|c^r - \overline{c}\|_*} + \lim_r \frac{\vartheta(\overline{c}, b^r) - \vartheta(\overline{\pi})}{\left\| b^r - \overline{b} \right\|_\infty} \\ &\geq \lim_r \|x^r\| + \overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) = \|\overline{x}\| + \overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) \ . \end{aligned}$$

which establishes the aimed equality $\overline{\mathrm{clm}}\vartheta^R(\overline{\pi}) = \overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + d(0_n, \mathcal{F}^{op}(\overline{\pi}))$.
■

Theorem 39 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$. Then*

$$\underline{\mathrm{clm}}\vartheta^R(\overline{\pi}) \leq \underline{\mathrm{clm}}\vartheta^R_{\overline{c}}\left(\overline{b}\right) + e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right) \ . \tag{1.10}$$

Proof. For simplicity, write $\alpha := \underline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$. Reasoning by contradiction, assume the existence of a sequence $\{\pi^r \equiv (c^r, b^r)\} \subset \mathrm{dom}\mathcal{F}^{op}$ converging to $\overline{\pi}$ such that

$$\vartheta(\overline{\pi}) - \vartheta(\pi^r) > (\alpha + \varepsilon) \|\pi^r - \overline{\pi}\| \ ,$$

for all $r \in \mathbb{N}$ and some $\varepsilon > 0$. According to Lemma 18(ii), pick any $\overline{x} \in \operatorname{Lim\,sup}_r \mathcal{E}^{op}(\pi^r)$, and take $r_1 < r_2 < ... < r_k < ...$ and associated $x^k \in \mathcal{E}^{op}(\pi^r) \subset \mathcal{F}^{op}(\pi^{r_k})$ such that $x^k \to \overline{x}$. According to Lemma 22 we may assume that $x^k \in \mathcal{F}^{op}(\overline{c}, b^{r_k})$ for all $k \in \mathbb{N}$. Then, for k large enough guaranteeing $\left\|x^k - \overline{x}\right\| \leq \varepsilon$ and $b^{r_k} \in U_{\overline{b}}$ (see again Lemma 25 and Corollary 27), and taking Theorem 28 into account, we attain the following contradiction:

$$\begin{aligned}
\vartheta(\overline{\pi}) - \vartheta(\pi^{r_k}) &= \vartheta(\overline{\pi}) - \vartheta(\overline{c}, b^{r_k}) + \overline{c}' x^k - (c^{r_k})' x^k \\
&\leq \underline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) \left\|\overline{b} - b^{r_k}\right\|_\infty + \|\overline{c} - c^{r_k}\|_* \left\|x^k\right\| \\
&\leq \left(\underline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + \|\overline{x}\| + \varepsilon\right) \|\pi^{r_k} - \overline{\pi}\| \\
&\leq (\alpha + \varepsilon) \|\pi^{r_k} - \overline{\pi}\| \ .
\end{aligned}$$

■

The following example shows that inequality (1.10) may be strict.

Example 40 (Example 37 revisited) Consider the nominal problem, in $\mathbb{R}^2$ with the Euclidean norm,

$$\begin{array}{lll}
\overline{\pi}: & \text{Min} & x_2 \\
 & \text{s.t.} & x_1 \leq -1 \ , \\
 & & -x_2 \leq 1 \ .
\end{array}$$

Recall that $\mathcal{E}^{op}(\overline{\pi}) = \left\{(-1,-1)'\right\}$. Let us see that the specification of (1.10) to this case reads as $2 \leq 1+\sqrt{2}$. For $\|\pi - \overline{\pi}\| < 1$ one has $\pi \equiv (c, b) \in \operatorname{dom}\mathcal{F}^{op}$ if and only if $c_1 \leq 0$, where $c = (c_1, c_2)'$. For convenience, let us write $c = (\varepsilon_1, 1 + \varepsilon_2)'$ and $b = (-1 + \varepsilon_3, 1 + \varepsilon_4)'$, with

$$\|\pi - \overline{\pi}\| = \max\left\{\sqrt{\varepsilon_1^2 + \varepsilon_2^2}, |\varepsilon_3|, |\varepsilon_4|\right\} =: \varepsilon < 1 \ .$$

Then we have, provided that $\varepsilon_1 \leq 0$,

$$\vartheta(\overline{\pi}) - \vartheta(\pi) = -1 - \varepsilon_1(-1+\varepsilon_3) - (1+\varepsilon_2)(-1-\varepsilon_4) \leq 2\varepsilon + \varepsilon^2 \ ,$$

and, accordingly, by letting $\varepsilon \downarrow 0$, we have $\underline{\mathrm{clm}}\vartheta^R(\overline{\pi}) \leq 2$. Indeed, by taking $\varepsilon_1 = \varepsilon_3 = 0$ and $\varepsilon_2 = \varepsilon_4 = \varepsilon$, we see that $\underline{\mathrm{clm}}\vartheta^R(\overline{\pi}) = 2$. A simpler calculation with $c = \overline{c}$ shows that $\underline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) = 1$.

The following corollary comes from Theorems 28, 38 and 39.

Corollary 41 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$. Then*

$$\mathrm{clm}\vartheta^R(\overline{\pi}) \leq \max\left\{k^- + e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right), k^+ + d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right)\right\} \ .$$

Theorem 42 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$ with $\mathcal{F}^{op}(\overline{\pi})$ bounded. Then*

$$\underline{\mathrm{clm}}\vartheta^R(\overline{\pi}) = \underline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right) \ .$$

Proof. Since $\mathcal{F}^{op}(\overline{\pi})$ is bounded, we apply the same reasoning as in proof of Theorem 36. So, we only have to prove that (1.10) holds as an equality in this case. Let us consider, similarly to the proof of Theorem 38, a sequence $\{b^r\}_{r\in\mathbb{N}} \subset \mathrm{dom}\mathcal{F}_{\overline{a}}$ such that

$$\underline{\mathrm{clm}}\vartheta^R_{\overline{c}}\left(\overline{b}\right) = \lim_r \frac{\vartheta(\overline{\pi}) - \vartheta(\overline{c}, b^r)}{\left\|b^r - \overline{b}\right\|_\infty} \ .$$

From Theorem 20 we easily deduce

$$e\left(\mathcal{F}^{op}(\overline{c}, b^r), 0_n\right) \to e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right) \ ,$$

as $r \to \infty$. Let us write, for each r, $e\left(\mathcal{F}^{op}(\overline{c}, b^r), 0_n\right) = \|x^r\|$ with $x^r \in \mathcal{F}^{op}(\overline{c}, b^r)$. For each r, let $u^r \in \mathbb{R}^n$ with $\|u^r\|_* = 1$ be such that $(u^r)' x^r =$

$\|x^r\|$ and let $c^r := \overline{c} - \left\|b^r - \overline{b}\right\|_\infty u^r$. Similarly to Remark 24, we have $\pi^r \equiv (c^r, b^r) \in \mathrm{dom}\mathcal{F}^{op}$ for r large enough (say for all r). Clearly $\|\pi^r - \overline{\pi}\| = \left\|b^r - \overline{b}\right\|_\infty$. Choose for each r any $\widehat{x}^r \in \mathcal{F}^{op}(\pi^r)$, which, for r large enough (say again for each r) satisfies $\widehat{x}^r \in \mathcal{F}^{op}(\overline{c}, b^r)$ by virtue of Lemma 22. Observe that

$$(u^r)' \, \widehat{x}^r \leq \|u^r\|_* \, \|\widehat{x}^r\| = \|\widehat{x}^r\| \leq \|x^r\| = (u^r)' \, x^r \ ,$$

due to the choice of x^r and u^r. Consequently,

$$\begin{aligned} (c^r)' \, x^r &= \overline{c}' x^r - \left\|b^r - \overline{b}\right\|_\infty (u^r)' \, x^r \\ &\leq \overline{c}' \widehat{x}^r - \left\|b^r - \overline{b}\right\|_\infty (u^r)' \, \widehat{x}^r = (c^r)' \, \widehat{x}^r = \vartheta\left(\pi^r\right) \ . \end{aligned}$$

In other words, $x^r \in \mathcal{F}^{op}(\pi^r)$. Thus,

$$\begin{aligned} \underline{\mathrm{clm}}\vartheta^R\left(\overline{\pi}\right) &\geq \limsup_r \frac{\vartheta\left(\overline{\pi}\right) - \vartheta\left(\pi^r\right)}{\|\pi^r - \overline{\pi}\|} \\ &= \lim_r \frac{\vartheta\left(\overline{\pi}\right) - \vartheta\left(\overline{c}, b^r\right)}{\left\|b^r - \overline{b}\right\|_\infty} + \lim_r \frac{\overline{c}' x^r - (c^r)' \, x^r}{\left\|b^r - \overline{b}\right\|_\infty} \\ &= \underline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + \lim_r \|x^r\| \\ &= \underline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + e\left(\mathcal{F}^{op}\left(\overline{\pi}\right), 0_n\right) \ . \end{aligned}$$

■

At this point, we observe that these calmness moduli can be splitted into two moduli each one, those of $\vartheta^R_{\overline{c}}$ and $\vartheta^R_{\overline{b}}$ addressed in the previous sections.

Corollary 43 *Let* $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$. *Then*

(i) $\overline{\mathrm{clm}}\vartheta^R(\overline{\pi}) = \overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + \overline{\mathrm{clm}}\vartheta^R_{\overline{b}}(\overline{c})$.

(ii) $\underline{\mathrm{clm}}\vartheta^R(\overline{\pi}) = \underline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + \underline{\mathrm{clm}}\vartheta^R_{\overline{b}}(\overline{c})$ *when* $\mathcal{F}^{op}(\overline{\pi})$ *is bounded.*

Hence, the calmness modulus under canonical perturbations is closely related with the calmness moduli of $\vartheta^R_{\overline{c}}$ and $\vartheta^R_{\overline{b}}$.

Corollary 44 *Let* $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$*. If* $\mathcal{F}^{op}(\overline{\pi})$ *is bounded, then*

$$\mathrm{clm}\vartheta^{R}(\overline{\pi}) \leq \mathrm{clm}\vartheta^{R}_{\overline{c}}(\overline{b}) + \mathrm{clm}\vartheta^{R}_{\overline{b}}(\overline{c}) \ . \tag{1.11}$$

When $\mathcal{F}^{op}(\overline{\pi}) = \{\overline{x}\}$*, then*

$$\mathrm{clm}\vartheta^{R}(\overline{\pi}) = \mathrm{clm}\vartheta^{R}_{\overline{c}}(\overline{b}) + \|\overline{x}\| \ .$$

The following example shows that inequality (1.11) could be strict.

Example 45 Consider the nominal problem in $\mathbb{R}^3$, endowed with the Euclidean norm:

$$\begin{array}{lll} \overline{\pi}: & \mathrm{Min} & x_1 \\ & \mathrm{s.t.} & -x_1 + x_2 \leq 0 \ , \quad (t=1) \\ & & -x_1 - x_2 \leq 0 \ , \quad (t=2) \\ & & -2x_1 \leq 0 \ , \qquad\ \ (t=3) \\ & & -x_3 \leq -1 \ , \qquad (t=4) \\ & & x_3 \leq 2 \ . \qquad\qquad (t=5) \end{array}$$

It is easy to check that $\mathcal{F}^{op}(\overline{\pi}) = \{0_2\} \times [1,2]$, which obviously is bounded, and $\mathcal{M}_{\overline{\pi}} = \{\{1,2\},\{3\}\}$. Using the results of Section 1.1 it is easy to see that $\mathrm{clm}\vartheta^{R}_{\overline{c}}(\overline{b}) = 1$. On the other hand, by Theorem 36 one obtains $\mathrm{clm}\vartheta^{R}_{\overline{b}}(\overline{c}) = 2$. However, $\mathrm{clm}\vartheta^{R}(\overline{\pi}) = \frac{5}{2}$ in virtue of Theorems 38 and 42.

Finally, under the SCQ at $\overline{b}$ together with the boundedness of $\mathcal{F}^{op}(\overline{\pi})$, equivalently, when $\overline{\pi} \in \mathrm{int}\,\mathrm{dom}\mathcal{F}^{op}$, we have the exact formulae for all moduli, which are gathered in the following theorem. In it, we have also taken into account the fact that $\overline{\pi} \in \mathrm{int}\,\mathrm{dom}\mathcal{F}^{op}$ turns out to be equivalent to the simultaneous nonemptiness and boundedness of both nominal optimal sets $\mathcal{F}^{op}(\overline{\pi})$ and $\Lambda^{op}(\overline{\pi})$; indeed, for $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$, the boundedness of

$\Lambda^{op}(\overline{\pi})$ is equivalent to SCQ (see [28, Theorem 6.1(v)]). In this case we can write $k^+ = \max_{\lambda \in \mathrm{extr}\Lambda^{op}(\overline{\pi})} \|\lambda\|_1 = \max_{\lambda \in \Lambda^{op}(\overline{\pi})} \|\lambda\|_1$. Moreover, one always has $k^- = \min_{\lambda \in \mathrm{extr}\Lambda^{op}(\overline{\pi})} \|\lambda\|_1 = \min_{\lambda \in \Lambda^{op}(\overline{\pi})} \|\lambda\|_1$, because of the linearity of $\|\cdot\|_1$ on Λ^{op}. So, according to these comments, the results previously obtained give rise to the following theorem.

Theorem 46 *Let $\overline{\pi} \in \operatorname{int} \operatorname{dom}\mathcal{F}^{op}$. Then, we have*

$$\begin{aligned}
\mathrm{clm}\vartheta(\overline{\pi}) &= \mathrm{clm}\vartheta^R(\overline{\pi}) \\
&= \max\{k^- + e\left(\mathcal{F}^{op}\left(\overline{\pi}\right), 0_n\right),\ k^+ + d\left(0_n, \mathcal{F}^{op}\left(\overline{\pi}\right)\right)\} \\
&= \max\left\{ \min_{\lambda \in \Lambda^{op}(\overline{\pi})} \|\lambda\|_1 + \max_{x \in \mathcal{F}^{op}(\overline{\pi})} \|x\|,\ \max_{\lambda \in \Lambda^{op}(\overline{\pi})} \|\lambda\|_1 + \min_{x \in \mathcal{F}^{op}(\overline{\pi})} \|x\| \right\}.
\end{aligned}$$

1.4 Calmness modulus and distance to infeasibility

To finish this chapter, we analyze the relationship between $\mathrm{clm}\vartheta(\overline{\pi})$ and the well analyzed concept of distance to infeasibility; the reader is addressed to [45, 46] for details on this distance in the context of conic linear systems and to [11] (where it is called *distance to ill-posedness)* in the linear semi-infinite setting. Specifically, [45, Theorem 1.1] provides a certain Lipschitz type inequality for ϑ which immediately yields an upper bound on $\mathrm{clm}\vartheta(\overline{\pi})$, provided that $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$. This upper bound has the distance to the infeasibility in the denominator. Let us recall some details: paper [45] deals with conic linear problems in the form

$$\begin{aligned} \text{Inf} \quad & c^*x \\ \text{s.t.} \quad & b - Ax \in C_Y, \\ & x \in C_X, \end{aligned} \tag{1.12}$$

where $C_X \subset X$ and $C_Y \subset Y$ are closed convex cones in X and Y, respectively. X is a reflexive normed space while Y is an arbitrary normed space. The norms in both spaces are denoted by $\|\cdot\|$. Here $b \in Y$, $A : X \to Y$ is a (continuous) linear operator, with norm $\|A\| := \sup\{\|Ax\| \mid \|x\| = 1\}$, and $c^* : X \to \mathbb{R}$ is an element of the dual space of X, i.e., a continuous linear functional, with $\|c^*\| := \sup\{c^*x \mid \|x\| = 1\}$. The parameter space of all problems (1.12) is endowed with the product norm

$$\|(A, b, c^*)\| := \max\{\|A\|, \|b\|, \|c^*\|\} .$$

Our parametrized problem (0.1) may be translated into the conic format, just by taking $X = C_X := \mathbb{R}^n$, $Y = \mathbb{R}^T$, $C_Y := \mathbb{R}^T_+$, and considering a fixed matrix A (which remains unperturbed). In this way, the results of [45] apply into our LP context, where we are considering $\|\cdot\|_\infty$ for measuring the perturbations of b (indeed, the reader is adressed to [15] for details about the convenience of this norm when dealing with polyhedral cones).

Following the notation of [45], we consider

$$Pri\emptyset := \mathbb{R}^T \setminus \mathrm{dom}\mathcal{F}_{\overline{a}} \text{ and } Dual\emptyset := \mathbb{R}^n \setminus \mathrm{dom}\Lambda,$$

corresponding, respectively, to the set of parameters b and c associated with primal and dual inconsistent problems. In this way,

$$d(b, Pri\emptyset) := \inf\{\left\|b - b^1\right\| \mid b^1 \in Pri\emptyset\}$$

represents the distance from $b \in \mathbb{R}^T$ to primal infeasibility, while $d\left(c, Dual\emptyset\right)$, analogously defined, denotes the corresponding distance to dual infeasibility. Observe that

$$\pi = (c, b) \in \operatorname{int}\operatorname{dom}\mathcal{F}^{op} \Leftrightarrow \min\left\{d\left(b, Pri\emptyset\right), d\left(c, Dual\emptyset\right)\right\} > 0.$$

Theorem 47 (see [45, Theorem 1.1(5)]) *Let $\overline{\pi} \in \operatorname{int}\operatorname{dom}\mathcal{F}^{op}$. Then, for any $\pi = (c, b) \in \mathbb{R}^n \times \mathbb{R}^T$ such that*

$$\left\|b - \overline{b}\right\| < d\left(\overline{b}, Pri\emptyset\right) \text{ and } \left\|c - \overline{c}\right\| < d\left(\overline{c}, Dual\emptyset\right),$$

we have

$$\begin{aligned} |\vartheta(\pi) - \vartheta(\overline{\pi})| \quad \leq \quad & \left\|b - \overline{b}\right\| \frac{\|\overline{c}\| + \|c - \overline{c}\|}{d\left(\overline{b}, Pri\emptyset\right) - \|\pi - \overline{\pi}\|} \frac{\|\overline{\pi}\|}{d\left(\overline{c}, Dual\emptyset\right)} \qquad (1.13) \\ & + \|c - \overline{c}\| \frac{\left\|\overline{b}\right\| + \left\|b - \overline{b}\right\|}{d\left(\overline{c}, Dual\emptyset\right) - \|\pi - \overline{\pi}\|} \frac{\|\overline{\pi}\|}{d\left(\overline{b}, Pri\emptyset\right)}. \end{aligned}$$

Now if we divide both members of (1.13) by $\|\pi - \overline{\pi}\|$ and let $\|\pi - \overline{\pi}\| \to 0$, we inmediately derive the following Corollary.

Corollary 48 *Let $\overline{\pi} \in \operatorname{int}\operatorname{dom}\mathcal{F}^{op}$. One has*

$$\operatorname{clm}\vartheta(\overline{\pi}) \leq \frac{\|\overline{c}\|}{d\left(\overline{b}, Pri\emptyset\right)} \frac{\|\overline{\pi}\|}{d\left(\overline{c}, Dual\emptyset\right)} + \frac{\left\|\overline{b}\right\|}{d\left(\overline{c}, Dual\emptyset\right)} \frac{\|\overline{\pi}\|}{d\left(\overline{b}, Pri\emptyset\right)}. \qquad (1.14)$$

Remark 49 Observe that Theorem 46 constitutes a refinement of Corollary 48, as far as inequality (1.14) can be strict. In fact, its right-hand side (upper bound on $\operatorname{clm}\vartheta(\overline{\pi})$) might be much greater that $\operatorname{clm}\vartheta(\overline{\pi})$ when $\overline{\pi}$ approaches the primal/dual infeasibility. Just consider the example, in $\mathbb{R}^2$ endowed with the Euclidean norm,

$$\begin{array}{lll} \pi^r: & \text{Min} & x_1 + \frac{1}{r}x_2 \\ & \text{s.t.} & -x_1 \leq 0 \ , \\ & & -x_2 \leq \frac{1}{r} \\ & & x_2 \leq \frac{1}{r} \ . \end{array}$$

One easily sees that $b^r \to 0_3$, $d\left(b^r, Pri\emptyset\right) \to 0$, and so the right-hand side of (1.14) goes to $+\infty$, while (appealing to Theorem 46)

$$\text{clm}\vartheta(\pi^r) = \left\| \left(1, \frac{1}{r}, 0\right)' \right\|_1 + \frac{1}{r} \to 1.$$

Chapter 2

Lipschitz continuity of the optimal value function

This chapter is a step forward in the study of the stability of the optimal value function started in Chapter 1. With a similar structure to the calmness work and taking advantage on the background on calmness, we intend now to study the Lipschitzian behavior of the optimal value function in different perturbation frameworks; specifically, our aim is focussed on the Lipschitzian behavior of ϑ, ϑ^R, $\vartheta^R_{\overline{c}}$ and $\vartheta^R_{\overline{b}}$. A key strategy here (inspired by [42, Sect. 2]) consists in computing the Lipschitz modulus through the computation of a uniform calmness constant. At this moment we advance that $\mathrm{lip}\vartheta^R_{\overline{c}}(\overline{b})$ is completely determined through a point-based formula (depending only on the nominal data) without any assumption (see Theorem 51), while $\mathrm{lip}\vartheta^R_{\overline{b}}(\overline{c})$ and $\mathrm{lip}\vartheta^R(\overline{\pi})$ are upper and lower estimated in general (see Theorems 54, 56, and 58). It is also shown that under the boundedness of $\mathcal{F}^{op}(\overline{\pi})$, both $\mathrm{lip}\vartheta^R_{\overline{b}}(\overline{c})$ and $\mathrm{lip}\vartheta^R(\overline{\pi})$ are also completely determined. We emphasize again that all the mentioned estimates (or exact values) are given exclusively in terms of the data of a nominal problem $\overline{\pi} \equiv \left(\overline{c}, \overline{b}\right)$. In the part *Conclusions and future work* at the end of this document we include, as well as in the calmness case, a table gathering the main results about the computed Lipschitz moduli. Since calmness and Lipschitz moduli are closely related, is not surprising that the ingredients in the formulae are the same. Hence, in the conclusions we also summarize the relationships between calmness and Lipschitz moduli of the optimal value function.

2.1 Lipschitz modulus under RHS perturbations

This section is devoted to compute the Lipschitz modulus of the optimal value under perturbations of b, $\text{lip}\vartheta_{\overline{c}}^{R}(\overline{b})$. Throughout this section, c is assumed to be fixed, i.e., $\overline{c}$, and it verifies

$$-\overline{c} \in \text{cone}\{\overline{a}_t, t \in T\} ,$$

so $(\overline{c}, b) \in \text{dom}\mathcal{F}^{op}$ if and only if $b \in \text{dom}\mathcal{F}_{\overline{a}}$ (see Proposition 12(ii)). Recall that

$$\lambda^D := \left(\lambda_t^D\right)_{t\in T} \in \mathbb{R}_+^T, \text{ for } D \in \mathcal{M}_{\overline{\pi}} ,$$

is the unique element such that $-\overline{c} = \sum_{t\in D} \lambda_t^D \overline{a}_t$ and $\lambda_t^D = 0, t \in T \setminus D$ (see Remark 10). Moreover we appeal to constant k^+ defined in (1.4) as:

$$k^+ = \max_{D\in\mathcal{M}_{\overline{\pi}}} \left\|\lambda^D\right\|_1 .$$

The next proposition follows an analogous argument to the one used for establishing Corollary 27. Nevertheless, due to its simplicity, and for completeness purposes, we include its proof. As in Section 1.1, for the sake of simplicity in the notation we usually write $\vartheta_{\overline{c}}^{R}$ when referring to the function itself and $\vartheta(\overline{c}, b)$, with $b \in \text{dom}\mathcal{F}_{\overline{a}}$, for its images.

Proposition 50 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$ and let $U_{\overline{b}}$ be as in Lemma 25. Then,*

$$\left|\vartheta(\overline{c}, b) - \vartheta(\overline{c}, \widetilde{b})\right| \leq k^{+}\|b - \widetilde{b}\|_{\infty} \textit{ for all } b, \widetilde{b} \in \mathrm{dom}\mathcal{F}_{\overline{a}} \cap U_{\overline{b}} \ .$$

Consequently

$$\mathrm{lip}\vartheta_{\overline{c}}^{R}(\overline{b}) \leq k^{+} \ .$$

Proof. Take $b, \widetilde{b} \in \mathrm{dom}\mathcal{F}_{\overline{a}} \cap U_{\overline{b}}$. Applying Corollary 26 we have

$$\vartheta(\overline{c}, b) - \vartheta(\overline{c}, \widetilde{b}) = \max_{D \in \mathcal{M}_{\overline{\pi}}} \left(-b'\lambda^{D}\right) - \max_{D \in \mathcal{M}_{\overline{\pi}}} \left(-\widetilde{b}'\lambda^{D}\right),$$

and let us assume the first maximum is reached at $\widehat{D} \in \mathcal{M}_{\overline{\pi}}$, then

$$\begin{aligned} \vartheta(\overline{c}, b) - \vartheta(\overline{c}, \widetilde{b}) &= -b'\lambda^{\widehat{D}} + \min_{D \in \mathcal{M}_{\overline{\pi}}} \widetilde{b}'\lambda^{D} \leq -b'\lambda^{\widehat{D}} + \widetilde{b}'\lambda^{\widehat{D}} \\ &= (\widetilde{b} - b)'\lambda^{\widehat{D}} \leq k^{+}\|b - \widetilde{b}\|_{\infty} \ . \end{aligned}$$

Since b and $\widetilde{b}$ have been arbitrarily chosen, switching them in the preceding argument we obtain the aimed inequality. ■

Theorem 51 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$. Then $\vartheta_{\overline{c}}^{R}$ is Lipschitz continuous at $\overline{b}$ and*

$$\mathrm{lip}\vartheta_{\overline{c}}^{R}(\overline{b}) = k^{+} \ . \tag{2.1}$$

Proof. According to the previous proposition, it remains to prove $\mathrm{lip}\vartheta_{\overline{c}}^{R}(\overline{b}) \geq k^{+}$. To do that take any $\overline{D} \in \mathcal{M}_{\overline{\pi}}$ such that $\|\lambda^{\overline{D}}\|_{1} = k^{+}$ and let us construct two sequences $\{b^{r}\}$, $\{\widetilde{b}^{r}\} \subset \mathrm{dom}\mathcal{F}_{\overline{a}}$ converging to $\overline{b}$ such that

$$\limsup_{r} \frac{\left|\vartheta\left(\overline{c}, b^{r}\right) - \vartheta(\overline{c}, \widetilde{b}^{r})\right|}{\left\|b^{r} - \widetilde{b}^{r}\right\|_{\infty}} = \|\lambda^{\overline{D}}\|_{1} \ ,$$

which will establish our aimed inequality. Let $\overline{x} \in \mathcal{F}^{op}(\overline{\pi})$. Fix an arbitrary $r \in \mathbb{N}$. Observe that

$$W_{r} := \left\{x \in \mathbb{R}^{n} \mid \overline{a}_{t}'x < \overline{b}_{t} + \frac{1}{r}, \ t \in T \backslash \overline{D}\right\} \ ,$$

is a neighborhood of $\overline{x}$. Now, since $\overline{a}_t'\overline{x} = \overline{b}_t,\ t \in \overline{D}$, and $\left\{\overline{a}_t,\ t \in \overline{D}\right\}$ is linearly independent, a standard argument in LP yields the existence of $0 < \delta_r < \frac{1}{r}$ such that the systems of linear equations

$$\left\{\overline{a}_t'x = \overline{b}_t - \delta_r,\ t \in \overline{D}\right\} \text{ and } \left\{\overline{a}_t'x = \overline{b}_t + \delta_r,\ t \in \overline{D}\right\} \tag{2.2}$$

have solutions inside W_r; pick x^r and $\widetilde{x}^r$ as solutions of the respective systems in (2.2) and such that $x^r,\ \widetilde{x}^r \in W_r$. Now, let us define $b^r = (b_t^r)_{t\in T}$ and $\widetilde{b}^r = (\widetilde{b}_t^r)_{t\in T}$ as follows

$$b_t^r := \begin{cases} \overline{b}_t - \delta_r & \text{if } t \in \overline{D}\ , \\ \overline{b}_t + \frac{1}{r} & \text{if } t \in T\backslash\overline{D}\ ; \end{cases} \quad \text{and } \widetilde{b}_t^r := \begin{cases} \overline{b}_t + \delta_r & \text{if } t \in \overline{D}\ , \\ \overline{b}_t + \frac{1}{r} & \text{if } t \in T\backslash\overline{D}\ . \end{cases}$$

In particular, $x^r \in \mathcal{F}_{\overline{a}}\left(b^r\right)$ and $\widetilde{x}^r \in \mathcal{F}_{\overline{a}}(\widetilde{b}^r)$; in fact, $x^r \in \mathcal{F}^{op}\left(\overline{c}, b^r\right)$ and $\widetilde{x}^r \in \mathcal{F}^{op}(\overline{c}, \widetilde{b}^r)$, since $\overline{D} \subset T_{b^r}\left(x^r\right) \cap T_{\widetilde{b}^r}\left(\widetilde{x}^r\right)$. Moreover, according to the KKT conditions and taking into account that $\lambda^{\overline{D}}$ is a vector of KKT multipliers associated with both problems $(\overline{c}, b^r)$ and $(\overline{c}, \widetilde{b}^r)$, by duality in LP we have that

$$\vartheta\left(\overline{c}, b^r\right) = -\left(b^r\right)'\lambda^{\overline{D}} \text{ and } \vartheta(\overline{c}, \widetilde{b}^r) = -(\widetilde{b}^r)'\lambda^{\overline{D}}\ . \tag{2.3}$$

In this way, and since clearly both sequences $\{b^r\}_{r\in\mathbb{N}}$ and $\{\widetilde{b}^r\}_{r\in\mathbb{N}}$ converge to $\overline{b}$, by applying (2.3) and recalling that $\lambda_t^{\overline{D}} = 0$ for $t \in T\backslash\overline{D}$, we have

$$\begin{aligned} \limsup_r \frac{\left|\vartheta\left(\overline{c}, b^r\right) - \vartheta(\overline{c}, \widetilde{b}^r)\right|}{\left\|b^r - \widetilde{b}^r\right\|_\infty} &= \limsup_r \frac{\left|-(b^r - \widetilde{b}^r)\lambda^{\overline{D}}\right|}{2\delta_r} \\ &= \limsup_r \frac{\left|-\sum_{t\in\overline{D}}(-2\delta_r\lambda_t^{\overline{D}})\right|}{2\delta_r} = \left\|\lambda^{\overline{D}}\right\|_1 . \end{aligned}$$

■

The following corollary is a direct consequence of the previous theorem together with Corollary 32.

Corollary 52 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$ and assume that SCQ holds at $\overline{b}$. Then we have*

$$\mathrm{lip}\vartheta_{\overline{c}}^{R}(\overline{b}) = \mathrm{clm}\vartheta_{\overline{c}}^{R}(\overline{b}) = k^{+}.$$

Remark 53 The specification of Example 30 for $\theta = \frac{1}{2}$ shows that shows that $\mathrm{clm}\vartheta_{\overline{c}}^{R}(\overline{b})$ can be strictly less than $\mathrm{lip}\vartheta_{\overline{c}}^{R}(\overline{b})$ when SCQ fails.

2.2 Lipschitz modulus under c-perturbations

This section is devoted to study the Lipschitz modulus under perturbations of the objective function where $\overline{b} \in \mathrm{dom}\mathcal{F}_{\overline{a}}$ is fixed. Recall that $c \in -\mathrm{cone}\{\overline{a}_t, t \in T\}$ if and only if $(c, \overline{b}) \in \mathrm{dom}\mathcal{F}^{op}$, and that $\vartheta_{\overline{b}}^{R}(c) := \vartheta\left(c, \overline{b}\right)$, for any $c \in -\mathrm{cone}\{\overline{a}_t, t \in T\}$.

The following theorem provides a lower and an upper estimate for $\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c})$. Moreover, it shows that the upper estimate becomes the exact modulus when $\mathcal{F}^{op}(\overline{\pi})$ is bounded.

Theorem 54 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$. Then*

$$d(0_n, \mathcal{F}^{op}(\overline{\pi})) \leq \mathrm{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) \leq \mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) \leq e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right).$$

Moreover, if we assume that $\mathcal{F}^{op}(\overline{\pi})$ is bounded, then

$$\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) = \mathrm{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) = e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right).$$

Proof. First, $\mathrm{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) \geq d(0_n, \mathcal{F}^{op}(\overline{\pi}))$ comes from Theorem 35. Also recall that $\mathrm{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) \leq \mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c})$ is always true. It rests to check that $\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) \leq e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$. The proof for this part follows the same structure as in proof of Theorem 36 but taking a second perturbation. Anyway, we incorporate it for the convenience of the reader. Write

$$\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) = \limsup_r \frac{\left|\vartheta(c^r, \overline{b}) - \vartheta(\widetilde{c}^r, \overline{b})\right|}{\|c^r - \widetilde{c}^r\|_*} , \tag{2.4}$$

for appropriate sequences $\{c^r\}_r$, $\{\widetilde{c}^r\}_r \subset -\mathrm{cone}\{\overline{a}_t, t \in T\}$ converging to $\overline{c}$. Because of the symmetry of the quotients in (2.4), it is not restrictive to assume $\vartheta(c^r, \overline{b}) - \vartheta(\widetilde{c}^r, \overline{b}) \geq 0$ for all r. According to Lemma 18(ii), there exists a certain $\overline{x} \in \mathrm{Lim\,sup}_r\, \mathcal{E}^{op}(\widetilde{c}^r, \overline{b})$ and associated $x^k \in \mathcal{E}^{op}(\widetilde{c}^{r_k}, \overline{b}) \subset \mathcal{F}^{op}(\widetilde{c}^{r_k}, \overline{b})$, for $r_1 < r_2 < \ldots < r_k < \ldots$, such that $x^k \to \overline{x} \in \mathcal{E}^{op}(\overline{\pi})$. Then, for all $k \in \mathbb{N}$ we have

$$0 \leq \vartheta(c^{r_k}, \overline{b}) - \vartheta(\widetilde{c}^{r_k}, \overline{b}) \leq (c^{r_k})' x^k - (\widetilde{c}^{r_k})' x^k \leq \|c^{r_k} - \widetilde{c}^{r_k}\|_* \|x^k\| ,$$

which implies

$$\begin{aligned} \mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) &= \limsup_k \frac{\vartheta(c^{r_k}, \overline{b}) - \vartheta(\widetilde{c}^{r_k}, \overline{b})}{\|c^{r_k} - \widetilde{c}^{r_k}\|_*} \\ &\leq \limsup_k \|x^k\| = \|\overline{x}\| \leq e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right) . \end{aligned}$$

Finally, let us assume that $\mathcal{F}^{op}(\overline{\pi})$ is bounded. From Theorem 36 we have

$$\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) \geq \mathrm{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) = e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right) \geq \mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c})$$

from which we obtain the aimed equality. ■

The next example is intended to show that the first two inequalities in the statement of Theorem 54 may be strict. At the moment we do not have an example where $\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) \leq e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$ holds strictly, so that the question of whether or not equality fulfills for all $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$ remains open.

Example 55 Consider the nominal problem, in $\mathbb{R}^3$ with the Euclidean norm,

$$\begin{array}{rll} \overline{\pi}: & \text{Min} & x_3 \\ & \text{s.t.} & x_1 \leq -1 \ , \ (t=1) \\ & & -x_2 \leq 2 \ , \ (t=2) \\ & & -x_3 \leq 0 \ . \ (t=3) \end{array}$$

Clearly $d(0_3, \mathcal{F}^{op}(\overline{\pi})) = 1$, $\mathcal{E}^{op}(\overline{\pi}) = \left\{(-1,-2,0)'\right\}$, and hence $e\left(\mathcal{E}^{op}(\overline{\pi}), 0_3\right) = \sqrt{5}$. Let us prove that $\mathrm{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) = 2$ and $\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) = \sqrt{5}$. Consider any $0 < \varepsilon < 1$ and any $c \in \mathbb{R}^3$ with $\|c - \overline{c}\|_* = \varepsilon$, which may be written as $c = (\varepsilon_1, \varepsilon_2, 1 + \varepsilon_3)'$ with $\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 = \varepsilon^2$. Then $\left(c, \overline{b}\right) \in \mathrm{dom}\mathcal{F}^{op}$ if and only if $\varepsilon_1 \leq 0$ and $\varepsilon_2 \geq 0$, in which case $\vartheta\left(c, \overline{b}\right) = c'\left(-1,-2,0\right)' = -\varepsilon_1 - 2\varepsilon_2$. Accordingly,

$$\min_{\substack{\|c-\overline{c}\|_*=\varepsilon \\ \left(c,\overline{b}\right)\in\mathrm{dom}\mathcal{F}^{op}}} \vartheta\left(c, \overline{b}\right) = \min_{\substack{\varepsilon_1^2+\varepsilon_2^2=\varepsilon^2 \\ \varepsilon_1\leq 0,\ \varepsilon_2\geq 0}} -\varepsilon_1 - 2\varepsilon_2 = -2\varepsilon \ , \tag{2.5}$$

attained at $c = (0, \varepsilon, 1)'$. The corresponding maximum equals ε and is attained at $c = (-\varepsilon, 0, 1)'$. Consequently, for any $0 < \varepsilon < 1$,

$$\max_{\substack{\|c-\overline{c}\|_*=\varepsilon \\ \left(c,\overline{b}\right)\in\mathrm{dom}\mathcal{F}^{op}}} \left|\vartheta\left(c, \overline{b}\right) - \vartheta\left(\overline{\pi}\right)\right| = 2\varepsilon \ ,$$

which, clearly entails $\mathrm{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) = 2$. Now let us compute the Lipschitz modulus of $\vartheta_{\overline{b}}^{R}$ at $\overline{c}$. As a motivation of such computation note that

$$\max_{\varepsilon_1^2+\varepsilon_2^2=\varepsilon^2} -\varepsilon_1 - 2\varepsilon_2 = \sqrt{5}\varepsilon \ ,$$

and this maximum is attained at $(\varepsilon_1, \varepsilon_2) = \left(-\varepsilon/\sqrt{5}, -2\varepsilon/\sqrt{5}\right)$. Let us consider $c := \left(-\varepsilon/\sqrt{5}, 0, 1\right)'$ and $\widetilde{c} := \left(0, 2\varepsilon/\sqrt{5}, 1\right)'$. Then

$$\frac{\left|\vartheta\left(c, \overline{b}\right) - \vartheta\left(\widetilde{c}, \overline{b}\right)\right|}{\|c - \widetilde{c}\|_*} = \frac{\varepsilon/\sqrt{5} - \left(-4\varepsilon/\sqrt{5}\right)}{\varepsilon} = \sqrt{5} \ .$$

Since this happens for all $0 < \varepsilon < 1$, we conclude $\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) \geq \sqrt{5}$. The converse inequality comes from Theorem 54.

2.3 Lipschitz modulus under canonical perturbations

The objective of this section is to compute (or at least estimate) the Lipschitz modulus of the optimal value function restricted to $\mathrm{dom}\mathcal{F}^{op}$, at a nominal parameter $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$ when the RHS of the constraints and the coefficients of the objective function can be simultaneously perturbed. The first result provides a lower bound of the Lipschitz modulus $\mathrm{lip}\vartheta^{R}(\overline{\pi})$.

Theorem 56 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$. Then*

$$\mathrm{lip}\vartheta^{R}(\overline{\pi}) \geq k^{+} + d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right).$$

Proof. The case $0_n \in \mathcal{F}^{op}(\overline{\pi})$ is trivial due to the fact that $\mathrm{lip}\vartheta^{R}(\overline{\pi}) \geq \mathrm{lip}\vartheta_{\overline{c}}^{R}(\overline{b})$. So, let us assume $0_n \notin \mathcal{F}^{op}(\overline{\pi})$. Take $\overline{x} \in \mathcal{F}^{op}(\overline{\pi})$ with $\|\overline{x}\| = d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right)$. Let us consider sequences $\{b^r\}_r, \{\widetilde{b}^r\}_r \subset \mathrm{dom}\mathcal{F}_{\overline{a}}$ such that

$$k^{+} = \mathrm{lip}\vartheta_{\overline{c}}^{R}(\overline{b}) = \lim_r \frac{\vartheta(\overline{c}, b^r) - \vartheta(\overline{c}, \widetilde{b}^r)}{\|b^r - \widetilde{b}^r\|_\infty}.$$

The next step is analogous to its counterpart for calmness in the proof of Theorem 38, so that we will focus on the differences. As in formula (1.8) in the referred proof, there exist sequences $\{x^r\}_r$ and $\{u^r\}_r$ in $\mathbb{R}^n$, with $\|x^r\| \to \|\overline{x}\|$, such that, for each r, $x^r \in \mathcal{F}^{op}(\overline{c}, b^r)$, $\|u^r\|_* = 1$ and

$$(u^r)'x \geq (u^r)'x^r = \|x^r\| = d\left(0_n, \mathcal{F}^{op}(\overline{c}, b^r)\right), \text{ whenever } x \in \mathcal{F}^{op}(\overline{c}, b^r).$$

Now we define $c^r := \overline{c} + \|b^r - \widetilde{b}^r\|_\infty u^r$. For $x \in \mathcal{F}^{op}(\overline{c}, b^r)$ one has

$$(c^r)'x = \overline{c}'x + \|b^r - \widetilde{b}^r\|_\infty (u^r)'x \geq \vartheta(\overline{c}, b^r) + \|b^r - \widetilde{b}^r\|_\infty \|x^r\|, \tag{2.6}$$

so $x \mapsto (c^r)'x$ is bounded from below on $\mathcal{F}^{op}(\overline{c}, b^r)$. Because of Lemma 23, there exists $r_0 \in \mathbb{N}$ such that $\pi^r \equiv (c^r, b^r) \in \mathrm{dom}\mathcal{F}^{op}$ for $r \geq r_0$. Then Lemma 22 yields $\mathcal{F}^{op}(\pi^r) \subset \mathcal{F}^{op}(\overline{c}, b^r)$ for $r \geq r_0$ large enough. Accordingly, by the restriction of (2.6) to points $x \in \mathcal{F}^{op}(\pi^r)$ we get

$$\vartheta(\pi^r) = (c^r)'x \geq \vartheta(\overline{c}, b^r) + \|b^r - \widetilde{b}^r\|_\infty \|x^r\| \ .$$

Let us define $\widetilde{\pi}^r :\equiv (\overline{c}, \widetilde{b}^r)$ which belongs to $\mathrm{dom}\mathcal{F}^{op}$ (because $\{\widetilde{b}^r\}_r \subset \mathrm{dom}\mathcal{F}_{\overline{a}}$ and $-\overline{c} \in \mathrm{cone}\{\overline{a}_t, t \in T\}$). Note that $\|\pi^r - \widetilde{\pi}^r\| = \|b^r - \widetilde{b}^r\|_\infty$. Then we have

$$\begin{aligned}
\mathrm{lip}\vartheta^R(\overline{\pi}) &\geq \limsup_r \frac{|\vartheta(\pi^r) - \vartheta(\widetilde{\pi}^r)|}{\|\pi^r - \widetilde{\pi}^r\|} \\
&\geq \limsup_r \frac{\vartheta(\pi^r) - \vartheta(\overline{c}, b^r) + \vartheta(\overline{c}, b^r) - \vartheta(\overline{c}, \widetilde{b}^r)}{\|b^r - \widetilde{b}^r\|_\infty} \\
&= \lim_r \frac{\vartheta(\overline{c}, b^r) - \vartheta(\overline{c}, \widetilde{b}^r)}{\|b^r - \widetilde{b}^r\|_\infty} + \limsup_r \frac{\vartheta(\pi^r) - \vartheta(\overline{c}, b^r)}{\|b^r - \widetilde{b}^r\|_\infty} \\
&\geq \mathrm{lip}\vartheta^R_{\overline{c}}(\overline{b}) + \lim_r \|x^r\| = k^+ + \|\overline{x}\| \ .
\end{aligned}$$

■

In order to establish an upper bound for the Lipschitz modulus of ϑ^R at $\overline{\pi}$, we appeal to the technique developed in [42, Section 2]. Specifically, Wu Li proved that if a set-valued mapping is Hausdorff lower semicontinuous, a uniform upper Lipschitz constant for that mapping in a convex neighborhood of the nominal parameter becomes a Lipschitz constant in such a neighborhood (see [42, Theorem 2.1] for details). Translating it into our context, a uniform calmness constant for ϑ^R in a neighborhood (relative to $\mathrm{dom}\mathcal{F}^{op}$) of $\overline{\pi}$ becomes a Lipschitz constant at $\overline{\pi}$. This technique was already applied in [8] for obtaining the so-called sharp Lipschitz constant for $\mathcal{F}^{op}$ under suitable hypotheses.

Lemma 57 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$. For all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$k^+ + e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right) + \varepsilon$$

is a calmness constant of ϑ^R at any $\pi \in (\mathrm{dom}\mathcal{F}^{op}) \cap B(\overline{\pi}, \delta)$ (the closed ball centered at $\overline{\pi}$ of radius δ).

Proof. We start by observing that, from Lemma 18 (ii), $\mathcal{E}^{op} : \mathrm{dom}\mathcal{F}^{op} \rightrightarrows \mathbb{R}^n$ is Hausdorff-upper semicontinuous at $\overline{\pi}$; i.e., $\lim_{\pi \to \overline{\pi}} e\left(\mathcal{E}^{op}(\pi), \mathcal{E}^{op}(\overline{\pi})\right) = 0$. Now, let us abuse the notation and identify also constant k^+ as a function $k^+ : \mathrm{dom}\mathcal{F}^{op} \longrightarrow \mathbb{R}^+$ defined as $k^+(\pi) = \max_{D \in \mathcal{M}_\pi} \left\|\lambda^D\right\|_1$, where $k^+(\overline{\pi})$ is our original k^+ as defined in (1.4). We need to prove that function k^+ is also upper semicontinuous at $\overline{\pi}$, that is, for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|\pi - \overline{\pi}\| < \delta$, for $\pi \in \mathrm{dom}\mathcal{F}^{op}$, then $k^+(\pi) \leq k^+(\overline{\pi}) + \varepsilon$. Reasoning by contradiction, suppose that there exists a sequence $\{\pi^r\}_r \subset \mathrm{dom}\mathcal{F}^{op}$ converging to $\overline{\pi}$ such that $k^+(\pi^r) \geq k^+(\overline{\pi}) + \varepsilon_0$ for a certain $\varepsilon_0 > 0$. Suppose that the maximum defining $k^+(\pi^r)$ is attained at a certain $D^r \in \mathcal{M}_{\pi^r}$. Since T is finite, we can assume the existence of a constant subsequence, say $D^r = D$ for all r. The fact that $-c^r \in \mathrm{cone}\{\overline{a}_t, t \in D\}$ entails $-\overline{c} \in \mathrm{cone}\{\overline{a}_t, t \in D\}$, although recall we cannot ensure the minimality of D for $\overline{\pi}$. Also recall that $\{\overline{a}_t', t \in D\}$ is linearly independent. Write

$$-c^r = \sum_{t \in D} \lambda_t^r \overline{a}_t \text{ for all } r, \text{ and } -\overline{c} = \sum_{t \in D} \lambda_t^D \overline{a}_t\ .$$

Using a standard argument it is easy to see that $\left\{\sum_{t\in D}\lambda_t^r\right\}_r$ is bounded so, taking a subsequence, if necessary, it may be assumed to converge to $\sum_{t\in D}\lambda_t^D$. Despite we cannot assume $D\in\mathcal{M}_{\overline{\pi}}$, we know that D contains at least a minimal element for $\overline{\pi}$, so let $\widetilde{D}\in\mathcal{M}_{\overline{\pi}}$ with $\widetilde{D}\subset D$ and $\lambda_t^D=0$ for all $t\notin\widetilde{D}$. Therefore we have

$$k^+(\pi^r)=\sum_{t\in D}\lambda_t^r\longrightarrow\sum_{t\in D}\lambda_t^D=\sum_{t\in\widetilde{D}}\lambda_t^D\leq k^+(\overline{\pi}),$$

hence we have attained a contradiction. Applying now the upper semicontinuity of both $\mathcal{E}^{op}$ and k^+, for any $\varepsilon>0$ there exists $\delta>0$ such that

$$e\left(\mathcal{E}^{op}(\pi),0_n\right)\leq e\left(\mathcal{E}^{op}(\overline{\pi}),0_n\right)+\varepsilon/2$$

$$\text{and } k^+(\pi)\leq k^+(\overline{\pi})+\varepsilon/2,$$

for all $\pi\in\mathrm{dom}\mathcal{F}^{op}$ with $\|\pi-\overline{\pi}\|<\delta$, and therefore, taking Corollary 41 into account,

$$\mathrm{clm}\vartheta^R(\overline{\pi})\leq k^+(\pi)+e\left(\mathcal{E}^{op}(\pi),0_n\right)\leq k^+(\overline{\pi})+e\left(\mathcal{E}^{op}(\overline{\pi}),0_n\right)+\varepsilon\ .$$

■

Theorem 58 *Let $\overline{\pi}\in\mathrm{dom}\mathcal{F}^{op}$. Then*

$$\mathrm{lip}\vartheta^R(\overline{\pi})\leq k^++e\left(\mathcal{E}^{op}(\overline{\pi}),0_n\right). \tag{2.7}$$

If, additionally, $\mathcal{F}^{op}(\overline{\pi})$ is bounded, then equality hods in (2.7), which reads as

$$\mathrm{lip}\vartheta^R(\overline{\pi})=k^++e\left(\mathcal{F}^{op}(\overline{\pi}),0_n\right).$$

Proof. Recall that $\mathrm{dom}\mathcal{F}^{op}$ is convex in $\mathbb{R}^n\times\mathbb{R}^T$ (see Corollary 13(ii)) and Theorem 19 establishes the continuity of ϑ^R on $\mathrm{dom}\mathcal{F}^{op}$. Then, the previous lemma and its preceding comments ensure that $k^{+}+e\left(\mathcal{E}(\overline{\pi}),0_n\right)+\varepsilon$ is a Lipschitz constant of ϑ^R at $\overline{\pi}$ for each $\varepsilon>0$. Letting $\varepsilon\downarrow 0$ we obtain (2.7).

Now assume that $\mathcal{F}^{op}(\overline{\pi})$ is bounded. In order to establish the converse inequality, consider sequences $\{b^r\}_r, \{\widetilde{b}^r\}_r \subset \mathrm{dom}\mathcal{F}_{\overline{a}}$ such that

$$k^{+}=\mathrm{lip}\vartheta^R_{\overline{c}}(\overline{b})=\lim_r \frac{\vartheta(\overline{c},\widetilde{b}^r)-\vartheta(\overline{c},b^r)}{\|\widetilde{b}^r-b^r\|_\infty}.$$

Apply Theorem 20 and Remark 21 to conclude that $\mathcal{F}^{op}(\overline{c},b^r)$ is nonempty and bounded for r large enough (say for all r). For each $r\in\mathbb{N}$ take $x^r\in\mathcal{F}^{op}(\overline{c},b^r)$ such that $\|x^r\|=e\left(\mathcal{F}^{op}(\overline{c},b^r),0_n\right)$ and let $u^r\in\mathbb{R}^n$ be such that $\|u^r\|_*=1$ and $(u^r)'\,x^r=\|x^r\|$.

The sequence $\{x^r\}_{r\in\mathbb{N}}$ may not converge, although it has for sure a convergent subsequence, but we can ensure, again by Theorem 20, that $\|x^r\|\to e\left(\mathcal{F}^{op}(\overline{\pi}),0_n\right)$.

For each r let us define $c^r:=\overline{c}-\|\widetilde{b}^r-b^r\|_\infty u^r$. Obviously $x\mapsto (c^r)'\,x$ is bounded from below on $\mathcal{F}^{op}(\overline{c},b^r)$, because this set is compact; so that, Lemma 23 yields $(c^r,b^r)\in\mathrm{dom}\mathcal{F}^{op}$ for r large enough, and then

$$\vartheta(\overline{c},b^r)-\vartheta(c^r,b^r)\geq(\overline{c}-c^r)'\,x^r=\|\widetilde{b}^r-b^r\|_\infty\|x^r\|.$$

Therefore

$$\begin{aligned}
\mathrm{lip}\vartheta^R(\overline{\pi}) &\geq \limsup_r \frac{\vartheta(\overline{c},\widetilde{b}^r)-\vartheta(c^r,b^r)}{\|(\overline{c},\widetilde{b}^r)-(c^r,b^r)\|}\\
&= \limsup_r \frac{\vartheta(\overline{c},\widetilde{b}^r)-\vartheta(\overline{c},b^r)+\vartheta(\overline{c},b^r)-\vartheta(c^r,b^r)}{\|\widetilde{b}^r-b^r\|_\infty}\\
&= \mathrm{lip}\vartheta^R_{\overline{c}}(\overline{b})+\limsup_r \frac{\vartheta(\overline{c},b^r)-\vartheta(c^r,b^r)}{\|\widetilde{b}^r-b^r\|_\infty}\\
&\geq k^{+}+\lim_r\|x^r\|=k^{+}+e\left(\mathcal{F}^{op}(\overline{\pi}),0_n\right).
\end{aligned}$$

■

Corollary 59 *Let* $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$*, with* $\mathcal{F}^{op}(\overline{\pi})$ *bounded. Then*

$$\mathrm{lip}\vartheta^{R}(\overline{\pi}) = \mathrm{lip}\vartheta^{R}_{\overline{c}}(\overline{b}) + \mathrm{lip}\vartheta^{R}_{\overline{b}}(\overline{c}).$$

Proof. It comes from Theorems 51, 54 and 58. ■

Chapter 3

Lipschitz lower semicontinuity of the feasible set mapping

Let us come back to the initial situation of system (0.2) introduced in the preliminaries:

$$\sigma = \left\{ a_t' x \leq b_t, \ \ t \in T \right\} \ ,$$

where T is arbitrary and there is no requirement to the functions $t \mapsto a_t$ and $t \mapsto b_t$. We also study the case when T is finite as a particularization.

In this last chapter, we change the functional view of previous ones and focus on studying the variability of multifunction $\mathcal{F}$ when we perturb the parameters of the system. Recall that we identify system σ with $(a, b) \in \Theta$ and it is the parameter to be perturbed around nominal one, say $\overline{\sigma} \equiv \left(\overline{a}, \overline{b}\right)$. We allow perturbations of a and b separately and both simultaneously.

As we commented in Section 0.2, the implications among Lipschitz type properties for a generic multifunction are strict in general. However, for a feasible set mapping the situation is quite different. In the following proposition we gather well-known results characterizing those properties for the feasible set mapping $\mathcal{F}$. We denote by

$$\operatorname{conv}(\sigma) := \operatorname{conv}\left(\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T\right\}\right) \ .$$

Proposition 60 *Let* $(\overline{\sigma}, \overline{x}) \in \operatorname{gph}\mathcal{F}$. *The following statements are equivalent:*

(i) $\mathcal{F}$ *is lsc at* $\overline{\sigma}$.

(ii) $\overline{\sigma} \in \operatorname{intdom}\mathcal{F}$.

(iii) $\overline{b} \in \operatorname{intdom}\mathcal{F}_{\overline{a}}$.

(iv) $0_{n+1} \notin \operatorname{cl}(\operatorname{conv}(\overline{\sigma}))$.

(v) $\overline{\sigma}$ *satisfies de SSCQ.*

(vi) $\mathcal{F}$ *satisfies the Aubin property at* $(\overline{\sigma}, \overline{x})$.

(vii) $\mathcal{F}_{\overline{a}}$ *satisfies the Aubin property at* $(\overline{b}, \overline{x})$.

(viii) $\mathcal{F}$ *is Lipschitz-lsc** *at* $(\overline{\sigma}, \overline{x})$.

(ix) $\mathcal{F}_{\overline{a}}$ *is Lipschitz-lsc** *at* $(\overline{b}, \overline{x})$.

(x) $\mathcal{F}$ *is Lipschitz-lsc at* $(\overline{\sigma}, \overline{x})$.

(xi) $\mathcal{F}_{\overline{a}}$ *is Lipschitz-lsc at* $(\overline{b}, \overline{x})$.

(xii) $\mathcal{F}^{-1}$ *is metrically regular at* $(\overline{\sigma}, \overline{x})$.

(xiii) There exist a neighborhood V of $\overline{\sigma}$ and a constant $\kappa \geq 0$ such that

$$d\left(\overline{x}, \mathcal{F}(\sigma)\right) \leq \kappa d\left(\sigma, \mathcal{F}^{-1}(\overline{x})\right) \text{ for all } \sigma \in V \ . \tag{3.1}$$

Proof. Equivalences (i)⇔(ii)⇔(iii)⇔(iv)⇔(v) come from [28, Theorem 6.1]. From [11, Corollary 5], we know that (ii)⇒(vi). Obviously, (vi)⇒(vii)⇒(ix)⇒(xi) and (vi)⇒(viii)⇒(x)⇒(i). From the definition of Lipschitz-lsc in (xi), we have that $\mathcal{F}_{\overline{a}}(b) \neq \emptyset$ for all b in a neighborhood of $\overline{b}$, which implies (iii). On the other hand, characterization (vi)⇔(xii) is a well-known result (see [37, Lemma 1.12]). Implication (xii)⇒(xiii) is trivial. Finally, (xiii)⇒(x) due to the fact that $d\left(\sigma, \mathcal{F}^{-1}(\overline{x})\right) \leq d\left(\sigma, \overline{\sigma}\right)$ for all $\sigma \in V$. ■

Remark 61 Property (xiii) of Proposition 60 can be seen as the counterpart of the metric regularity for the Lipschitz-lsc. Unlike what happens with the equality between modulus of metric regularity and Lipschitz modulus for a general multifunction (see Section 0.2), the picture is completely different for Lipschitz-lsc. The next example shows that, even for functions, the Lipschitz-lsc property does not imply its metric counterpart (3.1). However, in Theorem 78 we will prove that the infimum of constants κ for which (3.1) holds (for some associated neighborhoods) does coincide with the Lipschitz-lsc modulus when we deal with a feasible set mapping $\mathcal{F}$.

Example 62 Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by:

$$f(y) := \begin{cases} y \sin \frac{1}{y} & \text{if} \quad y \neq 0 \\ 0 & \text{if} \quad y = 0 \ . \end{cases}$$

We have that

$$f^{-1}(0) = \{0\} \cup \left\{ \frac{1}{k\pi} \ \mid \ k \in \mathbb{Z} \setminus \{0\} \right\} \ .$$

For $k \in \mathbb{N}$, consider the sequence $\{y_k\}$ defined as:

$$y_k := \frac{1}{2}\left(\frac{1}{2k\pi} + \frac{1}{(2k+1)\pi}\right) = \frac{4k+1}{4k(2k+1)\pi} \ .$$

Then,

$$d\left(y_k, f^{-1}(0)\right) = \frac{1}{2}\left(\frac{1}{2k\pi} - \frac{1}{(2k+1)\pi}\right) = \frac{1}{4k(2k+1)\pi} \ .$$

Hence,

$$\frac{d\left(0, f(y_k)\right)}{d\left(y_k, f^{-1}(0)\right)} = \frac{\frac{4k+1}{4k(2k+1)\pi}\sin\frac{4k(2k+1)\pi}{4k+1}}{\frac{1}{4k(2k+1)\pi}} = (4k+1)\sin\frac{2k\pi}{4k+1} \ ,$$

which tends to infinite when $k \to +\infty$. Accordingly, the counterpart for property (xiii) of Proposition 60 does not hold for f (instead of $\mathcal{F}$). However, f is trivially Lipschitz-lsc at $\overline{y} = 0$ (for functions Lipschitz-lsc and Lipschitz-lsc* are equivalent to calmness). More specifically,

$$d\left(f(0), f(y)\right) \leq d\left(0, y\right) \text{ for all } y \in \mathbb{R} \ .$$

Remark 63 Observe that all statements in Proposition 60 except (iv) and (v) refer to either $\mathcal{F}$ or $\mathcal{F}_{\overline{a}}$, and hence they work for, at least, RHS perturbations. The next examples show that, for mapping $\mathcal{F}_{\overline{b}}$ (i.e., LHS perturbations only), Lipschitz-lsc property does not imply Aubin continuity, and the latter does not imply SSCQ.

Example 64 Let $\mathcal{F}_0(a) := \{x \in \mathbb{R} \mid a'x \leq 0\}$. If $a < 0$, $\mathcal{F}_0(a) = \mathbb{R}_+$ while $\mathcal{F}_0(a) = \mathbb{R}_-$ when $a > 0$. Take $\overline{a} = 0$ and $\overline{x} = 0$, $\mathcal{F}_0$ is Lipschitz-lsc but not Aubin continuous. It is easy to check that $\text{liplsc}\mathcal{F}_0(\overline{a}, 0) = 0$ while $\text{lip}\mathcal{F}_0(\overline{a}, 0) = \text{liplsc}^*\mathcal{F}_0(\overline{a}, 0) = +\infty$.

Example 65 Consider the system $\{-x_1 + x_2 \leq 0; x_1 + x_2 \leq 0; -x_2 \leq 0\}$ where SSCQ fails. So $\mathcal{F}_{0_3}(\overline{a}) = \{0_2\}$ and $\mathcal{F}_{0_3}(a) = \{0_2\}$ for a close to $\overline{a}$. Then, $\mathcal{F}_0$ verifies the Aubin property and $\text{lip}\mathcal{F}_{0_3}(\overline{a}, 0_2) = \text{liplsc}^*\mathcal{F}_{0_3}(\overline{a}, 0_2) = \text{liplsc}\mathcal{F}_{0_3}(\overline{a}, 0_2) = 0$.

3.1 The Lipschitz-lsc* property

Taking into account the previous examples, we wonder which is the difference between the Lipschitz-lsc and the Lipschitz-lsc*. In order to highlight the real meaning of the last one, the following results are intended to characterize it in terms of the Aubin property for LHS perturbations only.

Remark 66 Note that:

(i) If $0_n \in \mathcal{F}_{\overline{b}}(\overline{a})$, then $\overline{b}_t \geq 0$ for all $t \in T$, hence $0_n \in \mathcal{F}_{\overline{b}}(a)$ for all $a \in (\mathbb{R}^n)^T$ and, accordingly, $\text{liplsc}\mathcal{F}_{\overline{b}}(\overline{a}, 0_n) = 0$.

(ii) If $\mathcal{F}_{\overline{b}}(\overline{a}) = \{0_n\}$ then, similarly as above, we have $\text{liplsc}^*\mathcal{F}_{\overline{b}}(\overline{a}, 0_n) = 0$ (recall Example 65).

Having the previous remark in mind, the next theorem deals with the remaining cases. First, we appeal to the following lemma which is a classical result.

Lemma 67 (**Extended Farkas Theorem**, cf. [28, Corollary 3.1.2]) *The inequality $a'x \leq b$ is a consequence of the consistent system $\{a_t' x \leq b_t, t \in T\}$ if and only if*

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \text{cl}\,\text{cone}\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T\,;\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} .$$

Theorem 68 *Let* $(\overline{a}, \overline{x}) \in \operatorname{gph} \mathcal{F}_{\overline{b}}$.

(i) Assume that $\overline{x} \neq 0_n$. *If* $\mathcal{F}_{\overline{b}}$ *is Lipschitz-lsc at* $(\overline{a}, \overline{x})$, *then* $\overline{\sigma}$ *satisfies the SSCQ.*

(ii) Assume that $\overline{x} = 0_n$ *and* $\mathcal{F}_{\overline{b}}(\overline{a}) \neq \{0_n\}$. *If* $\mathcal{F}_{\overline{b}}$ *is Lipschitz-lsc** *at* $(\overline{a}, \overline{x})$, *then* $\overline{\sigma}$ *satisfies the SSCQ.*

Proof. Since the fulfilment or not of Lipschitz-lsc or Lipschitz-lsc* of $\mathcal{F}_{\overline{b}}$ at $(\overline{a}, \overline{x})$ does not depend on the norm under consideration in $\mathbb{R}^n$, let us regard it endowed with de Euclidean norm. We proceed arguing by contradiction, that is, we assume in both assumptions that SSCQ fails at $\overline{\sigma}$ and want to show that $\mathcal{F}_{\overline{b}}$ fails to be Lipschitz-lsc and Lipschitz-lsc* at $(\overline{a}, \overline{x})$ in (i) and (ii), respectively.

(i) For any $\varepsilon > 0$ consider $a^{\varepsilon} \in (\mathbb{R}^n)^T$ given by

$$a_t^{\varepsilon} := \overline{a}_t + \varepsilon \overline{x} \text{ for all } t \in T \ .$$

Since SSCQ fails, we have

$$0_{n+1} \in \operatorname{cl} \operatorname{conv} \left\{ \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix}, \ t \in T \right\} \ ,$$

and can write

$$0_{n+1} = \lim_r \sum_{t \in T} \lambda_t^r \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix} \ ,$$

for some $\{\lambda^r\}_{r \in \mathbb{N}} \subset \mathbb{R}_+^{(T)}$ (i.e., λ_t^r is zero for all $t \in T$ except finitely many of them, which are positive) with $\sum_{t \in T} \lambda_t^r = 1$ for all $r \in \mathbb{N}$. Accordingly,

$$\lim_r \sum_{t \in T} \lambda_t^r \begin{pmatrix} a_t^{\varepsilon} \\ \overline{b}_t \end{pmatrix} = \begin{pmatrix} \varepsilon \overline{x} \\ 0 \end{pmatrix} \ .$$

For each $\varepsilon > 0$ such that $\mathcal{F}_{\overline{b}}(a^{\varepsilon}) \neq \emptyset$ (if any), Lemma 67 entails that the inequality $\varepsilon \overline{x}' x \leq 0$ is a consequence of system $\left(a^{\varepsilon}, \overline{b}\right)$, which implies

$\mathcal{F}_{\overline{b}}(a^{\varepsilon}) \subset \{x \in \mathbb{R}^n : \varepsilon\overline{x}'x \leq 0\}$ for all $\varepsilon > 0$ (whenever the system is feasible or not), and then, according to the well-known Ascoli formula,

$$d\left(\overline{x}, \mathcal{F}_{\overline{b}}(a^{\varepsilon})\right) \geq \frac{\varepsilon \|\overline{x}\|^2}{\varepsilon \|\overline{x}\|} = \|\overline{x}\| = \frac{1}{\varepsilon} d(\overline{a}, a^{\varepsilon}) \ .$$

Letting $\varepsilon \downarrow 0$ we see that $\mathcal{F}_{\overline{b}}$ is not Lipschitz-lsc at $(\overline{a}, \overline{x})$.

(ii) Take any $\widehat{x} \in \mathcal{F}_{\overline{b}}(\overline{a}) \backslash \{0_n\}$ and any $0 < \varepsilon < 1$. Define

$$\widehat{a}_t^{\varepsilon} := \overline{a}_t + \varepsilon^2 \widehat{x} \text{ for all } t \in T \ .$$

Then, by completely analogous arguments to (i) we conclude $\mathcal{F}_{\overline{b}}(\widehat{a}^{\varepsilon}) \subset \left\{x \in \mathbb{R}^n : \varepsilon^2 \widehat{x}'x \leq 0\right\}$ for all $\varepsilon > 0$ and

$$d\left(\varepsilon\widehat{x}, \mathcal{F}_{\overline{b}}(\widehat{a}^{\varepsilon})\right) \geq \frac{\varepsilon^3 \|\widehat{x}\|^2}{\varepsilon^2 \|\widehat{x}\|} = \varepsilon \|\widehat{x}\| = \frac{1}{\varepsilon} d(\overline{a}, \widehat{a}^{\varepsilon}) \ .$$

Note that, as $\varepsilon \downarrow 0$, $\varepsilon\widehat{x}$ becomes arbitrarily close to $\overline{x} = 0_n$ and $\widehat{a}^{\varepsilon} \to \overline{a}$, so that we have shown that $\mathcal{F}_{\overline{b}}$ is not Lipschitz-lsc* at $(\overline{a}, \overline{x})$. ■

As a corollary of previous results, we obtain:

Corollary 69 *Let $(\overline{\sigma}, \overline{x}) \in \operatorname{gph} \mathcal{F}$ with $\overline{\sigma} \equiv \left(\overline{a}, \overline{b}\right)$. Assume that $\mathcal{F}_{\overline{b}}(\overline{a}) \neq \{0_n\}$. Then, $\mathcal{F}_{\overline{b}}$ is Lipschitz-lsc* at $(\overline{a}, \overline{x})$ if and only if $\mathcal{F}$ is Aubin continuous at $(\overline{\sigma}, \overline{x})$ (see equivalence (vi) $\Leftrightarrow$ (ix) in Proposition 60).*

Next we show an example where $\mathcal{F}_{\overline{b}}(\overline{a}) = \{0_n\}$ (hence, $\mathcal{F}_{\overline{b}}$ is Lipschitz-lsc* at $(\overline{a}, 0_n)$) but the calmness of $\mathcal{F}_{\overline{b}}$ fails at $(\overline{a}, 0_n)$ (and hence, Aubin continuity also fails), even if T is a compact interval and $\binom{a_t}{b_t}$ depends continuously on $t \in T$. First we include a geometrical remark used in the example.

Remark 70 From a geometrical point of view, a point $x \in \mathbb{R}^n$ is feasible for a system $\sigma = \{a_t'x \leq b_t, \ t \in T\}$ if and only if all $\binom{a_t}{b_t}$ lie in the negative polar of $\binom{x}{-1}$. This comes just from rewriting $a_t'x \leq b_t$ as $\binom{a_t}{b_t}'\binom{x}{-1} \leq 0$.

Example 71 Let us consider the nominal system, in $\mathbb{R}$,

$$\overline{\sigma} := \left\{ tx \leq |t|^{3/2}\,,\ t \in [-1,1] \right\}\ ,$$

whose feasible set is $\{0\}$. For any $0 < \varepsilon < 1$ let us consider

$$\sigma_\varepsilon := \left\{ (t-\varepsilon)\, x \leq |t|^{3/2}\,,\ t \in [-1,1] \right\}\ .$$

Clearly $d\left(\overline{\sigma}, \sigma_\varepsilon\right) = \varepsilon$. Following our geometrical remark mentioned above, in order to find the feasible set of σ_ε we must look for the tangent lines to the curve

$$\left\{ \begin{pmatrix} t-\varepsilon \\ |t|^{3/2} \end{pmatrix},\ t \in [-1,1] \right\}\ ,$$

passing through 0_2. After routinary calculations we find two: one tangent at $\binom{-\varepsilon}{0}$ and another at $\binom{2\varepsilon}{(3\varepsilon)^{3/2}}$. If we write these tangent lines as $\binom{a}{b}' \binom{x}{-1} = 0$ (where $\binom{a}{b}$ runs over the line), we find that $x = 0$ and $x = \frac{3}{2}\sqrt{3\varepsilon}$ are feasible points of σ_ε. Indeed, $\mathcal{F}\left(\sigma_\varepsilon\right) = \left[0, \frac{3}{2}\sqrt{3\varepsilon}\right]$. This clearly implies that $\mathcal{F}_{\overline{b}}$ fails to be calm (and, hence, fails to be Aubin continuous) at $\overline{a} = (t)_{t\in[-1,1]}$. More in detail, for $a_\varepsilon := (t-\varepsilon)_{t\in[-1,1]}$ we have

$$\frac{d\left(\frac{3}{2}\sqrt{3\varepsilon}, \mathcal{F}_{\overline{b}}\left(\overline{a}\right)\right)}{d\left(a_\varepsilon, \overline{a}\right)} = \frac{\frac{3}{2}\sqrt{3\varepsilon}}{\varepsilon} \to +\infty \text{ as } \varepsilon \downarrow 0\ .$$

Now, we study the particular case when the index set T is finite. Next lemma stands out the difference with the infinite case and the key fact which allows us to characterize completely the Lipschitz-lsc* in the finite context by covering the residual case when $\mathcal{F}_{\overline{b}}\left(\overline{a}\right) = \{0_n\}$.

Lemma 72 *Assume that T is finite, say $T = \{1, \ldots, m\}$. Assume $\mathcal{F}_{\overline{b}}\left(\overline{a}\right) = \{0_n\}$, then $\mathcal{F}_{\overline{b}}$ is Aubin continuous at $(\overline{a}, 0_n)$.*

Proof. Since T is finite, we may assume that

$$\overline{b}_t = \overline{a}_t' 0_n = 0 \quad (t = 1, \ldots, m) \ , \tag{3.2}$$

because otherwise the according linear inequality constraints would be locally redundant and thus not affect the local behavior of $\mathcal{F}_{\overline{b}}$. Hence, $\overline{b} = 0_n$. Reasoning by contradiction, suppose that $\mathcal{F}_{\overline{b}}$ fails to be Aubin continuous at $(\overline{a}, 0_n)$. Proposition 60 yields that

$$0_{n+1} \in \operatorname{cl} \operatorname{conv} \{\overline{\sigma}_t | t = 1, \ldots, m\} = \operatorname{conv} \{\overline{\sigma}_t | t = 1, \ldots, m\} \ ,$$

due to the finiteness of T. As a consequence,

$$0_n \in \operatorname{conv} \{\overline{a}_t | t = 1, \ldots, m\} \ , \tag{3.3}$$

We can distinguish two cases:

(i) $0_n \in \operatorname{int} \operatorname{conv} \{\overline{a}_t | t = 1, \ldots, m\}$;

(ii) $0_n \in \operatorname{bd} \operatorname{conv} \{\overline{a}_t | t = 1, \ldots, m\}$.

(i) The fact that $0_n \in \operatorname{int} \operatorname{conv} \{\overline{a}_t | t = 1, \ldots, m\}$ implies that, for a close enough to $\overline{a}$,

$$0_n \in \operatorname{int} \operatorname{conv} \{a_t | t = 1, \ldots, m\} \ . \tag{3.4}$$

Let such a be arbitrarily given and assume that there exists some $x \in \mathcal{F}_{\overline{b}}(a)$, $x \neq 0_n$. Then, by (3.4), there exist $\varepsilon > 0$ and $\nu \geq 0$, such that

$$\varepsilon \frac{x}{\|x\|} = \sum_{t=1}^{m} \nu_t a_t \ ,$$

which entails by $a_t' x \leq \overline{b}_t = 0$ for $t = 1, \ldots, m$, that

$$\varepsilon \|x\| = \sum_{t=1}^{m} \nu_t a_t' x \leq 0 \ .$$

Whence the contradiction $x = 0_n$. It follows that $\mathcal{F}_{\overline{b}}(a) \subseteq \{0_n\}$. On the other hand, clearly, $0_n \in \mathcal{F}_{\overline{b}}(a)$. Consequently, we have shown that $\mathcal{F}_{\overline{b}}(a) = \{0_n\}$ for all a close enough to $\overline{a}$. It follows that $\mathcal{F}_{\overline{b}}$ is locally constant around $\overline{a}$ and, hence, contrary to our assumption, possesses the Aubin property at $(\overline{a}, 0_n)$.

(ii) If $0_n \in \operatorname{bd}\operatorname{conv}\{\overline{a}_t | t = 1, \ldots, m\}$, then by separation argument there exists some $u \neq 0_n$ with

$$u'y \leq 0 \quad \forall y \in \text{ conv } \{\overline{a}_t | t = 1, \ldots, m\} \ .$$

In particular, we have that

$$\overline{a}_t' u \leq 0 = \overline{b}_t \quad (t = 1, \ldots, m) \ ,$$

which yields the contradiction $0_n \neq u \in \mathcal{F}_{\overline{b}}(\overline{a})$. ■

Theorem 73 *Assume that T is finite and let $(\overline{a}, \overline{x}) \in \operatorname{gph} \mathcal{F}_{\overline{b}}$. Then, $\mathcal{F}_{\overline{b}}$ is Aubin continuous at $(\overline{a}, \overline{x})$ if and only if $\mathcal{F}_{\overline{b}}$ is Lipschitz-lsc* at $(\overline{a}, \overline{x})$.*

Proof. By definition, it is enough to assume that $\mathcal{F}_{\overline{b}}$ violates the Aubin property at $(\overline{a}, \overline{x})$ and to show that it fails to be Lipschitz-lsc* at $(\overline{a}, \overline{x})$. Due to the previous lemma, the case when $\mathcal{F}_{\overline{b}}(\overline{a}) = \{0_n\}$ is not considered because it is contrary to our assumption. Then, our aimed result follows from Corollary 69. ■

3.2 Lipschitz-lsc and Lipschitz-lsc* moduli of $\mathcal{F}$

Once we have studied the properties Lipschitz-lsc and Lipschitz-lsc* of $\mathcal{F}$, $\mathcal{F}_{\overline{a}}$ and $\mathcal{F}_{\overline{b}}$ in relation with the Aubin property, the next objective is to compute the corresponding moduli. As a first step in this part of our research, we compute in this section the Lipschitz-lsc and Lipschitz-lsc* moduli of $\mathcal{F}$ at $(\overline{\sigma}, \overline{x}) \in \operatorname{gph} \mathcal{F}$, including the particular case when T is finite.

Recall that the aimed moduli are lower bounds of the Lipschitz modulus (see Section 0.2) in the following way:

$$\operatorname{liplsc}\mathcal{F}(\overline{\sigma}, \overline{x}) \leq \operatorname{liplsc}^{*}\mathcal{F}(\overline{\sigma}, \overline{x}) \leq \operatorname{lip}\mathcal{F}(\overline{\sigma}, \overline{x}) \ . \tag{3.5}$$

We will see in Theorem 78 that, under certain conditions, the previous inequalities are actually equalities.

The Lipschitz modulus of $\mathcal{F}$ at $(\overline{\sigma}, \overline{x}) \in \operatorname{gph} \mathcal{F}$ has been previously computed in terms of the regularity modulus of $\mathcal{F}^{-1}$ in [7, Theorem 1]. To obtain $\operatorname{liplsc}\mathcal{F}(\overline{\sigma}, \overline{x})$ and $\operatorname{liplsc}^{*}\mathcal{F}(\overline{\sigma}, \overline{x})$ we follow in Theorem 78 the sketch of the proof given in [7, Theorem 1] adapted to our context and notation and also considering the case when T is finite.

The following two lemmas are technical results needed later on our computation. In them we appeal to the mapping $\mathcal{L} : \mathbb{R}^{n+1} \rightrightarrows \mathbb{R}^{n}$ defined as

$$\mathcal{L}\begin{pmatrix} a \\ b \end{pmatrix} := \left\{ x \in \mathbb{R}^{n} \middle| \, a'x \leq b \right\} \ .$$

Roughly speaking, $\mathcal{L}$ is like $\mathcal{F}$ but with one single inequality and the same for $\mathcal{L}^{-1}$ and $\mathcal{F}^{-1}$. We denote by $[\alpha]_{+} := \max\{0, \alpha\}$ the positive part of $\alpha \in \mathbb{R}$.

Lemma 74 [7, Lemma 1] *Let $\sigma \in \operatorname{dom}\mathcal{F}$ and $z \in \mathbb{R}^{n}$. Then, we have:*

$$d\left(z, \mathcal{F}(\sigma)\right) = \sup_{\binom{u}{v} \in \operatorname{conv}(\sigma)} d\left(z, \mathcal{L}\begin{pmatrix} u \\ v \end{pmatrix}\right) = \sup_{\binom{u}{v} \in \operatorname{conv}(\sigma)} \frac{[u'z - v]_{+}}{\|u\|_{*}} \ .$$

In the next lemma we use the dual norm in $\mathbb{R}^{n+1}$ of

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| = \max\left\{ \|u\|_{*}, |v| \right\} \ ,$$

defined in (0.5), which is given by

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_* := \|u\| + |v| \ ,$$

for all $\binom{u}{v} \in \mathbb{R}^{n+1}$.

Lemma 75 [11, Lemma 10] *Let $\sigma \in \Theta \equiv (\mathbb{R}^n \times \mathbb{R})^T$ and $z \in \mathbb{R}^n$. Then, we have:*

$$d\left(\sigma, \mathcal{F}^{-1}(z)\right) = \frac{\sup_{t \in T}[a_t' z - b_t]_+}{\|z\| + 1} \ .$$

Remark 76 Observe that $\sup_{t \in T}[a_t' z - b_t]_+$ can be understood as a measure of the infeasibility of point z with respect to system σ. Furthermore, it is worth recalling [7, Remark 1]:

$$\sup_{t \in T}[a_t' z - b_t]_+ = \sup_{\binom{u}{v} \in conv(\sigma)} [u' z - v]_+ \ .$$

Lemma 77 *Let $\left(\binom{u}{u'\overline{x}}, \overline{x}\right) \in \mathrm{gph}\mathcal{L}$ with $u \neq 0_n$. Then*

$$\mathrm{liplsc}\mathcal{L}\left(\begin{pmatrix} u \\ u'\overline{x} \end{pmatrix}, \overline{x}\right) = \frac{\|\overline{x}\| + 1}{\|u\|_*} \ .$$

Moreover, the latter modulus coincides with the modulus in property (3.1) for $\mathcal{L}^{-1}$ at $\left(\overline{x}, \binom{u}{u'\overline{x}}\right)$.

Proof. From [7, Proposition 2], $\mathcal{L}$ is Aubin continuous and hence, Lipschitz-lsc at $\left(\binom{u}{u'\overline{x}}, \overline{x}\right)$ (see also Proposition 60). We know that

$$\mathrm{liplsc}\mathcal{L}\left(\begin{pmatrix} u \\ u'\overline{x} \end{pmatrix}, \overline{x}\right) = \limsup_{\binom{a}{b} \to \binom{u}{u'\overline{x}}} \frac{d\left(\overline{x}, \mathcal{L}\binom{a}{b}\right)}{d\left(\binom{a}{b}, \binom{u}{u'\overline{x}}\right)} \leq \limsup_{\binom{a}{b} \to \binom{u}{u'\overline{x}}} \frac{d\left(\overline{x}, \mathcal{L}\binom{a}{b}\right)}{d\left(\binom{a}{b}, \mathcal{L}^{-1}(\overline{x})\right)} \ .$$

From Ascoli formula

$$d\left(\overline{x}, \mathcal{L}\begin{pmatrix} a \\ b \end{pmatrix}\right) = \frac{[a'\overline{x} - b]_+}{\|a\|_*} \ ,$$

and

$$d\left(\begin{pmatrix} a \\ b \end{pmatrix}, \mathcal{L}^{-1}(\overline{x})\right) = \frac{[a'\overline{x} - b]_+}{\|\overline{x}\| + 1} \ .$$

Moreover

$$a'\overline{x} - b = \left[\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} u \\ u'\overline{x} \end{pmatrix}\right]' \begin{pmatrix} \overline{x} \\ -1 \end{pmatrix} .$$

Therefore,

$$\begin{aligned} \operatorname{liplsc}\mathcal{L}\left(\binom{u}{u'\overline{x}}, \overline{x}\right) &= \limsup_{\binom{a}{b} \to \binom{u}{u'\overline{x}}} \frac{\frac{[a'\overline{x}-b]_+}{\|a\|_*}}{\left\|\binom{a}{b} - \binom{u}{u'\overline{x}}\right\|} \\ &= \limsup_{\binom{a}{b} \to \binom{u}{u'\overline{x}}} \frac{1}{\|a\|_*} \left[\begin{pmatrix} \overline{x} \\ -1 \end{pmatrix}' \frac{\binom{a}{b} - \binom{u}{u'\overline{x}}}{\left\|\binom{a}{b} - \binom{u}{u'\overline{x}}\right\|}\right]_+ . \end{aligned}$$

Note that $\frac{\binom{a}{b} - \binom{u}{u'\overline{x}}}{\left\|\binom{a}{b} - \binom{u}{u'\overline{x}}\right\|}$, maybe any vector of the unit sphere of $\mathbb{R}^{n+1}$. Hence,

$$\begin{aligned} \operatorname{liplsc}\mathcal{L}\left(\begin{pmatrix} u \\ u'\overline{x} \end{pmatrix}, \overline{x}\right) &= \frac{1}{\|u\|_*} \left\|\begin{pmatrix} \overline{x} \\ -1 \end{pmatrix}\right\|_* = \frac{\|\overline{x}\| + 1}{\|u\|_*} \\ &= \limsup_{\binom{a}{b} \to \binom{u}{u'\overline{x}}} \frac{d\left(\overline{x}, \mathcal{L}\binom{a}{b}\right)}{d\left(\binom{a}{b}, \mathcal{L}^{-1}(\overline{x})\right)} \ . \end{aligned}$$

■

In the following theorem we appeal to the notion of strong Slater element(SS element, in brief) introduced in Remark 6.

Theorem 78 *Let $(\overline{\sigma}, \overline{x}) \in \mathrm{gph}\mathcal{F}$. Assume that the SSCQ holds at $\overline{\sigma}$ and $\{\overline{a}_t, t \in T\}$ is bounded.*

(i) If $\overline{x}$ is an SS element of $\overline{\sigma}$, then

$$\mathrm{liplsc}\mathcal{F}(\overline{\sigma}, \overline{x}) = \mathrm{liplsc}^{*}\mathcal{F}(\overline{\sigma}, \overline{x}) = \mathrm{lip}\mathcal{F}(\overline{\sigma}, \overline{x}) = 0 \ . \tag{3.6}$$

(ii) Otherwise,

$$\mathrm{liplsc}\mathcal{F}(\overline{\sigma}, \overline{x}) = \mathrm{liplsc}^{*}\mathcal{F}(\overline{\sigma}, \overline{x}) = \mathrm{lip}\mathcal{F}(\overline{\sigma}, \overline{x})$$

$$= (\|\overline{x}\| + 1) \max\left\{ \frac{1}{\|u\|_{*}} \;\middle|\; \begin{pmatrix} u \\ u'\overline{x} \end{pmatrix} \in \mathrm{cl}\left(\mathrm{conv}(\overline{\sigma})\right) \right\}. \tag{3.7}$$

Moreover, the latter moduli also coincide with the corresponding one of property (3.1).

When T is finite, (3.7) reads as

$$\frac{\|\overline{x}\| + 1}{d_{*}\left(0_n, \mathrm{conv}\left\{\overline{a}_t, t \in T_{\overline{\sigma}}(\overline{x})\right\}\right)} \ .$$

Proof.

(i) From [7, Theorem 1(i)], and applying (3.5) we obtain (3.6).

(ii) $\left\{u \in \mathbb{R}^n \mid \binom{u}{u'\overline{x}} \in \mathrm{cl}\left(\mathrm{conv}(\overline{\sigma})\right)\right\}$ is nonempty and compact and the maximum in the modulus' expression is attained at some $\overline{u} \neq 0_n$ (see [7, proof Theorem 1(ii)]).

Inequality "$\leq$" is trivial, appealing again to [7, Theorem 1 (ii)], because $\mathrm{liplsc}\mathcal{F}(\overline{\sigma}, \overline{x}) \leq \mathrm{lip}\mathcal{F}(\overline{\sigma}, \overline{x})$.

Let us prove now the other inequality reasoning by contradiction. Suppose that there exists $\alpha > 0$ such that

$$\mathrm{liplsc}\mathcal{F}(\overline{\sigma}, \overline{x}) < \alpha < (\|\overline{x}\| + 1) \max\left\{ \frac{1}{\|u\|_{*}} \;\middle|\; \begin{pmatrix} u \\ u'\overline{x} \end{pmatrix} \in \mathrm{cl}\left(\mathrm{conv}(\overline{\sigma})\right) \right\} \ . \tag{3.8}$$

For $\binom{\overline{u}}{\overline{u}'\overline{x}} \in \mathrm{cl}\,(\mathrm{conv}(\overline{\sigma}))$, we have that $\overline{u} \neq 0_n$, so from Lemma 77 we have that

$$\alpha < \frac{\|\overline{x}\|+1}{\|u\|_*} = \mathrm{liplsc}\mathcal{L}\left(\begin{pmatrix}\overline{u}\\ \overline{u}'\overline{x}\end{pmatrix}, \overline{x}\right) .$$

Then, there exists a sequence $\left\{\binom{u^r}{v^r}\right\}$ converging to $\binom{\overline{u}}{\overline{u}'\overline{x}}$ such that

$$d\left(\overline{x}, \mathcal{L}\begin{pmatrix}u^r\\ v^r\end{pmatrix}\right) > \alpha d\left(\begin{pmatrix}u^r\\ v^r\end{pmatrix}, \begin{pmatrix}\overline{u}\\ \overline{u}'\overline{x}\end{pmatrix}\right) . \tag{3.9}$$

We will construct $\{\sigma^r\}$ converging to $\overline{\sigma}$ verifying, for each r, the following two conditions

$$\text{(C1):} \quad d\left(\overline{x}, \mathcal{F}(\sigma^r)\right) \geq d\left(\overline{x}, \mathcal{L}\tbinom{u^r}{v^r}\right) ,$$

$$\text{(C2):} \quad d\left(\sigma^r, \overline{\sigma}\right) = d\left(\tbinom{u^r}{v^r}, \tbinom{\overline{u}}{\overline{u}'\overline{x}}\right) .$$

In that case, we will have for all r

$$d\left(\overline{x}, \mathcal{F}(\sigma^r)\right) > \alpha d\left(\sigma^r, \overline{\sigma}\right) , \tag{3.10}$$

which contradicts the first inequality in (3.8). Now, we define $\sigma^r \equiv (a^r, b^r)$ as

$$a_t^r := \overline{a}_t + u^r - \overline{u} \quad \text{for all } t \in T ;$$

$$b_t^r := \overline{b}_t + v^r - \overline{u}'\overline{x} \quad \text{for all } t \in T .$$

We can write

$$\begin{pmatrix}\overline{u}\\ \overline{u}'\overline{x}\end{pmatrix} = \lim_k \sum_{t\in T} \lambda_t^k \begin{pmatrix}\overline{a}_t\\ \overline{b}_t\end{pmatrix}, \tag{3.11}$$

for certain $\lambda^k \equiv \left(\lambda_t^k\right)_{t\in T} \in {\mathbb{R}_+}^{(T)}$ such that $\sum_{t\in T} \lambda_t^k = 1$ for each k. Then,

$$\lim_k \sum_{t\in T} \lambda_t^k \begin{pmatrix}a_t^r\\ b_t^r\end{pmatrix} = \begin{pmatrix}u^r\\ v^r\end{pmatrix},$$

which, from Farkas Lemma, implies that $(u^r)'x \leq v^r$ is a consequence of σ^r (note that $\sigma^r \in \text{dom}\mathcal{F}$ for r large enough because of the SSCQ; see Proposition 60). Hence, (C1) follows. From the definition of σ^r it is easy to see that (C2) is also verified and $\sigma^r \to \overline{\sigma}$. Finally, due to (3.5) we obtain (3.7).

In order to see that the latter modulus coincide with the corresponding one in property (3.1) we just appeal to Lemmas 74 and 75 and Remark 76.

In the particular case when T is finite, for $\binom{u}{u'\overline{x}} \in \text{conv}(\overline{\sigma})$ there exist $\lambda \in \mathbb{R}_+^T$ with $\sum_{t \in T} \lambda_t = 1$ such that

$$0 = \begin{pmatrix} u \\ u'\overline{x} \end{pmatrix}' \begin{pmatrix} \overline{x} \\ -1 \end{pmatrix} = \sum_{t \in T} \lambda_t \left(\overline{a}_t' \overline{x} - \overline{b}_t \right) \ ,$$

which by a standard argument implies $\lambda_t = 0$ whenever $t \notin T_{\overline{\sigma}}(\overline{x})$. Therefore,

$$\begin{aligned} \max\left\{ \|u\|_* \,\middle|\, \tbinom{u}{u'\overline{x}} \in \text{conv}(\overline{\sigma}) \right\} &= \max\left\{ \|u\|_* \mid u \in \text{conv}\left\{ \overline{a}_t, t \in T_{\overline{\sigma}}(\overline{x}) \right\} \right\} \\ &= d_*\left(0_n, \text{conv}\left\{ \overline{a}_t, t \in T_{\overline{\sigma}}(\overline{x}) \right\} \right) \ . \end{aligned}$$

■

Conclusions and future work

The main original contributions of Chapters 1 and 2 are focused on the calmness moduli from below and above and the calmness and Lipschitz moduli of the following optimal value functions restricted to their domains (where the values are finite) in different parametric contexts: $\vartheta_{\overline{c}}^R$ in the context of RHS perturbations, $\vartheta_{\overline{b}}^R$ in the one of c-perturbations and ϑ^R for canonical perturbations. The analysis is developed around a nominal LP problem $\overline{\pi}$ (solvable) which is identified with the pair formed by a nominal vector of the objective function, $\overline{c}$, and a nominal RHS, $\overline{b}$. As a brief discussion about the convenience of dealing with such functions (restricted to their domains), we underline the fact that it allows us to avoid a typical interiority assumption under which some preliminary results are stated (see [28, Lemma 10.2] and [12]). Specifically, the nominal elements $\overline{b}$, $\overline{c}$, and $\overline{\pi}$ are not required to be in the interior of the respective domains of $\vartheta_{\overline{c}}^R$, $\vartheta_{\overline{b}}^R$ and ϑ^R. It is known that this interiority condition is equivalent to the simultaneous fulfilment of the SCQ at $\overline{b}$ and the boundedness (and nonemptiness) of the nominal optimal set $\mathcal{F}^{op}(\overline{\pi})$. We summarize in Tables 3.1 and 3.2 below the main results of the two first chapters and, at the same time, we try to clarify the role played by the two assumptions, SCQ and boundedness of $\mathcal{F}^{op}(\overline{\pi})$, separately, in relation to the computation/estimation of calmness and Lipschitz moduli. In those tables, inequalities marked with (1) and/or (2) mean that they become equalities under conditions SCQ and/or boundedness of $\mathcal{F}^{op}(\overline{\pi})$, respectively. Recall the ingredient

$$\mathcal{E}^{op}(\overline{\pi}) := \operatorname{extr}\left(\mathcal{F}^{op}(\overline{\pi}) \cap \operatorname{span}\{\overline{a}_t,\ t \in T\}\right) \ ,$$

introduced in (0.21), and

$$k^- := \min_{D\in\mathcal{M}_{\overline{\pi}}} \left\|\lambda^D\right\|_1 \text{ and } k^+ = \max_{D\in\mathcal{M}_{\overline{\pi}}} \left\|\lambda^D\right\|_1 \ ,$$

defined in (1.4), where $\mathcal{M}_{\overline{\pi}}$ is the family of minimal KKT subsets of indices at $\overline{\pi}$ (see (0.17)).

	Calmness of ϑ^R
Perturbing b	$\underline{\operatorname{clm}}\vartheta^R_{\overline{c}}(\overline{b}) = k^-$
	$\overline{\operatorname{clm}}\vartheta^R_{\overline{c}}(\overline{b}) \underset{(1)}{\leq} k^+$
	$k^- \leq \operatorname{clm}\vartheta^R_{\overline{c}}(\overline{b}) \underset{(1)}{\leq} k^+$
Perturbing c	$\underline{\operatorname{clm}}\vartheta^R_{\overline{b}}(\overline{c}) \underset{(2)}{\leq} e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$
	$\overline{\operatorname{clm}}\vartheta^R_{\overline{b}}(\overline{c}) = d(0_n, \mathcal{F}^{op}(\overline{\pi}))$
	$d(0_n, \mathcal{F}^{op}(\overline{\pi})) \leq \operatorname{clm}\vartheta^R_{\overline{b}}(\overline{c}) \underset{(2)}{\leq} e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$
Perturbing b **and** c	$\underline{\operatorname{clm}}\vartheta^R(\overline{\pi}) \underset{(2)}{\leq} k^- + e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$
	$\overline{\operatorname{clm}}\vartheta^R(\overline{\pi}) = \overline{\operatorname{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + d(0_n, \mathcal{F}^{op}(\overline{\pi}))$
	$d(0_n, \mathcal{F}^{op}(\overline{\pi})) \leq \operatorname{clm}\vartheta^R(\overline{\pi})$ $\underset{(1)\&(2)}{\leq} \max\left\{k^- + e\left(\mathcal{E}(\overline{\pi}), 0_n\right), k^+ + d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right)\right\}$

Table 3.1: Summary about calmness

	Lipschitz continuity of ϑ^R
Perturbing b	$\mathrm{lip}\vartheta^R_{\overline{c}}(\overline{b}) = k^+$
Perturbing c	$d(0_n, \mathcal{F}^{op}(\overline{\pi})) \leq \mathrm{lip}\vartheta^R_{\overline{b}}(\overline{c}) \underset{(2)}{\leq} e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$
Perturbing b **and** c	$k^+ + d(0_n, \mathcal{F}^{op}(\overline{\pi})) \leq \mathrm{lip}\vartheta^R(\overline{\pi}) \underset{(2)}{\leq} k^+ + e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$

Table 3.2: Summary about Lipschitz continuity

Consequently,

- $\vartheta^R_{\overline{c}}$, $\vartheta^R_{\overline{b}}$ and ϑ^R are always calm from below and above, hence calm, and also Lipschitz continuous at $\overline{\pi}$.

- If we assume SCQ, then calmness and Lipschitz moduli under RHS perturbations coincide:

$$\mathrm{clm}\vartheta^R_{\overline{c}}(\overline{b}) = \mathrm{lip}\vartheta^R_{\overline{c}}(\overline{b}) = k^+ \ .$$

- For c-perturbations, calmness and Lipschitz moduli coincide when $\mathcal{F}^{op}(\overline{\pi})$ is bounded:

$$\mathrm{clm}\vartheta^R_{\overline{b}}(\overline{c}) = \mathrm{lip}\vartheta^R_{\overline{b}}(\overline{c}) = e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right).$$

- While the SCQ does not play any role for the Lipschitz modulus of ϑ^R, the boundedness of $\mathcal{F}^{op}(\overline{\pi})$ is a sufficient condition to ensure that the upper estimate of that modulus turns out to be the exact one. Moreover, in this case:

$$\mathrm{lip}\vartheta^R(\overline{\pi}) = \mathrm{lip}\vartheta^R_{\overline{c}}(\overline{b}) + \mathrm{lip}\vartheta^R_{\overline{b}}(\overline{c}) \ .$$

- When both conditions are assumed (or equivalently, $\overline{\pi}$ is in the interior of the set of solvable problems), then, in addition to the previous statements, the calmness and Lipschitz moduli of ϑ do coincide with the corresponding ones of ϑ^R. Additionally, the reader can deduce from the tables that the calmness modulus of ϑ may be strictly less than the Lipschitz one.

To end with this part of the work, let us comment that all formulas obtained for computing or estimating our aimed moduli are point-based, in the sense that all ingredients used in them only involve the nominal elements (the problem's data), not appealing to parameters or points in a neighborhood. In this way they are implementable in practice.

Regarding Chapter 3, we have studied the significant difference between the Lipschitz-lsc* and the usual Lipschitz-lsc of the feasible set mapping $\mathcal{F}$. Whereas Lipschitz-lsc is equivalent to Aubin property only when RHS perturbations of the constraints are involved (recall Proposition 60, Remark 63 and counterexamples 64 and 65), the Lipschitz-lsc* case is different. For the initial situation of system (0.2) where T is arbitrary, we have concluded in Corollary 69 that when $\mathcal{F}_{\overline{b}}(\overline{a}) \neq \{0_n\}$ the Lipschitz-lsc* of $\mathcal{F}_{\overline{b}}$ is equivalent to the Aubin continuity of $\mathcal{F}$. Example 71 shows that it is no longer true if $\mathcal{F}_{\overline{b}}(\overline{a}) = \{0_n\}$. However, if T is finite we have proved the equivalence between the Lipschitz-lsc* and the Aubin continuity of $\mathcal{F}_{\overline{b}}$ is always held.

Going further, in Section 3.2 we have computed a tractable formula for the Lipschitz-lsc* and Lipschitz-lsc moduli of $\mathcal{F}$ at a given point $(\overline{\sigma}, \overline{x}) \in \operatorname{gph} \mathcal{F}$. In short, we have obtained them by proving that the Lipschitz-lsc modulus is equal to the Lipschitz one, hence the three moduli coincide (see Theorem 78). Such formula is given in the context when T is arbitrary but also, in particular, when T is finite.

Instinctively, the next step would be the computation of Lipschitz-lsc and Lipschitz-lsc* moduli under, separately, RHS and LHS perturbations, i.e., $\text{liplsc}\mathcal{F}_{\overline{a}}(\bar{b},\overline{x})$, $\text{liplsc}^*\mathcal{F}_{\overline{a}}(\bar{b},\overline{x})$, $\text{liplsc}\mathcal{F}_{\overline{b}}(\overline{a},\overline{x})$ and $\text{liplsc}^*\mathcal{F}_{\overline{b}}(\overline{a},\overline{x})$ so, it will be the first objective in our future work. Later on, we would like to focus on the behavior of $\mathcal{F}$ in terms of the Lipschitz-lsc* (and its relationship with Lipschitz-lsc and Aubin property) in nonlinear contexts. For this issue, we recall the works of D. Klatte and B. Kummer, [39] and [38]. Specifically, in [39] the authors study the Lipschitz-lsc of the feasible set mapping in the cases: (i) nonlinear optimization problems with differentiable data and finite index set and (ii) convex semi-infinite optimization problems. We pay special attention in [38, Lemma 1], where an equivalence between Lipschitz-lsc and Aubin property is given for a feasible set mapping with finitely many differentiable constraints. In this result it is also included the characterization between the Aubin property and the *Mangasarian-Fromowitz constraint qualification* (MFCQ, in brief), which is a well-known Robinson's result (see [48]).

Conclusiones y trabajo futuro

Las principales contribuciones de los Capítulos 1 y 2 se centran en los módulos de calmness por abajo y por arriba y en el módulo de Lipschitz de la función valor óptimo restringida a su dominio en diferentes contextos paramétricos: perturbando solo b, perturbando solo c o perturbando ambos (pertubaciones canonicas). Este tipo de análisis se desarrolla alrededor de un problema nominal de PL (resoluble), $\overline{\pi}$, que identificamos con el par $(\overline{c}, \overline{b})$. Sobre la conveniencia de trabajar con dichas funciones (restringidas a su dominio), destacamos que de esta forma evitamos el típico supuesto de interioridad usado anteriormente en la literatura (véanse [28, Lemma 10.2] y [12]). En concreto, no se requiere que los elementos nominales $\overline{b}$, $\overline{c}$ y $\overline{\pi}$ estén en el interior de los respectivos dominios de $\vartheta_{\overline{c}}^{R}$, $\vartheta_{\overline{b}}^{R}$ y ϑ^{R}. Es bien sabido que esta condición de interioridad es equivalente a que se verifice la SCQ junto con la acotación del conjunto óptimo nominal (no vacío) $\mathcal{F}^{op}(\overline{\pi})$. A continuación, las Tablas 3.3 y 3.4 resumen los resultados principales de los dos primeros capítulos y, al mismo tiempo, clarifican el papel que juegan dichas condiciones, SCQ y acotación de $\mathcal{F}^{op}(\overline{\pi})$, de forma independiente en relación al cálculo/estimación de los módulos de calmness y Lipschitz.

Las desigualdades marcadas con (1) y/o (2) en las tablas se verifican con igualdad bajo las condiciones de SCQ y/o $\mathcal{F}^{op}(\overline{\pi})$ acotado, respectivamente. Recuérdense los ingredientes

$$\mathcal{E}^{op}(\overline{\pi}) := \operatorname{extr}\left(\mathcal{F}^{op}(\overline{\pi}) \cap \operatorname{span}\{\overline{a}_t,\ t \in T\}\right) \ ,$$

introducido en (0.21), y

$$k^- := \min_{D \in \mathcal{M}_{\overline{\pi}}} \left\|\lambda^D\right\|_1 \text{ y } k^+ = \max_{D \in \mathcal{M}_{\overline{\pi}}} \left\|\lambda^D\right\|_1 \ ,$$

definidas en (1.4), donde $\mathcal{M}_{\overline{\pi}}$ es la familia de subconjuntos de índices KKT minimales en $\overline{\pi}$ (véase (0.17)).

	Calmness de ϑ^R
Perturbando b	$\underline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) = k^-$
	$\overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) \underset{(1)}{\leq} k^+$
	$k^- \leq \mathrm{clm}\vartheta^R_{\overline{c}}(\overline{b}) \underset{(1)}{\leq} k^+$
Perturbando c	$\underline{\mathrm{clm}}\vartheta^R_{\overline{b}}(\overline{c}) \underset{(2)}{\leq} e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$
	$\overline{\mathrm{clm}}\vartheta^R_{\overline{b}}(\overline{c}) = d(0_n, \mathcal{F}^{op}(\overline{\pi}))$
	$d(0_n, \mathcal{F}^{op}(\overline{\pi})) \leq \mathrm{clm}\vartheta^R_{\overline{b}}(\overline{c}) \underset{(2)}{\leq} e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$
Perturbando b **y** c	$\underline{\mathrm{clm}}\vartheta^R(\overline{\pi}) \underset{(2)}{\leq} k^- + e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$
	$\overline{\mathrm{clm}}\vartheta^R(\overline{\pi}) = \overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + d(0_n, \mathcal{F}^{op}(\overline{\pi}))$
	$d(0_n, \mathcal{F}^{op}(\overline{\pi})) \leq \mathrm{clm}\vartheta^R(\overline{\pi})$ $\underset{(1)\&(2)}{\leq} \max\left\{k^- + e\left(\mathcal{E}(\overline{\pi}), 0_n\right), k^+ + d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right)\right\}$

Tabla 3.3: Resumen sobre calmness

	Continuidad Lipschitz de ϑ^R
Perturbando b	$\mathrm{lip}\vartheta^R_{\overline{c}}(\overline{b}) = k^+$
Perturbando c	$d(0_n, \mathcal{F}^{op}(\overline{\pi})) \leq \mathrm{lip}\vartheta^R_{\overline{b}}(\overline{c}) \underset{(2)}{\leq} e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$
Perturbando b **y** c	$k^+ + d(0_n, \mathcal{F}^{op}(\overline{\pi})) \leq \mathrm{lip}\vartheta^R(\overline{\pi}) \underset{(2)}{\leq} k^+ + e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$

Tabla 3.4: Resumen sobre continuidad Lipschitz

En consecuencia,

- $\vartheta^R_{\overline{c}}$, $\vartheta^R_{\overline{b}}$ y ϑ^R son siempre "calm" por arriba y por abajo, por lo tanto "calm", y también Lipschitz continuas en $\overline{\pi}$.

- Si suponemos que la SCQ se verifica en $\overline{b}$, entonces los módulos de calmness y de Lipschitz bajo perturbaciones de b coinciden:
$$\mathrm{clm}\vartheta^R_{\overline{c}}(\overline{b}) = \mathrm{lip}\vartheta^R_{\overline{c}}(\overline{b}) = k^+ \ .$$

- Para perturbaciones de c, el módulo de Lipschitz coincide con el de calmness cuando $\mathcal{F}^{op}(\overline{\pi})$ está acotado:
$$\mathrm{clm}\vartheta^R_{\overline{b}}(\overline{c}) = \mathrm{lip}\vartheta^R_{\overline{b}}(\overline{c}) = e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right).$$

- Mientras que la condición SCQ no juega ningún papel para el módulo de Lipschitz de ϑ^R, la acotación de $\mathcal{F}^{op}(\overline{\pi})$ es una condición suficiente para asegurar que la cota superior se alcance. Más aún, en este caso
$$\mathrm{lip}\vartheta^R(\overline{\pi}) = \mathrm{lip}\vartheta^R_{\overline{c}}(\overline{b}) + \mathrm{lip}\vartheta^R_{\overline{b}}(\overline{c}) \ .$$

- Cuando se asumen ambas condiciones (o, equivalentemente, $\overline{\pi}$ está en el interior del conjunto de los problemas resolubles), entonces, además de lo anterior, los módulos de calmness y Lipschitz de ϑ coinciden con los correspondientes de ϑ^R. Además, el lector puede fácilmente deducir de las tablas anteriores que el módulo de calmness de ϑ puede ser estrictamente menor que el de Lipschitz.

Para cerrar esta parte queríamos remarcar que todas las fórmulas que se han obtenido para calcular o estimar los módulos son "point-based" en el sentido que todos los ingredientes que intervienen están dados en términos de los elementos nominales del problema, es decir, no hacen referencia a parámetros o puntos en un entorno. De esta forma, las fórmulas dadas son implementables en la práctica.

Con respecto al Capítulo 3, hemos estudiado la diferencia significativa que hay entre Lipschitz-lsc* y la usual Lipschitz-lsc de la multifunción conjunto factible $\mathcal{F}$. Mientras que Lipschitz-lsc es equivalente a la propiedad de Aubin solo cuando intervienen perturbaciones de b (recordemos Proposición 60, Observación 63 y los contraejemplos 64 y 65), el caso de Lipschitz-lsc* es distinto. Para el contexto inicial del sistema (0.2) donde T es arbitrario, hemos llegado a la conclusión en el Corolario 69 que cuando $\mathcal{F}_{\overline{b}}(\overline{a}) \neq \{0_n\}$, Lipschitz-lsc* de $\mathcal{F}_{\overline{b}}$ es equivalente a la continuidad de Aubin de $\mathcal{F}$. El Ejemplo 71 muestra que dicha afirmación no es cierta si $\mathcal{F}_{\overline{b}}(\overline{a}) = \{0_n\}$. Sin embargo, en el caso en que T sea finito, hemos probado que la equivalencia entre Lipschitz-lsc* y la propiedad de Aubin de $\mathcal{F}_{\overline{b}}$ siempre se verifica.

En la Sección 3.2 damos un paso más y hemos calculado una fórmula manejable para los módulos de Lipschitz-lsc* y Lipschitz-lsc de $\mathcal{F}$ en un punto dado $(\overline{\sigma}, \overline{x}) \in \operatorname{gph} \mathcal{F}$. De forma breve, hemos obtenido dichos módulos probando que el módulo de Lipschitz-lsc es igual que el de Lipschitz, y por tanto, los tres módulos coinciden (véase Teorema 78). Dicha expresión del módulo está dada en el contexto general con T arbitrario pero también en el caso particular cuando T es finito.

De forma intuitiva, nuestro siguiente paso será abordar el cálculo de los módulos de Lipschitz-lsc* y Lipschitz-lsc de $\mathcal{F}$ bajo perturbaciones de b y a de forma separada, es decir, tratar de estimar de la forma más precisa posible $\operatorname{liplsc}\mathcal{F}_{\overline{a}}(\overline{b},\overline{x})$, $\operatorname{liplsc}^*\mathcal{F}_{\overline{a}}(\overline{b},\overline{x})$, $\operatorname{liplsc}\mathcal{F}_{\overline{b}}(\overline{a},\overline{x})$ y $\operatorname{liplsc}^*\mathcal{F}_{\overline{b}}(\overline{a},\overline{x})$. Por tanto, este será nuestro primer objetivo dentro del plan de trabajo futuro. Con posterioridad, nos gustaría centrarnos en estudiar el comportamiento de $\mathcal{F}$ en términos de Lipschitz-lsc* (y su relación con Lipschitz-lsc y la propiedad de Aubin) en contextos no lineales. Para ello, apelamos a los trabajos de D. Klatte y B. Kummer, [39] y [38] como punto de partida. En [39] los autores estudian la Lipschitz-lsc de la multifunción conjunto factible en los casos: (i) problemas de optimización no lineal con datos diferenciables y un conjunto finito de índices y (ii) problemas de optimización semi-infinitos convexos. Cabe prestar especial atención a [38, Lema 1], donde se da una equivalencia entre Lipschitz-lsc y la propiedad de Aubin para una multifunción conjunto factible con un número finito de restricciones diferenciables. Además se incluye la conocida caracterización de Robinson entre la propiedad de Aubin y la *Mangasarian-Fromowitz constraint qualification* (véase [48]).

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Symbols and Abbreviations

Symbols

π : $\mathrm{Min}\{c'x \mid {a_t}'x \leq \ b_t,\ t \in T\}$ linear optimization problem

σ : $\{a_t'x \leq b_t\,,\ t \in T\}$ system of constraints associated with problem π

y' : transpose of column-vector y

$x'y$: usual scalar product of vectors x and y

T : index set associated with system σ and problem π

$a \equiv (a_t)_{t\in T}$: coefficients of the left-hand-side of the constraints in system σ

$b \equiv (b_t)_{t\in T}$: right-hand-side of the constraints in system σ and problem π

c : $\mathbb{R}^n$-vector of coefficients of the objective function in problem π

x : $\mathbb{R}^n$-vector of decision variables associated with problem π and system σ

$(\mathbb{R}^n)^T$: Cartesian product (functions from T to $\mathbb{R}^n$)

$\mathbb{R}_+^{(T)}$: space of functions $\lambda : T \to \mathbb{R}_+$ with finite support

$\pi \equiv (c, b)$: parameter associated with problem π when a is fixed

$\sigma \equiv (a, b)$: parameter associated with system σ

Θ : $(\mathbb{R}^n \times \mathbb{R})^T$ parameter space of σ

$\mathcal{F}$: feasible set mapping associated with problem π and system σ

$\mathcal{F}_{\overline{a}}$: feasible set mapping $\mathcal{F}$ when a is fixed in problem π and system σ

$\mathcal{F}_{\bar{b}}$: feasible set mapping $\mathcal{F}$ when b is fixed in problem π and system σ

$\mathcal{F}(\sigma)$: feasible set of problem π and system σ

$\mathcal{F}^{op}$: optimal set mapping associated with problem π

$\mathcal{F}^{op}(\pi)$: optimal set of problem π

ϑ : optimal value function associated with problem π

$\vartheta(\pi)$: optimal value of problem π

$T_\sigma(x)$: set of active indices at $x \in \mathcal{F}(\sigma)$

$T_b(x)$: set of active indices at $x \in \mathcal{F}_{\overline{a}}(b)$

$\mathrm{conv}\,(X)$: convex hull of $\emptyset \neq X \subset \mathbb{R}^p$, $p \in \mathbb{N}$

$\mathrm{cone}\,(X)$: conical convex hull of $X \subset \mathbb{R}^p$, $p \in \mathbb{N}$ with $\mathrm{cone}\emptyset = \{0_p\}$

0_p : zero vector in $\mathbb{R}^p$

$\mathrm{aff}\,(X)$: affine hull of $\emptyset \neq X \subset \mathbb{R}^p$, $p \in \mathbb{N}$

$\mathrm{span}\,(X)$: linear hull of $\emptyset \neq X \subset \mathbb{R}^p$, $p \in \mathbb{N}$

$X^\perp$: orthogonal complement of $\mathrm{span}\,(X)$

$\mathrm{extr}\,(X)$: set of extreme points of the convex set X

$\mathrm{int}\,(X)$: topological interior of X

$\mathrm{cl}\,(X)$: topological closure of X

$\mathrm{bd}\,(X)$: topological boundary of X

$\inf X$: infimum of $X \subset \mathbb{R}$

$\|\cdot\|$: arbitrary norm in $\mathbb{R}^n$ (Chapters 1 and 2) and $\mathbb{R}^{n+1}$ (Chapter 3)

$\|\cdot\|_*$: dual norm of $\|\cdot\|$

$\|\cdot\|_\infty$: Chebyshev norm (or maximum norm)

d : distance in $\mathbb{R}^n$

$d\,(\sigma_1, \sigma_2)$: extended distance between systems σ_1 and σ_2

$d\left(\sigma, \widetilde{\Theta}\right)$: distance between system σ and $\widetilde{\Theta} \subset \Theta$ (with $d\,(\sigma, \emptyset) = +\infty$)

$\mathcal{G}$: multifunction

$\mathcal{G}^{-1}$: inverse mapping of multifunction $\mathcal{G}$

gph$\mathcal{G}$: graph of $\mathcal{G}$

dom$\mathcal{G}$: domain of $\mathcal{G}$

lip$\mathcal{G}(\overline{y}, \overline{x})$: Lipschitz modulus of $\mathcal{G}$ at $(\overline{y}, \overline{x})$

reg$\mathcal{G}^{-1}(\overline{x}|\overline{y})$: regularity modulus of $\mathcal{G}^{-1}$ at $\overline{x}$ for $\overline{y} \in \mathcal{G}^{-1}(\overline{x})$

clm$\mathcal{G}(\overline{y}, \overline{x})$: calmness modulus of $\mathcal{G}$ at $(\overline{y}, \overline{x})$

liplsc$\mathcal{G}(\overline{y}, \overline{x})$: Lipschitz-lsc modulus of $\mathcal{G}$ at $(\overline{y}, \overline{x})$

liplsc$^*\mathcal{G}(\overline{y}, \overline{x})$: Lipschitz-lsc* modulus of $\mathcal{G}$ at $(\overline{y}, \overline{x})$

f : function

lip$f(\overline{z})$: Lipschitz modulus of f at $\overline{z}$

clm$f(\overline{z})$: calmness modulus of f at $\overline{z}$

$\underline{\text{clm}}f(\overline{z})$: calmness modulus from below of f at $\overline{z}$

$\overline{\text{clm}}f(\overline{z})$: calmness modulus from below of f at $\overline{z}$

ϑ^R : ϑ restricted to dom$\mathcal{F}^{op}$

$\mathbb{R}_+$: interval $[0, +\infty[$

$\mathbb{R}_-$: interval $]-\infty, 0]$

Λ : feasible set mapping associated to π's dual problem

$\Lambda(c)$: feasible set of π's dual problem (only depends on c)

Λ^{op} : optimal set mapping associated to π's dual problem

$\Lambda^{op}(\pi)$: optimal set of π's dual problem

$\mathcal{K}_\pi(x)$: family of subsets of indices associated to the KKT conditions

$\mathcal{M}_\pi(x) \equiv \mathcal{M}_\pi$: family minimal KKT subsets of indices

$|D|$: cardinality of set D

$\lambda^D := \left(\lambda_t^D\right)_{t \in T}$: vector of KKT multipliers associated to $D \in \mathcal{M}_\pi$

$\lim_r$: limit when r tends to $+\infty$

$\{x^r\}_{r \in \mathbb{N}}$ or $\{x^r\}$: sequence with general term x^r

$\liminf_r x^r$: lower limit of the sequence $\{x^r\}$

$\limsup_r x^r$: upper limit of the sequence $\{x^r\}$

$\{X_r\}_{r\in\mathbb{N}}$ or $\{X_r\}$: sequence of sets with general term X_r

$\operatorname{Lim\,inf}_r X_r$: Painlevé-Kuratowski lower/inner limit of $\{X_r\}$

$\operatorname{Lim\,sup}_r X_r$: Painlevé-Kuratowski upper/outer limit of $\{X_r\}$

$\operatorname{Lim}_r X_r$: Painlevé-Kuratowski limit of $\{X_r\}$

$\mu_r \downarrow a$: $\lim_r \mu_r = a$ and $\{\mu_r\}$ decreasing from certain $r_0 \in \mathbb{N}$

$\mathcal{E}(b)$: $\operatorname{extr}(\mathcal{F}_{\overline{a}}(b) \cap \operatorname{span}\{\overline{a}_t,\ t \in T\}),\ b \in \operatorname{dom}\mathcal{F}_{\overline{a}}$

$\mathcal{E}^{op}(\pi)$: $\operatorname{extr}(\mathcal{F}^{op}(\pi) \cap \operatorname{span}\{\overline{a}_t,\ t \in T\}),\ \pi \in \operatorname{dom}\mathcal{F}^{op}$

k^- : $\min_{D\in\mathcal{M}_{\overline{\pi}}} \left\|\lambda^D\right\|_1$

k^+ : $\max_{D\in\mathcal{M}_{\overline{\pi}}} \left\|\lambda^D\right\|_1$

$e(X, 0_n)$: the Hausdorff excess of $X \subset \mathbb{R}^n$ over $\{0_n\}$

$Pri\emptyset$: primal infeasibility

$Dual\emptyset$: dual infeasibility

$\operatorname{conv}(\sigma)$: convex hull of $\left\{\binom{a_t}{b_t}, t \in T\right\}$

$[\alpha]_+$: positive part of $\alpha \in \mathbb{R}$

$\mathcal{L}\binom{a}{b}$: solution set of inequality $a'x \leq b$

Abbreviations

LP : linear programming

LSIP : linear semi-infinite programming

Min : minimize

Max : maximize

Inf : to find the infimum of some functional

s.t : subject to

inf : infimum

sup : supremum

min : minimum

max : maximum

RHS: right-hand-side

LHS: left-hand-side

KKT : Karush-Kuhn-Tucker

lsc : lower semicontinuity/semicontinuous (in the sense of Berge)

usc : upper semicontinuity/semicontinuous (in the sense of Berge)

Lipschitz-lsc: Lipschitz lower semicontinuity/semicontinuous

Lipschitz-lsc*: Lipschitz lower semicontinuity/semicontinuous-star

SCQ : Slater constraint qualification

SSCQ : strong Slater constraint qualification

SS element : strong Slater element

Appendix

In this section, we include the original manuscripts of [25] and [26] with the submission confirmation of the last one to *Journal of Optimization Theory and Applications*.

Calmness of the optimal value in linear programming*

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Abstract

The final goal of the present paper is computing/estimating the calmness moduli from below and above of the optimal value function restricted to the set of solvable linear problems. Roughly speaking, these moduli provide measures of the maximum rates of decrease and increase of the optimal value under perturbations of the data (provided that solvability is preserved). This research is developed in the framework of (finite) linear optimization problems under canonical perturbations; i.e., under simultaneous perturbations of the right-hand-side (RHS) of the constraints and the coefficients of the objective function. As a first step, part of the work is developed in the context of RHS pertubations only, where a specific formulation for the optimal value function is provided. This formulation constitutes the starting point in obtaining exact formulae/estimations for the corresponding calmness moduli from below and above. We point out the fact that all expressions for the aimed calmness moduli are conceptually tractable (implementable) as far as they are given exclusively in terms of the nominal data.

Keywords. Calmness, Optimal Value, Linear programming

Mathematics Subject Classification: 90C31, 49J53, 49K40, 90C05

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1 Introduction

We consider the parameterized linear optimization problem

$$\begin{array}{rll} \pi: & \text{minimize} & c'x \\ & \text{subject to} & \overline{a}_t'x \leq b_t, \;\; t \in T := \{1, 2, ..., m\}, \end{array} \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, $\overline{a} \equiv (\overline{a}_t)_{t \in T} \in (\mathbb{R}^n)^T$ is fixed, and $\pi \equiv (c, b) \in \mathbb{R}^n \times \mathbb{R}^T$ is considered as the parameter to be perturbed around a nominal element denoted by $\overline{\pi} \equiv (\overline{c}, \overline{b})$. Observe that, for the sake of simplicity in the notation, we are identifying our parameter (c, b) with the associated optimization problem π. This is the context of the so-called *canonical perturbations,* where the right-hand-side (RHS) of the constraints and the objective function coefficients are allowed to be perturbed simultaneously. All elements in $\mathbb{R}^n$ are regarded as column-vectors and y' denotes the transpose of $y \in \mathbb{R}^n$.

Associated with the previous parameterized problem, we consider the *feasible set mapping* $\mathcal{F} : \mathbb{R}^T \rightrightarrows \mathbb{R}^n$, given by

$$\mathcal{F}(b) := \left\{x \in \mathbb{R}^n \mid \overline{a}_t'x \leq b_t, \;\; t \in T\right\}, \; b \in \mathbb{R}^T,$$

the *optimal value function* $\vartheta : \mathbb{R}^n \times \mathbb{R}^T \to [-\infty, +\infty]$, given by

$$\vartheta(\pi) := \inf\{c'x \mid x \in \mathcal{F}(b)\}, \; \pi \in \mathbb{R}^n \times \mathbb{R}^T,$$

(with the convention $\vartheta(\pi) := +\infty$ when $\mathcal{F}(b) = \emptyset$), and the *optimal set mapping* $\mathcal{F}^{op} : \mathbb{R}^n \times \mathbb{R}^T \rightrightarrows \mathbb{R}^n$, given by

$$\mathcal{F}^{op}(\pi) := \left\{x \in \mathcal{F}(b) \mid c'x = \vartheta(\pi)\right\}, \; \pi \in \mathbb{R}^n \times \mathbb{R}^T.$$

The present paper is mainly focussed on the calmness of ϑ at a nominal parameter $\overline{\pi}$ such that $\vartheta(\overline{\pi})$ is finite. As a first stage, part of this work (Section 3) is developed in the setting of RHS perturbations; i.e., where c is assumed to be fixed, say $c = \overline{c}$.

The concept of calmness for a function $f : \mathbb{R}^p \to [-\infty, +\infty]$ $(p \in \mathbb{N})$ may be introduced through the simultaneous fulfilment of the so-called calmness from below and calmness from above (see, e.g., [37, Section 8.F]). Let $\overline{z} \in \mathbb{R}^p$ be such that $f(\overline{z})$ is finite; recall that f is *calm at* $\overline{z}$ *from below* if there exist a constant $\kappa_1 \geq 0$ and a neighborhood U_1 of $\overline{z}$ such that

$$f(\overline{z}) - f(z) \leq \kappa_1 \|z - \overline{z}\|, \text{ for all } z \in U_1. \tag{2}$$

Respectively, f is *calm at $\overline{z}$ from above* if

$$f(z) - f(\overline{z}) \leq \kappa_2 \|z - \overline{z}\|, \text{ for all } z \in U_2, \tag{3}$$

for some constant $\kappa_2 \geq 0$ and some neighborhood U_2 of $\overline{z}$. Along this paper, the infimum of those constants κ_1 and κ_2 for which (2) and (3), respectively, hold (for some associated neighborhoods) are called the *calmness modulus from below* and *above* of f at $\overline{z}$, and they are denoted by $\underline{\text{clm}}f(\overline{z})$ and $\overline{\text{clm}}f(\overline{z})$, respectively; these moduli can alternatively be expressed as

$$\underline{\text{clm}}f(\overline{z}) = \limsup_{z \to \overline{z}} \frac{f(\overline{z}) - f(z)}{\|z - \overline{z}\|} \text{ and } \overline{\text{clm}}f(\overline{z}) = \limsup_{z \to \overline{z}} \frac{f(z) - f(\overline{z})}{\|z - \overline{z}\|}. \tag{4}$$

In both expressions the possibility of approaching $\overline{z}$ by constant sequences is allowed under the convention $\frac{0}{0} := 0$; so, $\underline{\text{clm}}f(\overline{z})$ and $\overline{\text{clm}}f(\overline{z})$ are always non-negative. Alternatively, in order to ensure the nonnegativity of $\underline{\text{clm}}f(\overline{z})$ and $\overline{\text{clm}}f(\overline{z})$, we could define these moduli as the maximum between 0 and the corresponding 'lim sup' in (4) with $z \to \overline{z}$, $z \neq \overline{z}$. Observe that, for $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(z) := |z|$ and $\overline{z} := 0$, we have $\limsup_{z \to \overline{z}, z \neq \overline{z}} \frac{f(\overline{z}) - f(z)}{|z - \overline{z}|} = -1$, while, under our convention, $\limsup_{z \to \overline{z}} \frac{f(\overline{z}) - f(z)}{|z - \overline{z}|} = 0$.

Finally, f is said to be *calm at $\overline{z}$* if it is calm from below and above at $\overline{z}$, and the *calmness modulus* of f at $\overline{z}$, $\text{clm}f(\overline{z})$, is defined as

$$\text{clm}f(\overline{z}) := \limsup_{z \to \overline{z}} \frac{|f(z) - f(\overline{z})|}{\|z - \overline{z}\|} = \max\left\{\underline{\text{clm}}f(\overline{z}), \overline{\text{clm}}f(\overline{z})\right\}.$$

Note that $\underline{\text{clm}}f(\overline{z})$ is nothing else but the *strong slope* of f at $\overline{z}$, while $\overline{\text{clm}}f(\overline{z})$ corresponds to that of $-f$ (see, e.g., [2]). Roughly speaking, they respectively provide measures of maximum rates of decrease and increase of f at $\overline{z}$.

Coming back to our optimal value function ϑ, it is well-known that $\vartheta(\pi)$ is finite if and only if $\mathcal{F}^{op}(\pi) \neq \emptyset$; i.e., if and only if $\pi \in \text{dom}\mathcal{F}^{op}$ (the domain of $\mathcal{F}^{op}$). The following remark motivates the fact of considering the calmness property of ϑ restricted to $\text{dom}\mathcal{F}^{op}$, denoted by ϑ^R (the notation is inspired by [10], where the feasible set mapping restricted to its domain is analyzed); so, $\vartheta^R : \text{dom}\mathcal{F}^{op} \to]-\infty, +\infty[$ and the corresponding calmness modulus at $\overline{\pi} \in \text{dom}\mathcal{F}^{op}$ is given by

$$\text{clm}\vartheta^R(\overline{\pi}) = \limsup_{\pi \underset{\text{dom}\mathcal{F}^{op}}{\to} \overline{\pi}} \frac{|\vartheta(\pi) - \vartheta(\overline{\pi})|}{\|\pi - \overline{\pi}\|},$$

(the norm in question is introduced in Section 2) where this notation reflects that $\pi \rightarrow \overline{\pi}$ with $\pi \in \mathrm{dom}\mathcal{F}^{op}$. The calmness moduli of ϑ^R from below and above are analogously defined. In the following remark we also appeal to the *Slater constraint qualification* (SCQ, in brief) which is satisfied at $b \in \mathrm{dom}\mathcal{F}$ if there exists $\widehat{x} \in \mathbb{R}^n$ (called a *Slater point*) such that $\overline{a}_t' \widehat{x} < b_t$ for all $t \in T$.

Remark 1 It is clear from the definitions that if $\overline{\pi}$ belongs to the interior of $\mathrm{dom}\mathcal{F}^{op}$ (equivalently, SCQ holds at $\overline{b}$ and $\mathcal{F}^{op}(\overline{\pi})$ is nonempty and bounded; see e.g. [19, Th. 6.1 and Cor. 10.2]), then

$$\mathrm{clm}\vartheta^R(\overline{\pi}) = \mathrm{clm}\vartheta(\overline{\pi}).$$

Theorem 8 provides an exact formula for $\mathrm{clm}\vartheta(\overline{\pi})$ in such a case. Otherwise, if $\overline{\pi}$ is in the boundary of $\mathrm{dom}\mathcal{F}^{op}$, $\mathrm{clm}\vartheta(\overline{\pi}) = +\infty$ (ϑ is not calm at $\overline{\pi}$), while we will show that $\mathrm{clm}\vartheta^R(\overline{\pi})$ is always finite (see Section 6). In this way, $\mathrm{clm}\vartheta^R(\overline{\pi})$ still represents a certain measure of the stability of our problem $\overline{\pi}$ when either SCQ fails at $\overline{b}$ or $\mathcal{F}^{op}(\overline{\pi})$ is unbounded.

According to the previous notations, the main goal of this work consists in computing (or estimating) $\mathrm{clm}\vartheta^R(\overline{\pi})$ via the computation of the corresponding calmness moduli from below and above. At this moment we point out the fact that the present paper establishes point-based formulae for the aimed calmness moduli (sometimes estimations), i.e., formulae which only involve the nominal data $\overline{\pi}$ (not appealing to parameters in a neighborhood). In relation to this point, the main contributions of the paper are gathered in theorems 5, 6, and 7 (the last one under the boundedness of $\mathcal{F}^{op}(\overline{\pi})$); see also the announced Theorem 8 (stated for $\overline{\pi}$ in the interior of $\mathrm{dom}\mathcal{F}^{op}$). Our first step will be developed in the context of RHS perturbations, in which case the corresponding optimal value function is specially tractable; in fact, an explicit formula for computing the optimal values around $\overline{\pi}$ is provided (Corollary 1) and it is used as a starting point for deriving the results about the calmness modulus of the optimal value under RHS perturbations (Theorem 4 and Corollary 3).

In order to integrate the new contributions into the existent literature, first let us comment that exact formulae for the calmness moduli of multifunctions $\mathcal{F}$ and $\mathcal{F}^{op}$ have been already established, respectively, in [8] and [6] (see [24] and [28] in relation to the calmness of $\mathcal{F}$ and $\mathcal{F}^{op}$ in nonlinear contexts). In general, the concept of calmness for a function $f : \mathbb{R}^p \rightarrow [-\infty, +\infty]$ does not coincide with the corresponding one to the multifunction $z \longmapsto \{f(z)\}$. The latter does not entail the continuity of function f. Calmness property constitutes an important concept in the field of variational analysis; with respect to this point, the reader is addressed to the monographs [12, 27, 31, 37] and references therein.

The present research could also be integrated in the widely explored field of sensitivity analysis in linear programming (LP for short), where, from different approaches, one tries to answer the natural question of *what* happens *if* one modifies the nominal problem's data. Specifically, our focus is on a *local aspect* of sensitivity analysis in contrast to the classical theory of parametric linear optimization, which usually concerns the behavior of ϑ^R and $\mathcal{F}^{op}$ on $\mathrm{dom}\mathcal{F}^{op}$ or some of its subsets.

The theory of parametric linear optimization goes back to the early time of LP (see, e.g., [16] and [38]). A systematic development of LP with canonical perturbations started in the 1970s. One direction of research was focussed on the behavior of ϑ^R. Specifically, the continuity of ϑ^R was proved through different approaches: via parametric analysis (see [32]), by a parametric approach using Berge's theory (see [3, 5]), and by a primal-dual approach (see [25] and [41]). A second direction of development of sensitivity analysis in LP starting in the late 1960s was the analysis of semicontinuity and Lipschitz semicontinuity properties, which was based on approaches of variational analysis like Berge's theory or Hoffman's error bounds (see [3, 11, 13, 29, 35, 36, 39, 41]). Along this paper the continuity in the Painlevé-Kuratowski sense of multifunctions $\mathcal{F}$ and $\mathcal{F}^{op}$ restricted to their domains plays a crucial role; see Section 2 for details and specific references on these results.

In the 1990s and continuing until today, both directions became of great interest; see the survey [40] on different approaches to sensitivity, and the monograph [15]; see also [17] for the study of regions in which ϑ is affine, and [1, 14, 22, 23] for an approach to the sensitivity analysis from an optimal partition perspective, related to support set invariancy. For extensions to linear semi-infinite optimization, where T is infinite, the reader is addressed to [18], [20], and [21]. In the context of conic linear systems (which includes our framework as a particular case), the pioneer works [33] and [34] provide a quantitative approach to the stability of optimization problems, by using as an ingredient the *distance to infeasibility.*

To the authors' knowledge, the contributions of this paper about the computation (or estimation) of calmness moduli, which are contained in theorems from 4 to 8 and corollaries 3 and 4, are new. As immediate antecedents we refer to as [33] (see also [34]) and [40]. Specifically, from [33, Theorem 1.1(5)] one immediately derives an upper bound for $\mathrm{clm}\vartheta\left(\overline{\pi}\right)$, provided that $\overline{\pi}$ belongs to the interior of $\mathrm{dom}\mathcal{F}^{op}$, in terms of the distances to primal and dual infeasibility; the details are gathered in the last section at the end of the paper (in Theorem 9 and Corollary 5). In the same case, our Theorem 8 provides an exact formula for $\mathrm{clm}\vartheta\,(\overline{\pi})$, which constitutes a refinement of Corollary 5 as far as the referred upper bound might be far from the exact value of $\mathrm{clm}\vartheta\,(\overline{\pi})$ (see Remark 7). On the other hand, [40, Theorem 18], translated into our notation, provides a particular constant k_1 involved in the calmness of ϑ from below (2) in the context of RHS perturbations (vector $\overline{c}$ remains fixed).

The structure of the paper is as follows. Section 2 presents the necessary notation, key tools, and preliminary results on the continuity of $\mathcal{F}$, ϑ, and $\mathcal{F}^{op}$ restricted to their domains. Sections 3 and 4 contain the original contributions of this paper. Section 3 is concerned with the calmness modulus of the optimal value function under RHS perturbations only, while Section 4 considers canonical perturbations. We finish the paper with a section of conclusions, where all the new results about the calmness moduli are gathered in Table 1. By combining the results collected in this table, we compute (among others) the aimed $\mathrm{clm}\vartheta\,(\overline{\pi})$, provided that $\overline{\pi}$ is in the interior of $\mathrm{dom}\mathcal{F}^{op}$ (see Theorem 8). Finally, in order to better integrate the current work in the literature, a comparative analysis between Theorem 8 and a certain consequence of [33, Theorem 1.1(5).] is developed in Subsection 5.1.

2 Preliminaries

In this section we introduce some necessary notation and results which are used later on. Given $X \subset \mathbb{R}^p$, $p \in \mathbb{N}$, we denote by $\mathrm{conv}X$, $\mathrm{cone}X$, $\mathrm{aff}X$, and $\mathrm{span}X$, the *convex hull*, the *conical convex hull,* the *affine hull,* and the *linear hull* of X, respectively. Moreover, $X^{\perp}$ denotes the orthogonal complement of $\mathrm{span}X$, and, provided that X is convex, $\mathrm{extr}X$ stands for the set of extreme points of X. Recall that $x \in \mathrm{extr}X$ means that it is impossible to express x as a convex combination of two points of $X\backslash\{x\}$. It is assumed that $\mathrm{cone}X$ always contains the zero-vector 0_p, in particular $\mathrm{cone}\,\emptyset = \{0_p\}$.

From the topological side, if X is a subset of any topological space, $\mathrm{int}X$, $\mathrm{cl}X$ and $\mathrm{bd}X$ stand, respectively, for the interior, the closure, and the boundary of X.

Throughout the paper, the *parameter spaces* $\mathbb{R}^T$ and $\mathbb{R}^n \times \mathbb{R}^T$ (associated with the contexts of RHS and canonical perturbations) are endowed, respectively, with the norms

$$\|b\|_\infty := \max_{t\in T} |b_t| \text{ and } \|\pi\| := \max\{\|c\|_*, \|b\|_\infty\}, \tag{5}$$

where $\mathbb{R}^n$ is equipped with an arbitrary norm, $\|\cdot\|$, with *dual norm* given by $\|u\|_* = \max_{\|x\|\leq 1} |u'x|$. Note that the choice of the dual norm $\|\cdot\|_*$ for measuring the perturbations of c comes from the fact that it is seen as the functional $x \mapsto c'x$. Moreover, the use of supremum (maximum indeed) norm for both b and π is a usual choice for measuring errors, and it is followed, for instance, in previous works on calmness of feasible and optimal solutions in the same parametric context, as [6] and [8].

Recall that the dual problem of $\pi \equiv (c, b)$ (introduced in (1)) is given by

$$\begin{array}{ll} \text{maximize} & -b'\lambda \\ \text{subject to} & \sum_{t\in T} \lambda_t \overline{a}_t = -c, \\ & \lambda = (\lambda_t)_{t\in T} \in \mathbb{R}_+^T. \end{array} \tag{6}$$

From now on, let us denote by $\Lambda : \mathbb{R}^n \rightrightarrows \mathbb{R}^T$ and $\Lambda^{op} : \mathbb{R}^n \rightrightarrows \mathbb{R}^T$ the feasible and optimal set mappings corresponding to the family of (dual) problems (6); i.e., $\Lambda(c)$ is the feasible set of (6), which does not depend on b, and $\Lambda^{op}(\pi)$ denotes the optimal set of (6).

2.1 Minimal KKT subsets of indices

Hereinafter, we use the following notation associated with any $\pi \equiv (c, b) \in \mathrm{dom}\mathcal{F}^{op}$: The *set of active indices* at $x \in \mathcal{F}(b)$, for $b \in \mathbb{R}^T$, is denoted by $T_b(x)$; i.e.,

$$T_b(x) := \left\{ t \in T \mid \overline{a}_t' x = b_t \right\}.$$

We denote by $\mathcal{K}_\pi(x)$ the following family of subsets of indices involved in the Karush-Kuhn-Tucker (KKT in brief) conditions:

$$\mathcal{K}_\pi(x) := \{ D \subset T_b(x) \mid |D| \leq n, \ -c \in \mathrm{cone}\{\overline{a}_t, \ t \in D\} \}.$$

(The condition '$|D| \leq n$' comes from Carathéodory's Theorem.) Moreover, we shall appeal to the family of *minimal KKT subsets of indices*

$$\mathcal{M}_\pi(x) := \{ D \in \mathcal{K}_\pi(x) \mid D \text{ is minimal for the inclusion order} \},$$

which constitutes a key ingredient in the formula of the calmness modulus of $\mathcal{F}^{op}$ established in [6]. Trivially, $\mathcal{M}_\pi(x) = \{\emptyset\}$ when $c = 0_n$.

Remark 2 Recall the stardard fact of LP theory: the dual optimal set $\Lambda^{op}(\pi)$ does coincide with the set of KKT multipliers associated with any primal solution $x \in \mathcal{F}^{op}(\pi)$. As a direct consequence, $\mathcal{M}_\pi(x)$ does not depend on point x. Formally,

$$\mathcal{M}_\pi(x) = \mathcal{M}_\pi(y), \text{ whenever } x, y \in \mathcal{F}^{op}(\pi),$$

and, accordingly, we may remove the optimal point in the notation of the minimal KKT subsets of indices. So, from now on, we just denote

$$\mathcal{M}_\pi := \mathcal{M}_\pi(x), \text{ for any } x \in \mathcal{F}^{op}(\pi),$$

provided that $\pi \in \mathrm{dom}\mathcal{F}^{op}$.

Remark 3 Observe that, a standard argument of linear algebra (in the line of Carathéodory's Theorem) yields the linear independence of $\{\overline{a}_t,\ t \in D\}$, whenever $D \in \mathcal{M}_\pi$. Specifically, arguing by contradiction, assume that $\sum_{t\in D} \mu_t \overline{a}_t = 0_n$ for some $(\mu_t)_{t\in D}$; without loss of generality, $\mu_t > 0$, for some $t \in D$. Write $\sum_{t\in D} \lambda_t \overline{a}_t = -c$, for certain $(\lambda_t)_{t\in D}$, and consider $t_0 \in D$ such that $\frac{\lambda_{t_0}}{\mu_{t_0}} = \min\left\{\frac{\lambda_t}{\mu_t} : \mu_t > 0, t \in D\right\}$. Then, $-c = \sum_{t\in D\setminus\{t_0\}} \left(\lambda_t - \frac{\lambda_{t_0}}{\mu_{t_0}}\mu_t\right) \overline{a}_t \in \mathrm{cone}\{\overline{a}_t,\ t \in D \setminus \{t_0\}\}$, which contradicts the minimality of D.

In this way, for any $D \in \mathcal{M}_\pi$, we define $\lambda^D := \left(\lambda_t^D\right)_{t\in T}$ as the unique element in $\mathbb{R}_+^T$ verifying

$$\sum_{t\in D} \lambda_t^D \overline{a}_t = -c, \textit{ and } \lambda_t^D = 0, \textit{ whenever } t \in T \setminus D. \tag{7}$$

Observe that the minimality of D entails $\lambda_t > 0$ for all $t \in D$. In the case $c = 0_n$, we have $\lambda^\emptyset = 0_T$.

Lemma 1 *Let $\pi \in \mathrm{dom}\mathcal{F}^{op}$. We have*

$$\left\{\lambda^D,\ D \in \mathcal{M}_\pi\right\} = \mathrm{extr}\Lambda^{op}(\pi).$$

Proof. Consider the nontrivial case $\mathcal{M}_{\overline{\pi}} \neq \{\emptyset\}$ (otherwise, $c = 0$ and it is clear that $\mathrm{extr}\Lambda^{op}(\bar{b}) = \{0_T\}$). One easily sees (according to the previous remark) that $D \in \mathcal{M}_\pi$ if and only if $D \subset T_b(x)$, for any $x \in \mathcal{F}^{op}(\overline{\pi})$, the set of vectors $\{\overline{a}_t,\ t \in D\}$ is linearly independent, and

$$\sum_{t\in D} \lambda_t \overline{a}_t = -c,$$

for some $\lambda_t > 0$, $t \in D$. The latter condition (with the componets λ_t there) is equivalent to

$$\lambda^D \in \mathrm{extr}\Lambda^{op}(\pi),$$

for $\lambda_t^D := \lambda_t > 0, t \in D$, $\lambda_s^D := 0$, $s \in T \setminus D$. ■

2.2 On the continuity of $\mathcal{F}$, ϑ, and $\mathcal{F}^{op}$ restricted to their domains

Recall that a multifunction between metric spaces, $\mathcal{G} : Y \rightrightarrows X$, is said to be lower semicontinuous in the sense of Berge at $\overline{y} \in \mathrm{dom}\mathcal{G}$ (*lsc*, in brief) if for any open set $V \subset X$ such that $\mathcal{G}(\overline{y}) \cap V \neq \emptyset$, there exists $\varepsilon > 0$ such that

$$\mathcal{G}(y) \cap V \neq \emptyset, \text{ whenever } d(y, \overline{y}) < \varepsilon.$$

It is well-known that the lower semicontinuity of $\mathcal{G}$ at $\overline{y}$ can be characterized in terms of the Painlevé-Kuratowski lower/inner limit as follows: $\mathcal{G}$ is *lsc* at $\overline{y} \in \mathrm{dom}\mathcal{G}$ if and only if

$$\mathcal{G}(\overline{y}) \subset \ \operatorname{Lim\,inf}_r \mathcal{G}(y^r), \tag{8}$$

for any $\{y^r\} \subset Y$ converging to $\overline{y}$. Since we are restricting our mappings $\mathcal{F}$ and $\mathcal{F}^{op}$ to their domains, we may confine ourselves to the case $\mathcal{G}(y^r) \neq \emptyset$ for all r. In such a case, recall that the Painlevé-Kuratowski lower/inner limit, $\operatorname{Lim\,inf}_r \mathcal{G}(y^r)$, is formed by all possible limits of sequences $\{x^r\}$, with $x^r \in \mathcal{G}(y^r)$, for all r. Recall also that the Painlevé-Kuratowski upper/outer limit, $\operatorname{Lim\,sup}_r \mathcal{G}(y^r)$, consists of all the cluster points (limits of subsequences) of sequences $\{x^r\}$, with $x^r \in \mathcal{G}(y^r)$, for all r. It is clear that

$$\operatorname{Lim\,inf}_r \mathcal{G}(y^r) \subset \operatorname{Lim\,sup}_r \mathcal{G}(y^r).$$

When these two sets coincide, we say that there exists the limit of $\{\mathcal{G}(y^r)\}_{r\in\mathbb{N}}$ in the Painlevé-Kuratowski sense, and we write

$$\operatorname{Lim}_r \mathcal{G}(y^r) = \operatorname{Lim\,inf}_r \mathcal{G}(y^r) = \operatorname{Lim\,sup}_r \mathcal{G}(y^r).$$

Remark 4 In this paper we opt for stating Painlevé-Kuratowski convergence results in a sequential form, following [37, p. 109]. Functional expressions of the type $\operatorname{Lim\,inf}_{y \to \overline{y}} \mathcal{G}(y)$ can be found, for instance, in [12, p. 142] or [31, p. 13]. In the latter, the notation Lim inf, with capital L, is used for multifunctions in order to distinguish this concept from its counterpart for real-valued functions.

The following theorem, which can be traced out from the literature, establishes the Painlevé-Kuratowski continuity of $\mathcal{F}$ restricted to $\mathrm{dom}\mathcal{F}$ (recall that it is closed in $\mathbb{R}^T$). Indeed, it can be found under different approaches. It comes from [11, Corollary II.3.1] (dealing with the continuity of $\mathcal{F}$ in the Hausdorff metric); see also [3, Theorem 3.4.1] for a proof of this result in terms of the representation of $\mathcal{F}(b)$ as a compact polyhedron plus its recession cone, similar to [30, Lemma 3.3].

Theorem 1 *Let* $\{b^r\} \subset \mathrm{dom}\mathcal{F}$ *be a sequence converging to* $\overline{b}$. *Then*

$$\mathcal{F}\left(\overline{b}\right) = \mathrm{Lim}_r\, \mathcal{F}\left(b^r\right).$$

Remark 5 The situation is different when dealing with $\mathcal{F}^{op}$. Specifically, one have

$$\mathrm{Lim\,sup}_r\, \mathcal{F}^{op}\left(\pi^r\right) \subset \mathcal{F}^{op}\left(\overline{\pi}\right), \tag{9}$$

for any $\{\pi^r\} \subset \mathrm{dom}\mathcal{F}^{op}$ converging to $\overline{\pi}$, as it follows from the Berge's theory (see [3, Theorem 5.5.1]) or from the upper Lipschitz property for polyhedral multifunctions, which is the case of $\mathcal{F}^{op}$ (see [36]). However $\mathcal{F}^{op}\left(\overline{\pi}\right)$ may not be included in $\mathrm{Lim\,inf}_r\, \mathcal{F}^{op}\left(\pi^r\right)$. Just consider the counterexample, in $\mathbb{R}$, *minimize* cx s.t. $x \in [-1, 1]$ around $\overline{c} = 0$.

The following lemma will be used later in the computation of our aimed calmness modulus of the optimal value function. In it we use the notation

$$\begin{aligned} \mathcal{E}\left(b\right) &:= \mathrm{extr}\left(\mathcal{F}\left(b\right) \cap \mathrm{span}\left\{\overline{a}_t,\ t \in T\right\}\right),\ b \in \mathrm{dom}\mathcal{F}, \\ \mathcal{E}^{op}\left(\pi\right) &:= \mathrm{extr}\left(\mathcal{F}^{op}\left(\pi\right) \cap \mathrm{span}\left\{\overline{a}_t,\ t \in T\right\}\right),\ \pi \in \mathrm{dom}\mathcal{F}^{op}. \end{aligned} \tag{10}$$

In order to motivate the use of mappings $\mathcal{E}$ and $\mathcal{E}^{op}$ from a geometrical point of view, it is easy to see that $\left\{\overline{a}_t,\ t \in T\right\}^{\perp}$ is the lineality space of both $\mathcal{F}\left(b\right)$ and $\mathcal{F}^{op}\left(\pi\right)$, provided that $\pi \equiv (c, b) \in \mathrm{dom}\mathcal{F}^{op}$; i.e., $\left\{\overline{a}_t,\ t \in T\right\}^{\perp}$ consists of those 'directions' $d \in \mathbb{R}^n$ such that $x + \mu d \in \mathcal{F}\left(b\right)$ (resp. $x + \mu d \in \mathcal{F}^{op}\left(\pi\right)$) for all $x \in \mathcal{F}\left(b\right)$ (resp. $x \in \mathcal{F}^{op}\left(\pi\right)$) and all $\mu \in \mathbb{R}$. It is easy to see that, for $\pi \in \mathrm{dom}\mathcal{F}^{op}$, we have $\mathrm{extr}\mathcal{F}^{op}\left(\pi\right) = \emptyset$, equivalently $\mathrm{extr}\mathcal{F}\left(b\right) = \emptyset$, if and only if $\left\{\overline{a}_t,\ t \in T\right\}^{\perp} \neq \{0_n\}$. In such a case, a way to ensure the existence of extreme points is intersecting $\mathcal{F}\left(b\right)$, and $\mathcal{F}^{op}\left(\pi\right)$, with $\left(\left\{\overline{a}_t,\ t \in T\right\}^{\perp}\right)^{\perp} = \mathrm{span}\left\{\overline{a}_t,\ t \in T\right\}$. This construction is inspired by the definition of multifunction F_0 considered in [30, p. 142].

In fact, in the case when $\mathrm{span}\left\{\overline{a}_t,\ t \in T\right\} \subsetneq \mathbb{R}^n$, we can take a basis of $\left\{\overline{a}_t,\ t \in T\right\}^{\perp}$, $\{u_1, ..., u_p\}$, and form the matrix Q whose rows are u_i', $i = 1, ..., p$; then, in order to apply the results of [30] we consider the following convenient representation of $\mathcal{E}\left(b\right)$ and $\mathcal{E}^{op}\left(\pi\right)$, for $\pi = (c, b) \in \mathrm{dom}\mathcal{F}^{op}$: take any $D \in \mathcal{M}_\pi$, and write

$$\mathcal{E}\left(b\right) = \mathrm{extr}\left\{x \in \mathbb{R}^n \mid \overline{a}_t' x \leq b_t,\ t \in T;\ Qx = 0\right\}, \tag{11}$$

and

$$\mathcal{E}^{op}\left(\pi\right) = \mathrm{extr}\left\{x \in \mathbb{R}^n \mid \overline{a}_t' x \leq b_t,\ t \in T \setminus D;\ \overline{a}_t' x = b_t,\ t \in D;\ Qx = 0\right\}. \tag{12}$$

(In the case when span$\{\overline{a}_t,\ t\in T\}=\mathbb{R}^n$, we just omit equation '$Qx=0$'.) Then, as a consequence of [30, Lemma 3.3] we derive the following lemma. In it, and throughout the paper, π^r is identified with $(c^r,b^r)\in\mathbb{R}^n\times\mathbb{R}^T$ for all $r\in\mathbb{N}$ and the nominal problem $\overline{\pi}$ is with parameter $(\overline{c},\overline{b})$.

Lemma 2 *Let $\{\pi^r\}\subset\mathrm{dom}\mathcal{F}^{op}$ converge to $\overline{\pi}$. We have:*
(i) $\{\mathcal{E}(b^r)\}_{r\in\mathbb{N}}$ is uniformly bounded and $\emptyset\neq\mathrm{Lim}\,\mathcal{E}(b^r)=\mathcal{E}\left(\overline{b}\right)$,
(ii) $\emptyset\neq\mathrm{Lim\,sup}_r\,\mathcal{E}^{op}(\pi^r)\subset\mathcal{E}^{op}(\overline{\pi})$.

Proof. (i) According to (11), $\mathcal{E}(b)$ is nothing else but extr$F_0(b)$ in [30, Lemma 3.3] (here we omit d since we have no equations). So, the current statement is a direct consequence of [30, Lemma 3.3] where the Lipschitz contituity of $\mathcal{E}$ in the Hausdorff metric is established.

(ii) First, for each r, take any $D^r\in\mathcal{M}_{\pi^r}$ and write (by (12))

$$\mathcal{E}^{op}(\pi^r)=\mathrm{extr}\left\{x\in\mathbb{R}^n\mid\overline{a}_t'x\leq b_t^r,\ t\in T\setminus D^r;\ \overline{a}_t'x=b_t^r,\ t\in D^r;\ Qx=0\ \right\}.$$

The finiteness of T entails the existence of a constant subsequence $\{D^{r_k}\}_{k\in\mathbb{N}}$; say $D^{r_k}=D$ for $r_1<r_2<...$, so $\mathcal{E}^{op}(\pi^r)$ coincides with the set of extreme points of feasible sets corresponding to the same feasible set mapping. Then, we can apply [30, Lemma 3.3] for deriving, in particular,

$$\begin{aligned}\emptyset &\neq \mathrm{Lim}_k\,\mathcal{E}^{op}(\pi^{r_k})\\ &= \mathrm{extr}\left\{x\in\mathcal{F}(\overline{b})\mid\overline{a}_t'x=\overline{b}_t,\ t\in D;\ Qx=0\right\}\subset\mathcal{E}^{op}(\overline{\pi}),\end{aligned}$$

where the last inclusion comes from the fact that $-\overline{c}\in\mathrm{cone}\{\overline{a}_t,\ t\in D\}$, which follows from $-c^r\in\mathrm{cone}\{\overline{a}_t,\ t\in D\}$, for all r (although the minimality of D in relation to $-c^r$ does not entail the minimality of D for $-\overline{c}$). Since obviously $\mathrm{Lim}_k\,\mathcal{E}^{op}(\pi^{r_k})\subset\mathrm{Lim\,sup}_r\,\mathcal{E}^{op}(\pi^r)$, the latter turns out to be nonempty.

Now, if we consider any element of $\mathrm{Lim\,sup}_r\,\mathcal{E}^{op}(\pi^r)$, say $z=\lim_s z^{r_s}$ with $z^{r_s}\in\mathcal{E}^{op}(\pi^{r_s})$, then, by repeating the previous argument for an appropriate subsequence $\{\pi^{r_{s_k}}\}$ of $\{\pi^{r_s}\}$, we obtain $z\in\mathrm{Lim}_k\,\mathcal{E}^{op}(\pi^{r_{s_k}})\subset\mathcal{E}^{op}(\overline{\pi})$. ■

The next result is well-known in the literature. The reader is addressed to [3, Theorem 4.5.2] for a proof based on the Berge's theory, or to [25, Satz 2.7] and [41, Theorem 14] for a primal-dual approach to the continuity of ϑ^R (see also [32] for a parametric analysis). Indeed, one can find stronger versions: ϑ^R is Lipschitz on bounded subsets of dom$\mathcal{F}^{op}$; see [36, p. 214] in the context of canonically perturbed convex quadratic problems (see also [41, p. 25]). On the other hand, [29] proved the continuity of the optimal value function for a (generally non-convex) quadratic program under canonical perturbations.

Nevertheless, for the reader's convenience, we include here a direct proof based on the previous lemma.

Theorem 2 ϑ^R *is continuous on* $\mathrm{dom}\mathcal{F}^{op}$*.*

Proof. Let $\{\pi^r\} \subset \mathrm{dom}\mathcal{F}^{op}$ be convergent to $\overline{\pi}$ (belonging to $\mathrm{dom}\mathcal{F}^{op}$ because of the closedness of this set) and let us see that

$$\lim\nolimits_r \vartheta(\pi^r) = \vartheta(\overline{\pi}).$$

Take any $\overline{x} \in \mathcal{F}^{op}(\overline{\pi})$ and, appealing to Theorem 1, take any sequence $\{x^r\}$ converging to $\overline{x}$, with $x^r \in \mathcal{F}(b^r)$ for all r. Then,

$$\vartheta(\overline{\pi}) = \overline{c}'\overline{x} = \lim\nolimits_r (c^r)' x^r \geq \limsup\nolimits_r \vartheta(\pi^r).$$

Now, reasoning by contradiction, assume that $\liminf_r \vartheta(\pi^r) < \vartheta(\overline{\pi})$. Write $\liminf_r \vartheta(\pi^r)$ as $\lim_k \vartheta(\pi^{r_k})$ for an appropriate subsequence. By Lemma 2(ii), without loss of generality (taking a subsequence if necessary), we can assume the existence of $x^k \in \mathcal{E}^{op}(\pi^{r_k})$, for all k, such that $\{x^k\}$ converges to some $\overline{x} \in \mathcal{E}^{op}(\overline{\pi})$. Therefore, we attain the contradiction

$$\overline{c}'\overline{x} = \lim\nolimits_k (c^{r_k})' x^k = \lim_k \vartheta(\pi^{r_k}) < \vartheta(\overline{\pi}).$$

■

Finally, recall that the restriction of $\mathcal{F}^{op}$ to its domain is not continuous (in the Painlevé-Kuratowski sense) as shown in Remark 5. However, it is if we only perturb b, as the following theorem asserts. In fact, it is a well-known result of stability theory in LP. Specifically, it can be derived from the fact that $\mathcal{F}^{op}(\overline{c}, \cdot)$ is Lipschitzian on $\mathrm{dom}\mathcal{F}$, provided that

$$-\overline{c} \in \mathrm{cone}\{\overline{a}_t,\ t \in T\} \tag{13}$$

(in which case $(\overline{c}, b) \in \mathrm{dom}\mathcal{F}^{op}$ if and only if $b \in \mathrm{dom}\mathcal{F}$); see, e.g. [26, p. 232] or [13, Chapter IX (Sec. 7)].

Theorem 3 *Let* $\overline{c} \in \mathbb{R}^n$ *verify* (13)*. For any* $\{b^r\} \subset \mathrm{dom}\mathcal{F}$ *converging to* $\overline{b}$ *we have*

$$\mathcal{F}^{op}(\overline{\pi}) = \mathrm{Lim}_r\, \mathcal{F}^{op}(\overline{c}, b^r).$$

3 Calmness modulus under RHS perturbations

Along this section we deal with linear optimization problems with a fixed c, say $\overline{c}$, which is assumed to verify (13). So, the only parameter to be considered here is $b \in \mathrm{dom}\mathcal{F}$ (equivalently $(\overline{c}, b) \in \mathrm{dom}\mathcal{F}^{op}$). Formally, we consider the new optimal value function $\vartheta_{\overline{c}}^{R} : \mathrm{dom}\mathcal{F} \to]-\infty, +\infty[$ defined as

$$\vartheta_{\overline{c}}^{R}(b) := \vartheta(\overline{c}, b), \text{ for all } b \in \mathrm{dom}\mathcal{F}.$$

The final goal of this section is to compute/estimate the calmness modulus of $\vartheta_{\overline{c}}^{R}$ at $\overline{b} \in \mathrm{dom}\mathcal{F}$, which is given by

$$\mathrm{clm}\vartheta_{\overline{c}}^{R}(\overline{b}) = \limsup_{b \underset{\mathrm{dom}\mathcal{F}}{\to} \overline{b}} \frac{|\vartheta(\overline{c}, b) - \vartheta(\overline{\pi})|}{\left\|b - \overline{b}\right\|_{\infty}},$$

(recall $\overline{\pi} \equiv (\overline{c}, \overline{b})$) and the corresponding calmness moduli from below and above (which are analogously defined). Observe that we use indistinctly the notation $\vartheta_{\overline{c}}^{R}(b)$ and $\vartheta(\overline{c}, b)$ whenever $b \in \mathrm{dom}\mathcal{F}$. In fact, for clarity, we usually write $\vartheta_{\overline{c}}^{R}$ when talking about the function itself and write $\vartheta(\overline{c}, b)$ for the image of $\vartheta_{\overline{c}}^{R}$ at $b \in \mathrm{dom}\mathcal{F}$.

To start with, we have the well-known expression for $\vartheta_{\overline{c}}^{R}$ as a piecewise linear function (see, e.g., [4, p. 214]) given by

$$\vartheta(\overline{c}, b) = \max_{\lambda \in \mathrm{extr}\Lambda(\overline{c})} -b'\lambda, \text{ for all } b \in \mathrm{dom}\mathcal{F}.$$

The following results are devoted to refine the previous expression in a neighborhood of $\overline{b}$ by appealing to the family of minimal KKT subsets of indices, $\mathcal{M}_{\overline{\pi}}$; specifically, to replace $\mathrm{extr}\Lambda(\overline{c})$ with a smaller set written in terms of $\mathcal{M}_{\overline{\pi}}$.

The following result is standard (the finiteness of $\mathrm{extr}\Lambda(\overline{c})$ is a key fact).

Lemma 3 *Let $\overline{b} \in \mathrm{dom}\mathcal{F}$. There exists a neighborhood $U_{\overline{b}} \subset \mathbb{R}^T$ of $\overline{b}$ such that*

$$\mathrm{extr}\Lambda^{op}(\overline{c}, b) \subset \mathrm{extr}\Lambda^{op}(\overline{\pi}), \textit{ whenever } b \in \mathrm{dom}\mathcal{F} \cap U_{\overline{b}}.$$

As a consequence of the previous lemma, and taking into account the obvious fact that

$$\vartheta(\overline{c}, b) = \max_{\lambda \in \mathrm{extr}\Lambda^{op}(\overline{c}, b)} -b'\lambda, \text{ for all } b \in \mathrm{dom}\mathcal{F},$$

we derive the following corollary.

Corollary 1 *Let $\bar{b} \in \mathrm{dom}\mathcal{F}$ and let $U_{\bar{b}}$ be as in Lemma 3. Then*

$$\vartheta(\bar{c}, b) = \max_{D \in \mathcal{M}_{\overline{\pi}}} -b'\lambda^D, \text{ for all } b \in \mathrm{dom}\mathcal{F} \cap U_{\bar{b}}.$$

Now, applying the previous corollary, and taking the fact that $\vartheta(\overline{\pi}) = -\bar{b}'\lambda^D$ for all $D \in \mathcal{M}_{\overline{\pi}}$ into account, we deduce

$$\vartheta(\bar{c}, b) - \vartheta(\overline{\pi}) = \left(\max_{D \in \mathcal{M}_{\overline{\pi}}} -b'\lambda^D\right) - \vartheta(\overline{\pi}) = \max_{D \in \mathcal{M}_{\overline{\pi}}} -\left(b - \bar{b}\right)'\lambda^D, \tag{14}$$

while

$$\vartheta(\overline{\pi}) - \vartheta(\bar{c}, b) = \vartheta(\overline{\pi}) - \max_{D \in \mathcal{M}_{\overline{\pi}}} -b'\lambda^D = \min_{D \in \mathcal{M}_{\overline{\pi}}} (b - \bar{b})'\lambda^D, \tag{15}$$

for all $b \in \mathrm{dom}\mathcal{F} \cap U_{\bar{b}}$, where $U_{\bar{b}}$ is as in Lemma 3. Consequently, if we denote

$$k^- := \min_{D \in \mathcal{M}_{\overline{\pi}}} \left\|\lambda^D\right\|_1 \text{ and } k^+ = \max_{D \in \mathcal{M}_{\overline{\pi}}} \left\|\lambda^D\right\|_1,$$

(where, as usual, $\left\|\lambda^D\right\|_1 = \sum_{t \in D} \lambda_t^D$) we deduce the following result saying that k^- and k^+ are, respectively, a calmness constant from below and above for our optimal value function $\vartheta_{\bar{c}}^R$.

Corollary 2 *Let $\bar{b} \in \mathrm{dom}\mathcal{F}$ and let $U_{\bar{b}}$ be as in Lemma 3. Then, for $b \in \mathrm{dom}\mathcal{F} \cap U_{\bar{b}}$, one has*

(i) $\vartheta(\overline{\pi}) - \vartheta(\bar{c}, b) \leq k^- \left\|b - \bar{b}\right\|_\infty$;

(ii) $\vartheta(\bar{c}, b) - \vartheta(\overline{\pi}) \leq k^+ \left\|b - \bar{b}\right\|_\infty$.

So, k^- and k^+ are, respectively, upper bounds on the calmness moduli from below and above of $\vartheta_{\bar{c}}^R$ at $\bar{b}$, given by

$$\underline{\mathrm{clm}}\vartheta_{\bar{c}}^R(\bar{b}) = \limsup_{b \underset{\mathrm{dom}\mathcal{F}}{\rightarrow} \bar{b}} \frac{\vartheta(\overline{\pi}) - \vartheta(\bar{c}, b)}{\left\|b - \bar{b}\right\|_\infty} \text{ and } \overline{\mathrm{clm}}\vartheta_{\bar{c}}^R(\bar{b}) = \limsup_{b \underset{\mathrm{dom}\mathcal{F}}{\rightarrow} \bar{b}} \frac{\vartheta(\bar{c}, b) - \vartheta(\overline{\pi})}{\left\|b - \bar{b}\right\|_\infty}.$$

The following theorem shows that k^- is always attained as the calmness modulus from below of $\vartheta_{\bar{c}}^R$. The counterpart for k^+ is no longer true, as Example 1 shows.

Theorem 4 *Let $\bar{b} \in \mathrm{dom}\mathcal{F}$. One has:*

$$\underline{\mathrm{clm}}\vartheta_{\bar{c}}^R(\bar{b}) = k^- \text{ and } \overline{\mathrm{clm}}\vartheta_{\bar{c}}^R(\bar{b}) \leq k^+.$$

Consequently,

$$k^- \leq \mathrm{clm}\vartheta_{\bar{c}}^R(\bar{b}) \leq k^+.$$

Proof. As commented above, it is clear that $\underline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) \leq k^{-}$ and $\overline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) \leq k^{+}$.

So, we only need to prove that $\underline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) \geq k^{-}$. Consider the sequence

$$b^{r} := \overline{b} + \frac{1}{r}1_{T}, \text{ for all } r,$$

where $1_{T} \in \mathbb{R}^{T}$ represents the vector having all its coordinates equal to 1. Clearly, $\{b^{r}\} \subset \text{dom}\mathcal{F}$. Then, appealing to (15) we have

$$\begin{aligned} \underline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) &\geq \limsup_{r} \frac{\vartheta\left(\overline{c},\overline{b}\right) - \vartheta\left(\overline{c},b^{r}\right)}{\left\|b^{r}-\overline{b}\right\|_{\infty}} \\ &= \limsup_{r} \frac{\min_{D\in\mathcal{M}_{\overline{\pi}}}\left(b^{r}-\overline{b}\right)'\lambda^{D}}{\left\|b^{r}-\overline{b}\right\|_{\infty}} \\ &= \lim_{r} \frac{\frac{1}{r}\min_{D\in\mathcal{M}_{\overline{\pi}}}\left\|\lambda^{D}\right\|_{1}}{\frac{1}{r}} = k^{-}. \end{aligned}$$

■

The following proposition is intended to provide an alternative approach for determining $\left\|\lambda^{D}\right\|_{1}$, with $D \in \mathcal{M}_{\overline{\pi}}$. In it, u_{D} represents the projection of 0_{n} on aff $\{\overline{a}_{t},\ t \in D\}$ in the Euclidean norm, provided that $D \in \mathcal{M}_{\overline{\pi}}$ (observe that $0_{n} \notin$ aff $\{\overline{a}_{t},\ t \in D\}$ as a consequence of the linear independence of $\{\overline{a}_{t},\ t \in D\}$).

Proposition 1 *For each $D \in \mathcal{M}_{\overline{\pi}}$, one has*

$$\left\|\lambda^{D}\right\|_{1} = \frac{-\overline{c}'u_{D}}{\|u_{D}\|_{2}^{2}}. \tag{16}$$

Proof. Take any $D \in \mathcal{M}_{\overline{\pi}}$. Since $-\left\|\lambda^{D}\right\|_{1}^{-1}\overline{c} \in$ aff $\{\overline{a}_{t},\ t \in D\}$, the definition of u_{D} yields

$$\left(-\left\|\lambda^{D}\right\|_{1}^{-1}\overline{c} - u_{D}\right)' u_{D} = 0,$$

which entails the aimed equality (16). ■

The following example shows that the calmness modulus from above of $\vartheta_{\overline{c}}^{R}$ can take any positive value less than or equal to k^{+}.

Example 1 Consider the problem in $\mathbb{R}$ given by

$$\begin{array}{lll} \overline{\pi}: & \text{minimize} & x_{1} \\ & \text{subject to} & -x_{1} \leq 0,\ t=1, \\ & & -2x_{1} \leq 0,\ t=2, \\ & & \theta x_{1} \leq 0,\ t=3, \end{array}$$

where $\theta > 0$. Trivially, $\mathcal{M}_{\overline{\pi}} := \{\{1\},\{2\}\}$, $\lambda^{\{1\}} = 1 = k^{+}$, $\lambda^{\{2\}} = \frac{1}{2} = k^{-}$. Let us check that

$$\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) = \min\left\{1, \frac{1}{\theta}\right\}.$$

Observe that $\bar{b} = 0_3$. According to Corollary 1, in some neighborhood $U_{\overline{b}}$ of $\overline{b}$ we have

$$\vartheta\left(\overline{c}, b\right) = \max\{-b_1, -\tfrac{1}{2}b_2\}, \text{ for } b = (b_1, b_2, b_3)' \in \mathrm{dom}\mathcal{F} \cap U_{\overline{b}}.$$

Moreover, it is immediate that

$$b \in \mathrm{dom}\mathcal{F} \Leftrightarrow \max\{-b_1, -\tfrac{1}{2}b_2\} \leq \tfrac{1}{\theta}b_3.$$

So,

$$\frac{\vartheta\left(\overline{c}, b\right) - \vartheta\left(\overline{\pi}\right)}{\left\|b - \overline{b}\right\|_{\infty}} = \frac{\max\{-b_1, -\frac{1}{2}b_2\}}{\|b\|_{\infty}} \leq \frac{\frac{1}{\theta}b_3}{\|b\|_{\infty}} \leq \tfrac{1}{\theta}, \text{ for all } b \in \mathrm{dom}\mathcal{F} \cap U_{\overline{b}}.$$

Consequently, one always have

$$\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) \leq \frac{1}{\theta}.$$

Now, we distinguish two cases:

Case 1: $\theta > 1$. Just consider the sequence

$$b^{r} = \left(-\frac{1}{\theta r}, -\frac{1}{\theta r}, \frac{1}{r}\right)', \ r = 1, 2...$$

It is clear that $b^{r} \in \mathrm{dom}\mathcal{F}$ for all r. Moreover, for r large enough (to ensure $b^{r} \in U_{\overline{b}}$) one has

$$\frac{\vartheta\left(\overline{c}, b^{r}\right) - \vartheta\left(\overline{\pi}\right)}{\left\|b^{r} - \overline{b}\right\|_{\infty}} = \frac{\frac{1}{\theta r}}{\frac{1}{r}} = \frac{1}{\theta},$$

So, $\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) \geq \frac{1}{\theta}$.

Case 2: $0 < \theta \leq 1$, yielding $\frac{1}{\theta} \geq 1 = k^{+} \geq \overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b})$. Consider the sequence

$$b^{r} = \left(-\frac{1}{r}, -\frac{1}{r}, \frac{1}{r}\right)', \ r = 1, 2...$$

One has $b^{r} \in \mathrm{dom}\mathcal{F}$ for all r and, for r large enough,

$$\frac{\vartheta\left(\overline{c}, b^{r}\right) - \vartheta\left(\overline{\pi}\right)}{\left\|b^{r} - \overline{b}\right\|_{\infty}} = \frac{\frac{1}{r}}{\frac{1}{r}} = 1.$$

Inspired by the previous example, the following proposition provides a sufficient condition for having the equality $\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) = k^{+}$.

Proposition 2 *Let $\overline{b} \in \mathrm{dom}\mathcal{F}$. Assume that there exists some $\overline{D} \in \mathcal{M}_{\overline{\pi}}$, with $\left\|\lambda^{\overline{D}}\right\|_1 = k^+$, and some $\varepsilon > 0$ such that $b^\varepsilon \in \mathrm{dom}\mathcal{F}$, with*

$$b_t^\varepsilon := \begin{cases} \overline{b}_t - \varepsilon & \text{if } t \in \overline{D}, \\ \overline{b}_t + \varepsilon & \text{if } t \in T \backslash \overline{D}. \end{cases} \tag{17}$$

Then,

$$\overline{\mathrm{clm}}\vartheta_{\overline{c}}^R(\overline{b}) = \mathrm{clm}\vartheta_{\overline{c}}^R(\overline{b}) = k^+.$$

Proof. Take $\overline{D} \in \mathcal{M}_{\overline{\pi}}$ and $\varepsilon > 0$ verifying the assumptions of the current proposition. First, observe that for any $0 < \widetilde{\varepsilon} < \varepsilon$, the associated $b^{\widetilde{\varepsilon}}$ defined as in (17) (replacing ε with $\widetilde{\varepsilon}$) also verifies that $b^{\widetilde{\varepsilon}} \in \mathrm{dom}\mathcal{F}$, as far as $\mathrm{dom}\mathcal{F}$ is a convex set. Just observe that $b^{\widetilde{\varepsilon}} = \left(1 - \frac{\widetilde{\varepsilon}}{\varepsilon}\right)\overline{b} + \frac{\widetilde{\varepsilon}}{\varepsilon} b^\varepsilon$.

Let $U_{\overline{b}}$ be as in Lemma 3, and consider the sequence $\{b^{\frac{1}{r}}\}$, where $b^{\frac{1}{r}}$ comes again from replacing ε with $\frac{1}{r}$ in (17). Let r_0 be large enough to guarantee $\frac{1}{r_0} < \varepsilon$ (so, $b^{\frac{1}{r}} \in \mathrm{dom}\mathcal{F}$, $r \geq r_0$) and $b^{\frac{1}{r}} \in U_{\overline{b}}$, for all $r \geq r_0$. Then

$$\begin{aligned} \mathrm{clm}\vartheta_{\overline{c}}^R(\overline{b}) &\geq \limsup_r \frac{\vartheta\left(\overline{c}, b^{\frac{1}{r}}\right) - \vartheta\left(\overline{\pi}\right)}{\left\|b^{\frac{1}{r}} - \overline{b}\right\|_\infty} \\ &= \limsup_r \frac{\max_{D \in \mathcal{M}_{\overline{\pi}}}\left(-\left(b^{\frac{1}{r}} - \overline{b}\right)' \lambda^D\right)}{\frac{1}{r}} = \left\|\lambda^{\overline{D}}\right\|_1, \end{aligned}$$

where the last equality comes from the fact that

$$\left|-\left(b^{\frac{1}{r}} - \overline{b}\right)' \lambda^D\right| \leq \left\|b^{\frac{1}{r}} - \overline{b}\right\|_\infty \left\|\lambda^D\right\|_1 = \frac{1}{r}\left\|\lambda^D\right\|_1,$$

for all $D \in \mathcal{M}_{\overline{\pi}}$, and $-\left(b^{\frac{1}{r}} - \overline{b}\right)' \lambda^{\overline{D}} = \frac{1}{r}\left\|\lambda^{\overline{D}}\right\|_1$. ■

As a consequence of the previous proposition we have the following corollary under SCQ. It is well-known that SCQ at $\overline{b} \in \mathrm{dom}\mathcal{F}$ is equivalent to $\overline{b} \in \mathrm{int}\,\mathrm{dom}\mathcal{F}$.

Corollary 3 *Let $\overline{b} \in \mathrm{dom}\mathcal{F}$ and assume that SCQ holds at $\overline{b}$. Then*

$$\overline{\mathrm{clm}}\vartheta_{\overline{c}}^R(\overline{b}) = \mathrm{clm}\vartheta_{\overline{c}}^R(\overline{b}) = k^+.$$

4 Calmness modulus under canonical perturbations

This section is devoted to compute/estimate the calmness moduli from below and above of the optimal value function restricted to $\text{dom}\mathcal{F}^{op}$, $\vartheta^R : \text{dom}\mathcal{F}^{op} \rightarrow]-\infty, +\infty[$, at $\overline{\pi} \in \text{dom}\mathcal{F}^{op}$. Recall that they are respectively given by

$$\underline{\text{clm}}\vartheta^R(\overline{\pi}) = \limsup_{\pi \underset{\text{dom}\mathcal{F}^{op}}{\rightarrow} \overline{\pi}} \frac{\vartheta(\overline{\pi}) - \vartheta(\pi)}{\|\pi - \overline{\pi}\|} \text{ and } \overline{\text{clm}}\vartheta^R(\overline{\pi}) = \limsup_{\pi \underset{\text{dom}\mathcal{F}^{op}}{\rightarrow} \overline{\pi}} \frac{\vartheta(\pi) - \vartheta(\overline{\pi})}{\|\pi - \overline{\pi}\|},$$

and, roughly speaking, provide a measure of the maximum rate of decrease and increase, respectively, under perturbations of the data (regarding solvable problems only). These calmness moduli are shown to be closely related to the corresponding calmness moduli of $\vartheta^R_{\overline{c}}$ (where only perturbations of b are allowed). To start with, we establish the following lemma.

Lemma 4 *There exists* $\overline{\delta} > 0$ *such that if* $\pi, \overline{\pi} \in \text{dom}\mathcal{F}^{op}$ *satisfy* $\|\pi - \overline{\pi}\| < \overline{\delta}$, *with* $\pi \equiv (c, b)$ *and* $\overline{\pi} \equiv \left(\overline{c}, \overline{b}\right)$, *then*

$$\mathcal{F}^{op}(\pi) \subset \mathcal{F}^{op}(\overline{c}, b).$$

Proof. Reasoning by contradiction, assume the existence of a sequence of problems $\{\pi^r \equiv (c^r, b^r)\} \subset \text{dom}\mathcal{F}^{op}$ converging to $\overline{\pi}$ and a sequence of points $\{x^r\} \subset \mathbb{R}^n$ such that $x^r \in \mathcal{F}^{op}(\pi^r) \setminus \mathcal{F}^{op}(\overline{c}, b^r)$ for all r. We have that

$$-c^r \in \text{cone}\{\overline{a}_t,\ t \in T_{b^r}(x^r)\}, \text{ for all } r.$$

We may assume (by taking a subsequence if necessary) that $T_{b^r}(x^r) = D$ for all r (not depending on r). Then

$$-\overline{c} \in \text{cone}\{\overline{a}_t,\ t \in D\},$$

(by closedness) which yields the contradiction $x^r \in \mathcal{F}^{op}(\overline{c}, b^r)$ for each r. ■

Theorem 5 *Let* $\overline{\pi} \in \text{dom}\mathcal{F}^{op}$. *Then*

$$\overline{\text{clm}}\vartheta^R(\overline{\pi}) = \overline{\text{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right).$$

Proof. Take $\overline{x} \in \mathcal{F}^{op}(\overline{\pi})$ with $\|\overline{x}\| = d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right)$. Fix arbitrarily $\varepsilon > 0$ and let $\pi \equiv (c, b) \in \text{dom}\mathcal{F}^{op}$ be such that $\|\pi - \overline{\pi}\| < \delta$ for a certain $\delta > 0$

satisfying the following:

$$\begin{array}{rcl} \delta & \leq & \overline{\delta} \text{ (the one from Lemma 4),} \\ \left\| b-\overline{b} \right\|_{\infty} & < & \delta \Rightarrow \left\{ \begin{array}{l} d\left(\overline{x}, \mathcal{F}^{op}\left(\overline{c}, b\right)\right) < \varepsilon \text{ (by Theorem 3),} \\ b \in U_{\overline{b}} \text{ (that of Lemma 3),} \\ \vartheta\left(\overline{c}, b\right) - \vartheta\left(\overline{\pi}\right) \leq \left(\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) + \varepsilon\right) \left\| b-\overline{b} \right\|_{\infty}. \end{array} \right. \end{array}$$

Now pick an arbitrary $\widehat{x} \in \mathcal{F}^{op}(\pi) \subset \mathcal{F}^{op}(\overline{c}, b)$ (because $\delta \leq \overline{\delta}$) and $\widetilde{x} \in \mathcal{F}^{op}(\overline{c}, b)$ with $\|\widetilde{x}-\overline{x}\| < \varepsilon$. Clearly $\vartheta(\pi) = c'\widehat{x} \leq c'\widetilde{x}$ and $\vartheta(\overline{c}, b) = \overline{c}'\widehat{x} = \overline{c}'\widetilde{x}$. Then we have

$$\begin{array}{rcl} \vartheta(\pi) - \vartheta(\overline{\pi}) & = & c'\widehat{x} - \overline{c}'\widehat{x} + \vartheta(\overline{c}, b) - \vartheta(\overline{\pi}) \\ & \leq & c'\widetilde{x} - \overline{c}'\widetilde{x} + \left(\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) + \varepsilon\right) \left\| b-\overline{b} \right\|_{\infty} \\ & \leq & \|c-\overline{c}\|_{*} \left(\|\overline{x}\| + \varepsilon\right) + \left(\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) + \varepsilon\right) \left\| b-\overline{b} \right\|_{\infty} \\ & \leq & \left(\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) + \|\overline{x}\| + 2\varepsilon\right) \|\pi-\overline{\pi}\|. \end{array}$$

Since $\varepsilon > 0$ has been arbitrarily chosen, we get $\overline{\mathrm{clm}}\vartheta^{R}(\overline{\pi}) \leq \overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) + d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right)$.

Next we show that the previous inequality holds as an equality. The case $0_n \in \mathcal{F}^{op}(\overline{\pi})$ is trivial, since in such a case we have $\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) \leq \overline{\mathrm{clm}}\vartheta^{R}(\overline{\pi}) \leq \overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) + 0$; i.e., $\overline{\mathrm{clm}}\vartheta^{R}(\overline{\pi}) = \overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b})$. Hence, let us assume $0_n \notin \mathcal{F}^{op}(\overline{\pi})$. Let us consider a sequence $\{b^r\}_{r \in \mathbb{N}} \subset \mathrm{dom}\mathcal{F}$ such that

$$\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) = \lim_r \frac{\vartheta(\overline{c}, b^r) - \vartheta(\overline{\pi})}{\left\| b^r - \overline{b} \right\|_{\infty}}.$$

Because of Theorem 3 we may assume $0_n \notin \mathcal{F}^{op}(\overline{c}, b^r)$ for all r. The same theorem ensures the existence of $x^r \in \mathcal{F}^{op}(\overline{c}, b^r)$ with $\|x^r\| = d\left(0_n, \mathcal{F}^{op}(\overline{c}, b^r)\right)$, for all r, and $\|x^r\| \to \|\overline{x}\|$ (we do not need to guarantee $x^r \to \overline{x}$). More in detail, replacing $\{b^r\}$ with an appropriate subsequence (we do not relabel for simplicity) we could choose $x^r \in B\left(0_n, \|\overline{x}\| + \frac{1}{r}\right)$, i.e., the open ball centered at 0_n with radius $\|\overline{x}\| + \frac{1}{r}$, which is an open set containing $\overline{x}$; again considering an appropriate subsequence we may assume that $\{x^r\}$ converges to certain $z \in \mathrm{cl}B\left(0_n, \|\overline{x}\|\right)$, and if $\|z\| < \|\overline{x}\|$ we attain a contradiction with Theorem 3. Now, for each r, we appeal to [7, Lemma 9] to ensure the existence of $u^r \in \mathbb{R}^n$ with $\|u^r\|_* = 1$ such that

$$(u^r)' x \geq (u^r)' x^r = \|x^r\| \text{ for all } x \in \mathcal{F}^{op}(\overline{c}, b^r). \tag{18}$$

Let us define $c^r := \overline{c} + \left\| b^r - \overline{b} \right\|_{\infty} u^r$, which entails $\|c^r - \overline{c}\|_* = \left\| b^r - \overline{b} \right\|_{\infty}$. First we note that, for all $x \in \mathcal{F}^{op}(\overline{c}, b^r)$, we have

$$(c^r)' x = \overline{c}' x + \left\| b^r - \overline{b} \right\|_{\infty} (u^r)' x \geq \vartheta(\overline{c}, b^r) + \|c^r - \overline{c}\|_* \|x^r\|. \tag{19}$$

Our next step consists of establishing the existence of $r_0 \in \mathbb{N}$ such that

$$\pi^r \equiv (c^r, b^r) \in \mathrm{dom}\mathcal{F}^{op}, \text{ for } r \geq r_0. \tag{20}$$

Assuming for the moment that (20) holds, it yields $\mathcal{F}^{op}(\pi^r) \subset \mathcal{F}^{op}(\overline{c}, b^r)$ for $r \geq r_0$ large enough (to apply Lemma 4). Then, by repeating inequality (19) with any $x \in \mathcal{F}^{op}(\pi^r)$, we deduce $\vartheta(\pi^r) \geq \vartheta(\overline{c}, b^r) + \|c^r - \overline{c}\|_* \|x^r\|$ and therefore, recalling $\|c^r - \overline{c}\|_* = \left\|b^r - \overline{b}\right\|_\infty$, we have

$$\begin{aligned}
\overline{\mathrm{clm}}\vartheta^R(\overline{\pi}) &\geq \limsup_r \frac{\vartheta(\pi^r) - \vartheta(\overline{\pi})}{\|\pi^r - \overline{\pi}\|} \\
&= \limsup_r \frac{\vartheta(\pi^r) - \vartheta(\overline{c}, b^r)}{\|c^r - \overline{c}\|_*} + \lim_r \frac{\vartheta(\overline{c}, b^r) - \vartheta(\overline{\pi})}{\left\|b^r - \overline{b}\right\|_\infty} \\
&\geq \lim_r \|x^r\| + \overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) = \|\overline{x}\| + \overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}),
\end{aligned}$$

which establishes the aimed equality $\overline{\mathrm{clm}}\vartheta^R(\overline{\pi}) = \overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + d(0_n, \mathcal{F}^{op}(\overline{\pi}))$.

So, it remains to prove (20). From [30, Lemma 4.1], we can write

$$\mathcal{F}(b^r) = \mathrm{conv}\,\mathcal{E}(b^r) + \left\{d \in \mathbb{R}^n : \overline{a}_t' d \leq 0,\ t \in T\right\},$$

(recall that $\mathcal{E}(b^r) := \mathrm{extr}(\mathcal{F}(b^r) \cap \mathrm{span}\{\overline{a}_t,\ t \in T\}$ for all r) where the last term is the recession cone of $\mathcal{F}(b^r)$, which does not depend on r (only on the fact that $b^r \in \mathrm{dom}\mathcal{F}$). Let us write this (polyhedral) recession cone as $\mathrm{cone}\{d_1, ..., d_p\}$. On the other hand, Lemma 2(i) ensures that $\{\mathcal{E}(b^r)\}_{r \in \mathbb{N}}$ is a sequence of uniformly bounded nonempty compact sets. Assume, reasoning by contradiction, that $\vartheta(\pi^r) = -\infty$ for all r (replacing, if necessary, the sequence with an appropriate subsequence). Because of the compactness of $\mathrm{conv}\mathcal{E}(b^r)$, $\vartheta(\pi^r) = -\infty$ implies (again considering an appropriate subsequence, if necessary) that $(c^r)' d_k < 0$ for all r and some fixed $k \in \{1, ..., p\}$. Letting $r \to \infty$ we obtain $\overline{c}' d_k \leq 0$, which entails that d_k is not only a recession direction of $\mathcal{F}(b^r)$, but also of $\mathcal{F}^{op}(\overline{c}, b^r)$, for all r. This, together with $(c^r)' d_k < 0$ ensures that, for each $r \in \mathbb{N}$, $x \mapsto (c^r)' x$ is not bounded from below on $\mathcal{F}^{op}(\overline{c}, b^r)$, contradicting (19). ■

Remark 6 With respect to the proof of the previous theorem, when $\mathcal{F}^{op}(\overline{\pi})$ is bounded, one immediately has (20) as a consequence of [19, Lemma 10.2].

Theorem 6 *Let* $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$. *Then*

$$\underline{\mathrm{clm}}\vartheta^R(\overline{\pi}) \leq \underline{\mathrm{clm}}\vartheta^R_{\overline{c}}\left(\overline{b}\right) + e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right), \tag{21}$$

where the last term represents the Hausdorff excess of $\mathcal{E}^{op}(\overline{\pi})$ *over* $\{0_n\}$, *which may alternatively be written as* $\max\{\|x\| \mid x \in \mathcal{E}^{op}(\overline{\pi})\}$; *i.e., the maximum norm in a finite set.*

Proof. For simplicity, write $\alpha := \underline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) + e\left(\mathcal{E}^{op}\left(\overline{\pi}\right), 0_n\right)$. Reasoning by contradiction, assume the existence of a sequence $\{\pi^r \equiv (c^r, b^r)\} \subset \mathrm{dom}\mathcal{F}^{op}$ converging to $\overline{\pi}$ such that

$$\vartheta\left(\overline{\pi}\right) - \vartheta\left(\pi^r\right) > (\alpha + \varepsilon) \left\|\pi^r - \overline{\pi}\right\|$$

for all $r \in \mathbb{N}$ and some $\varepsilon > 0$. According to Lemma 2(ii), pick any $\overline{x} \in \mathrm{Lim\,sup}_r\, \mathcal{E}^{op}\left(\pi^r\right)$, and take $r_1 < r_2 < ... < r_k < ...$ and associated $x^k \in \mathcal{F}^{op}\left(\pi^{r_k}\right)$ such that $x^k \to \overline{x}$. According to Lemma 4 we may assume that $x^k \in \mathcal{F}^{op}\left(\overline{c}, b^{r_k}\right)$ for all $k \in \mathbb{N}$. Then, for k large enough guaranteeing $\left\|x^k - \overline{x}\right\| \leq \varepsilon$ and $b^{r_k} \in U_{\overline{b}}$ (see again Lemma 3 and Corollary 2), and taking Theorem 4 into account, we attain the following contradiction:

$$\begin{aligned}
\vartheta\left(\overline{\pi}\right) - \vartheta\left(\pi^{r_k}\right) &= \vartheta\left(\overline{\pi}\right) - \vartheta\left(\overline{c}, b^{r_k}\right) + \overline{c}' x^k - \left(c^{r_k}\right)' x^k \\
&\leq \underline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) \left\|\overline{b} - b^{r_k}\right\|_\infty + \left\|\overline{c} - c^{r_k}\right\|_* \left\|x^k\right\| \\
&\leq \left(\underline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) + \left\|\overline{x}\right\| + \varepsilon\right) \left\|\pi^{r_k} - \overline{\pi}\right\| \\
&\leq (\alpha + \varepsilon) \left\|\pi^{r_k} - \overline{\pi}\right\|.
\end{aligned}$$

■

The following example shows that inequality (21) may be strict.

Example 2 Consider the nominal problem, in $\mathbb{R}^2$ with the Euclidean norm,

$$\overline{\pi} : \text{minimize } x_2 \text{ s.t. } x_1 \leq -1, \quad -x_2 \leq 1.$$

Clearly $\mathcal{E}^{op}\left(\overline{\pi}\right) = \left\{(-1,-1)'\right\}$. Let us see that the specification of (21) to this case reads as $2 \leq 1 + \sqrt{2}$. For $\left\|\pi - \overline{\pi}\right\| < 1$ one has $\pi \equiv (c, b) \in \mathrm{dom}\mathcal{F}^{op}$ if and only if $c_1 \leq 0$, where $c = (c_1, c_2)'$. For convenience, let us write $c = (\varepsilon_1, 1 + \varepsilon_2)'$ and $b = (-1 + \varepsilon_3, 1 + \varepsilon_4)'$, with

$$\left\|\pi - \overline{\pi}\right\| = \max\left\{\sqrt{\varepsilon_1^2 + \varepsilon_2^2}, \left|\varepsilon_3\right|, \left|\varepsilon_4\right|\right\} =: \varepsilon < 1.$$

Then, provided that $\varepsilon_1 \leq 0$, we have

$$\vartheta\left(\overline{\pi}\right) - \vartheta\left(\pi\right) = -1 - \varepsilon_1\left(-1 + \varepsilon_3\right) - \left(1 + \varepsilon_2\right)\left(-1 - \varepsilon_4\right) \leq 2\varepsilon + \varepsilon^2,$$

and, accordingly, by letting $\varepsilon \searrow 0$, we have $\underline{\mathrm{clm}}\vartheta^R(\overline{\pi}) \leq 2$. Indeed, by taking $\varepsilon_1 = \varepsilon_3 = 0$ and $\varepsilon_2 = \varepsilon_4 = \varepsilon$, we see that $\underline{\mathrm{clm}}\vartheta^R(\overline{\pi}) = 2$. A simpler calculation with $c = \overline{c}$ shows that $\underline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) = 1$.

Theorem 7 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$ with $\mathcal{F}^{op}(\overline{\pi})$ bounded. Then*

$$\begin{aligned}\underline{\mathrm{clm}}\vartheta^{R}(\overline{\pi}) &= \underline{\mathrm{clm}}\vartheta^{R}_{\overline{c}}\left(\overline{b}\right) + e\left(\mathrm{extr}\mathcal{F}^{op}(\overline{\pi}), 0_n\right) \\ &= \underline{\mathrm{clm}}\vartheta^{R}_{\overline{c}}(\overline{b}) + e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right).\end{aligned}$$

Proof. The last equality follows a standard argument by using the convexity of the norm. Let us observe that the boundedness of $\mathcal{F}^{op}(\overline{\pi})$ entails $\mathrm{span}\{\overline{a}_t,\ t \in T\} = \mathbb{R}^n$, so that we have to prove that (21) holds as an equality in this case. Let us consider, similarly to the proof of Theorem 5, a sequence $\{b^r\}_{r\in\mathbb{N}} \subset \mathrm{dom}\mathcal{F}$ such that

$$\underline{\mathrm{clm}}\vartheta^{R}_{\overline{c}}\left(\overline{b}\right) = \lim_r \frac{\vartheta(\overline{\pi}) - \vartheta(\overline{c}, b^r)}{\left\|b^r - \overline{b}\right\|_{\infty}}.$$

From Theorem 3 we easily deduce

$$e\left(\mathcal{F}^{op}(\overline{c}, b^r), 0_n\right) \to e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right),$$

as $r \to \infty$. Let us write, for each r, $e\left(\mathcal{F}^{op}(\overline{c}, b^r), 0_n\right) = \|x^r\|$ with $x^r \in \mathcal{F}^{op}(\overline{c}, b^r)$. For each r, let $u^r \in \mathbb{R}^n$ with $\|u^r\|_* = 1$ be such that $(u^r)' x^r = \|x^r\|$ and let $c^r := \overline{c} - \left\|b^r - \overline{b}\right\|_{\infty} u^r$. Similarly to Remark 6, we have $\pi^r \equiv (c^r, b^r) \in \mathrm{dom}\mathcal{F}^{op}$ for r large enough (say for all r). Clearly $\|\pi^r - \overline{\pi}\| = \left\|b^r - \overline{b}\right\|_{\infty}$. Choose for each r any $\widehat{x}^r \in \mathcal{F}^{op}(\pi^r)$, which, for r large enough (say again for each r) satisfies $\widehat{x}^r \in \mathcal{F}^{op}(\overline{c}, b^r)$ by virtue of Lemma 4. Observe that

$$(u^r)' \widehat{x}^r \leq \|u^r\|_* \|\widehat{x}^r\| = \|\widehat{x}^r\| \leq \|x^r\| = (u^r)' x^r,$$

due to the choice of x^r and u^r. Consequently,

$$\begin{aligned}(c^r)' x^r &= \overline{c}' x^r - \left\|b^r - \overline{b}\right\|_{\infty} (u^r)' x^r \\ &\leq \overline{c}' \widehat{x}^r - \left\|b^r - \overline{b}\right\|_{\infty} (u^r)' \widehat{x}^r = (c^r)' \widehat{x}^r = \vartheta(\pi^r).\end{aligned}$$

In other words, $x^r \in \mathcal{F}^{op}(\pi^r)$. Thus,

$$\begin{aligned}\underline{\mathrm{clm}}\vartheta^{R}(\overline{\pi}) &\geq \limsup_r \frac{\vartheta(\overline{\pi}) - \vartheta(\pi^r)}{\|\pi^r - \overline{\pi}\|} \\ &= \lim_r \frac{\vartheta(\overline{\pi}) - \vartheta(\overline{c}, b^r)}{\left\|b^r - \overline{b}\right\|_{\infty}} + \lim_r \frac{\overline{c}' x^r - (c^r)' x^r}{\left\|b^r - \overline{b}\right\|_{\infty}} \\ &= \underline{\mathrm{clm}}\vartheta^{R}_{\overline{c}}(\overline{b}) + \lim_r \|x^r\| \\ &= \underline{\mathrm{clm}}\vartheta^{R}_{\overline{c}}(\overline{b}) + e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right).\end{aligned}$$

■

5 Conclusions

First, we summarize in Table 1 the main contributions of the present work in relation to the calmness moduli from below and above of the optimal value functions ϑ^R and $\vartheta^R_{\overline{c}}$, where $\vartheta^R : \mathrm{dom}\mathcal{F}^{op} \to \left]-\infty, +\infty\right[$ is the restriction of ϑ to the set of solvable (equivalently, bounded) problems and $\vartheta^R_{\overline{c}} : \mathrm{dom}\mathcal{F} \to \left]-\infty, +\infty\right[$ is the optimal value function depending only on parameter b. Recall that

$$k^- := \min_{D \in \mathcal{M}_{\overline{\pi}}} \left\|\lambda^D\right\|_1 \text{ and } k^+ := \max_{D \in \mathcal{M}_{\overline{\pi}}} \left\|\lambda^D\right\|_1,$$

where $\mathcal{M}_{\overline{\pi}}$ is the set of minimal KKT subsets of indices at $\overline{\pi}$.

Table 1: Summary of results

	Calmness from below	Calmness from above
Perturbing b	$\underline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) = k^-$	$\overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) \underset{(1)}{\leq} k^+$
Perturbing b and c	$\underline{\mathrm{clm}}\vartheta^R(\overline{\pi})$ $\underset{(2)}{\leq} k^- + e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$	$\overline{\mathrm{clm}}\vartheta^R(\overline{\pi})$ $= \overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right)$ $\leq k^+ + d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right)$

(with $\mathcal{E}^{op}(\overline{\pi})$ being defined in (10)).

So, to start with, we observe that $\vartheta^R_{\overline{c}}$ and ϑ^R are always calm from below and above, and hence calm. Moreover, by combining the previous results in the table, we have

$$k^- \leq \mathrm{clm}\vartheta^R_{\overline{c}}(\overline{b}) \leq k^+$$

and

$$d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right) \leq \mathrm{clm}\vartheta^R(\overline{\pi}) \leq \max\{k^- + e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right),\ k^+ + d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right)\}.$$

Secondly, we comment that inequalities (1) and (2) in Table 1 might be strict as examples 1 and 2 show. Going further, the paper shows that (1) is held as an equality under SCQ, while (2) becomes an equality when $\mathcal{F}^{op}(\overline{\pi})$ is bounded.

Consequently, under these two conditions (SCQ at $\bar{b}$ together with the boundedness of $\mathcal{F}^{op}(\overline{\pi})$), equivalently, when $\overline{\pi} \in \operatorname{int}\operatorname{dom}\mathcal{F}^{op}$, we have the exact formulae for all moduli, which are gathered in the following theorem. In it, we have also taken into account the fact that $\overline{\pi} \in \operatorname{int}\operatorname{dom}\mathcal{F}^{op}$ turns out to be equivalent to the simultaneous nonemptiness and boundedness of both nominal optimal sets $\mathcal{F}^{op}(\overline{\pi})$ and $\Lambda^{op}(\overline{\pi})$; indeed, for $\overline{\pi} \in \operatorname{dom}\mathcal{F}^{op}$, the boundedness of $\Lambda^{op}(\overline{\pi})$ is equivalent to SCQ (see [19, Th. 6.1(v)]). In this case we can write $k^{+} = \max_{\lambda \in \operatorname{extr}\Lambda^{op}(\overline{\pi})} \|\lambda\|_1 = \max_{\lambda \in \Lambda^{op}(\overline{\pi})} \|\lambda\|_1$. Moreover, one always has $k^{-} = \min_{\lambda \in \operatorname{extr}\Lambda^{op}(\overline{\pi})} \|\lambda\|_1 = \min_{\lambda \in \Lambda^{op}(\overline{\pi})} \|\lambda\|_1$, because of the linearity of $\|\cdot\|_1$ on Λ. Recall also that the boundedness of $\mathcal{F}^{op}(\overline{\pi})$ entails $\mathcal{E}^{op}(\overline{\pi}) = \operatorname{extr}\mathcal{F}^{op}(\overline{\pi})$ and $e(\operatorname{extr}\mathcal{F}^{op}(\overline{\pi}), 0_n) = e(\mathcal{F}^{op}(\overline{\pi}), 0_n) = \max_{x \in \mathcal{F}^{op}(\overline{\pi})} \|x\|$. So, according to these comments, the results in Table 1 give rise to the following theorem.

Theorem 8 *Let $\overline{\pi} \in \operatorname{int}\operatorname{dom}\mathcal{F}^{op}$. Then, we have*

$$
\begin{aligned}
\operatorname{clm}\vartheta(\overline{\pi}) &= \operatorname{clm}\vartheta^{R}(\overline{\pi}) \\
&= \max\{k^{-} + e(\mathcal{F}^{op}(\overline{\pi}), 0_n),\ k^{+} + d(0_n, \mathcal{F}^{op}(\overline{\pi}))\} \\
&= \max\left\{ \min_{\lambda \in \Lambda^{op}(\overline{\pi})} \|\lambda\|_1 + \max_{x \in \mathcal{F}^{op}(\overline{\pi})} \|x\|,\ \max_{\lambda \in \Lambda^{op}(\overline{\pi})} \|\lambda\|_1 + \min_{x \in \mathcal{F}^{op}(\overline{\pi})} \|x\| \right\}.
\end{aligned}
$$

Looking again at the previous summary of results in Table 1, we immediately derive the following corollary under the uniqueness of optimal solution. It is stated without the SCQ assumption. In fact, if we have both, $\mathcal{F}^{op}(\overline{\pi}) = \{\overline{x}\}$ and SCQ at $\overline{\pi}$, then we additionally have an exact formulae for $\overline{\operatorname{clm}}\vartheta^{R}_{\overline{c}}(\bar{b})$ $(= k^{+})$.

Corollary 4 *Let $\overline{\pi} \in \operatorname{dom}\mathcal{F}^{op}$ with $\mathcal{F}^{op}(\overline{\pi}) = \{\overline{x}\}$. Then*

$$
\underline{\operatorname{clm}}\vartheta^{R}(\overline{\pi}) = \underline{\operatorname{clm}}\vartheta^{R}_{\overline{c}}(\bar{b}) + \|\overline{x}\| \ \ \textit{and} \ \ \overline{\operatorname{clm}}\vartheta^{R}(\overline{\pi}) = \overline{\operatorname{clm}}\vartheta^{R}_{\overline{c}}(\bar{b}) + \|\overline{x}\|.
$$

Consequently,

$$
\operatorname{clm}\vartheta^{R}(\overline{\pi}) = \operatorname{clm}\vartheta^{R}_{\overline{c}}(\bar{b}) + \|\overline{x}\|.
$$

5.1 Calmness modulus and distance to infeasibility

To finish this work, we analyze the relationship between $\operatorname{clm}\vartheta(\overline{\pi})$ and the well studied concept of distance to infeasibility; the reader is addressed to [33, 34] for details on this distance in the context of conic linear systems and to [7] (where it is called *distance to ill-posedness)* in the linear semi-infinite setting. Specifically, [33, Theorem 1.1] provides a certain Lipschitz type inequality for ϑ which immediately yields an upper bound on $\text{clm}\vartheta(\overline{\pi})$, provided that $\overline{\pi} \in \text{dom}\mathcal{F}^{op}$. This upper bound has the distance to infeasibiltiby in the denominator. Let us recall some details: paper [33] deals with conic linear problems in the form

$$\begin{array}{rl} \text{Inf} & c^* x \\ \text{s.t.} & b - Ax \in C_Y, \\ & x \in C_X, \end{array} \tag{22}$$

where $C_X \subset X$ and $C_Y \subset Y$ are closed convex cones in X and Y, respectively. X is a reflexive normed space while Y is an arbitrary normed space. The norms in both spaces are denoted by $\|\cdot\|$. Here $b \in Y$, $A : X \to Y$ is a (continuous) linear operator, with norm $\|A\| := \sup\{\|Ax\| \mid \|x\| = 1\}$, and $c^* : X \to \mathbb{R}$ is an element of the dual space of X, i.e., a continuous linear functional, with $\|c^*\| := \sup\{c^* x \mid \|x\| = 1\}$. The parameter space of all problems (22) is endowed with the product norm

$$\|(A, b, c^*)\| := \max\{\|A\|, \|b\|, \|c^*\|\}.$$

Our parametrized problem (1) may be translated into the conic format, just by taking $X = C_X := \mathbb{R}^n$, $Y = \mathbb{R}^T$, $C_Y := \mathbb{R}^T_+$, and considering a fixed matrix A (which remains unperturbed). In this way, the results of [33] apply into our LP context, where we are considering $\|\cdot\|_\infty$ for measuring the perturbations of b (indeed, the reader is addressed to [9] for details about the convenience of this norm when dealing with polyhedral cones).

Following the notation of [33], we consider

$$Pri\emptyset := \mathbb{R}^T \setminus \text{dom}\mathcal{F} \text{ and } Dual\emptyset := \mathbb{R}^n \setminus \text{dom}\Lambda,$$

corresponding, respectively, to the set of parameter b and c associated with primal and dual inconsistent (infeasible) problems. In this way,

$$d(b, Pri\emptyset) := \inf\{\|b - b^1\| \mid b^1 \in Pri\emptyset\}$$

represents the distance from $b \in \mathbb{R}^T$ to primal infeasibility, while $d(c, Dual\emptyset)$, analogously defined, denotes the corresponding distance to dual infeasibility. Observe that

$$\pi = (c, b) \in \text{int}\,\text{dom}\mathcal{F}^{op} \Leftrightarrow \min\{d(b, Pri\emptyset), d(c, Dual\emptyset)\} > 0.$$

Theorem 9 (See [33, Theorem 1.1(5)]) *Let* $\overline{\pi} \in \operatorname{int}\operatorname{dom}\mathcal{F}^{op}$. *Then, for any* $\pi = (c, b) \in \mathbb{R}^n \times \mathbb{R}^T$ *such that*

$$\left\|b - \overline{b}\right\| < d\left(\overline{b}, Pri\emptyset\right) \text{ and } \|c - \overline{c}\| < d\left(\overline{c}, Dual\emptyset\right),$$

we have

$$\begin{aligned} |\vartheta(\pi) - \vartheta(\overline{\pi})| \quad \leq \quad & \left\|b - \overline{b}\right\| \frac{\|\overline{c}\| + \|c - \overline{c}\|}{d\left(\overline{b}, Pri\emptyset\right) - \|\pi - \overline{\pi}\|} \; \frac{\|\overline{\pi}\|}{d\left(\overline{c}, Dual\emptyset\right)} \\ & + \|c - \overline{c}\| \frac{\left\|\overline{b}\right\| + \left\|b - \overline{b}\right\|}{d\left(\overline{c}, Dual\emptyset\right) - \|\pi - \overline{\pi}\|} \; \frac{\|\overline{\pi}\|}{d\left(\overline{b}, Pri\emptyset\right)}. \end{aligned} \tag{23}$$

Now if we divide both members of (23) by $\|\pi - \overline{\pi}\|$ and let $\|\pi - \overline{\pi}\| \to 0$, we inmediately derive the following Corollary.

Corollary 5 *Let* $\overline{\pi} \in \operatorname{int}\operatorname{dom}\mathcal{F}^{op}$. *One has*

$$\operatorname{clm}\vartheta(\overline{\pi}) \leq \frac{\|\overline{c}\|}{d\left(\overline{b}, Pri\emptyset\right)} \frac{\|\overline{\pi}\|}{d\left(\overline{c}, Dual\emptyset\right)} + \frac{\left\|\overline{b}\right\|}{d\left(\overline{c}, Dual\emptyset\right)} \frac{\|\overline{\pi}\|}{d\left(\overline{b}, Pri\emptyset\right)}. \tag{24}$$

Remark 7 Observe that Theorem 8 constitutes a refinement of Corollary 5, as far as inequality (24) can be strict. In fact, its right-hand side (upper bound on $\operatorname{clm}\vartheta(\overline{\pi})$) might be much greater than $\operatorname{clm}\vartheta(\overline{\pi})$ when $\overline{\pi}$ approaches the primal/dual infeasibility. Just consider the example, in $\mathbb{R}^2$ endowed with the Euclidean norm,

$$\pi^r : \text{minimize } x_1 + \frac{1}{r} x_2 \text{ s.t. } -x_1 \leq 0, \; -x_2 \leq \frac{1}{r}, x_2 \leq \frac{1}{r}.$$

One easily sees that $b^r \to 0_3$, $d\left(b^r, Pri\emptyset\right) \to 0$, and so the right-hand side of (24) goes to $+\infty$, while (appealing to Theorem 8)

$$\operatorname{clm}\vartheta(\pi^r) = \left\|\left(1, \frac{1}{r}, 0\right)'\right\|_1 + \frac{1}{r} \to 1.$$

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Lipschitz modulus of the optimal value in linear programming

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Lipschitz modulus of the optimal value in linear programming*

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Abstract

The present paper is devoted to the computation of the Lipschitz modulus of the optimal value function restricted to its domain in linear programming (LP for short) under different type of perturbations. In a first stage, we study separately peturbations of the right-hand side (RHS in brief) of the constraints and perturbations of the coefficients of the objective function. Secondly, we deal with canonical perturbations, i.e., RHS perturbations together with linear perturbations of the objective. We advance that an exact formula for the Lipschitz modulus in the context of RHS perturbations is provided, and lower and upper estimates for the corresponding moduli are also established in the other two perturbation frameworks. In both cases, the corresponding upper estimates are shown to provide the exact moduli when the nominal (original) optimal set is bounded. A key strategy here consists in taking advantage of the background on calmness in LP and providing the aimed Lipschitz modulus through the computation of a uniform calmness constant.

Keywords: Lipschitz modulus, Optimal Value, Linear programming, Variational Analysis, Calmness
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1 Introduction

This paper deals with the Lipschitz continuity of the optimal value in LP. Specifically, we consider the optimal value function restricted to its domain

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(where the value is finite), denoted by ϑ^R, and analyze its behavior around a fixed (referred to as nominal) LP problem. Along this work different type of perturbations of the nominal problem are considered and, in each of these perturbation frameworks, our goal is to compute (or at least estimate) the *Lipschitz modulus* of the corresponding optimal value (see Sections 2.1 and 2.2 for the formal definitions). Roughly speaking, this Lipschitz modulus provides a local measure of the greatest rate of variation of the optimal value with respect to data perturbation. In this sense, the present research is focussed on a local aspect of the *sensitivity analysis* in LP, in contrast to the classical theory of parametric linear optimization (see, e.g., [1] and [2]).

First, we consider the case of RHS perturbations of the constraints, where a formula for the exact Lipschitz modulus of ϑ^R at a nominal problem is obtained. Secondly, we deal with linear perturbations of the objective function (c-perturbations, for simplicity). After that, we tackle the problem of computing the Lipschitz modulus of ϑ^R in the setting of the so-called *canonical perturbations*, i.e., RHS perturbations together with c-perturbations. In the last two settings, lower and upper estimates for the aimed moduli are derived. In both cases the upper estimates turn out to be the exact moduli when the nominal optimal set is bounded (and nonempty).

The systematic study of stability in LP with canonical perturbations started in the 1970s. Specifically, the continuity of ϑ^R was proved through different approaches (see [3, 4, 5, 6]). One can find a second line of research based on variational analysis like Berge's theory or Hoffman's error bounds; see [4, 7, 8, 9, 10, 11, 12, 6].

The immediate antecedents of this work can be traced out from [13] and [14]. The first one, instead of ϑ^R, deals with the optimal value function, ϑ, defined on the whole space (and, so, taking values in the extended real line). As a counterpart, the local study is made around a problem which is in the interior of the domain of ϑ. This interiority condition characterizes the Lipschitz continuity of ϑ at such a problem (this fact is held in the more general setting of linear semi-infinite optimization; see [15, Lemma 10.2]) and it is equivalent to the well-known Slater constraint qualification together with the boundedness (and nonemptiness) of the nominal optimal set. Specifically, [13, Theorem 4.3] provides a formula for a particular Lipschitz constant for ϑ in terms of the so-called *distance to ill-posedness* (see also the pioneer works [16] and [17], developed in the context of conic linear problems). Let us point out that the new results of the current paper constitute an improvement of [13, Theorem 4.3] in different directions: first, here we do not require any interiority assumption; moreover, the Lipschitz modulus provides –roughly speaking– the more accurate Lipschitz constant; and, finally, we also tackle the case of partial perturbations (RHS or c-perturbations).

Paper [14] is focussed on the calmness of ϑ^R, which is known to be a weaker property than Lipschitz continuity. In that paper, the calmness of ϑ^R is approached through the *calmness from above* and *calmness from below*, which roughly speaking, measure the local rate of increase and decrease, respectively, with respect to the nominal problem. While calmness property compares the nominal optimal value with the optimal value of a perturbed problem, Lipschitz property involves the optimal values of two different perturbed problems around the nominal one. This fact entails notable differences between both properties and their moduli. In particular, the new contributions of the current paper are not direct consequences of the ones of [14], as we shall emphasize in the corresponding proofs. In any case, we take advantage of the background on calmness. In particular, a key strategy (inspired by [18, Sect. 2]) based on computing the aimed Lipschitz modulus through a uniform calmness constant is appealed to.

Finally, let us comment that both, calmness and Lipschitz properties have extensions for multifunctions, closely related to metric regularity notions, which are important concepts in the field of variational analysis; see the monographs [19, 20, 21, 22] for additional references and details. The analysis of pseudo-Lipschitz (Aubin) property for the particular case of the *argmin mapping* (resp. the *feasible set mapping*) has been addressed in [23, 24] (resp. [25]).

The structure of the paper is as follows. Section 2 introduces the model we are dealing with, the main goals of this work, as well as the necessary notation and preliminary results on calmness (from [14]) which are used later on. Section 3 is devoted to the study of the Lipschitz modulus of ϑ^R under RHS perturbations. The main result of this section is Theorem 4. Section 4 is developed in the context of c-perturbations, and mainly consists of Theorem 5, where the announced lower and upper estimates (exact value when the nominal optimal set is nonempty and bounded) for the aimed modulus are provided. Section 5 deals with canonical perturbations. Theorem 6 provides a lower estimate of the corresponding Lipschitz modulus, while Theorem 7 provides an upper estimate based on a certain uniform calmness constants which is established in Lemma 5. The last theorem also provides the exact Lipschitz modulus under the boundedness (and nonemptiness) of the nominal optimal set. Finally, Section 6 gathers some conclusions.

2 Preliminaries and main goals

This section is devoted to formalize the main goal of the paper and to connect it with the immediate antecedents. The section is divided into three subsections: first, we introduce the parameterized optimization model and the mappings which are dealt with in the paper; secondly, we make precise the main goal of this work, which consists in computing (or estimating) the Lipschitz modulus of the optimal value function under different type of perturbations. The third subsection gathers some results about calmness of the same function traced out from [14].

2.1 The parameterized model

We consider a parameterized linear optimization problem, in $\mathbb{R}^n$, given in the form

$$\begin{array}{lll} \pi \ : & \text{minimize} & c'x \\ & \text{subject to} & \overline{a}_t'x \leq b_t, \quad t \in T := \{1, 2, \ldots, m\}, \end{array} \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, $\overline{a} \equiv (\overline{a}_t)_{t \in T} \in (\mathbb{R}^n)^T$ is fixed, $c \in \mathbb{R}^n$ and $b \equiv (b_t)_{t \in T} \in \mathbb{R}^T$. Any $z \in \mathbb{R}^n$ is considered as a column vector and z' denotes its transpose. Our problem π is identified with the pair $(c, b) \in \mathbb{R}^n \times \mathbb{R}^T$, which constitutes our parameter to be perturbed. So, as mentioned above, we are working in the setting of the so-called *canonical perturbations.*

The space of variables, $\mathbb{R}^n$, is endowed with an arbitrary norm, $\|\cdot\|$, while the parameter space $\mathbb{R}^n \times \mathbb{R}^T$ is endowed with the norm

$$\|\pi\| := \max\left\{\|c\|_*, \|b\|_\infty\right\}, \ \pi \equiv (c, b) \in \mathbb{R}^n \times \mathbb{R}^T,$$

where $\|u\|_* := \max_{\|x\| \leq 1} |u'x|$, and $\|b\|_\infty := \max_{t \in T} |b_t|$. Observe that, in relation to vector c of the objective function, we use the dual norm since it is seen as a functional.

Along the paper we deal with the following mappings: The *feasible set mapping,* $\mathcal{F} : \mathbb{R}^T \rightrightarrows \mathbb{R}^n$, defined as

$$\mathcal{F}(b) := \{x \in \mathbb{R}^n \mid \overline{a}_t'x \leq b_t, \ t \in T\}, \ b \in \mathbb{R}^T;$$

the *optimal value function*, $\vartheta : \mathbb{R}^n \times \mathbb{R}^T \to [-\infty, +\infty]$, given by

$$\vartheta(\pi) := \inf\{c'x \mid x \in \mathcal{F}(b)\},$$

(with the convention $\vartheta(\pi) := +\infty$ when $\mathcal{F}(b) = \emptyset$); and the *optimal set mapping*, $\mathcal{F}^{op} : \mathbb{R}^n \times \mathbb{R}^T \rightrightarrows \mathbb{R}^n$, which assigns to each problem $\pi \equiv (c, b)$ its optimal set

$$\mathcal{F}^{op}(\pi) := \{x \in \mathcal{F}(b) \mid c'x = \vartheta(\pi)\}.$$

The domain of $\mathcal{F}$, denoted by dom$\mathcal{F}$, is formed by all $b \in \mathbb{R}^T$ whose associated linear inequality systems are consistent; formally,

$$\mathrm{dom}\mathcal{F} := \left\{b \in \mathbb{R}^T \mid \mathcal{F}(b) \neq \emptyset\right\}.$$

Analogously, the domain of $\mathcal{F}^{op}$, dom$\mathcal{F}^{op}$, is formed by all problems $\pi \equiv (c, b) \in \mathbb{R}^n \times \mathbb{R}^T$ having a nonempty optimal set. It is known from standard arguments in LP that dom$\mathcal{F}^{op}$ coincides with the domain of ϑ. It is also known that both dom$\mathcal{F} \subset \mathbb{R}^T$ and dom$\mathcal{F}^{op} \subset \mathbb{R}^n \times \mathbb{R}^T$ are closed and convex sets.

This paper mainly deals with the *optimal value function restricted to its domain*, $\vartheta^R : \mathrm{dom}\mathcal{F}^{op} \to \left]-\infty, +\infty\right[$, i.e.,

$$\vartheta^R := \vartheta|_{\mathrm{dom}\mathcal{F}^{op}},$$

and two other functions coming from considering perturbations of b and c independently. Specifically, given a nominal (fixed) $\overline{\pi} \equiv \left(\overline{c}, \overline{b}\right) \in \mathrm{dom}\mathcal{F}^{op}$ we define

$$\vartheta^R_{\overline{c}} : \mathrm{dom}\mathcal{F} \to]-\infty, +\infty[\text{ and } \vartheta^R_{\overline{b}} : C \to]-\infty, +\infty[,$$

with

$$C = -\mathrm{cone}\{\overline{a}_t, t \in T\}, \tag{2}$$

(where 'cone' means conical convex hull) given, respectively, by

$$\vartheta^R_{\overline{c}}(b) = \vartheta^R(\overline{c}, b) \text{ and } \vartheta^R_{\overline{b}}(c) = \vartheta^R(c, \overline{b}).$$

Observe that the previous two functions are finite valued since we are not perturbing $\overline{a}$, which entails that $\{\overline{c}\} \times \mathrm{dom}\mathcal{F}$ and $C \times \{\overline{b}\}$ are both included in dom$\mathcal{F}^{op}$ (recall that, in LP, optimality is equivalent to primal and dual feasibility).

One can find different proofs (from different approaches) for the next theorem; see, e.g., [4, Theorem 4.5.2], [5, Theorem 2.7], and [6, Theorem 14]); see also [11, p. 214], for a stronger version (ϑ^R is Lipschitz on bounded subsets of dom$\mathcal{F}^{op}$) in the more general context of canonically perturbed convex quadratic problems; see also [6, p. 25], and [9] for (generally non-convex) quadratic programs.

Theorem 1 *ϑ^R is continuous on* $\text{dom}\mathcal{F}^{op}$.

Finally, the following theorem is a well-known result of stability theory in LP (see, e.g. [26, p. 232] or [8, Chapter IX (Section 7)]). In it, we appeal to the Painlevé-Kuratowski convergence of sequences of sets. More in detail, given $X_r \subset \mathbb{R}^n$, $r \in \mathbb{N}$, $\text{Lim}\inf_r X_r$ consists of all points which may be written as $\lim_r x^r$ with $x^r \in X_r$ for r large enough; whereas elements of $\text{Lim}\sup_r X_r$ are those of the form $\lim_k x^k$ with $x^k \in X_{r_k}$ for some subsequence $r_1 < r_2 < ...$ Obviously $\text{Lim}\inf_r X_r \subset \text{Lim}\sup_r X_r$, and when both coincide we just write $\text{Lim}_r X_r$.

Theorem 2 *Let $\overline{c} \in C$. For any $\{b^r\}_{r\in\mathbb{N}} \subset \text{dom}\mathcal{F}$ converging to $\overline{b}$ we have*

$$\mathcal{F}^{op}(\overline{\pi}) = \text{Lim}_r\, \mathcal{F}^{op}\left(\overline{c}, b^r\right).$$

Remark 1 In general, the boundedness of a Painlevé-Kuratowski limit of sets does not imply the boundedness of those sets. For instance, $\text{Lim}_r\{1, r\} = \{1\}$. Nevertheless, in the previous theorem the boundedness of $\mathcal{F}^{op}(\overline{\pi})$ does imply the uniform boundedness of $\{\mathcal{F}^{op}\left(\overline{c}, b^r\right)\}_{r\in\mathbb{N}}$. This follows from the convexity of each $\mathcal{F}^{op}\left(\overline{c}, b^r\right)$ or, alternatively, from [15, Corollary 6.2.1] together with Theorem 1.

2.2 Main goals

This subsection is devoted to formalize the main goals of the current work, and to integrate them in the existing literature. At this moment, we advance that our aim is focussed on the Lipschitzian behavior of the optimal value function in different frameworks of perturbations; specifically, on the Lipschitzian behavior of ϑ, ϑ^R, $\vartheta^R_{\overline{c}}$, and $\vartheta^R_{\overline{b}}$.

Recall that a function $f : A \subset \mathbb{R}^p \to [-\infty, +\infty]$, $p \in \mathbb{N}$, is said to be *Lipschitz continuous* at $\overline{z} \in A$, with $f(\overline{z})$ finite, if there exist a constant $\kappa \geq 0$ (called *Lipschitz constant*) and a neighborhood U of $\overline{z}$ such that

$$|f(z) - f(\widetilde{z})| \leq \kappa \left\|z - \widetilde{z}\right\|, \text{ for all } z, \widetilde{z} \in U \cap A. \tag{3}$$

The infimum of constants κ for which (3) holds, for some associated neighborhood, is the *Lipschitz modulus* of f at $\overline{z}$, denoted by $\text{lip}f(\overline{z})$. Observe that the Lipschitz modulus can be expressed as

$$\text{lip}f(\overline{z}) = \limsup_{\substack{z,\widetilde{z}\to\overline{z} \\ z,\widetilde{z}\in A}} \frac{|f(z) - f(\widetilde{z})|}{\|z - \widetilde{z}\|}. \tag{4}$$

(In the previous expressions, we do not exclude coincidences among z, $\widetilde{z}$, and $\overline{z}$, under the convention $\frac{0}{0} := 0$ and $\infty - \infty := 0$.)

In relation to the optimal value function, it is well-known that ϑ is Lipschitz continuous at $\overline{\pi} \equiv (\overline{c}, \overline{b})$ if and only if $\overline{\pi} \in \text{int dom}\mathcal{F}^{op}$ (the interior of dom$\mathcal{F}^{op}$). In fact, as commented above, this characterization is held in the more general framework of linear semi-infinite systems (with –possibly– infinitely many inequalities); see, [15, Lemma 10.2]. Moreover, it is also known (see, e.g., [15, Theorem 6.1 and Lemma 10.2]) that condition '$\overline{\pi} \in \text{int dom}\mathcal{F}^{op}$' is equivalent to the simultaneous fulfilment of two conditions: $\mathcal{F}^{op}(\overline{\pi})$ is nonempty and bounded, and the *Slater constraint qualification* (SCQ, in brief) is satisfied at $\overline{b}$. Recall that SCQ is satisfied at $\overline{b}$ if there exists $\widehat{x} \in \mathbb{R}^n$, called a *Slater point*, such that $\overline{a}_t' \widehat{x} < \overline{b}_t$ for all $t \in T$.

Remark 2 Observe that, in the case when $\overline{\pi} \in \text{int dom}\mathcal{F}^{op}$ one clearly has $\text{lip}\vartheta(\overline{\pi}) = \text{lip}\vartheta^R(\overline{\pi})$. On the other hand, if $\overline{\pi} \in \text{bd dom}\mathcal{F}^{op}$ (bd standing for boundary), one has $\text{lip}\vartheta(\overline{\pi}) = +\infty$, whereas $\text{lip}\vartheta^R(\overline{\pi})$ is still finite, as it is shown in the current work (as a consequence of Theorem 7).

The previous remark motivates that we focus this paper on computing (or at least estimating) $\text{lip}\vartheta^R(\overline{\pi})$. For solvable problems $\text{lip}\vartheta^R(\overline{\pi})$ is always finite and provides a quantitative measure of the stability of the optimal value under data perturbations (provided that they yield solvable problems).

We advance that $\text{lip}\vartheta^R_{\overline{c}}(\overline{b})$ is completely determined through a point-based formula (depending only on the nominal data) without any assumption (see Theorem 4), while $\text{lip}\vartheta^R_{\overline{b}}(\overline{c})$ and $\text{lip}\vartheta^R(\overline{\pi})$ are upper and lower estimated in general (see Theorems 5, 6, and 7). It is also shown that under the boundedness of $\mathcal{F}^{op}(\overline{\pi})$, both $\text{lip}\vartheta^R_{\overline{b}}(\overline{c})$ and $\text{lip}\vartheta^R(\overline{\pi})$ are also completely determined. All the mentioned estimates (or exact values) are given exclusively in terms of $\overline{\pi} \equiv (\overline{c}, \overline{b})$.

2.3 Antecedents on calmness

This subsection mainly gathers some results about the calmness of ϑ^R, traced out from [14], which are used in the remaining sections.

Recall that the calmness property is weaker than the Lipschitz one, as far as it comes from fixing $\widetilde{z} = \overline{z}$ in (3). With the notation before (3), the *calmness modulus* of f at $\overline{z}$ is given by

$$\text{clm}f(\overline{z}) = \limsup_{z \to \overline{z},\ z \in A} \frac{|f(z) - f(\overline{z})|}{\|z - \overline{z}\|}.$$

Obviously, $\text{clm}f(\overline{z}) \leq \text{lip}f(\overline{z})$.

At this moment we introduce some necessary notation used along the paper. To start with, given $X \subset \mathbb{R}^p$, $p \in \mathbb{N}$, we denote by conv$X$, span$X$, and extr$X$ the *convex hull*, the *linear hull* of X, and the set of extreme points of X, respectively. Recall that cone$X$ stands for the *conical convex hull* of X.

For $b \in \text{dom}\mathcal{F}$ and $x \in \mathcal{F}(b)$, we denote by $T_b(x)$ the *set of active indices* at x; i.e.,

$$T_b(x) := \left\{ t \in T \mid \overline{a}_t' x = b_t \right\}.$$

Associated with $\pi \equiv (c, b) \in \text{dom}\mathcal{F}^{op}$ we consider the following *family of minimal Karush-Kunh-Tucker (KKT) subsets of indices*

$$\mathcal{M}_\pi := \left\{ D \subset T_b(x) \;\middle|\; \begin{array}{c} -c \in \text{cone}\{\overline{a}_t,\ t \in D\}, \\ D \text{ is minimal for the inclusion order} \end{array} \right\}, \tag{5}$$

for some $x \in \text{dom}\mathcal{F}^{op}$. Observe that $\mathcal{M}_\pi$ is correctly defined since the right member of (5) indeed does not depend on x (this comes from a standard fact in LP; see [14, Remark 2]). It is also standard that $\{\overline{a}_t, t \in D\}$ is linearly independent for any $D \in \mathcal{M}_\pi$, and this fact justifies the well-definedness of the following elements associated with our nominal problem $\overline{\pi} \equiv \left(\overline{c}, \overline{b}\right) \in \text{dom}\mathcal{F}^{op}$, which were already introduced in [14]:

$$k^- := \min_{D \in \mathcal{M}_{\overline{\pi}}} \left\| \lambda^D \right\|_1 \text{ and } k^+ := \max_{D \in \mathcal{M}_{\overline{\pi}}} \left\| \lambda^D \right\|_1, \tag{6}$$

where, for $D \in \mathcal{M}_{\overline{\pi}}$, $\lambda^D = \left(\lambda_t^D\right)_{t \in T} \in \mathbb{R}_+^T$ is the unique element such that $-\overline{c} = \sum_{t \in D} \lambda_t^D \overline{a}_t$ and $\lambda_t^D = 0$ for all $t \in T \setminus D$.

Paper [14] analyzes the calmness modulus of the optimal value function under right-hand-side perturbations, $\text{clm}\vartheta_{\overline{c}}^R(\overline{b})$, as well as the calmness modulus under canonical perturbations, $\text{clm}\vartheta^R(\overline{\pi})$. In that paper, each of the moduli is studied by splitting it into the so-called calmness from above and calmness from below moduli. The reader is addressed to [14] for details, since these concepts do not have their counterpart for the Lipschitz modulus. Nevertheless, we need some tools from that paper.

Recall (see, e.g., [27, p. 65]) that any non-empty convex set F can be decomposed as the direct sum

$$F = L_F + \left(F \cap L_F^\perp\right),$$

where L_F is the lineality space of F and $L_F^\perp$ is the orthogonal complement of L_F. In our case, when either $F = \mathcal{F}(b)$ for $b \in \text{dom}\mathcal{F}$ or $F = \mathcal{F}^{op}(\pi)$ for

$\pi \in \mathrm{dom}\mathcal{F}^{op}$, one has that $L_F^{\perp} = \mathrm{span}\{\overline{a}_t, t \in T\}$. In [14, Section 2.2] we appeal to the following set of extreme points:

$$\mathcal{E}^{op}(\pi) := \mathrm{extr}\left(\mathcal{F}^{op}(\pi) \cap \mathrm{span}\{\overline{a}_t,\ t \in T\}\right),\ \pi \in \mathrm{dom}\mathcal{F}^{op}, \tag{7}$$

which is clearly nonempty and finite.

The following lemmas will be used later on. The first one comes from [14, Lemma 2] together with a standard argument of LP. Specifically, the uniform boundedness of the sequence $\{\mathcal{E}^{op}(\pi^r)\}_{r\in\mathbb{N}}$ comes from the fact that any point of $\mathcal{E}^{op}(\pi^r)$, $r \in \mathbb{N}$, is the unique solution of a Cramer's system. The proof of the second one can be directly extracted from the proof of [14, Theorem 5] (see equation (20) therein). The third comes from [14, Lemma 4].

Lemma 1 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$. For any $\{\pi^r\}_{r\in\mathbb{N}} \subset \mathrm{dom}\mathcal{F}^{op}$ converging to $\overline{\pi}$, we have that $\{\mathcal{E}^{op}(\pi^r)\}_{r\in\mathbb{N}}$ is uniformly bounded and*

$$\emptyset \neq \mathrm{Lim\,sup}_r\, \mathcal{E}^{op}(\pi^r) \subset \mathcal{E}^{op}(\overline{\pi}).$$

Lemma 2 *Let $\overline{\pi} \equiv (\overline{c}, \overline{b}) \in \mathrm{dom}\mathcal{F}^{op}$ and $\{\pi^r \equiv (c^r, b^r)\}_{r\in\mathbb{N}}$ be a sequence converging to $\overline{\pi}$, with $b^r \in \mathrm{dom}\mathcal{F}$ for all $r \in \mathbb{N}$. If $x \mapsto (c^r)'x$ is bounded from below on $\mathcal{F}^{op}(\overline{c}, b^r)$ for all r, then*

$$\pi^r \in \mathrm{dom}\mathcal{F}^{op}$$

for r large enough.

Lemma 3 *Let $\overline{\pi} \equiv (\overline{c}, \overline{b}) \in \mathrm{dom}\mathcal{F}^{op}$ and $\{\pi^r \equiv (c^r, b^r)\}_{r\in\mathbb{N}} \subset \mathrm{dom}\mathcal{F}^{op}$ be a sequence converging to $\overline{\pi}$. Then*

$$\mathcal{F}^{op}(\pi^r) \subset \mathcal{F}^{op}(\overline{c}, b^r)$$

for r large enough.

From now on $e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$ denotes the Hausdorff excess of $\mathcal{E}^{op}(\overline{\pi})$ over $\{0_n\}$, which may be written alternatively as $\max_{x\in\mathcal{E}^{op}(\overline{\pi})} \|x\|$. On the other hand, $d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right)$ represents the distance from the origin to the set $\mathcal{F}^{op}(\overline{\pi})$; i.e., $d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right) = \min_{x\in\mathcal{F}^{op}(\overline{\pi})} \|x\|$.

Theorem 3 [14, Theorem 4, Corollary 3, and Section 5] *Let $\overline{\pi} \equiv (\overline{c}, \overline{b}) \in \mathrm{dom}\mathcal{F}^{op}$. Then*

(i) $\mathrm{clm}\vartheta_{\overline{c}}^R(\overline{b}) \leq k^+$, and equality holds when SCQ is satisfied at $\overline{b}$.

(ii) $\mathrm{clm}\vartheta^R(\overline{\pi}) \leq \max\{k^- + e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right), k^+ + d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right)\}$, and equality holds when $\overline{\pi} \in \mathrm{int\ dom}\mathcal{F}^{op}$.

3 Lipschitz modulus under RHS perturbations

This section is devoted to compute the Lipschitz modulus of the optimal value under perturbations of b (RHS-perturbations); i.e., to compute $\mathrm{lip}\vartheta_{\overline{c}}^{R}(\overline{b})$. First, we recall a useful result which provides an explicit expression (as the maximum of a finite amount of linear functions) for the optimal value function in the current perturbation setting. Recall that we are considering a nominal problem $\overline{\pi} \equiv \left(\overline{c}, \overline{b}\right)$.

Lemma 4 [14, Lemma 3 and Corollary 1] *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$. There exists a neighborhood $U_{\overline{b}} \subset \mathbb{R}^T$ of $\overline{b}$ such that*

$$\vartheta(\overline{c}, b) = \max_{D \in \mathcal{M}_{\overline{\pi}}} -b'\lambda^D, \text{ for all } b \in \mathrm{dom}\mathcal{F} \cap U_{\overline{b}}.$$

Observe that, by the KKT conditions, with respect to the nominal problem we have

$$\vartheta(\overline{\pi}) = -\overline{b}'\lambda^D, \text{ for all } D \in \mathcal{M}_{\overline{\pi}}. \tag{8}$$

The next proposition follows an analogous argument to the one used for establishing [14, Corollary 2]. Nevertheless, due to its simplicity, and for completeness purposes, we include its proof. Along this section we use indistinctly $\vartheta_{\overline{c}}^{R}(b)$ or $\vartheta(\overline{c}, b)$, provided that $b \in \mathrm{dom}\mathcal{F}$. Indeed, for the sake of simplicity in the notation we usually write $\vartheta_{\overline{c}}^{R}$ when referring to the function itself and $\vartheta(\overline{c}, b)$, $b \in \mathrm{dom}\mathcal{F}$, for its images.

Proposition 1 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$ and let $U_{\overline{b}}$ be as in the previous lemma. Then,*

$$\left|\vartheta(\overline{c}, b) - \vartheta(\overline{c}, \widetilde{b})\right| \leq k^+ \|b - \widetilde{b}\|_\infty \text{ for all } b, \widetilde{b} \in \mathrm{dom}\mathcal{F} \cap U_{\overline{b}}.$$

Consequently

$$\mathrm{lip}\vartheta_{\overline{c}}^{R}(\overline{b}) \leq k^+.$$

Proof. Take $b, \widetilde{b} \in \mathrm{dom}\mathcal{F} \cap U_{\overline{b}}$. Applying the previous lemma we have

$$\vartheta(\overline{c}, b) - \vartheta(\overline{c}, \widetilde{b}) = \max_{D \in \mathcal{M}_{\overline{\pi}}} \left(-b'\lambda^D\right) - \max_{D \in \mathcal{M}_{\overline{\pi}}} \left(-\widetilde{b}'\lambda^D\right),$$

and let us assume the first maximum is reached at $\widehat{D} \in \mathcal{M}_{\overline{\pi}}$, then

$$\begin{aligned} \vartheta(\overline{c}, b) - \vartheta(\overline{c}, \widetilde{b}) &= -b'\lambda^{\widehat{D}} + \min_{D \in \mathcal{M}_{\overline{\pi}}} \widetilde{b}'\lambda^D \leq -b'\lambda^{\widehat{D}} + \widetilde{b}'\lambda^{\widehat{D}} \\ &= (\widetilde{b} - b)'\lambda^{\widehat{D}} \leq k^+ \|b - \widetilde{b}\|_\infty. \end{aligned}$$

Since b and $\widetilde{b}$ have been arbitrarily chosen, switching them in the preceding argument we obtain the aimed inequality. ∎

Theorem 4 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$. Then $\vartheta_{\overline{c}}^{R}$ is Lipschitz continuous at $\overline{b}$ and*

$$\mathrm{lip}\vartheta_{\overline{c}}^{R}(\overline{b}) = k^{+}. \tag{9}$$

Proof. According to the previous proposition, it remains to prove $\mathrm{lip}\vartheta_{\overline{c}}^{R}(\overline{b}) \geq k^{+}$. To do that take any $\overline{D} \in \mathcal{M}_{\overline{\pi}}$ such that $\left\|\lambda^{\overline{D}}\right\|_1 = k^{+}$ and let us construct two sequences $\{b^r\}, \{\widetilde{b}^r\} \subset \mathrm{dom}\mathcal{F}$ converging to $\overline{b}$ such that

$$\limsup_r \frac{\left|\vartheta\left(\overline{c}, b^r\right) - \vartheta\left(\overline{c}, \widetilde{b}^r\right)\right|}{\left\|b^r - \widetilde{b}^r\right\|_\infty} = \left\|\lambda^{\overline{D}}\right\|_1, \tag{10}$$

which will establish our aimed inequality.

Let $\overline{x} \in \mathcal{F}^{op}(\overline{\pi})$. Fix an arbitrary $r \in \mathbb{N}$. Observe that

$$W_r := \left\{x \in \mathbb{R}^n \mid \overline{a}_t' x < \overline{b}_t + \frac{1}{r},\ t \in T \backslash \overline{D}\right\}$$

is a neighborhood of $\overline{x}$. Now, since $\overline{a}_t'\overline{x} = \overline{b}_t,\ t \in \overline{D}$, and $\left\{\overline{a}_t,\ t \in \overline{D}\right\}$ is linearly independent, a standard argument in LP yields the existence of $0 < \delta_r < \frac{1}{r}$ such that the systems of linear equations

$$\left\{\overline{a}_t' x = \overline{b}_t - \delta_r,\ t \in \overline{D}\right\} \text{ and } \left\{\overline{a}_t' x = \overline{b}_t + \delta_r,\ t \in \overline{D}\right\} \tag{11}$$

have solutions inside W_r; pick x^r and $\widetilde{x}^r$ as solutions of the respective systems in (11) and such that $x^r, \widetilde{x}^r \in W_r$.

Now, let us define $b^r = (b_t^r)_{t\in T}$ and $\widetilde{b}^r = \left(\widetilde{b}_t^r\right)_{t\in T}$ as follows

$$b_t^r := \begin{cases} \overline{b}_t - \delta_r & \text{if } t \in \overline{D}, \\ \overline{b}_t + \frac{1}{r} & \text{if } t \in T\backslash\overline{D}, \end{cases} \quad \text{and } \widetilde{b}_t^r := \begin{cases} \overline{b}_t + \delta_r & \text{if } t \in \overline{D}, \\ \overline{b}_t + \frac{1}{r} & \text{if } t \in T\backslash\overline{D}. \end{cases}$$

In particular, $x^r \in \mathcal{F}(b^r)$ and $\widetilde{x}^r \in \mathcal{F}\left(\widetilde{b}^r\right)$; in fact, $x^r \in \mathcal{F}^{op}(\overline{c}, b^r)$ and $\widetilde{x}^r \in \mathcal{F}^{op}\left(\overline{c}, \widetilde{b}^r\right)$, since $\overline{D} \subset T_{b^r}(x^r) \cap T_{\widetilde{b}^r}(\widetilde{x}^r)$. Moreover, according to the KKT conditions and taking into account that $\lambda^{\overline{D}}$ is a vector of KKT multipliers associated with both problems $(\overline{c}, b^r)$ and $\left(\overline{c}, \widetilde{b}^r\right)$, by duality in LP we have that

$$\vartheta(\overline{c}, b^r) = -(b^r)'\lambda^{\overline{D}} \text{ and } \vartheta\left(\overline{c}, \widetilde{b}^r\right) = -\left(\widetilde{b}^r\right)'\lambda^{\overline{D}}. \tag{12}$$

In this way, and since clearly both sequences $\{b^r\}_{r\in\mathbb{N}}$ and $\{\widetilde{b}^r\}_{r\in\mathbb{N}}$ converge to $\bar{b}$, by applying (12) and, recalling that $\lambda_t^{\overline{D}} = 0$ for $t \in T \setminus \overline{D}$, we have

$$\begin{aligned}
\limsup_r \frac{\left|\vartheta\left(\bar{c}, b^r\right) - \vartheta\left(\bar{c}, \widetilde{b}^r\right)\right|}{\left\|b^r - \widetilde{b}^r\right\|_\infty} &= \limsup_r \frac{\left|-\left(b^r - \widetilde{b}^r\right)\lambda^{\overline{D}}\right|}{2\delta_r} \\
&= \limsup_r \frac{\left|-\sum_{t\in\overline{D}}\left(-2\delta_r\lambda_t^{\overline{D}}\right)\right|}{2\delta_r} = \left\|\lambda^{\overline{D}}\right\|_1.
\end{aligned}$$

■

The following corollary is a direct consequence of the previous theorem, together with Theorem 3(i).

Corollary 1 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$ and assume that SCQ holds at $\bar{b}$. Then we have*

$$\mathrm{lip}\vartheta_{\bar{c}}^R(\bar{b}) = \mathrm{clm}\vartheta_{\bar{c}}^R(\bar{b}) = k^+.$$

The next example, inspired in [14, Example 1], shows that $\mathrm{clm}\vartheta_{\bar{c}}^R(\bar{b})$ can be strictly less than $\mathrm{lip}\vartheta_{\bar{c}}^R(\bar{b})$ when SCQ fails.

Example 1 Consider the problem in $\mathbb{R}$ given by

$$\begin{array}{lll}
\overline{\pi}: & \text{minimize} & x \\
& \text{subject to} & -x \leq 0, \ t = 1, \\
& & -2x \leq 0, \ t = 2, \\
& & 2x \leq 0, \ t = 3.
\end{array}$$

Observe that $\bar{c} = 1$ and $\bar{b} = 0_3$. Obviously, $\vartheta(\overline{\pi}) = 0$, $\mathcal{M}_{\overline{\pi}} = \{\{1\},\{2\}\}$, $\lambda^{\{1\}} = 1$, $\lambda^{\{2\}} = \frac{1}{2}$, and so $k^+ = 1 = \mathrm{lip}\vartheta_{\bar{c}}^R(\bar{b})$. Let us check that $\mathrm{clm}\vartheta_{\bar{c}}^R(\bar{b}) = \frac{1}{2}$.

According to Lemma 4 we have

$$\mathrm{clm}\vartheta_{\bar{c}}^R(\bar{b}) = \limsup_{b\to\bar{b},\ b\in\mathrm{dom}\mathcal{F}} \frac{\left|\vartheta\left(\bar{c}, b\right) - \vartheta\left(\overline{\pi}\right)\right|}{\left\|b - \bar{b}\right\|_\infty} = \limsup_{b\to\bar{b},\ b\in\mathrm{dom}\mathcal{F}} \frac{\left|\max\{-b_1, -\frac{1}{2}b_2\}\right|}{\|b\|_\infty} \leq \frac{1}{2},$$

where we have appealed to the fact that $b \in \mathrm{dom}\mathcal{F}$ implies $-b_1 \leq \frac{1}{2}b_3$. We may attain $\frac{1}{2}$ by considering $(b^r) = \left(\frac{1}{r}, \frac{1}{r}, \frac{1}{r}\right)'$, $r \in \mathbb{N}$.

4 Lipschitz modulus under c -perturbations

This section is devoted to study $\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c})$, where $\overline{b}\in\mathrm{dom}\mathcal{F}$ is fixed. Recall the notation $C=-\mathrm{cone}\{\overline{a}_t,t\in T\}$, and the standard fact (in LP) that $c\in C$ if and only if $(c,\overline{b})\in\mathrm{dom}\mathcal{F}^{op}$. Recall also that $\vartheta_{\overline{b}}^{R}(c):=\vartheta\left(c,\overline{b}\right)$, for any $c\in C$.

The following theorem provides a lower and an upper estimate for the aimed Lipschitz modulus. Moreover, it shows that the upper estimate becomes the exact modulus when $\mathcal{F}^{op}(\overline{\pi})$ is bounded.

Theorem 5 *Let $\overline{\pi}\in\mathrm{dom}\mathcal{F}^{op}$. Then*

$$d(0_n,\mathcal{F}^{op}(\overline{\pi}))\leq\mathrm{clm}\vartheta_{\overline{b}}^{R}(\overline{c})\leq\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c})\leq e\left(\mathcal{E}^{op}(\overline{\pi}),0_n\right).$$

Moreover, if we assume that $\mathcal{F}^{op}(\overline{\pi})$ is bounded, then

$$\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c})=\mathrm{clm}\vartheta_{\overline{b}}^{R}(\overline{c})=e\left(\mathcal{F}^{op}(\overline{\pi}),0_n\right).$$

Proof. First, let us see $\mathrm{clm}\vartheta_{\overline{b}}^{R}(\overline{c})\geq d(0_n,\mathcal{F}^{op}(\overline{\pi}))$ in the nontrivial case $d(0_n,\mathcal{F}^{op}(\overline{\pi}))>0$. Let $\overline{x}\in\mathcal{F}^{op}(\overline{\pi})$ with $\|\overline{x}\|=d\left(0_n,\mathcal{F}^{op}(\overline{\pi})\right)$. According to [28, Lemma 9], there exists $u\in\mathbb{R}^n$ with $\|u\|_*=1$ such that $u'x\geq u'\overline{x}=\|\overline{x}\|$ for all $x\in\mathcal{F}^{op}(\overline{\pi})$. Define

$$c^r:=\overline{c}+\frac{1}{r}u,\ \text{ for each } r\in\mathbb{N}.$$

For all $x\in\mathcal{F}^{op}(\overline{\pi})$ we have

$$\left(c^r\right)'x=\overline{c}'x+\frac{1}{r}u'x\geq\overline{c}'\overline{x}+\frac{1}{r}u'\overline{x}=\left(c^r\right)'\overline{x}. \tag{13}$$

This implies, that $x\mapsto\left(c^r\right)'x$ is bounded from below on $\mathcal{F}^{op}(\overline{\pi})$ and, by Lemma 2, $(c^r,\overline{b})\in\mathrm{dom}\mathcal{F}^{op}$ for r large enough (say for all r). Lemma 3 entails $\mathcal{F}^{op}(c^r,\overline{b})\subset\mathcal{F}^{op}(\overline{\pi})$, for r large enough, and indeed (13) yields $\overline{x}\in\mathcal{F}^{op}(c^r,\overline{b})$. Then, we have

$$\begin{aligned}\mathrm{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) &\geq \limsup_r\frac{\vartheta(c^r,\overline{b})-\vartheta(\overline{\pi})}{\|c^r-\overline{c}\|_*}\\ &= \limsup_r\frac{\left(c^r-\overline{c}\right)'\overline{x}}{\frac{1}{r}\|u\|_*}=u'\overline{x}=\|\overline{x}\|=d\left(0_n,\mathcal{F}^{op}(\overline{\pi})\right).\end{aligned}$$

Recall that $\mathrm{clm}\vartheta_{\overline{b}}^{R}(\overline{c})\leq\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c})$ is always true. Now let us check $\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c})\leq e\left(\mathcal{E}^{op}(\overline{\pi}),0_n\right)$. Write

$$\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c})=\limsup_r\frac{\left|\vartheta(c^r,\overline{b})-\vartheta(\widetilde{c}^r,\overline{b})\right|}{\|c^r-\widetilde{c}^r\|_*}, \tag{14}$$

for appropriate sequences $\{c^r\}_r$, $\{\widetilde{c}^r\}_r \subset C$ converging to $\overline{c}$. Because of the symmetry of the quotients in (14), it is not restrictive to assume $\vartheta(c^r, \overline{b}) - \vartheta(\widetilde{c}^r, \overline{b}) \geq 0$ for all r.

According to Lemma 1, there exists a certain $\overline{x} \in \operatorname{Lim\,sup}_r \mathcal{E}^{op}(\widetilde{c}^r, \overline{b})$ and associated $x^k \in \mathcal{E}^{op}(\widetilde{c}^{r_k}, \overline{b}) \subset \mathcal{F}^{op}(\widetilde{c}^{r_k}, \overline{b})$, for $r_1 < r_2 < \ldots < r_k < \ldots$, such that $x^k \to \overline{x} \in \mathcal{E}^{op}(\overline{\pi})$. Then, for all $k \in \mathbb{N}$ we have

$$0 \leq \vartheta(c^{r_k}, \overline{b}) - \vartheta(\widetilde{c}^{r_k}, \overline{b}) \leq (c^{r_k})' x^k - (\widetilde{c}^{r_k})' x^k \leq \|c^{r_k} - \widetilde{c}^{r_k}\|_* \|x^k\|,$$

which implies

$$\begin{aligned} \operatorname{lip}\vartheta^R_{\overline{b}}(\overline{c}) &= \limsup_k \frac{\vartheta(c^{r_k}, \overline{b}) - \vartheta(\widetilde{c}^{r_k}, \overline{b})}{\|c^{r_k} - \widetilde{c}^{r_k}\|_*} \\ &\leq \limsup_k \|x^k\| = \|\overline{x}\| \leq e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right). \end{aligned}$$

Finally, let us assume that $\mathcal{F}^{op}(\overline{\pi})$ is bounded, which entails $\operatorname{span}\{\overline{a}_t, t \in T\} = \mathbb{R}^n$, hence $\mathcal{E}^{op}(\overline{\pi}) = \operatorname{extr}\mathcal{F}^{op}(\overline{\pi})$ and $e\left(\operatorname{extr}\mathcal{F}^{op}(\overline{\pi}), 0_n\right) = e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right)$ (this last follows a standard argument by using the convexity of the norm). Observe that we only have to prove $\operatorname{clm}\vartheta^R_{\overline{b}}(\overline{c}) \geq e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right)$. Let $\overline{x} \in \mathcal{F}^{op}(\overline{\pi})$ with $\|\overline{x}\| = e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right)$. Take $u \in \mathbb{R}^n$ with $\|u\|_* = 1$ be such that $u'\overline{x} = \|\overline{x}\|$. Define the perturbation $c^r := \overline{c} - \frac{1}{r}u$ for all r. Since $\mathcal{F}^{op}(\overline{\pi})$ is bounded, from [15, Lemma 10.2] $(c^r, \overline{b}) \in \operatorname{dom}\mathcal{F}^{op}$ for r large enough. Then, since both problems $(c^r, \overline{b})$ and $\overline{\pi}$ have the same feasible set, we have

$$\vartheta(c^r, \overline{b}) \leq (c^r)'\overline{x} = \overline{c}'\overline{x} - \frac{1}{r}u'\overline{x} = \vartheta(\overline{\pi}) - \|c^r - \overline{c}\|_* \|\overline{x}\|.$$

Therefore,

$$\operatorname{clm}\vartheta^R_{\overline{b}}(\overline{c}) \geq \limsup_r \frac{\vartheta(\overline{\pi}) - \vartheta(c^r, \overline{b})}{\|\overline{c} - c^r\|_*} \geq \|\overline{x}\| = e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right).$$

Finally, the aimed equality comes from

$$\operatorname{lip}\vartheta^R_{\overline{b}}(\overline{c}) \geq \operatorname{clm}\vartheta^R_{\overline{b}}(\overline{c}) \geq e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right) \geq \operatorname{lip}\vartheta^R_{\overline{b}}(\overline{c}).$$

■

The next example is intended to show that the first two inequalities in the statement of Theorem 5 may be strict. At the moment we do not have an example where $\operatorname{lip}\vartheta^R_{\overline{b}}(\overline{c}) \leq e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right)$ holds strictly, so that the question of whether or not equality fulfills for all $\overline{\pi} \in \operatorname{dom}\mathcal{F}^{op}$ remains open.

Example 2 Consider the nominal problem, in $\mathbb{R}^3$ with the Euclidean norm,

$$\overline{\pi} : \text{minimize } x_3 \text{ s.t. } x_1 \leq -1, \ -x_2 \leq 2, \ -x_3 \leq 0.$$

Clearly $d(0_3, \mathcal{F}^{op}(\overline{\pi})) = 1$, $\mathcal{E}^{op}(\overline{\pi}) = \left\{(-1,-2,0)'\right\}$, and hence $e\left(\mathcal{E}^{op}(\overline{\pi}), 0_3\right) = \sqrt{5}$. Let us prove that $\mathrm{clm}\vartheta^R_{\overline{b}}(\overline{c}) = 2$ and $\mathrm{lip}\vartheta^R_{\overline{b}}(\overline{c}) = \sqrt{5}$. Consider any $0 < \varepsilon < 1$ and any $c \in \mathbb{R}^3$ with $\|c - \overline{c}\|_* = \varepsilon$, which may be written as $c = (\varepsilon_1, \varepsilon_2, 1 + \varepsilon_3)'$ with $\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 = \varepsilon^2$. Then $\left(c, \overline{b}\right) \in \mathrm{dom}\mathcal{F}^{op}$ if and only if $\varepsilon_1 \leq 0$ and $\varepsilon_2 \geq 0$, in which case $\vartheta\left(c, \overline{b}\right) = c'(-1,-2,0)' = -\varepsilon_1 - 2\varepsilon_2$. Accordingly,

$$\min_{\substack{\|c-\overline{c}\|_*=\varepsilon \\ (c,\overline{b})\in\mathrm{dom}\mathcal{F}^{op}}} \vartheta\left(c, \overline{b}\right) = \min_{\substack{\varepsilon_1^2+\varepsilon_2^2=\varepsilon^2 \\ \varepsilon_1\leq 0,\ \varepsilon_2\geq 0}} -\varepsilon_1 - 2\varepsilon_2 = -2\varepsilon, \tag{15}$$

attained at $c = (0, \varepsilon, 1)'$. The corresponding maximum equals ε and is attained at $c = (-\varepsilon, 0, 1)'$. Consequently, for any $0 < \varepsilon < 1$,

$$\max_{\substack{\|c-\overline{c}\|_*=\varepsilon \\ (c,\overline{b})\in\mathrm{dom}\mathcal{F}^{op}}} \left|\vartheta\left(c, \overline{b}\right) - \vartheta\left(\overline{\pi}\right)\right| = 2\varepsilon,$$

which, clearly entails $\mathrm{clm}\vartheta^R_{\overline{b}}(\overline{c}) = 2$. Now let us compute the Lipschitz modulus of $\vartheta^R_{\overline{b}}$ at $\overline{c}$. As a motivation of such computation note that

$$\max_{\varepsilon_1^2+\varepsilon_2^2=\varepsilon^2} -\varepsilon_1 - 2\varepsilon_2 = \sqrt{5}\varepsilon,$$

and this maximum is attained at $(\varepsilon_1, \varepsilon_2) = \left(-\varepsilon/\sqrt{5}, -2\varepsilon/\sqrt{5}\right)$. Let us consider $c := \left(-\varepsilon/\sqrt{5}, 0, 1\right)'$ and $\widetilde{c} := \left(0, 2\varepsilon/\sqrt{5}, 1\right)'$. Then

$$\frac{\left|\vartheta\left(c, \overline{b}\right) - \vartheta\left(\widetilde{c}, \overline{b}\right)\right|}{\|c - \widetilde{c}\|_*} = \frac{\varepsilon/\sqrt{5} - \left(-4\varepsilon/\sqrt{5}\right)}{\varepsilon} = \sqrt{5}.$$

Since this happens for all $0 < \varepsilon < 1$, we conclude $\mathrm{lip}\vartheta^R_{\overline{b}}(\overline{c}) \geq \sqrt{5}$. The converse inequality comes from Theorem 5.

5 Lipschitz modulus under canonical perturbations

The objective of this section is to compute (or at least estimate) the Lipschitz modulus of the optimal value function, restricted to $\mathrm{dom}\mathcal{F}^{op}$, at a nominal parameter $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$ under canonical perturbations, i.e., when the RHS of the constraints and the coefficients of the objective function can be simultaneously perturbed.

The following theorem provides a lower bound of the Lipschitz modulus $\mathrm{lip}\vartheta^R(\overline{\pi})$.

Theorem 6 *Let $\overline{\pi} \in \text{dom}\mathcal{F}^{op}$. Then*

$$\text{lip}\vartheta^R(\overline{\pi}) \geq k^+ + d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right).$$

Proof. The case $0_n \in \mathcal{F}^{op}(\overline{\pi})$ is trivial due to the fact that $\text{lip}\vartheta^R(\overline{\pi}) \geq \text{lip}\vartheta^R_{\overline{c}}(\overline{b})$. So, let us assume $0_n \notin \mathcal{F}^{op}(\overline{\pi})$. Take $\overline{x} \in \mathcal{F}^{op}(\overline{\pi})$ with $\|\overline{x}\| = d\left(0_n, \mathcal{F}^{op}(\overline{\pi})\right)$. Let us consider sequences $\{b^r\}_r, \{\widetilde{b^r}\}_r \subset \text{dom}\mathcal{F}$ such that

$$k^+ = \text{lip}\vartheta^R_{\overline{c}}(\overline{b}) = \lim_r \frac{\vartheta(\overline{c}, b^r) - \vartheta(\overline{c}, \widetilde{b^r})}{\|b^r - \widetilde{b^r}\|_\infty}.$$

The next step is analogous to its counterpart for calmness in the proof of [14, Theorem 5], so that we will focus on the differences. As in formula (18) in the referred proof, there exist sequences $\{x^r\}_r$ and $\{u^r\}_r$ in $\mathbb{R}^n$, with $\|x^r\| \to \|\overline{x}\|$, such that, for each r, $x^r \in \mathcal{F}^{op}(\overline{c}, b^r)$, $\|u^r\|_* = 1$, and

$$(u^r)'x \geq (u^r)'x^r = \|x^r\| = d\left(0_n, \mathcal{F}^{op}(\overline{c}, b^r)\right), \text{ whenever } x \in \mathcal{F}^{op}(\overline{c}, b^r).$$

Now we define $c^r := \overline{c} + \|b^r - \widetilde{b^r}\|_\infty u^r$. For $x \in \mathcal{F}^{op}(\overline{c}, b^r)$ one has

$$(c^r)'x = \overline{c}'x + \left\|b^r - \widetilde{b^r}\right\|_\infty (u^r)'x \geq \vartheta(\overline{c}, b^r) + \left\|b^r - \widetilde{b^r}\right\|_\infty \|x^r\|, \qquad (16)$$

so $x \mapsto (c^r)'x$ is bounded from below on $\mathcal{F}^{op}(\overline{c}, b^r)$. Because of Lemma 2, there exists $r_0 \in \mathbb{N}$ such that $\pi^r \equiv (c^r, b^r) \in \text{dom}\mathcal{F}^{op}$ for $r \geq r_0$. Then Lemma 3 yields $\mathcal{F}^{op}(\pi^r) \subset \mathcal{F}^{op}(\overline{c}, b^r)$ for $r \geq r_0$ large enough. Accordingly, by the restriction of (16) to points $x \in \mathcal{F}^{op}(\pi^r)$ we get

$$\vartheta(\pi^r) = (c^r)'x \geq \vartheta(\overline{c}, b^r) + \left\|b^r - \widetilde{b^r}\right\|_\infty \|x^r\|.$$

Let us define $\widetilde{\pi}^r :\equiv (\overline{c}, \widetilde{b^r})$ which belongs to $\text{dom}\mathcal{F}^{op}$ (because $\{\widetilde{b^r}\}_r \subset \text{dom}\mathcal{F}$ and $\overline{c} \in C$). Note that $\|\pi^r - \widetilde{\pi}^r\| = \left\|b^r - \widetilde{b^r}\right\|_\infty$.

Then we have

$$\begin{aligned}
\text{lip}\vartheta^R(\overline{\pi}) &\geq \limsup_r \frac{|\vartheta(\pi^r) - \vartheta(\widetilde{\pi}^r)|}{\|\pi^r - \widetilde{\pi}^r\|} \\
&\geq \limsup_r \frac{\vartheta(\pi^r) - \vartheta(\overline{c}, b^r) + \vartheta(\overline{c}, b^r) - \vartheta(\overline{c}, \widetilde{b^r})}{\|b^r - \widetilde{b^r}\|_\infty} \\
&= \lim_r \frac{\vartheta(\overline{c}, b^r) - \vartheta(\overline{c}, \widetilde{b^r})}{\|b^r - \widetilde{b^r}\|_\infty} + \limsup_r \frac{\vartheta(\pi^r) - \vartheta(\overline{c}, b^r)}{\|b^r - \widetilde{b^r}\|_\infty} \\
&\geq \text{lip}\vartheta^R_{\overline{c}}(\overline{b}) + \lim_r \|x^r\| = k^+ + \|\overline{x}\|.
\end{aligned}$$

$\blacksquare$

In order to establish an upper bound for the Lipschitz modulus of ϑ^R at $\overline{\pi}$, we appeal to the technique developed in [18, Section 2]. Specifically, Wu Li proved that if a set-valued mapping is Hausdorff lower semicontinuous, a uniform upper Lipschitz constant for that mapping in a convex neighborhood of the nominal parameter becomes a Lipschitz constant in such a neighborhood (see [18, Theorem 2.1] for details). Translating it into our context, a uniform calmness constant for ϑ^R in a neighborhood (relative to $\mathrm{dom}\mathcal{F}^{op}$) of $\overline{\pi}$ becomes a Lipschitz constant at $\overline{\pi}$. This technique was already applied in [24] for obtaining the so-called sharp Lipschitz constant for $\mathcal{F}^{op}$ under suitable hypotheses.

Lemma 5 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$. For all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$k^+ + e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right) + \varepsilon$$

is a calmness constant of ϑ^R at any $\pi \in (\mathrm{dom}\mathcal{F}^{op}) \cap B(\overline{\pi}, \delta)$ (the closed ball centered at $\overline{\pi}$ of radius δ).

Proof. We start by observing that, from Lemma 1, $\mathcal{E}^{op} : \mathrm{dom}\mathcal{F}^{op} \rightrightarrows \mathbb{R}^n$ is Hausdorff-upper semicontinuous at $\overline{\pi}$; i.e., $\lim_{\pi \to \overline{\pi}} e\left(\mathcal{E}^{op}(\pi), \mathcal{E}^{op}(\overline{\pi})\right) = 0$.

Now, let us abuse the notation and identify also constant k^+ as a function $k^+ : \mathrm{dom}\mathcal{F}^{op} \longrightarrow \mathbb{R}^+$ defined as $k^+(\pi) = \max_{D \in \mathcal{M}_\pi} \left\|\lambda^D\right\|_1$, where $k^+(\overline{\pi})$ is our original k^+ as defined in (6). We need to prove that function k^+ is also upper semicontinuous at $\overline{\pi}$, that is, for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|\pi - \overline{\pi}\| < \delta$, for $\pi \in \mathrm{dom}\mathcal{F}^{op}$, then $k^+(\pi) \leq k^+(\overline{\pi}) + \varepsilon$. Reasoning by contradiction, suppose that there exists a sequence $\{\pi^r\}_r \subset \mathrm{dom}\mathcal{F}^{op}$ converging to $\overline{\pi}$ such that $k^+(\pi^r) \geq k^+(\overline{\pi}) + \varepsilon_0$ for a certain $\varepsilon_0 > 0$. Suppose that the maximum defining $k^+(\pi^r)$ is attained at a certain $D^r \in \mathcal{M}_{\pi^r}$. Since T is finite, we can assume the existence of a constant subsequence, say $D^r = D$ for all r. The fact that $-c^r \in \mathrm{cone}\{\overline{a}_t, t \in D\}$ entails $-\overline{c} \in \mathrm{cone}\{\overline{a}_t, t \in D\}$, although we cannot ensure the minimality of D for $\overline{\pi}$. Recall that $\{\overline{a}_t', t \in D\}$ is linearly independent. Write

$$-c^r = \sum_{t \in D} \lambda_t^r \overline{a}_t \text{ for all } r, \text{ and } -\overline{c} = \sum_{t \in D} \lambda_t^D \overline{a}_t \;.$$

Using a standard argument it is easy to see that $\left\{\sum_{t \in D} \lambda_t^r\right\}_r$ is bounded so, taking a subsequence, if necessary, it may be assumed to converge to $\sum_{t \in D} \lambda_t^D$. Despite we cannot assume $D \in \mathcal{M}_{\overline{\pi}}$, we know that D contains at least a minimal element for $\overline{\pi}$, so let $\widetilde{D} \in \mathcal{M}_{\overline{\pi}}$ with $\widetilde{D} \subset D$ and $\lambda_t^D = 0$ for all $t \notin \widetilde{D}$. Therefore we have

$$k^+(\pi^r) = \sum_{t \in D} \lambda_t^r \longrightarrow \sum_{t \in D} \lambda_t^D = \sum_{t \in \widetilde{D}} \lambda_t^D \leq k^+(\overline{\pi}),$$

hence we attain a contradiction.

Applying the upper semicontinuity of both, $\mathcal{E}^{op}$ and k^+, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$e\left(\mathcal{E}^{op}(\pi), 0_n\right) \leq e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right) + \varepsilon/2$$

$$\text{and } k^+(\pi) \leq k^+(\overline{\pi}) + \varepsilon/2,$$

for all $\pi \in \text{dom}\mathcal{F}^{op}$ with $\|\pi - \overline{\pi}\| < \delta$, and therefore, taking Theorem 3(ii) into account,

$$\text{clm}\vartheta^R(\pi) \leq k^+(\pi) + e\left(\mathcal{E}^{op}(\pi), 0_n\right) \leq k^+(\overline{\pi}) + e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right) + \varepsilon\ .$$

■

Theorem 7 *Let $\overline{\pi} \in \text{dom}\mathcal{F}^{op}$. Then*

$$\text{lip}\vartheta^R(\overline{\pi}) \leq k^+ + e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right). \tag{17}$$

If, additionally, $\mathcal{F}^{op}(\overline{\pi})$ is bounded, then equality holds in (17), which reads as

$$\text{lip}\vartheta^R(\overline{\pi}) = k^+ + e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right).$$

Proof. Recall that $\text{dom}\mathcal{F}^{op}$ is convex in $\mathbb{R}^n \times \mathbb{R}^T$ and Theorem 1 establishes the continuity of ϑ^R on $\text{dom}\mathcal{F}^{op}$. Then, the previous lemma and its preceding comments ensure that $k^+ + e\left(\mathcal{E}(\overline{\pi}), 0_n\right) + \varepsilon$ is a Lipschitz constant of ϑ^R at $\overline{\pi}$ for each $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ we obtain (17).

Now assume that $\mathcal{F}^{op}(\overline{\pi})$ is bounded. In order to establish the converse inequality, consider sequences $\{b^r\}_r, \{\widetilde{b}^r\}_r \subset \text{dom}\mathcal{F}$ such that

$$k^+ = \text{lip}\vartheta^R_{\overline{c}}(\overline{b}) = \lim_r \frac{\vartheta(\overline{c}, \widetilde{b}^r) - \vartheta(\overline{c}, b^r)}{\|\widetilde{b}^r - b^r\|_\infty}.$$

Apply Theorem 2 and Remark 1 to conclude that $\mathcal{F}^{op}(\overline{c}, b^r)$ is nonempty and bounded for r large enough (say for all r). For each $r \in \mathbb{N}$ take $x^r \in \mathcal{F}^{op}(\overline{c}, b^r)$ such that $\|x^r\| = e\left(\mathcal{F}^{op}(\overline{c}, b^r), 0_n\right)$ and let $u^r \in \mathbb{R}^n$ be such that $\|u^r\|_* = 1$ and $(u^r)' x^r = \|x^r\|$.

The sequence $\{x^r\}_{r\in\mathbb{N}}$ may not converge, although it has for sure a convergent subsequence, but we can ensure, again by Theorem 2, that $\|x^r\| \to e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right)$.

For each r let us define $c^r := \overline{c} - \|\widetilde{b}^r - b^r\|_\infty u^r$. Obviously $x \mapsto (c^r)' x$ is bounded from below on $\mathcal{F}^{op}(\overline{c}, b^r)$, because this set is compact; so that, Lemma 2 yields $(c^r, b^r) \in \mathrm{dom}\mathcal{F}^{op}$ for r large enough, and then

$$\vartheta(\overline{c}, b^r) - \vartheta(c^r, b^r) \geq (\overline{c} - c^r)' x^r = \|\widetilde{b}^r - b^r\|_\infty \|x^r\|.$$

Therefore

$$\begin{aligned}
\mathrm{lip}\vartheta^R(\overline{\pi}) &\geq \limsup_r \frac{\vartheta(\overline{c}, \widetilde{b}^r) - \vartheta(c^r, b^r)}{\|(\overline{c}, \widetilde{b}^r) - (c^r, b^r)\|} \\
&= \limsup_r \frac{\vartheta(\overline{c}, \widetilde{b}^r) - \vartheta(\overline{c}, b^r) + \vartheta(\overline{c}, b^r) - \vartheta(c^r, b^r)}{\|\widetilde{b}^r - b^r\|_\infty} \\
&= \mathrm{lip}\vartheta_{\overline{c}}^R(\overline{b}) + \limsup_r \frac{\vartheta(\overline{c}, b^r) - \vartheta(c^r, b^r)}{\|\widetilde{b}^r - b^r\|_\infty} \\
&\geq k^+ + \lim_r \|x^r\| = k^+ + e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right).
\end{aligned}$$

■

Corollary 2 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$, with $\mathcal{F}^{op}(\overline{\pi})$ bounded. Then*

$$\mathrm{lip}\vartheta^R(\overline{\pi}) = \mathrm{lip}\vartheta_{\overline{c}}^R(\overline{b}) + \mathrm{lip}\vartheta_{\overline{b}}^R(\overline{c}).$$

Proof. It comes from Theorems 4, 5 and 7. ■

6 Conclusions

The main original contributions of the present paper are focused on the Lipschitz moduli of the optimal value functions restricted to their domains in different parametric contexts ($\vartheta_{\overline{c}}^R$, in the context of RHS perturbations, $\vartheta_{\overline{b}}^R$, in the one of c-perturbations, and ϑ^R, for canonical perturbations; see Section 2.1 for the definitions). The analysis is developed around a nominal LP problem $\overline{\pi}$ which is identified with the pair formed by a nominal vector of the objective function, $\overline{c}$, and a nominal RHS, $\overline{b}$. As a brief discussion about the convenience of dealing with such functions, restricted to their domains, we underline the fact that it allows us to avoid a typical interiority assumption under which some preliminary results are stated (see [15, Lemma

10.2] and [13]). Specifically, the nominal elements $\overline{b}$, $\overline{c}$, and $\overline{\pi}$ are not required to be in the interior of the respective domains of $\vartheta_{\overline{c}}^{R}$, $\vartheta_{\overline{b}}^{R}$, and ϑ^{R}.

In contrast, [15, Lemma 10.2] and [13] deal with the optimal function ϑ defined on the whole space, and in this case the condition '$\overline{\pi}$ is in the interior of the domain of ϑ' is not avoidable as far as it characterizes the Lipschitz continuity of ϑ at $\overline{\pi}$ (so, the Lipschitz modulus of ϑ is infinite when the interiority condition does not hold). It is known that this interiority condition is equivalent to the simultaneous fulfilment of the Slater CQ and the boundedness (and nonemptiness) of the nominal optimal set. In the next paragraphs we comment the most important contributions of this work and, at the same time, we try to clarify the role played by the two assumptions, Slater CQ and boundedness, separately, in relation to the computation/estimation of our Lipschitz moduli. The boundedness of the optimal set does play an important role:

- When $\overline{\pi}$ is a solvable problem (without any extra assumption), the Lipschitz modulus of $\vartheta_{\overline{c}}^{R}$ is completely determined (Theorem 4), and the corresponding moduli for $\vartheta_{\overline{b}}^{R}$, and ϑ^{R} are lower and upper estimated (Theorems 5, 6, and 7). In particular, all these functions are always Lipschitz continuous at $\overline{\pi}$.

- When $\overline{\pi}$ is solvable and Slater CQ holds, we additionally have that the Lipschitz modulus of $\vartheta_{\overline{c}}^{R}$ does coincide with its calmness modulus (Corollary 1).

- When $\overline{\pi}$ is solvable and the nominal optimal set is bounded, the upper estimates of $\vartheta_{\overline{b}}^{R}$ and ϑ^{R} turn out to be the exact moduli. Moreover, in this case, the Lipschitz modulus of ϑ^{R} coincides with the sum of the corresponding moduli of $\vartheta_{\overline{c}}^{R}$ and $\vartheta_{\overline{b}}^{R}$ (Corollary 2).

- When $\overline{\pi}$ is in the interior of solvable problems (Slater CQ together with boundedness of the nominal optimal set), then, in addition to the previous statements, the Lipschitz modulus of ϑ does coincide with the one of ϑ^{R}. Moreover, the reader can easily check that the calmness modulus of ϑ may be strictly less than the Lipschitz one from the exact expressions of both moduli (Theorems 3 and 7).

Finally, let us comment that all formulas obtained in this work for computing or estimating our aimed moduli are point-based, in the sense that all ingredients used in them only involve the nominal elements (the nominal point and problem's data), not appealing to parameters or points in a neighborhood. In this way they are implementable in practice.

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