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FINITE CHAIN CALCULUS IN DISTRIBUTIVE LATTICES AND ELEMENTARY KRULL DIMENSION

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To Mirian Andrés Gómez in memoriam

Resumen. En este artículo consideramos cadenas finitas de elementos en retículos distributivos acotados L, que pueden ser representadas como morfismos $Ch_n \to L$ con dominio una n-cadena estándar con $n+2$ elementos. Consideramos también cocadenas, esto es, morfismos de retículos $L \to Ch_n$ asociados a cadenas de ideales primos de L. Una cadena pertenece a una cocadena de la misma longitud si la correspondiente composición de morfismos $Ch_n \to L \to Ch_n$ es la identidad. Definimos cadenas enlazadas mediante una relación ecuacional entre cadenas de la misma longitud, estableciendo algunas propiedades de dicha relación. Finalmente, probamos que una cadena pertenece a una cocadena sí y sólo si no están enlazadas. Este resultado significa que la dimensión de Krull de retículos distributivos acotados puede definirse en términos de cadenas enlazadas de elementos.

Abstract. In this paper we consider finite chains of elements in bounded distributive lattices L, which can be represented as morphisms $Ch_n \to L$ with domain a standard n-chain with $n+2$ elements. We consider also cochains, that is, lattice morphisms $L \to Ch_n$, which are associated to chains of prime ideals of L. A chain belongs to a cochain of the same length if the composition $Ch_n \to L \to Ch_n$ is the identity. We define linked chains as an equational relation between chains of the same length, and we establish some properties of this relation. Finally we prove that a chain belongs to a cochain if and only if it is non-linked. This result means that Krull dimension of bounded distributive latices can be defined in terms of linked chains of elements.

1. Introduction

We consider distributive lattices with a bottom element 0 and a top element 1, and lattice morphisms preserving 0 and 1. The Krull dimension of a distributive lattice L, denoted $KdimL$, is the greatest length of non degenerated chains of prime ideals in L. In other words, $KdimL$ is the chain dimension of the ordered set $Spec(L)$ of all prime ideals of L, called the prime spectrum of L. This notion is imported from commutative algebra [2, 11], where the prime spectrum $Spec(A)$ of a commutative ring A is extensively used. But to work with prime ideals is not constructive, hence mathematics is also interested on new notions of prime spectrum and Krull dimension depending on neither excluded middle nor axiom

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of choice. In this paper we does not deal with the relation among bounded distributive latices and unitary and commutative rings [9, 10]. Following a proposal of A. Joyal, the author gave [5, 6] a constructive characterization of the Krull dimension of a distributive lattice L in terms of elements of the free boolean algebra $L_\neg$ generated by L (in this sense one say "elementary" Krull dimension). A few years later, T. Coquand and H. Lombardi [4] characterized Krull dimension of distributive lattices by using finite subsets $\{a_1, \dots, a_n\}$ of L such that a system of inequations like $a_i \vee x_i \geq a_{i+1} \wedge x_{i+1}$ has a solution. These kind of systems were used by F. W. Anderson and R. L. Blair [1] to characterize those distributive lattices which are subdirect product of chains, but they applied them to finite chains of elements $a_1 \leq \dots \leq a_n$ (see [3, pp. 122–123]). The work of Anderson and Blair was classical, but Coquand and Lombardi worked out a constructive Krull dimension.

In this paper we advance the elements for a variation of the constructive characterization of Krull dimension given by Coquand and Lombardi, in which we use *finite chains of elements* $a_1 \leq \dots \leq a_n$ such that a system of *equations* like $a_i \vee x_i = a_{i+1} \wedge x_{i+1}$ has a solution. We leave for a second paper the autonomous development of a so based Krull dimension theory of distributive lattices, and its direct comparison with that of Coquand and Lombardi. Here we only establish our definition and compare it with the classical one based on primed ideal, so that our working is not constructive.

After this introduction, in section 2 we fix notation and terminology, and recall some basic results on distributive lattices. In section 3 we study finite chains of elements in distributive lattices L, which can be represented as morphisms $\mathbf{a} : Ch_n \to L$ with domain a standard n-chain with $n+2$ elements, so $Ch_0 = 2 = \{0 < 1\}$, $Ch_1 = \{0 < t < 1\}$, etc. In particular, we give a recursive property of the family Ch_n of distributive lattices in terms of free adjunction of one element and a quotient to guarantee the total order. Then we compare chains and cochains of the same length, where cochains are lattice morphisms $\mathbf{P} : L \to Ch_n$, which are associated to chains of prime ideals of L, that is, a chain in the prime spectrum $Spec(L)$. Recall that the prime ideals P of L are in bijection with the morphisms $\varphi : L \to 2$ in $\mathcal{L}$, with $P = \varphi^{-1}(0)$, so $Spec(L) \cong \mathcal{L}(L, 2)$. A chain belongs to a cochain of the same length if the composition $\mathbf{P} \circ \mathbf{a} : Ch_n \to L \to Ch_n$ is the identity. It is clear that this realtion requires the cochain to be onto, and we prove that the converse is also true (proposition 3.7).

Section 4 is devoted to our basic notion of link of chains of elements in distributive lattices (definition 4.1). A 1-chain in L is an element in L, and a linked 1-chain is a complemented element. This definition produce (proposition 4.2) a family $Link_n$ of distributive lattices, $Link_1 = 2 \times 2$, which we relate with the family Ch_n. Some properties of linked chains are given. The notion of locally separated chain given by Anderson and Blair (see [3, pp. 122–123]) approximately corresponds to our non-linked chains, also called independent. Finally, in section 5 we define a notion of dimension of a bounded distributive lattice as the greatest length of non-linked chains (definition 5.1), and we prove (theorem 5.3) that this dimension and Krull dimension are equal.

2. Preliminaries

In this first section we fix the notation and terminology to be used in the following sections, and we recall some basic results that we shall explicitly need in the sequel. We always consider bounded distributive lattices, that is, distributive lattices with a bottom element 0 and a top element 1, and lattice homomorphisms preserving 0 and 1. The corresponding category will be denoted $\mathcal{L}$. For general notions of lattices we refer to [3] or [7]. We consider the category Set of sets and maps and the forgetful functor $O : \mathcal{L} \to Set$, which has as left adjoint the free functor $Fr : Set \to \mathcal{L}$. If Fr_n denotes the distributive lattice freely generated by n generators, we have $Fr_0 = 2 = \{0 < 1\}$, $Fr_1 = \{0 < t < 1\}$, and the following order diagram for the case $n = 2$:

$$Fr_2 : \begin{tikzcd} & 1 \arrow[d, dash] & \\ & x_1 \vee x_2 \arrow[dl, dash] \arrow[dr, dash] & \\ x_1 \arrow[dr, dash] & & x_2 \arrow[dl, dash] \\ & x_1 \wedge x_2 \arrow[d, dash] & \\ & 0 & \end{tikzcd}$$

In the next section a recursive construction of Fr_n will be given.

Given an element $a \in L$, the boolean complement of a in L, if it exists, shall be denoted $\neg a$. A boolean algebra is a complemented distributive lattice. In general, the complemented elements of L form the *center* of L, which is a boolean algebra denoted $\mathcal{C}(L)$. For any distributive lattice L we have also the boolean algebra freely generated by L, denoted $L_\neg$. The inclusion morphism $l : L \hookrightarrow L_\neg$ is mono and epi in $\mathcal{L}$, and it is an isomorphism if and only if L is a boolean algebra ($L \cong \mathcal{C}(L)$). If $\mathcal{B}$ is the full subcategory of $\mathcal{L}$ defined by the boolean algebras, the inclusion functor $\iota : \mathcal{B} \hookrightarrow \mathcal{L}$ has adjoints $(-)_\neg \dashv \iota \dashv \mathcal{C}$. The forgetful functor $Ob : \mathcal{B} \to Set$ has also as left adjoint the free functor $Frb : Set \to \mathcal{B}$. The free boolean algebra Frb_n with n generators has 2^{2^n} elements, in particular $Frb_0 = 2$, $Frb_1 = 2 \times 2$, and $Frb_n = (Fr_n)_\neg$ for $n \geq 0$.

A general result in universal algebra says that any distributive lattice is a quotient of a free distributive lattice. In the category $\mathcal{L}$ quotients are related with *ideals* and *filters*. Recall that an ideal is a subset $I \subseteq L$ such that $0 \in I$, $I \vee I \subseteq I$, and $I \wedge L \subseteq I$. A filter is a subset $F \subseteq L$ with the dual properties. For every morphism $f : L \to D$ in $\mathcal{L}$, $I = f^{-1}(0)$ is an ideal and $F = f^{-1}(1)$ is a filter, and both subsets are disjoint (except if D is trivial with 0=1). When $D = 2$, the ideal I is a *prime ideal* ($x \wedge y \in I$ implies $x \in I$ or $y \in I$), F is a *prime filter* (dual condition), and I, F are complementary subsets of L. The non-constructive *prime ideal theorem* says (see [3] or [7]):

Theorem 2.1. *Given in a distributive lattice L, an ideal I and a filter F such that $F \cap I \neq \emptyset$, there exists a prime ideal $P \subseteq L$ such that $I \subseteq P$, and $F \cap P \neq \emptyset$.*

Recall that the *Krull dimension* of a distributive lattice L, denoted $KdimL$, is the chain dimension of the ordered set $Spec(L)$ of all prime ideals of L, called the *prime spectrum* of L. This means the greatest n such that there exists a non-degenerated chain $P_0 \subseteq \cdots \subseteq P_n$ of prime ideals of L. To assure the existence of prime ideals, the prime ideal theorem (theorem 2.1) is extensively used in classical Krull dimension theory.

A distributive lattice is said *integer* if it satisfies $0 \neq 1$ and the formula: $x \wedge y = 0$ implies $x = 0$ or $y = 0$ (that is, the ideal $\{0\}$ is prime), and it is said *local* if it satisfies $0 \neq 1$ and the dual formula: $x \vee y = 1$ implies $x = 1$ or $y = 1$ (that is, the filter $\{1\}$ is prime). It is clear that Fr_n is integer and local for $0 \leq n \leq 2$ (in the next section we shall prove that this property is true for all n), but 2×2 is neither integer nor local. Either if L is integer or local we have $\mathcal{C}(L) = \{0, 1\}$. Hence L integer (resp. local) and boolean implies $L = 2$.

The *ideal generated* by an element $a \in L$ is $(a) = \{x \in L; x \leq a\}$, also denoted $(a]$. The filter generated by an element $a \in L$ is $[a) = \{x \in L; x \geq a\}$. The *interval* defined by $a, b \in L$ is $[a, b] = \{x \in L; a \leq x \leq b\}$. It is clear that any interval is the intersection of an ideal and a filter, $[a, b] = [a, 1] \wedge [0, b]$, and it is non-empty if and only if $a \leq b$. An order diagram of the form

$$\begin{array}{ccccc} & q & & & p = a \wedge b \\ / & & \backslash & & \\ a & & b & & \\ \backslash & & / & & \\ & p & & & q = a \vee b \end{array}$$

is called a *diamond*, explicitly the diamond *generated* by the elements a, b, so that a is complemented in the interval $[p, q]$ with complement b. We remark that there is an isomorphism $[p, a] \to [b, q]$ sending x to $x \vee b$. By symmetry, $[p, b] \cong [a, q]$. We can translate these isomorphisms to equalities in the boolena algebra $L_\neg$ provided with the boolean addition (symmetrical difference) $x + y = (\neg x \wedge y) \vee (x \wedge \neg y)$. If $x \leq y$ then $x + y = \neg x \wedge y$. A simple boolean calculus gives the next result:

Lemma 2.2. *In the diamond generated by a, b in a distributive lattice L, the following equalities hold:*
(i) $p + a = b + q$, $p + b = a + q$
(ii) $p + q = a + b$.

An ideal $I \subseteq L$ defines a congruence in L by $x \cong y$ if $x \vee a = y \vee a$ for some $a \in I$. An equivalent condition is $x \vee y = (x \wedge y) \vee a$ for some $a \in I$. If $\overline{x}$ denotes the class of x, then $I = \overline{0}$, and $F = \overline{1}$ is a filter. We remark that the relation between I and F is not symmetrical, because F defines a congruence in L by $x \sim y$ if $x \wedge b = y \wedge b$ for some $b \in I$, with $F = \overline{1}$, but in general the ideal $I' = \overline{0}$ is only a subset of I. Consider for instance $L = Fr_1$, $I = (t)$; then $F = \{1\}$ and $I' = (0)$.

The set $\mathcal{I}(L)$ of all ideal of a distributive lattice L is a complete Heyting algebra (see [3, 7, 8]), in particular a distributive lattice, and principal ideals (a) determine a lattice homomorphism $(-) : L \to \mathcal{I}(L)$, which is universal among the morphism in $\mathcal{L}$ with domain L and codomain a complete Heyting algebra. Meets in $\mathcal{I}(L)$ are intersections, and joins are ideals generated by unions, in particular, $x \in I \vee J$ if and only if $x \leq a \vee b, a \in I, b \in J$. Given an element $a \in L$ and an ideal $I \subseteq L$, we have $(a) \wedge I = \{a \wedge x; x \in I\}$, so that we can use the notation $a \wedge I = (a) \wedge I$.

Dually we shall write $a \vee I = (a) \vee I = \{a \vee x; x \in I\}$. Implication in $\mathcal{I}(L)$ is $I \to J = \{x \in L; x \wedge I \subseteq J\}$, hence $\neg I = I \to (0) = \{x \in L; \forall x \in I, x \wedge y = 0\}$. We shall write $a \to I = (a) \to I = \{x \in J; a \wedge x \in I\}$. Note that $a \wedge I \subseteq a \to I$, but the converse is false.

We can force $a = b$ in a distributive lattice L to get a new distributive lattice $L_{(a=b)}$. Because $a = b$ if and only if $a \wedge b = a \vee b$ we can solve only the case $a \leq b$. With this condition, the congruence generated by (a, b) is given by

$$x \cong y \qquad \text{if} \qquad \begin{cases} x \wedge a = y \wedge a \\ x \vee b = y \vee b \end{cases}$$

and the corresponding quotient morphism $l_{(a=b)} : L \to L_{(a=b)}$ is universal among the morphisms $f : L \to D$ in $\mathcal{L}$ such that $f(a) = f(b)$. The kernel of $l_{(a=b)}$ is the ideal $I = \neg(a) \cap (b)$.

3. Chains and cochains in a distributive lattice

We will deal with n-chains $a_1 \leq \cdots \leq a_n$ in a distributive lattice L, and we suppose that any chain is completed with 0 at the bottom and 1 at the top. When necessary, in some formulas we write $0 = a_0, a_{n+1} = 1$. For $n \geq 1$, we shall consider the distributive lattice

$$Ch_n = \{0 < t_1 < t_2 < \cdots < t_{n-1} < t_n < 1\}.$$

It is useful to denote $Ch_0 = 2 = \{0 < 1\}$, and to write Ch_{-1} for the trivial distributive lattice with $0 = 1$, the final object in $\mathcal{L}$. Note that for $n \geq 0$ Ch_n is integer and local. Because $Spec(Ch_n) = \{(0) \subset [0, t_1] \subset \cdots \subset [0, t_n]\}$, we have $KdimCh_n = n$ for all $n \geq 0$. Ch_n can be realized as the lattice of all the chains $x_0 \leq \cdots \leq x_n$ in the ordered set 2, and there is the inclusion $Ch_n \hookrightarrow 2^{n+1} = (Ch_n)_\neg$. Note that in the cube 2^{n+1} the points of Ch_n are the vertex of the $(n+1)$-simplex. A n-chain in a distributive lattice L is a lattice morphisms

$$\mathbf{a} : Ch_n \to L, \ \ \mathbf{a}(t_i) = a_i, \ \ 1 \leq i \leq n,$$

and, of course, $\mathbf{a}(0) = 0, \mathbf{a}(1) = 1$. We will remark that Ch_n can be obtained from $Ch_1 = \{0 < t < 1\}$ by a universal construction:

Proposition 3.1. *For any $1 \leq i \leq n$, the chain $\mathbf{t}_i \leq \cdots \leq \mathbf{t}_n : Ch_1 \to Ch_n$ of morphisms in $\mathcal{L}$, given by $\mathbf{t}_i(t) = t_i$, is universal among the chains $\mathbf{a}_1 \leq \cdots \leq \mathbf{a}_n : Ch_1 \to L$ of morphisms in $\mathcal{L}$.*

$$\mathbf{a}_1 \leq \cdots \leq \mathbf{a}_n \qquad \begin{tikzcd} Ch_1 \arrow[r, "\mathbf{t}_i"] \arrow[d, "\mathbf{a}_i"'] & Ch_n \arrow[dl, dotted, "\mathbf{a}"] \\ L & \end{tikzcd}$$

Proof. For any chain of morphisms $\mathbf{a}_1 \leq \cdots \leq \mathbf{a}_n : Ch_1 \to L$ (chain of 1-chains) there exists a unique n-chain $\mathbf{a} : Ch_n \to L$, given by $\mathbf{a}(t_i) = \mathbf{a_i}(t)$, such that $\mathbf{a} \circ \mathbf{t}_i = \mathbf{a}_i, 1 \leq i \leq n$. □

Now we will consider Ch_n is a kind of free distributive lattice with a chain of generators. To explain this fact we start with a general construction.

We take the sublattice $L[x] \subseteq L \times L$ formed by all the 2-chains $a \leq b$ in L. Because we have (componentwise) $(a, b) = (a, a) \vee ((b, b) \wedge (0, 1))$, if we identify L with the diagonal of $L \times L$, and we denote $x = (0, 1)$, then $(a, b) = a \vee (b \wedge x)$, so the polynomial notation is justified (see [8, pp. 32-33]). Note that, because $a \leq b$, each element $z = (a, b) \in L[x]$ has two expansions $z = a \vee (b \wedge x) = (a \vee x) \wedge b$.

We define recursively $L[x_1, \ldots, x_n] \cong L[x_1, \ldots, x_{n-1}][x]$, but a direct definition is also possible. The following well known universal property shows that $L[x_1, \ldots, x_n]$ is the *free* L-algebra in $\mathcal{L}$ with n generators.

Proposition 3.2. *For any morphism $f : L \to D$ in $\mathcal{L}$, and elements $u_1, \ldots, u_n \in D$, there exists a unique morphism $g : L[x_1, \ldots, x_n] \to D$ in $\mathcal{L}$ such that $g(x) = f(x)$ for all $x \in L$, and $g(x_i) = u_i$ for $1 \leq i \leq n$.*

$$
\begin{tikzcd}
 & L \arrow[r, hook] \arrow[d, "f"'] & L[x_1, \ldots, x_n] \arrow[dl, dotted, "g"] \\
u_1, \ldots, u_n \in D & D &
\end{tikzcd}
$$

Taking $L = 2$ in proposition 3.2 we get $Fr_n = 2[x_1, \ldots, x_n] \cong Fr_{n-1}[x], n \geq 1$. Moreover in $\mathcal{L}$ we have the coproduct $L[x_1, \ldots, x_n] = L + Fr_n$. When $D = 2$ in proposition 3.2, a morphism $g : L[x] \to 2$ corresponds to a prime ideal $g^{-1}(0)$ of $L[x]$, hence these prime ideals are of the form (P, u), with P a prime ideal of L and $u \in 2$. Actually, with $z = a \vee (b \wedge x), a \leq b$, we have $(P, 0) = \{z \in L[x]; a \in P\}$, and $(P, 1) = \{z \in L[x]; b \in P\}$, so that $(P, 1) \subset (P, 0)$ and $Spec(L[x]) \cong Spec(L) \times 2^{op}$, where $2^{op} = \{1 < 0\}$. Hence $KdimL[x] = 1 + dimL$, and by induction $KdimL[x_1, \ldots, x_n] = n + dimL$.

Ch_n and Fr_2 are particular families of distributive lattices obtained as a *tower* $L = L_1 * \cdots * L_n$ of distributive lattices, in which $0_i \in L_i$ is identified with $1_i \in L_{i+1}$ and we take all the relations $x \leq y$ for $x \in L_i$ and $y \in L_j, i \leq j$ (see [3, p. 186]). The n elements $u_i = 1_i = 0_{i+1}$ are the *proper nodes* of the tower L, so that every L_i is the *interval* of L limited by nodes $[u_{i-1}, u_i] = \{x \in L; u_{i-1} \leq x \leq u_i\}$. If L is the trivial distributive lattice $(0 = 1)$ then $L' = L * L' = L' * L$. It is clear that, for $n \geq 0$, Ch_n is a tower of $n + 1$ boolean algebras isomorphic to 2, and Fr_2 is a tower of the form $2 * (2 \times 2) * 2$.

If in $Fr_2 \cong Ch_1[x]$ we force $x_1 \leq x_2$ then Fr_2 becomes Ch_2. Now we see this transformation in general.

Proposition 3.3. *For any distributive lattice L and $a, b \in L$, there is a lattice homomorphism $l_{(a \leq b)} : L \to L_{(a \leq b)}$ which is universal among the morphisms in $\mathcal{L}$ $f : L \to D$ such that $f(a) \leq f(b)$.*

$$f(a) \leq f(b) \qquad \begin{tikzcd} L \arrow[r, "l"] \arrow[d, "f"'] & L_{(a\leq b)} \arrow[dl, dotted, "g"] \\ D & \end{tikzcd}$$

Proof. To force $a \leq b$ is equivalent to force $a \wedge b = a$, so we define the relation

$$x \cong y \qquad \text{if} \qquad \begin{cases} x \wedge a \wedge b = y \wedge a \wedge b \\ x \vee a = y \vee a \end{cases}$$

It is easy to verify that $x \cong y$ is a congruence, so that we have the quotient morphism $l_{(a\leq b)} : L \to L_{(a\leq b)}$. If $\overline{x} = l_{(a\leq b)}(x)$ denotes the class of equivalence of $x \in L$, then $\overline{a} = [a \wedge b, a] = \overline{a \wedge b}$, that is, $l_{(a\leq b)}(a) \leq l_{(a\leq b)}(b)$. Given $f : L \to D$ in $\mathcal{L}$ such that $f(a) \leq f(b)$, we can define $g : L_a \to D$ by $g(\overline{x}) = f(x)$ since $x \cong y$ implies $f(x) \wedge f(a) = f(y) \wedge f(a)$, $f(x) \vee f(a) = f(y) \vee f(a)$, and both conditions in a distributive lattice are equivalent to $f(x) = f(y)$. Then g is the lattice homomorphism we need for this universal construction, to verify the details is straightforward. $\square$

If we take $D = 2$ in proposition 3.3, it results that $Spec(L_{(a\leq b)})$ is the set of all prime ideals of L such that $b \in P$ implies $a \in P$.

The generalization of proposition 3.3 to forcing $a_1 \leq \cdots \leq a_n$ uses the intersection of $n-1$ congruences, and $L_{(a_1\leq\cdots\leq a_n)}$ is the jointly pushout in $\mathcal{L}$ of the family $l_{(a_i\leq a_{i+1})} : L \to L_{(a_i\leq a_{i+1})}$. The details of this construction and the proof of the following theorem are an easy exercise in category calculus.

Theorem 3.4. *For $n \geq 2$:*
(i) $Ch_n \cong Ch_{n-1}[x]_{(t_{n-1}\leq x)}$.
(ii) $Ch_n \cong 2[x_1, \ldots, x_n]_{(x_1\leq\cdots\leq x_n)}$.

In a distributive lattice L, jointly with chains $\mathbf{a} : Ch_n \to L$ we will define *cochains* as lattice homomorphisms $\mathbf{P} : L \to Ch_n$. In the case $n = 0$, we have a unique 0-chain $\mathbf{a} : 2 \to L$, and to give a 0-cochain $\mathbf{P} : L \to 2$ is equivalent to give the prime ideal $\mathbf{P}^{-1}(0)$. In this case we have $\mathbf{P} \circ \mathbf{a} = \mathbf{id}_2$. Now we study the case $n \geq 1$, in which we relate cochains with chains of prime ideal of L following the work of Anderson and Blair (see [1] or [3, pp. 122–123]).

Proposition 3.5. *For $n \geq 1$, a cochain $\mathbf{P} : L \to Ch_n$ is completely determined by a chain $P_0 \subseteq \cdots \subseteq P_n$ of prime ideals of L. The cochain is onto if and only if the chain of prime ideals is non-degenerated.*

Proof. Given a cochain $\mathbf{a} : L \to Ch_n$ the chain $\{0\} \subset [0, t_1] \subset \cdots \subset [0, t_n]$ of all prime ideal of Ch_n determines a chain $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$ of prime ideals of L by inverse image: $P_0 = \mathbf{P}^{-1}(0)$, $P_i = \mathbf{P}^{-1}([0, t_i])$, $1 \leq i \leq n$. Conversely, given a chain $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$ of prime ideals we can define a map $\mathbf{P} : L \to Ch_n$ by $\mathbf{P}(x) = 0$ if and only if $x \in P_0$, $\mathbf{P}(x) = t_i$ if and only if $x \in P_i$ and $x \notin P_{i-1}$, $1 \leq i \leq n$, and $\mathbf{P}(x) = 1$ if and only if $x \notin P_n$. It is clear that $\mathbf{P}(0) = 0$, $\mathbf{P}(1) = 1$, and $\mathbf{P}$ is a monotone map. To prove that $\mathbf{P}(x \vee y) \leq \mathbf{P}(x) \vee \mathbf{P}(y) = \max\{\mathbf{P}(x), \mathbf{P}(y)\}$, and $\mathbf{P}(x \wedge y) \geq \mathbf{P}(x) \wedge \mathbf{P}(y) = \min\{\mathbf{P}(x), \mathbf{P}(y)\}$ is an easy exercise by using that $x \vee y \in P_i$ if and only if $x \in P_i$ and $y \in P_i$ (P_i ideal), and $x \wedge y \in P_i$ if and only if $x \in P_i$ or $y \in P_i$ (P_i prime ideal). Finally, note that $\mathbf{P}^{-1}(t_i) = P_i \setminus P_{i-1}$, so that the chain of prime ideals is non degenerated if and only if the cochain $\mathbf{P}$ is onto. □

After proposition 3.5, we can say that $KdimL$ is the greatest n such that there exists a onto cochain $\mathbf{P} : L \to Ch_n$.

Definition 3.6. *In a distributive lattive L we say that a n-chain $\mathbf{a} : Ch_n \to L$* belongs to *a n-cochain $\mathbf{P} : L \to Ch_n$, $n \geq 1$, and we write $\mathbf{a} \in \mathbf{P}$, if $\mathbf{P} \circ \mathbf{a} = \mathbf{id}$.*

Remark that the relation $\mathbf{a} \in \mathbf{P}$ means that $\mathbf{a}$ is a section with retraction $\mathbf{P}$. Hence $\mathbf{a}$ is injective and gives us a non degenerated chain of elements $0 < a_1 < \cdots < a_n < 1$. On the other hand, $\mathbf{P}$ is onto and gives us (proposition 3.5) a related non-degenerated chain of prime ideals $P_0 \subset P_1 \subset \cdots \subset P_n$.

Proposition 3.7. *For $n \geq 1$, a n-cochain $\mathbf{P} : L \to Ch_n$ is a retraction if and only if it is onto.*

Proof. It is clear that retraction implies onto. Conversely, given a onto n-cochain $\mathbf{P} : L \to Ch_n$, by proposition 3.5 we have a non degenerated chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of prime ideals of L, so that we can take elements $a'_i \in P_i \setminus P_{i-1}$, $1 \leq i \leq n$. Now it is easy to verify that the elements $a_i = a'_1 \vee \cdots \vee a'_i$, $1 \leq i \leq n$, determine a n-chain $\mathbf{a}$ that belongs to $\mathbf{P}$. □

4. Linked chains in distributive lattices

In this section we shall study a relation $\mathbf{a} \diamond \mathbf{b}$ between chains with the same domain, which is related to a new sequence $Link_n$ of distributive lattices.

Definition 4.1. *For $n \geq 1$, a n-*link *in a distributive lattice L is a pair $\mathbf{a}, \mathbf{b}$ of n-chains such that*

$$(*) \begin{cases} a_1 \wedge b_1 = 0 \\ a_i \vee b_i = a_{i+1} \wedge b_{i+1}, i = 1, \ldots, n-1 \\ a_n \vee b_n = 1 \end{cases}$$

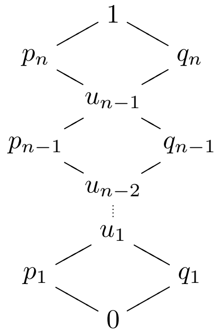
Then we say that the chains $\mathbf{a}, \mathbf{b}$ are linked, *and write $\mathbf{a} \diamond \mathbf{b}$. A n-chain $\mathbf{a}$ is said* linked *if there exists a n-link $\mathbf{a} \diamond \mathbf{b}$, otherwise the chain is said* independent.

We must remark that an independent chain is a *locally separated* chain in the sense used by Anderson and Blair [1] to characterize those distributive lattices which are subdirect product of chains (see [3, pp. 122–123]). Note that $(*)$ gives a *link* for any chain $x_1 \leq \cdots \leq x_n$ with $x_i = a_i$ or $x_i = b_i$ in each index.

Proposition 4.2. *There exists a universal n-link $\mathbf{p} \diamond \mathbf{q} : Ch_n \to Link_n$.*

Proof. Let us denote $Link_n$, $n \geq 1$, the distributive lattice defined by the link conditions, that is, all the elements are different and

$$\begin{cases} p_1 \wedge q_1 = 0 \\ u_i = p_i \vee q_i = p_{i+1} \wedge q_{i+1}, i = 1, \dots, n-1 \\ p_n \vee q_n = 1 \end{cases}$$



There are n-chains $\mathbf{p}, \mathbf{q} : Ch_n \hookrightarrow Link_n$, $\mathbf{p}(t_i) = p_i$, $\mathbf{q}(t_i) = q_i$, $1 \leq i \leq n$, such that $\mathbf{p} \diamond \mathbf{q}$. Finally, for any n-link $\mathbf{a} \diamond \mathbf{b} : Ch_n \to L$ there exists a unique lattice homomorphism $\mathbf{l} : Link_n \to L$ such that $\mathbf{l} \circ \mathbf{p} = \mathbf{a}, \mathbf{l} \circ \mathbf{q} = \mathbf{b}$.

$$\begin{tikzcd} Ch_n \arrow[r, "\mathbf{p}\diamond\mathbf{q}"] \arrow[d, "\mathbf{a}\diamond\mathbf{b}"'] & Link_n \arrow[dl, dotted, "\mathbf{l}"] \\ L & \end{tikzcd}$$

□

$Link_n$ is a tower of n boolean algebras isomorphic to 2×2, and it is neither integer nor local. $Link_1$ is a boolean algebra, but $\mathcal{C}(Link_n) \cong 2$ if $n \geq 2$. $Spec(Link_n)$ is formed by two similar chains of subsets: $[0, p_1] \subset \cdots \subset [0, p_n]$ and $[0, q_1] \subset \cdots \subset [0, q_n]$, so $KdimLink_n = n - 1$. The next results are an easy consequence:

Lemma 4.3. *(i) Does not exist onto cochains $Link_n \to Ch_n$.*
(ii) The monomorphisms $\mathbf{p}, \mathbf{q} : Ch_n \hookrightarrow Link_n$ are not sections in $\mathcal{L}$.

Corollary 4.4. *A linked chain $\mathbf{a} : Ch_n \to L$ is not a section in $\mathcal{L}$.*

Proof. Given a linked chain $\mathbf{a} : Ch_n \to L$ we have a factorization through the canonical monomorphism $\mathbf{p}$ (or $\mathbf{q}$), $\mathbf{a} = \mathbf{l} \circ \mathbf{p}$. If $\mathbf{a}$ is a section then there exists a cochain $\mathbf{P} : L \to Ch_n$ such that $\mathbf{P} \circ \mathbf{a} = \mathbf{id}$, so that $(\mathbf{P} \circ \mathbf{l}) \circ \mathbf{p} = \mathbf{id}$, a contradiction by lemma 4.3(ii). □

In the diamond generated by the elements a, b, we have the lattice isomorphism $[p, a] \times [a, q] \cong [p, b] \times [b, q]$. Note that an element $(x, y) \in [p, a] \times [a, q]$ is a chain $x \leq y$ in $[p, q]$ with a *interpolated*, that is, $p \leq x \leq a \leq y \leq q$. To this element it corresponds in the above isomorphism $(y \wedge b, x \vee b) \in [p, b] \times [b, q]$, and the chain $x \leq y$ is linked in $[p, q]$ by the chain $y \wedge b \leq x \vee b$, the node of the link being $u = x \vee (y \wedge b) = y \wedge (x \vee b)$. As a corollary we obtain that if $x \leq a \leq y$ with a

complemented then $x \leq y$ is linked. Now we give more general examples of linked and independent chains.

Proposition 4.5. *(i) Given a n-chain $a_1 \leq \cdots \leq a_n$, $n \geq 1$, if the k-chain $a_1 \leq \cdots \leq a_k$ is linked in the interval $[0, a_{k+1}]$, $1 \leq k < n$, then the complete n-chain is also linked.*

(ii) Given a chain, if a subchain is linked then the given chain is also linked.

(iii) The n-chain $a_1 \leq \cdots \leq a_n$, $n \geq 1$, is linked when there exists a complemented element b such that $a_i \leq b \leq a_j$ for some $1 \leq i \leq j \leq n$.

Proof. (i) By induction, it is enough to prove the case $k = n - 1$. Given a link of $a_1 \leq \cdots \leq a_{n-1}$ in $[0, a_n]$ we complete the link adding at the top the diamond generated by $a_n, 1$. (ii) By induction, only three cases must be considered. First, suppose $a_1 \leq \cdots \leq a_n$ linked, and $a_n \leq a_{n+1}$. If the top diamond of the given link is of the form showed bellow at left, then we have a link for $a_1 \leq \cdots \leq a_{n+1}$ finishing as it is showed bellow at right:

$$\begin{tikzcd}[column sep=small]
& 1 \arrow[dl, dash] \arrow[dr, dash] & \\
a_n \arrow[dr, dash] & & b_n \arrow[dl, dash] \\
& u_n &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
& 1 \arrow[dl, dash] \arrow[dr, dash] & \\
a_{n+1} \arrow[dr, dash] & & 1 \arrow[dl, dash] \\
& a_{n+1} \arrow[dl, dash] \arrow[dr, dash] & \\
a_n \arrow[dr, dash] & & a_{n+1} \wedge b_n \arrow[dl, dash] \\
& u_n &
\end{tikzcd}$$

The second case, when $a_1 \leq \cdots \leq a_n$ is linked and $a_0 \leq a_1$ is dual of the first one. Third case, suppose $a_1 \leq \cdots \leq a_n$ linked, and $a_i \leq a \leq a_{i+1}$, $1 \leq i < n$. The given link has the two diamonds showed at left, which we can change by the three diamonds at right to link the chain $a_1 \leq \cdots \leq a_i \leq a \leq a_{i+1} \leq \cdots \leq a_n$.

$$\begin{tikzcd}[column sep=small]
& u_{i+2} \arrow[dl, dash] \arrow[dr, dash] & \\
a_{i+1} \arrow[dr, dash] & & b_{i+1} \arrow[dl, dash] \\
& u_{i+1} \arrow[dl, dash] \arrow[dr, dash] & \\
a_i \arrow[dr, dash] & & b_i \arrow[dl, dash] \\
& u_i &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
& u_{i+2} \arrow[dl, dash] \arrow[dr, dash] & \\
a_{i+1} \arrow[dr, dash] & & a \vee b_{i+1} \arrow[dl, dash] \\
& a \vee u_{i+1} \arrow[dl, dash] \arrow[dr, dash] & \\
a \arrow[dr, dash] & & u_{i+1} \arrow[dl, dash] \\
& a \wedge u_{i+1} \arrow[dl, dash] \arrow[dr, dash] & \\
a_i \arrow[dr, dash] & & a \wedge b_i \arrow[dl, dash] \\
& u_i &
\end{tikzcd}$$

(iii) It is easy to check that $a_i \leq a_j$ is linked by $a_j \wedge \neg b \leq a_i \vee \neg b$, and then we apply (ii). $\square$

Corollary 4.6. *(i) Every degenerated n-chain, $n \geq 2$, is linked.*

(ii) The n-chain $a_1 \leq \cdots \leq a_n$, $n \geq 1$, is linked when some a_i is complemented.

(iii) In a boolean algebra any chain is linked.

Proof. (i) By proposition 4.5(ii), because $a_i \leq a_i$ is linked by $0 \leq 1$. (ii) It is a particular case of (i), and also follows from proposition 4.5(ii) because a_i is a linked subchain. (iii) It is clear from (ii). □

By proposition 4.5 every independent chain $\mathbf{a} : Ch_n \to L$ must be a monomorphism in $\mathcal{L}$ (injective). Of course, monomorphic chains can be linked, but a linked chain is not a section (corollary 4.4). Note that two different links of a chain must have different nodes because, if the nodes are given, then the link is unique if it exists. In fact, given a link in L (see $(*)$ above), every a_i is complemented in the interval $[u_{i-1}, u_i]$, with x_i as the complement of a_i in that interval.

In a general distributive lattice L, a *node* is an element $u \in L$ such that $x \leq u$ or $u \leq x$ for any $x \in L$. A node u is *proper* if $0 \neq u \neq 1$. If there exists a proper node then $\mathcal{C}(L) \cong 2$. The proper nodes of a distributive lattice L form a chain.

Proposition 4.7. *Any non degenerated n-chain of nodes is independent.*

Proof. After proposition 4.5 we can suppose a chain $u_1 < \cdots < u_n$ of proper nodes. If we have a link with $x_1 \leq \cdots \leq x_n$ then $u_n \vee x_n = 1$ implies $x_n = 1$, and so $x_{n-1} = u_n, \ldots, x_1 = u_2$, so that $0 = u_1 \wedge x_1 = u_1 \wedge u_2 = u_1$, a contradiction. □

Note that in proposition 4.5(iii) it is enough to consider a complemented element between a_1 and a_n, that is, in the interval $[a_1, a_n]$. Hence $[a_1, a_n] \cap \mathcal{C}(L) \neq \emptyset$ implies that the chain a is linked, and any independent chain is contained in $L \setminus \mathcal{C}(L)$. If both a and b are complemented then $a \leq b$ can be linked (in $\mathcal{C}(L)$) by $\neg a \wedge b \leq 1$ or $0 \leq a \vee \neg b$; hence the link of a chain (the solution of the system of equations $(*)$ if it exists) is not unique. Following with this case, we observe that $a \leq b$ can also be linked by a chain $x \leq y$ not in $\mathcal{C}(L)$ (for instance $0 \leq 1$ is linked with any $a \leq a$), and then the chain $\neg b \leq \neg a$ is also linked by $x \leq y$. Finally, if x, y are also complemented, then $\neg b \leq \neg a$ is also linked by $\neg y \leq \neg x$ with node $\neg u$.

$$
\begin{array}{ccccccccccccc}
 & & 1 & & = & & 1 & & = & & 1 & & \\
 & / & & \backslash & & / & & \backslash & & / & & \backslash & \\
b & & & & y & & & & \neg a & & & & \neg x \\
 & \backslash & & / & & \backslash & & / & & \backslash & & / & \\
 & & u & & & & v & & & & \neg u & & \\
 & / & & \backslash & & / & & \backslash & & / & & \backslash & \\
a & & & & x & & & & \neg b & & & & \neg y \\
 & \backslash & & / & & \backslash & & / & & \backslash & & / & \\
 & & 0 & & = & & 0 & & = & & 0 & &
\end{array}
$$

For any distributive lattice L we shall consider the subset $\mathcal{C}_1(L) \subseteq L_\neg$ of all elements of the form $\neg a \wedge b, a, b \in L$. Note that L and $\neg L = \{\neg a; a \in L\}$ belong to $\mathcal{C}_1(L)$, which is closed by finite meets. Because $\neg a \wedge b = \neg(a \wedge b) \wedge b = \neg a \wedge (a \vee b)$, we can consider these elements with the condition $a \leq b$ in L; then $\mathcal{C}_1(L) = \{a + b; a, b \in L\}$ where $a + b$ is the boolean addition.

Proposition 4.8. *Given $x \in \mathcal{C}_1(L)$ of the form $x = \neg a \wedge b$, $a \leq b$ in L, the complement $\neg x$ belongs to $\mathcal{C}_1(L)$ if and only if $a \leq b$ is linked in L.*

Proof. An easy boolean calculus shows that $\neg x = \neg c \wedge d$, $c \leq d$ in L if and only if $a \leq b$ is linked by $c \leq d$. □

5. Elementary Krull dimension

In this last section we introduce an elementary Krull dimension for distributive lattices, that is, we give an elementary definition of dimension, $dimL$, and we prove that $dimL = KdimL$.

Definition 5.1. *The* dimension *of a distributive lattice L, denoted* dimL*, is the greatest n such that there exists an independent chain $\mathbf{a} : Ch_n \to L$.*

Lemma 5.2. *Given a n-chain $\mathbf{a} : Ch_n \to L$, $n \geq 1$, we define the ideals $I_0 = \neg(a_1)$, $I_{i+1} = a_{i+2} \to (a_{i+1} \vee I_i)$, $0 \leq i \leq n-2$. The following statements hold:*
(i) If $\mathbf{a} \diamond \mathbf{b}$ then $b_i \in I_{i-1}$, $1 \leq i < n$.
(ii) If $a_i \in I_{i-1}$ for some $i = 1, \ldots, n$ then the chain $\mathbf{a}$ is linked.
(iii) $\mathbf{a}$ is independent if and only if $a_n \notin I_{n-1}$ and $a_i \in I_i \setminus I_{i-1}$, $i = 1, \ldots, n-1$.

Proof. First we note that $a_{i+1} \vee I_i \subseteq I_{i+1}$, that is, $a_{i+1} \in I_{i+1}$ and $I_i \subseteq I_{i+1}$. Now we proceed: (i) By induction: it is clear that $b_1 \in I_0 = \neg(a_1)$. Now $b_i \in I_{i-1}$ is equivalent to $a_i \wedge b_i \in a_{i-1} \vee I_i$, and that is true because $a_i \wedge b_i = a_{i-1} \vee b_{i-1}$, with $b_{i-1} \in I_i$ by the inductive hypothesis. (ii) Case $i = 1$: $a_1 \in I_0 = \neg(a_1)$ means that $a_1 = 0$, so the chain a is linked by proposition 4.5. For $i \geq 1$, by using the note above, we have: $a_i \in I_{i-1}$ means that $a_i = a_{i-1} \vee x_{i-1}$, with $x_{i-1} \in I_{i-2}$, but this condition means that $a_{i-1} \wedge x_{i-1} = a_{i-2} \vee x_{i-2}$, with $x_{i-2} \in I_{i-3}$, and so on. Hence the chain $a_1 \leq \cdots \leq a_{i-1}$ is linked in $[0, a_i]$, so that the complete chain a is linked by proposition 4.5. (iii) follows from (ii). □

Theorem 5.3. *Given a distributive lattice L:*
(i) For $n \geq 1$, $\mathbf{a} : Ch_n \to L$ is a section if and only if is independent.
(ii) $dimL = KdimL$.

Proof. (i) Given a section $\mathbf{a} : Ch_n \to L$, with $\mathbf{P} \circ \mathbf{a} = \mathbf{id}$, then $\mathbf{a}$ is mono (non-degenerated), and $\mathbf{P}$ is onto, so by proposition 3.5 we have a non-degenerated chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of prime ideals such that $a_i \in P_i \setminus P_{i-1}$. Given a link $(*)$, $a_1 \wedge x_1 = 0 \in P_0$ with $a_1 \notin P_0$ implies $x_1 \in P_0$, and so $a_1 \vee x_1 \in P_1$; but then $a_2 \wedge x_2 = a_1 \vee x_1 \in P_1$ implies $x_2 \in P_1$, and so on. Finally $x_n \in P_{n-1} \subset P_n$ implies $1 = a_n \vee x_n \in P_n$, a contradiction. Hence $\mathbf{a}$ is independent.

Conversely, suppose $\mathbf{a} : Ch_n \to L$ independent. By lemma 5.2(iii) we have a chain of ideals $I_0 \subset \cdots \subset I_{n-1}$ such that $a_n \notin I_{n-1}$ and $a_i \in I_i \setminus I_{i-1}$, $i = 1, \ldots, n-1$. Then we can determine a chain of prime ideals $P_0 \subset \cdots \subset P_n$ such that $a_i \in P_i \setminus P_{i-1}$, $i = 1, \ldots, n$, by using the prime ideal theorem (theorem 2.1, for details see [3, p. 123]).

(ii) It follows from (i) and propositions 3.5 and 3.7. □

Theorem 5.3(ii) shows that an elementary Krull dimension theory of distributive lattices based on definition 5.1 is possible. We leave for a second paper the autonomous development of this link theory, which is an equivalent variation of the elementary constructive Krull dimension theory of Coquand and Lombardi [4].

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