TESIS DOCTORAL

Acotaciones con pesos de operadores relacionados con las series de Fourier-Bessel

Luz Roncal Gómez

UNIVERSIDAD DE LA RIOJA

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Weighted boundedness of operators related to the Fourier-Bessel expansions

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To my parents,
who always trusted and supported me.

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Introduction

Let $(\Omega, \mathcal{M}, d\mu)$ be a measure space ($\mu \geq 0$). We consider

$$L^p(\Omega, d\mu) = \{f : \Omega \to \mathbb{C},\ f \text{ measurable } : \|f\|_{L^p(\Omega,d\mu)} < \infty\},$$

where

$$\|f\|_{L^p(\Omega,d\mu)} = \left(\int_\Omega |f|^p \, d\mu\right)^{1/p},$$

for $1 \leq p < \infty$, and

$$\|f\|_{L^\infty(\Omega,d\mu)} = \operatorname{ess\,sup}\{|f(x)| : x \in \Omega\}.$$

Let $\{\psi_j\}_{j\in\Lambda}$ be a sequence of functions, where Λ is a set of indexes contained in $\mathbb{Z}$ (or in $\mathbb{Z}^d$), orthonormal on $L^2(\Omega, d\mu)$, that is,

$$\int_\Omega \psi_j \overline{\psi_m} \, d\mu = \delta_{j,m}.$$

The Fourier expansion of an appropriate function f associated to the system $\{\psi_j\}_{j\in\Lambda}$, is given by

$$f \sim \sum_{j\in\Lambda} a_j(f)\psi_j,$$

where

$$a_j(f) = \int_\Omega f\overline{\psi_j} \, d\mu.$$

Convergence properties of these series in spaces $L^p(\Omega, d\mu)$ have been studied deeply for particular sets of $\{\psi_j\}_{j\in\Lambda}$. If we define the partial sums for the Fourier expansions for each $f \in L^2(\Omega, d\mu)$ as

$$S_n f = \sum_{\substack{j\in\Lambda \\ |j|\leq n}} a_j(f)\psi_j,$$

then, Hilbert spaces theory ensures that

$$\|S_n f\|_{L^2(\Omega,d\mu)} = \left(\sum_{\substack{j\in\Lambda \\ |j|\leq n}} |a_j(f)|^2\right)^{1/2} \leq \|f\|_{L^2(\Omega,d\mu)}$$

and

$$\lim_{n\to\infty} \int_\Omega |f - S_n f|^2 \, d\mu = 0,$$

for each $f \in L^2(\Omega, d\mu)$ whenever the system $\{\psi_j\}_{j\in\Lambda}$ is complete in the aforementioned space. The first question to be asked immediately is related to the study about mean convergence, that is, for which values of $p, 1 \le p < \infty$, we have that, for every $f \in L^p(\Omega, d\mu)$,

$$\lim_{n\to\infty} \int_\Omega |f - S_n f|^p \, d\mu = 0$$

holds. When this last equality is verified, then we say that the sequence $\{\psi_j\}_{j\in\Lambda}$ forms a basis for the space of these functions. In order to solve the problem it is only necessary to study the uniform boundedness of the operator S_n.

This kind of problems were first investigated by Riesz [**49**], who analyzed the convergence of the classical Fourier series, associated to the system $\{e^{ikx}\}_{k\in\mathbb{Z}}$. Other orthonormal systems have been studied in mathematical literature through the years.

For instance, Pollard [**45, 46, 47, 48**], Muckenhoupt [**33**], Badkov [**3**], Pérez [**43**], Varona [**56**] and Guadalupe-Pérez-Varona [**26**] have depeloped research into the convergence of Fourier expansions of Jacobi polynomials, orthonormal on $L^2([-1,1], (1-x)^\alpha(1+x)^\beta \, dx)$.

Concerning similar results about Laguerre polynomials and functions, orthonormal on $L^2((0,\infty), e^{-x}x^\alpha \, dx)$ and on $L^2((0,\infty), dx)$ respectively, we can see the works by Askey-Wainger [**2**] and Muckenhoupt [**34, 35**]. The same references can be consulted for Hermite polynomials and functions, orthonormal on $L^2(\mathbb{R}, e^{-x^2} \, dx)$ and on the space $L^2(\mathbb{R}, dx)$ respectively.

With regard to the Fourier-Bessel system, consisting of Bessel functions and orthonormal on $L^2([0,1], x\, dx)$, we can check the results by Wing [**62**], Benedek-Panzone [**7, 8**], Pérez [**43**], Varona [**56**] and Guadalupe-Pérez-Ruiz-Varona [**25**].

Finally, we refer results by Benedek-Panzone [**6**], Barceló-Córdoba [**5**], Ciaurri [**14**], Ciaurri-Guadalupe-Pérez-Varona [**15**] and Generozov [**24**] about certain orthonormal systems formed by eigenfunctions of second order differential operators.

It is common to consider other summation methods for the Fourier series when the convergence of the partial sum operator fails. There exist very interesting results involving Cesàro means with potential weights, obtained by Muckenhoupt-Webb [**39, 40**] in the setting of Fourier-Laguerre and Fourier-Hermite expansions, respectively. These results extend a previous work developed by Poiani [**44**]. Recently, Ciaurri-Varona have been checking up on the same question [**21**] in the context of Fourier series of generalized Hermite functions, extending the results in [**40**]. An exhaustive analysis for the Cesàro means of Fourier-Jacobi series has been carried out in [**12**], in particular the weak behavior of such means is investigated.

Bochner-Riesz means are another important summation method used frequently in harmonic analysis. Bochner-Riesz summability has been treated, for example, for Fourier and Hankel transforms. In the case of multidimensional Fourier transform, the problem of convergence of this summability method has been solved completely only for dimension two, see [**22**], and for radial functions, see [**60**]. The general case continues open. The Fourier transform for multidimensional radial functions turns into the Hankel transform of a fixed order. The study of Bochner-Riesz means for the Hankel transform has been developed, for any order, in [**20, 19**].

For $\delta > 0$, we define the Bochner-Riesz means for an orthonormal system $\{\psi_j\}_{j\in\Lambda}$ by means of the identity

$$B_R^\delta(f,x) = \sum_{j\in\Lambda} \left(1 - \frac{r_j^2}{R^2}\right)_+^\delta a_j(f)\psi_j(x),$$

where $R > 0$, $(1-s^2)_+ = \max\{1-s^2, 0\}$ and r_j is a sequence properly chosen. As in the case of partial sum operators, it is easy to see that the convergence of this summation method follows from the uniform boundedness of B_R^δ, whenever we have the density of the system $\{\psi_j\}_{j\in\Lambda}$.

The first goal of this dissertation is to analyze the convergence, for radial functions, of Bochner-Riesz means for the multidimensional Fourier-Bessel series. This orthonormal system consists of the eigenfunctions of Laplace operator in the multidimensional unit ball. The analysis of the partial sums associated to this system has been studied in [**4**] for Lebesgue spaces with a mixed norm. In the radial case, and without further comments, we will refer to these series as the Fourier-Bessel series.

There is another type of convergence that is frequently dealt with in the literature, the almost everywhere convergence. The origin of this question can be found in the conjecture of Lusin (1915). It states that the classical Fourier series converges almost everywhere to f, for every $f \in L^2$. As known, the proof of this conjecture is due to Carleson [**10**], and the extension to L^p, $1 < p < \infty$, to Hunt [**30**]. In general, if the orthonormal system $\{\psi_j\}_{j\in\Lambda}$ is dense, the boundedness of the corresponding maximal operator of the partial sums of Fourier series in the $L^p(\Omega, d\mu)$ space implies the almost everywhere convergence of the Fourier series.

Our second aim will be to analyze the supremum of the Bochner-Riesz means for Fourier-Bessel expansions. We will study weighted L^p norm inequalities for this operator when $1 < p \le \infty$ and weak type inequalities for $p = 1$.

Partial sums and Bochner-Riesz means can be seen as particular cases of multipliers for Fourier-Bessel series. There exists a very helpful tool that is frequently applied in the study of multipliers, the square functions or g_k-functions. The theory of g_k-functions was first developed through thirties of the last century by Littlewood and Paley, Zygmund, and Marcinkiewicz. In these first approaches, the g_k-functions were defined for the Fourier series. They are non-linear operators which allow us to give a useful characterization of the L^p norm of a function in terms of the behavior of its Poisson integral. Among other applications, the g_k-functions are used to obtain results for multipliers with Hörmander conditions and estimates for the Riesz transform. Theorems for multipliers using g_k-functions have been proved in [**51**] for the d-dimensional Fourier transform, in [**38**] for the ultraspherical expansions, in [**54**] for Hermite expansions, in [**32**] for general semigroups, and in [**23**] for Laguerre expansions and including potential weights. The g_k-functions appear in results related to the Riesz transforms in [**27**] for the Ornstein-Uhlenbeck semigroup, in [**28**] for the Hermite semigroup, and in [**41**] for the Laguerre expansions.

As a last target, g_k-functions related to the Poisson semigroup of Fourier-Bessel expansions will be defined for each $k \geq 1$. It will be proved that these g_k-functions are Calderón-Zygmund operators in the sense of the associated space of homogeneous type, and then we will deduce mapping properties from the general theory. This last result is a first step towards a very general theorem about multipliers for Fourier-Bessel expansions, and therefore a future research remains open for this problem.

This report is divided into three chapters. The first chapter has an introductory nature and we describe there the system of orthogonal functions which will be analized in this report. Besides, in this chapter we will deal with several topics about the partial sum operator. Known results for partial sums related to the multidimensional Fourier-Bessel series will be shown, in the general and in the radial case.

Chapter 2 is devoted to the study of the convergence, for radial functions, of the Bochner-Riesz means of multidimensional Fourier-Bessel series. In the first section we will start by checking that this fact is equivalent to the weighted boundedness of Bochner-Riesz means for Fourier-Bessel series. Before introducing the principal results, we will need to show our conditions on the weights, and several definitions. We will proceed to enunciate the main result, an inequality with general weigths and its main corollary, a similar result with power weights. In the second section we will obtain a pointwise estimate for the kernel of the Bochner-Riesz means for the Fourier-Bessel expansions. Section 3 will contain the proof of the Main Theorem. In the fourth section, the proof of the Main Corollary is shown. In Section 5, some consequences of the Main Corollary are obtained, such as the convergence of the

Bochner-Riesz means and the boundedness of other operators related to the Fourier-Bessel series. We prove weak type inequalities for $p = 1$ in Section 6 and in the last section we study the almost everywhere convergence.

In the third chapter, a study of g_k-functions for Fourier-Bessel expansions is carried out. The main result is a characterization of the L^p norm of a function in terms of the behavior of the Poisson integral corresponding to the Fourier-Bessel series including weights. Before stating our result about g_k-functions, we introduce some topics concerning these operators and, on the other hand, about the Calderón-Zygmund theory. In the second section, we go on to state that the g_k-functions can be seen as vector-valued Calderón-Zygmund operators. The rest of the chapter is devoted to proving this fact, showing several sharp technical lemmas that are needed to check that the kernel associated to the g_k-functions is a standard kernel in the properly defined Banach space.

CHAPTER 1

The Fourier-Bessel system

In this chapter we will describe the multidimensional Fourier-Bessel system. Although the main results in this dissertation will be focused on the radial case, we are going to show first the complete family of functions. This orthonormal system will consist of the eigenfunctions of the Laplacian operator on the d-dimensional unit ball B^d with $d \geq 2$. Each one of the functions of the orthonormal system will be, essentially, the product of a Bessel function times a spherical harmonic.

Before going on, a remark is in turn. There exists a large number of families of orthogonal functions that appear as eigenfunctions of differential operators of second order. For instance, the functions $\{e^{ik\pi x}\}_{k\in\mathbb{Z}}$ associated to the classical Fourier series are the eigenfunctions of the operator $\frac{d^2}{dx^2}$ in the interval $[-1,1]$. Therefore, it seems natural to think about the analysis of the behavior of the Fourier series associated to the eigenfunctions of the Laplacian operator Δ on the d-dimensional unit ball.

Let us begin with several definitions and concepts that will be significant in our work. Let $\mathcal{P}_k$ be the space of homogeneous polynomials of degree k in $\mathbb{R}^d$, and let

$$H_k = \{P|_{S^{d-1}} : P \in \mathcal{P}_k \text{ and } \Delta P = 0\}.$$

The elements in H_k are called spherical harmonics of degree k. The dimension of this space will be denoted by d_k

$$d_k = \dim H_k = (2k+d-2)\frac{(k+d-3)!}{k!(d-2)!}.$$

We now define the Bessel function of order ν, with $\nu > -1$, by

$$J_\nu(t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(\nu+1+j)}\left(\frac{t}{2}\right)^{\nu+2j}, \qquad t > 0.$$

By $\{s_{j,\nu}\}_{j\geq 1}$ we denote the sequence of successive positive zeros of J_ν, in growing order.

Let $\{Y^{m,k}\}$ be, with $1 \leq m \leq d_k$, an orthonormal basis for H_k in $L^2(S^{d-1}, d\sigma)$, where S^{d-1} is the unit sphere and $d\sigma$ is the normalized measure on it. For each $d \geq 2$, we define the functions $\Phi^{(d)}_{j,k,m}(x)$, with $j \geq 1, k \geq 0, 1 \leq m \leq d_k$, as

$$\Phi^{(d)}_{j,k,m}(x) = |x|^{(1-d)/2}\varphi_j^{k+(d-2)/2}(|x|)Y^{m,k}\left(\frac{x}{|x|}\right),$$

where

$$\varphi_j^\nu(r) = \frac{\sqrt{2r}}{|J_{\nu+1}(s_{j,\nu})|} J_\nu(s_{j,\nu} r), \quad j = 1, 2, \dots . \tag{1}$$

This system is the solution of the Dirichlet problem in the unit ball

$$\Delta u = -\lambda^2 u \text{ in } B^d, \qquad u = 0 \text{ in } S^{d-1},$$

as we can see in the following result, that is well known (see [**52**]).

Theorem 1.1. *If $d \geq 2$, it is verified that*

$$\Delta\Phi_{j,k,m}^{(d)}(x) = -(s_{j,k+\frac{d-2}{2}})^2 \Phi_{j,k,m}^{(d)}(x), \quad x \in B^d,$$

and

$$\Phi_{j,k,m}^{(d)}(x) = 0, \quad x \in S^{d-1}.$$

Furthermore, the set of functions

$$\{\Phi_{j,k,m}^{(d)} : j \geq 1, k \geq 0, 1 \leq m \leq d_k\}$$

is orthonormal and complete in $L^2(B^d, dx)$.

We will develop our research in the analysis of the radial case, which corresponds with the case $k = 0$. In this case, the Laplacian operator is reduced to an ordinary differential equation called radial part of Laplacian, that is, if $f(x) = \phi(r)$ where $x \in \mathbb{R}^d$ and $r = |x|$,

$$\Delta f(x) = \phi''(r) + \frac{d-1}{r}\phi'(r).$$

Then, the eigenfunctions to be dealt with are

$$\begin{aligned}\Phi_{j,0,m}^{(d)}(x) &= |x|^{(1-d)/2} \varphi_j^{0+(d-2)/2}(|x|) Y^{m,0}\left(\frac{x}{|x|}\right) \\ &= |x|^{(1-d)/2} \varphi_j^{(d-2)/2}(|x|).\end{aligned}$$

Before describing the known results about partial sums for multidimensional Fourier-Bessel series, we are going to introduce some topics related to this kind of operators in the spaces $L^p(\Omega, d\mu)$ with weights.

The results about convergence can be generalized to a larger family of spaces. Given a family $\{\psi_j\}_{j\in\Lambda}$, orthonormal in $L^2(\Omega, d\mu)$, and w a nonnegative function μ-measurable, then $\{w\psi_j\}_{j\in\Lambda}$ is an orthonormal system in $L^2(\Omega, w^{-2} d\mu)$:

$$\int_\Omega (w\psi_j)(\overline{w\psi_m}) w^{-2}\, d\mu = \int_\Omega \psi_j \overline{\psi_m}\, d\mu = \delta_{j,m}.$$

Let S_n and $\mathbf{S}_n$ be the partial sum operators with respect to $\{\psi_j\}_{j\in\Lambda}$ and $\{w\psi_j\}_{j\in\Lambda}$ respectively. Then,

$$\mathbf{S}_n f = \sum_{\substack{j\in\Lambda \\ |j|\leq n}} \left(\int_\Omega f w \overline{\psi_j} w^{-2}\, d\mu\right) w\psi_j = w S_n(w^{-1} f).$$

From this formula, the following relation between the boundedness of S_n and $\mathbf{S}_n$ can be deduced, that is, $\forall f \in L^p(\Omega, w^{-2}d\mu)$,

$$\|\mathbf{S}_n f\|_{L^p(\Omega,w^{-2}d\mu)} \le C\|f\|_{L^p(\Omega,w^{-2}d\mu)} \\ \Leftrightarrow \|S_n(w^{-1}f)\|_{L^p(\Omega,w^{p-2}d\mu)} \le C\|w^{-1}f\|_{L^p(\Omega,w^{p-2}d\mu)}$$

holds, and taking $g = w^{-1}f$, this is satisfied if and only if

$$\|S_n g\|_{L^p(\Omega,w^{p-2}d\mu)} \le C\|g\|_{L^p(\Omega,w^{p-2}d\mu)}, \quad \forall g \in L^p(\Omega, w^{p-2}d\mu).$$

This fact leads us to the study of boundedness of the partial sums for Fourier series with weights (measurable and nonnegative functions); that is, this kind of inequalities:

$$\|S_n f\|_{L^p(\Omega,u^p d\mu)} \le C\|f\|_{L^p(\Omega,u^p d\mu)}.$$

More generally, we can study two-weight inequalities:

$$\|S_n f\|_{L^p(\Omega,u^p d\mu)} \le C\|f\|_{L^p(\Omega,v^p d\mu)}, \quad \forall f \in L^p(\Omega, v^p d\mu), \quad \forall n \ge 0. \tag{2}$$

The condition required for the system $\{\psi_j\}_{j\in\Lambda}$ so that $S_n f$ can be defined in $L^p(\Omega, u^p d\mu)$ for every $f \in L^p(\Omega, v^p d\mu)$ and $\forall n \ge 0$ is

$$\psi_j \in L^p(\Omega, u^p d\mu) \cap L^q(\Omega, v^{-q}d\mu), \quad \forall j \in \Lambda.$$

We can wonder again if the boundedness (2) is equivalent to the mean convergence, that is, to

$$S_n f \longrightarrow f \text{ in } L^p(\Omega, u^p d\mu), \quad \forall f \in L^p(\Omega, v^p d\mu).$$

For this, we need that $L^p(\Omega, v^p d\mu)$ is contained in $L^p(\Omega, u^p d\mu)$; this is satisfied if and only if $u \le Cv$ μ–a.e. for certain constant $C > 0$. The following well-known result is going to tell us that our two questions are actually the same:

THEOREM 1.2. *Let $\{\psi_j\}_{j\in\Lambda}$ be an orthonormal sistem in $L^2(\Omega, d\mu)$, $1 < p < \infty$. Let u and v be two weights such that $u \le Cv$ μ-a.e. with certain positive constant C, the linear closure of $\{\psi_j\}_{j\in\Lambda}$ is dense in $L^p(\Omega, v^p d\mu)$ and*

$$\psi_j \in L^p(\Omega, u^p d\mu) \cap L^q(\Omega, v^{-q}d\mu), \quad \forall j \in \Lambda.$$

Then, the following statements are equivalent:

(a) *$S_n f \to f$ in $L^p(\Omega, u^p d\mu)$, $\forall f \in L^p(\Omega, v^p d\mu)$.*
(b) *There exists $C > 0$ such that $\|S_n f\|_{L^p(\Omega,u^p d\mu)} \le C\,\|f\|_{L^p(\Omega,v^p d\mu)}$, $\forall f \in L^p(\Omega, v^p d\mu)$, $\forall n \ge 0$.*

Once we have introduced all the topics about the partial sum operator in general, we are going to comment some aspects related to our system in particular.

First of all, we will show two results concerning the multidimensional Fourier-Bessel series in Lebesgue spaces with a mixed norm.

Recall that the functions

$$\Phi_{j,k,m}^{(d)}(x) = |x|^{(1-d)/2}\varphi_j^{k+(d-2)/2}(|x|)Y^{m,k}\left(\frac{x}{|x|}\right),$$

with $j \geq 1, k \geq 0, 1 \leq m \leq d_k$, form an orthonormal basis in $L^2(B^d, dx)$ and the Fourier series of an appropriate function f defined in B^d is given by

$$\begin{aligned} f(x) &\sim \sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\sum_{m=1}^{d_k} c_{j,k}^m(f)\Phi_{j,k,m}^{(d)}(x) \\ &= \sum_{k=0}^{\infty}\sum_{m=1}^{d_k} Y^{m,k}\left(\frac{x}{|x|}\right)\sum_{j=1}^{\infty} c_{j,k}^m(f)|x|^{(1-d)/2}\varphi_j^{k+(d-2)/2}(|x|), \end{aligned}$$

where

$$c_{j,k}^m(f) = \int_{B^d} f(x)\overline{\Phi_{j,k,m}^{(d)}(x)}\,dx.$$

Consider the partial sums $S_{n,K}^d f$ given by

$$S_{n,K}^d(f,x) = \sum_{k=0}^{K}\sum_{m=1}^{d_k} Y^{m,k}\left(\frac{x}{|x|}\right)\sum_{j=1}^{n} c_{j,k}^m(f)|x|^{(1-d)/2}\varphi_j^{k+(d-2)/2}(|x|).$$

The question to be asked is to know if $S_{n,K}^d f$ converges to f when $n, K \to \infty$, in the spaces

$$L^{p,2}(B^d, dx) = \{f : \|f\|_{L^{p,2}(B^d,dx)} < \infty\},$$

where

$$\|f\|_{L^{p,2}(B^d,dx)} = \left(\int_0^1 \left(\int_{S^{d-1}} |f(r,x')|^2\,d\sigma(x')\right)^{p/2} r^{d-1}\,dr\right)^{1/p}$$

or, what is the same, to guess for which $p \in [1,\infty)$ there exists $C_p < \infty$ such that

$$\|S_{n,K}^d f\|_{L^{p,2}(B^d,dx)} \leq C_p\|f\|_{L^{p,2}(B^d,dx)}$$

for all $n, K \geq 0$. The answer to this question was given by Balodis-Córdoba [**4**] and is found in the following theorems:

THEOREM 1.3 (**Uniform boundedness of the partial sum operators**). *Let $d \geq 2$; then*

$$\|S_{n,K}^d f\|_{L^{p,2}(B^d,dx)} \leq C\|f\|_{L^{p,2}(B^d,dx)} \Leftrightarrow \frac{2d}{d+1} < p < \frac{2d}{d-1},$$

with $n \geq AK+1$, where A is a constant depending only on d and C is independent of n and K.

From here, they obtained immediately the result of convergence of the partial sums:

THEOREM 1.4 (**Convergence of partial sums**). *If* $d \geq 2$ *and*

$$\frac{2d}{d+1} < p < \frac{2d}{d-1},$$

then, if n_i *and* K_i *are sequences of natural numbers such that*

$$n_i \geq AK_i + 1 \text{ and } K_i \to \infty,$$

where A *is the constant of the former theorem, we have*

$$\|f - S^d_{n_i,K_i} f\|_{L^{p,2}(B^d,dx)} \longrightarrow 0, \quad i \to \infty,$$

for every $f \in L^{p,2}(B^d, dx)$.

Now we are going to focus on the radial case, therefore we will show some results related to the Fourier-Bessel series for the d-dimensional radial functions. We shall observe that

$$L^{p,2}_{\text{rad}}(B^d, dx) = \{f \in L^p(B^d, dx) : f \text{ radial }\} = L^p((0,1), r^{d-1}dr).$$

In this case, we have to consider $\Phi^{(d)}_{j,k,m}$ with $k = 0$, therefore, we will take $c^m_{j,k}(f) = 0$, $\forall k \geq 1$. The set of functions with which we shall work is

$$\{r^{(1-d)/2}\varphi_j^{(d-2)/2}(r)\}_{j\geq 1}. \tag{3}$$

More in general, we can consider the Fourier series associated to the system of functions $\{\varphi^\nu_j\}_{j\geq 1}$ as defined in (1) for $\nu > -1$.

Lommel's formula (see [**58**, Ch. 5, p. 134]; in this reference this identity is not named Lommel's formula but this name appears in the literature commonly, for instance in [**57**]) states that

$$\int_0^z J_\nu(as)J_\nu(bs)s\,ds = z\frac{aJ_{\nu+1}(za)J_\nu(zb) - bJ_\nu(za)J_{\nu+1}(zb)}{a^2 - b^2},$$

therefore we have that

$$\int_0^1 J_\nu(s_{j,\nu}r)J_\nu(s_{k,\nu}r)r\,dr = \frac{1}{2}(J_{\nu+1}(s_{j,\nu}))^2\delta_{j,k}, \quad j,k = 1,2,\dots. \tag{4}$$

From this fact we deduce that the sequence of functions $\{\varphi^\nu_j\}_{j\geq 1}$ form an orthonormal system in $L^2((0,1), dr)$ with $\nu > -1$. The Fourier series associated to this system will be named the Fourier-Bessel series and is given by

$$f \sim \sum_{j=1}^\infty a_j(f)\varphi^\nu_j,$$

where

$$a_j(f) = \int_0^1 f(r)\varphi^\nu_j(r)\,dr. \tag{5}$$

The condition $\nu > -1$ ensures the integrability in (4). The system $\{\varphi^\nu_j\}_{j\geq 1}$ is also complete in $L^2((0,1), dr)$, as we can see in [**29**] with $\nu \geq -1/2$ and in [**7**] with $-1 < \nu < -1/2$.

Taking into account the above facts, we have that the set (3) is orthonormal and dense in $L^2((0,1), r^{d-1}dr)$. We will define $\mathbf{S}_n^d$, the partial sum of the Fourier series associated to this orthonormal system

$$\mathbf{S}_n^d f = \sum_{j=1}^{n} c_j(f) r^{(1-d)/2} \varphi_j^{(d-2)/2},$$

where

$$c_j(f) = \int_0^1 f(r) r^{(1-d)/2} \varphi_j^{(d-2)/2}(r) r^{d-1}\, dr. \tag{6}$$

By Theorem 1.3, since

$$L^{p,2}_{\text{rad}}(B^d, dx) = L^p((0,1), r^{d-1}dr)$$

and $\mathbf{S}_n^d = S_{n,0}^d$, it is clear that

$$\|\mathbf{S}_n^d f\|_{L^p((0,1),r^{d-1}dr)} \le \|f\|_{L^p((0,1),r^{d-1}dr)}$$

if and only if

$$\frac{2d}{d+1} < p < \frac{2d}{d-1}.$$

If we denote by $\mathcal{S}_n^\nu$ the partial sum associated to Fourier-Bessel series

$$\mathcal{S}_n^\nu f = \sum_{j=1}^{n} a_j(f) \varphi_j^\nu,$$

it is clear that

$$\|\mathbf{S}_n^d f\|_{L^p((0,1),r^{d-1}dr)} \le C\|f\|_{L^p((0,1),r^{d-1}dr)} \Leftrightarrow$$
$$\|r^{\frac{(d-1)}{p}+\frac{(1-d)}{2}} \mathcal{S}_n^{\frac{(d-2)}{2}} f\|_{L^p((0,1),dr)} \le C\|r^{\frac{(d-1)}{p}+\frac{(1-d)}{2}} f\|_{L^p((0,1),dr)}.$$

Therefore the boundedness for radial functions of the multidimensional Fourier-Bessel series is equivalent to the weighted boundedness of the Fourier-Bessel series.

The first results of convergence in $L^p((0,1), dr)$ of the Fourier-Bessel series are due to Wing. In his work [**62**], he proves the convergence of the partial sums when $\nu \ge -1/2$. In this case, if $f \in L^p((0,1), dr)$, for $1 < p < \infty$, then

$$\lim_{n\to\infty} \int_0^1 |f(r) - \mathcal{S}_n^\nu(f,r)|^p\, dr = 0.$$

Wing bases his proof on the uniform boundedness of the partial sums,

$$\|\mathcal{S}_n^\nu f\|_{L^p((0,1),dr)} \le C\|f\|_{L^p((0,1),dr)},$$

which follows from an appropriate estimate of the kernel associated to the Fourier-Bessel series.

Later, Benedek-Panzone, in [**7**] extend the result by Wing for the Fourier-Bessel series of order $\nu \in (-1, -\frac{1}{2})$ and show the convergence of this series for functions $f \in L^p((0,1),\, dr)$, whenever $1/(\nu+\frac{3}{2}) < p < 1/(-\nu-\frac{1}{2})$.

In [**25**], Guadalupe-Pérez-Ruiz-Varona studied the uniform boundedness of the partial sum operators associated to the Fourier-Bessel series in L^p spaces with extra weights, obtaining necessary and sufficient conditions for this boundedness in terms of the weights. These results are gathered therein in the following theorems:

THEOREM 1.5 (**Sufficiency**). *Let $\nu > -1$, $1 < p < \infty$ and the weights*

$$U(r) = r^{a+1/p}(1-r)^b \prod_{k=1}^{m} |r-r_k|^{b_k}, \quad V(r) = r^{A+1/p}(1-r)^B \prod_{k=1}^{m} |r-r_k|^{B_k},$$

where $0 < r_1 < \cdots < r_m < 1$ and $a, A, b, B, b_k, B_k \in \mathbb{R}$. If the following conditions are satisfied

$$\begin{aligned} &B \le b, && pB < p-1, && -1 < pb; \\ &B_k \le b_k, && pB_k < p-1, && -1 < pb_k \quad (1 \le k \le m); \end{aligned}$$

$$\left|\frac{1}{p} - \frac{1}{2} + \frac{a+A}{2}\right| < \frac{a-A}{2} + \min\left\{\frac{1}{2}, \nu+1\right\},$$
$$A \le a,$$

then there exists a constant C such that

$$\|\mathcal{S}_n^\nu f\|_{L^p((0,1),U^p(r)dr)} \le C\, \|f\|_{L^p((0,1),V^p(r)dr)}\,.$$

THEOREM 1.6 (**Necessity**). *If the inequality*

$$\|\mathcal{S}_n^\nu f\|_{L^p((0,1),U^p(r)dr)} \le C\, \|f\|_{L^p((0,1),V^p(r)dr)}$$

holds, then U, V must satisfy the following conditions

$$\int_0^1 U^p(r) r^{\nu p}\, dr < \infty, \qquad \int_0^1 U^p(r) r^{-p/2}\, dr < \infty,$$
$$\int_0^1 V^{-q}(r) r^{q/p+\nu q+1}\, dr < \infty, \qquad \int_0^1 V^{-q}(r) r^{q/p+1-q/2}\, dr < \infty,$$
$$U(r) \le C V(r) \ \text{a.e.}$$

It is interesting to observe that this last result implies the necessity of conditions imposed over a, A, b, B, b_k, B_k in the former result.

On the other hand, they also studied weak type inequalities for the partial sum operators in the end points of the interval of strong boundedness. By interpolation, the range of p's such that the uniform boundedness holds is always an interval. When $U(r) = V(r) = 1$ (for example), this interval is open, say (p_0, p_1), and then the natural question is if $\mathcal{S}_n^\nu$ are "uniformly" of weak type (p_0, p_0) or/and (p_1, p_1). That is, if we denote

$$\|f\|_{L^p_*((0,1),U^p(r)dr)} = \left\{\sup_{\lambda>0} \lambda^p \int_{\{r:|f(r)|>\lambda\}} U(r)^p \, dr\right\}^{1/p}$$

then the question is if

$$\|\mathcal{S}_n^\nu f\|_{L^p_*((0,1),U^p(r)dr)} \le C \, \|f\|_{L^p((0,1),V^p(r)dr)} \, ,$$

with a constant C independent of n, and $p = p_0$ or p_1. They gave necessary conditions on U, V for this last inequality to be true:

THEOREM 1.7 (**Weak type**). *Let $\nu > -1$ and $1 < p < \infty$. If the inequality*

$$\|\mathcal{S}_n^\nu f\|_{L^p_*((0,1),U^p(r)dr)} \le C \, \|f\|_{L^p((0,1),V^p(r)dr)}$$

holds, then U, V must satisfy the following conditions

$$\int_0^1 V^{-q}(r) r^{q/p+\nu q+1} \, dr < \infty, \quad \int_0^1 V^{-q}(r) r^{q/p+1-q/2} \, dr < \infty,$$

$$U(r) \le C V(r) \ \text{a.e.}$$

and

$$\sup_{\lambda>0} \lambda^p \int_{\{r:r^\nu>\lambda\}} U(r)^p \, dr < \infty, \qquad \sup_{\lambda>0} \lambda^p \int_{\{r:r^{-1/2}>\lambda\}} U(r)^p \, dr < \infty.$$

CHAPTER 2

Weighted convergence of Bochner-Riesz means for Fourier-Bessel expansions

1. Introduction

As we have already commented in the introduction, it is common to consider other summation methods different from partial sums in order to enlarge the class of functions for which the convergence of Fourier series associated to an orthonormal system holds. One of the most classical summation methods within the harmonic analysis are the Bochner-Riesz means. We define the Bochner-Riesz means for the multidimensional Fourier-Bessel series, for each $\delta > 0$, as

$$(7)\quad \mathcal{B}^{\delta,d}_{R,K}(f,x) = \sum_{k=0}^{K}\sum_{j\geq 1}\sum_{m=1}^{d_k}\left(1-\frac{\left(s_{j,k+(d-2)/2}\right)^2}{R^2}\right)_+^{\delta} c^m_{j,k}(f)\Phi^{(d)}_{j,k,m}(x),$$

where $R > 0$, $(1-s^2)_+ = \max\{1-s^2, 0\}$ and $\{s_{j,k+(d-2)/2}\}_{j\geq 1}$ are the zeros of Bessel function J_ν, with $\nu = k+(d-2)/2$.

The study of the convergence of this operator to f is, technically, rather complex and it requires the use of very sharp estimations for the kernel that define it in terms of R and K. A previous step to attack the general problem is the analysis of the radial case. In this situation, the problem boils down to studying the Bochner-Riesz means associated to the system $\{r^{(1-d)/2}\varphi_j^{(d-2)/2}(r)\}_{j\geq 1}$ in the spaces $L^p((0,1), r^{d-1}dr)$. In this way, (7) comes down to considering the case $K = 0$; this operator will be denoted simply by $\mathcal{B}^{\delta,d}_R$ and it is given by

$$\mathcal{B}^{\delta,d}_R(f,r) = \sum_{j\geq 1}\left(1-\frac{\left(s_{j,(d-2)/2}\right)^2}{R^2}\right)_+^{\delta} c_j(f) r^{(1-d)/2}\varphi_j^{(d-2)/2}(r),$$

where $c_j(f)$ are defined as in (6). A first target to obtain might be the proof of that

$$\|\mathcal{B}^{\delta,d}_R f\|_{L^p((0,1),r^{d-1}dr)} \leq C\|f\|_{L^p((0,1),r^{d-1}dr)}$$

if and only if

$$(8)\qquad \frac{2d}{d+1+2\delta} < p < \frac{2d}{d-1-2\delta}.$$

A standard consequence of this fact is that

$$\lim_{R\to\infty}\mathcal{B}^{\delta,d}_R(f,r) = f(r) \quad \text{in} \quad L^p((0,1), r^{d-1}dr)$$

in the interval of values for p given in (8).

As in the case of partial sums, if we define the Bochner-Riesz means for the Fourier-Bessel by

$$B_R^{\delta,\nu}(f,r) = \sum_{j\geq 1} \left(1 - \frac{(s_{j,\nu})^2}{R^2}\right)_+^{\delta} a_j(f)\varphi_j^{\nu}(r),$$

with $a_j(f)$ as in (5), it is verified that

$$\|\mathcal{B}_R^{\delta,d}(f,r)\|_{L^p((0,1),r^{d-1}dr)} \leq C\|f\|_{L^p((0,1),r^{d-1}dr)} \Leftrightarrow \\ \|r^{\frac{d-1}{p}+\frac{1-d}{2}} B_R^{\delta,(d-2)/2} f\|_{L^p((0,1),dr)} \leq C\|r^{\frac{d-1}{p}+\frac{1-d}{2}} f\|_{L^p((0,1),dr)},$$

that is, we will need a weighted inequality for the Bochner-Riesz means related to the Fourier-Bessel expansions.

Our target will be the analysis of the inequality

$$\left\|u(r)B_R^{\delta,\nu}(f,r)\right\|_{L^p((0,1),dr)} \leq C\,\|u(r)f(r)\|_{L^p((0,1),dr)}\,, \tag{9}$$

with a weight $u(r)$ in the most general conditions.

First, we have to describe the conditions on the weights that are involved in it. We define the function

$$\Psi_\nu(t) = \begin{cases} t^{\nu+1/2}, & \text{if } 0 < t \leq 4, \\ 1, & \text{if } t > 4. \end{cases} \tag{10}$$

In the definition of $\Psi_\nu(t)$, the point $t = 4$ can be modified by another value greater than one. We will use this fact tacitly in several places throughout this work.

For $1 < p < \infty$, p' is its adjoint, $1/p + 1/p' = 1$. Given a weight function $u(r)$ on $(0,1)$, consider the following set of conditions:

$$\sup_{0<R}\sup_{0<s<1}\left(\int_s^1 \left(\frac{u(r)\chi_{[4/R,1)}(r)}{R^\delta r^{\delta+1}}\right)^p dr\right)^{\frac{1}{p}} \\ \times \left(\int_0^s \left(\frac{\Psi_\nu(Rr)}{u(r)}\right)^{p'} dr\right)^{\frac{1}{p'}} < \infty, \tag{11}$$

$$\sup_{0<R}\sup_{0<s<1}\left(\int_0^s (u(r)\Psi_\nu(Rr))^p\,dr\right)^{\frac{1}{p}} \\ \times \left(\int_s^1 \left(\frac{\chi_{[4/R,1)}(r)}{u(r)R^\delta r^{(\delta+1)}}\right)^{p'} dr\right)^{\frac{1}{p'}} < \infty, \tag{12}$$

(13)

$$\sup_{0<w<v<\min\{1,2w\}} \frac{1}{v-w}\left(\int_w^v u(r)^p\,dr\right)^{1/p}\left(\int_w^v u(r)^{-p'}\,dr\right)^{1/p'} < \infty,$$

$$\sup_{0<R} R^{2(\nu+1)} \left(\int_0^{4/R} (u(r) r^{\nu+1/2})^p \, dr \right)^{1/p} \times \left(\int_0^{4/R} \left(\frac{r^{\nu+1/2}}{u(r)} \right)^{p'} dr \right)^{1/p'} < \infty. \tag{14}$$

For a weight u satisfying (13) we write $u^p \in A_{p,\mathrm{loc}}(0,1)$ and say that u^p is a local A_p weight. The left side of (13) is then called the $A_{p,\mathrm{loc}}$ norm of u^p.

We recall now some definitions. Let $f(r)$ be a non-negative function defined for r on $(0,1)$. Let η be a real number, Hardy operators are defined by

$$H_\eta(f,r) = r^{-\eta} \int_0^r f(t)\, dt, \tag{15}$$

and

$$H^*_\eta(f,r) = r^{-\eta} \int_r^1 f(t)\, dt. \tag{16}$$

From now on, $H_0(f,r)$ and $H^*_0(f,r)$ will be denoted as $H(f,r)$ and $H^*(f,r)$, respectively. We will also need a local version of the one-dimensional Hardy-Littlewood maximal operator,

$$M(f,r) = \sup_{|r-y| \le r/2} \frac{1}{y-r} \int_r^y |f(t)|\, dt.$$

The following theorems, due to Muckhenhoupt (see Theorems 1 and 2 in [**36**]) are well-known. The original results are stated for the interval $(0,\infty)$, but they are also valid for $(0,1)$.

Theorem 2.1 (**Boundedness of H**). *If $1 \le p \le \infty$, there is a finite C for which*

$$\|U(r)H(f,r)\|_{L^p((0,1),dr)} \le C \, \|V(r)f(r)\|_{L^p((0,1),dr)}$$

is true if and only if

$$\sup_{0<s<1} \left(\int_s^1 U(r)^p \, dr \right)^{\frac{1}{p}} \left(\int_0^s V(r)^{-p'} \, dr \right)^{\frac{1}{p'}} < \infty,$$

Theorem 2.2 (**Boundedness of H^***). *If $1 \le p \le \infty$, there is a finite C such that*

$$\|U(r)H^*(f,r)\|_{L^p((0,1),dr)} \le C \, \|V(r)f(r)\|_{L^p((0,1),dr)}$$

if and only if

$$\sup_{0<s<1} \left(\int_0^s U(r)^p \, dr \right)^{\frac{1}{p}} \left(\int_s^1 V(r)^{-p'} \, dr \right)^{\frac{1}{p'}} < \infty,$$

We can reach uniform weighted inequalities by including a proper supremum in the conditions given in the results above. With a slight modification of the proofs of Theorems 1 and 2 in [**36**], we can show the following

PROPOSITION 2.3. *If $1 \le p \le \infty$ and*

$$\sup_{R>0} \sup_{0<s<1} \left(\int_s^1 U_R(r)^p \, dr \right)^{\frac{1}{p}} \left(\int_0^s V_R(r)^{-p'} \, dr \right)^{\frac{1}{p'}} < \infty,$$

then the inequality

$$\|U_R(r) H(f,r)\|_{L^p((0,1),dr)} \le C \, \|V_R(r) f(r)\|_{L^p((0,1),dr)}$$

holds with a constant C independent of R.

Analogously, we also state

PROPOSITION 2.4. *If $1 \le p \le \infty$ and*

$$\sup_{R>0} \sup_{0<s<1} \left(\int_0^s U_R(r)^p \, dr \right)^{\frac{1}{p}} \left(\int_s^1 V_R(r)^{-p'} \, dr \right)^{\frac{1}{p'}} < \infty,$$

then the inequality

$$\|U_R(r) H^*(f,r)\|_{L^p((0,1),dr)} \le C \, \|V_R(r) f(r)\|_{L^p((0,1),dr)}$$

holds with a constant C independent of R.

From these statements, we have that the condition (11) is sufficient for the weighted Hardy's inequality

$$\left(\int_0^1 \left| \frac{u(r) \chi_{[4/R,1)}(r)}{R^\delta r^{\delta+1}} \int_0^s f(t) \, dt \right|^p dr \right)^{1/p} \le C \left(\int_0^1 \left| \frac{u(r)}{\Psi_\nu(Rr)} f(r) \right|^p dr \right)^{1/p}$$

to hold with a constant C independent of R, while the condition (12) is sufficient for its dual version

$$\left(\int_0^1 \left| u(r) \Psi_\nu(Rr) \int_s^1 f(t) \, dt \right|^p dr \right)^{1/p} \le C \left(\int_0^1 \left| u(r) R^\delta r^{(\delta+1)} \chi_{[4/R,1)}(r) f(r) \right|^p dr \right)^{1/p}$$

to be satisfied, also with the constant C not depending on R. On the other hand, the local A_p condition (13) for u^p is, for $1 < p < \infty$, necessary and sufficient for the estimate

$$\int_0^1 |M(f,r) u(r)|^p \, dr \le C \int_0^1 |f(r) u(r)|^p \, dr \tag{17}$$

to hold. Sufficiency part above is just a version of [**37**, Lemma 6.1]. Necessity of (13) is provided in [**42**, Section 6]. Finally, condition (14) is a technical requirement that we need to estimate our operator in a square close to the origin.

The main result of this chapter, that will be proved in Section 3, is contained in the following:

THEOREM 2.5 (Main Theorem). *Let* $\nu > -1$*,* $\delta > 0$*,* $1 < p < \infty$ *and* $R > 0$*. Let* $u(r)$ *be a weight that satisfies the conditions* (11)*,* (12)*,* (13) *and* (14)*. Then*

$$\|u(r)B_R^{\delta,\nu}(f,r)\|_{L^p((0,1),dr)} \leq C\|u(r)f(r)\|_{L^p((0,1),dr)} \tag{18}$$

for all $f \in L^p((0,1),dr)$*, with a constant* C *independent of* R *and* f*.*

In Section 4, we will apply the result to obtain a corollary with power weights $u(r) = r^a$. To simplify the notation in the corollary, we consider the following definition: for each $\nu > -1$, $\delta > 0$, and $1 < p < \infty$, we say that the parameters (a,ν,δ) satisfy the c_p conditions if

$$-1/p - (\nu + 1/2) < a < 1 - 1/p + (\nu + 1/2),$$
$$-\delta - 1/p < a < 1 + \delta - 1/p.$$

COROLLARY 2.6 (Main Corollary). *Let* $\nu > -1$*,* $\delta > 0$*,* $1 < p < \infty$*, and* $R > 0$*. Then*

$$\left\| r^a B_R^{\delta,\nu}(f,r)\right\|_{L^p((0,1),dr)} \leq C \left\| r^a f(r)\right\|_{L^p((0,1),dr)}, \tag{19}$$

with a constant C *independent of* R *and* f *if and only if* (a,ν,δ) *satisfy the* c_p *conditions.*

The operators $B_R^{\delta,\nu}$ can be described by the expression

$$B_R^{\delta,\nu}(f,r) = \int_0^1 f(y)K_R^{\delta,\nu}(r,y)\,dy$$

where

$$K_R^{\delta,\nu}(r,y) = \sum_{j\geq 1}\left(1 - \frac{(s_{j,\nu})^2}{R^2}\right)_+^{\delta} \varphi_j^{\nu}(r)\varphi_j^{\nu}(y).$$

To obtain the boundedness of the Bochner-Riesz means in (18), we will need a sharp pointwise estimate for the kernel of the operator that will be obtained in the next section.

2. An estimate for the kernel

This section contains an estimate for the kernel of the Bochner-Riesz means related to the Fourier-Bessel expansions. Before introducing this estimate, we need to recall some topics about Bessel functions. The definition of the different types of Bessel functions that appear (Y_ν

and $H_\nu^{(1)}$) are taken from Chapter 3 in [**58**] and the asymptotics (20), (21) and (23) from Chapter 7 also in [**58**].

For Bessel functions it is verified

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} + O(z^{\nu+2}), \tag{20}$$

where $|z| < 1$ and $|\arg(z)| \leq \pi$; and

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left[\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(e^{\mathrm{Im}(z)} z^{\frac{-3}{2}}) \right], \tag{21}$$

where $|z| \geq 1$ and $|\arg(z)| \leq \pi - \theta$.

Note that, for $t > 0$

$$|\sqrt{t} J_\nu(t)| \leq C_\nu \Psi_\nu(t), \tag{22}$$

where $\Psi_\nu(t)$ was defined in (10). This estimate is a simple consequence of (20) and (21). The Hankel function of first type, denoted by $H_\nu^{(1)}$, is defined as

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z),$$

where Y_ν denotes the Weber function, given by

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}, \ \nu \notin \mathbb{Z},$$

and

$$Y_n(z) = \lim_{\nu\to n} \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}, \ n \in \mathbb{Z}.$$

From these definitions, we have

$$H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - e^{-\nu\pi i} J_\nu(z)}{i\sin\nu\pi}, \ \nu \notin \mathbb{Z},$$

and

$$H_n^{(1)}(z) = \lim_{\nu\to n} \frac{J_{-\nu}(z) - e^{-\nu\pi i} J_\nu(z)}{i\sin\nu\pi}, \ n \in \mathbb{Z}.$$

The function $H_\nu^{(1)}$ verifies

(23)

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z-\nu\pi/2-\pi/4)}[A + O(z^{-1})], |z| > 1, -\pi < \arg(z) < 2\pi,$$

for certain constant A.

We will consider the decomposition of the $(0,1)\times(0,1)$ square into five regions (see Figure 1)

$$\begin{aligned}
A_1 &= \left\{(r,y) : 0<r,y\le \frac{4}{R}\right\},\\
A_2 &= \left\{(r,y) : \frac{4}{R}<\max\{r,y\}<1,\ |r-y|\le \frac{4}{3R}\right\},\\
A_3 &= \left\{(r,y) : \frac{4}{R}\le r<1,\ 0<y\le \frac{r}{2}\right\},\\
A_4 &= \left\{(r,y) : 0<r\le \frac{y}{2},\ \frac{4}{R}\le y<1\right\},\\
A_5 &= \left\{(r,y) : \frac{4}{R}<r<1,\ \frac{r}{2}<y<r-\frac{4}{3R}\right\}\\
&\qquad \cup \left\{(r,y) : \frac{y}{2}<r\le y-\frac{4}{3R},\ \frac{4}{R}\le y<1\right\}.
\end{aligned}$$

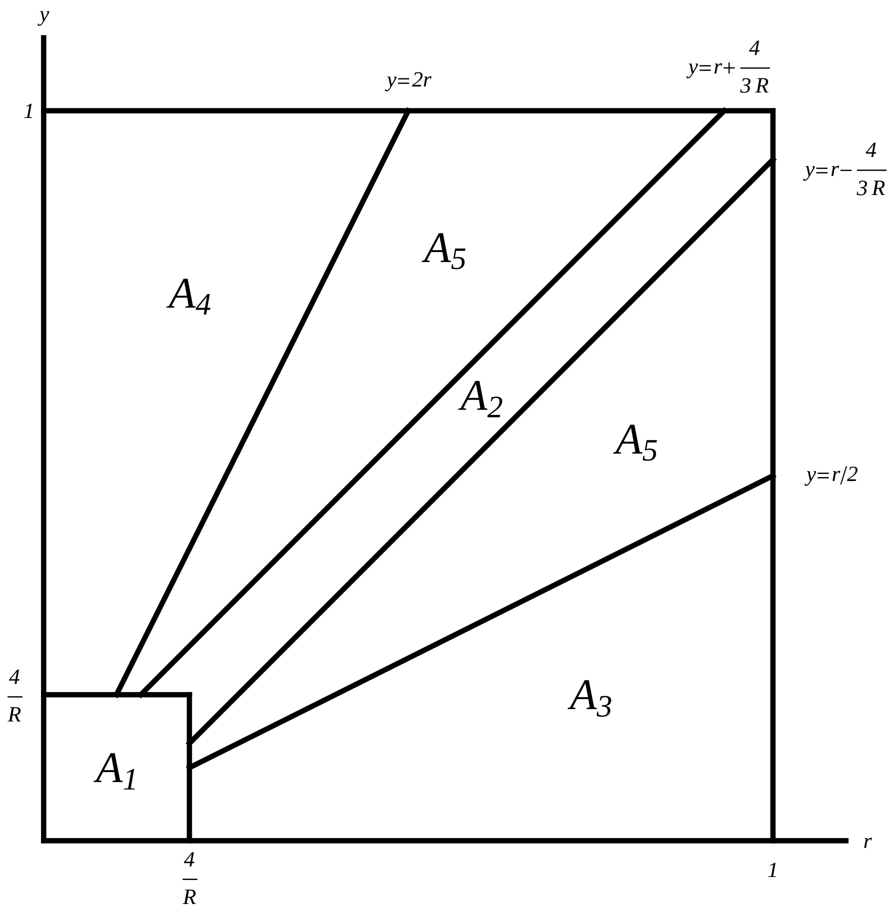

FIGURE 1. Decomposition of the $(0,1)\times(0,1)$ square.

With the previous notation, the estimate for the kernel is contained in the following lemma

LEMMA 2.1. *For $\nu > -1, \delta > 0$, and $R > 8$, the following holds:*

$$|K_R^{\delta,\nu}(r,y)| \le C \begin{cases} (ry)^{\nu+1/2} R^{2(\nu+1)}, & (r,y) \in A_1, \\ R, & (r,y) \in A_2 \\ G_\nu(r,y), & (r,y) \in A_3 \cup A_4 \cup A_5, \end{cases} \tag{24}$$

with

$$G_\nu(r,y) = \frac{\Psi_\nu(Rr)\Psi_\nu(Ry)}{R^\delta |r-y|^{\delta+1}}.$$

For the estimates in A_1 and A_2 we need an appropriate upper bound for the functions $\varphi_j^\nu(t)$. From (22),

$$|\varphi_j^\nu(t)| \le C \left|\sqrt{s_{j,\nu}} J_{\nu+1}(s_{j,\nu})\right|^{-1} \Psi_\nu(s_{j,\nu} t).$$

In [**9**], we find that

$$\left|\sqrt{s_{j,\nu}} J_{\nu+1}(s_{j,\nu})\right|^{-1} \sim 1,$$

and so

$$|\varphi_j^\nu(t)| \le C\Psi_\nu(s_{j,\nu} t). \tag{25}$$

Let ℓ be an integer such that $s_\ell < R \le s_{\ell+1}$ ($\ell \sim R$). Then

$$K_R^{\delta,\nu}(r,y) = \sum_{j=1}^{\ell} \left(1 - \frac{(s_{j,\nu})^2}{R^2}\right)^\delta \varphi_j^\nu(r)\varphi_j^\nu(y).$$

From the inequality (25) we have

$$|\varphi_j^\nu(r)\varphi_j^\nu(y)| \le C\Psi_\nu(s_{j,\nu} r)\Psi_\nu(s_{j,\nu} y.)$$

Now, for $(r,y) \in A_1$ and taking into account the definition of Ψ_ν, for each $j = 1, \dots, \ell$, it follows that

$$|\varphi_j^\nu(r)\varphi_j^\nu(y)| \le C s_{j,\nu}^{2\nu+1} (ry)^{\nu+1/2}.$$

From the estimate $s_{j,\nu} \sim j$ (see [**58**]), we obtain

$$\frac{1}{R} \sum_{j=1}^{\ell} \left(1 - \frac{(s_{j,\nu})^2}{R^2}\right)^\delta \left(\frac{s_{j,\nu}}{R}\right)^{2\nu+1} \le C,$$

and in this way

$$\begin{aligned} |K_R^{\delta,\nu}(r,y)| &\le (ry)^{\nu+1/2} R^{2(\nu+1)} \frac{1}{R} \sum_{j=1}^{\ell} \left(1 - \frac{(s_{j,\nu})^2}{R^2}\right)^\delta \left(\frac{s_{j,\nu}}{R}\right)^{2\nu+1} \\ &\le C(ry)^{\nu+1/2} R^{2(\nu+1)}, \end{aligned}$$

giving us the bound in (24) for the region A_1.

For $(r,y) \in A_2$, from (25) and the definition of Ψ_ν, we obtain

$$|\varphi_j^\nu(r)\varphi_j^\nu(y)| \le C.$$

From this, it is easily seen that

$$|K_R^{\delta,\nu}(r,y)| \le C \sum_{j=1}^{\ell} \left(1 - \frac{s_{j,\nu}^2}{R^2}\right)^{\delta} \le CR.$$

The estimate of $K_R^{\delta,\nu}$ for $(r,y) \in A_3 \cup A_4 \cup A_5$ is the most delicate part of the proof. We start by giving an appropriate integral expression for the kernel.

LEMMA 2.2. *For $R > 0$ and $\nu > -1$, the following holds:*

$$K_R^{\delta,\nu}(r,y) = I_{R,1}^{\delta,\nu}(r,y) + I_{R,2}^{\delta,\nu}(r,y),$$

with

$$I_{R,1}^{\delta,\nu}(r,y) = (ry)^{1/2} \int_0^R z \left(1 - \frac{z^2}{R^2}\right)^{\delta} J_\nu(zr) J_\nu(zy)\, dz$$

and

$$I_{R,2}^{\delta,\nu}(r,y) = \lim_{\varepsilon \to 0} \frac{(ry)^{1/2}}{2} \int_{\mathbf{S}_\varepsilon} \left(1 - \frac{z^2}{R^2}\right)^{\delta} \frac{z H_\nu^{(1)}(z) J_\nu(zr) J_\nu(zy)}{J_\nu(z)}\, dz,$$

where, for each $\varepsilon > 0$, $\mathbf{S}_\varepsilon$ is the path of integration given by the interval $R + i[\varepsilon, \infty)$ in the direction of increasing imaginary part and the interval $-R + i[\varepsilon, \infty)$ in the opposite direction.

PROOF. For $\nu \notin \mathbb{Z}$, we consider the function

$$H_{r,y}^{\delta,\nu}(z) = (ry)^{1/2} \left(1 - \frac{z^2}{R^2}\right)^{\delta} \frac{z H_\nu^{(1)}(z) J_\nu(zr) J_\nu(zy)}{J_\nu(z)}.$$

In the case $\nu = n \in \mathbb{Z}$, we take the definition corresponding to the limit $\nu \to n$. The proof of the result will be done for $\nu \notin \mathbb{Z}$, the other case can be deduced considering the limit.

The function $H_{r,y}^{\delta,\nu}(z)$ is analytic in

$$\mathbb{C} \setminus ((-\infty, -R] \cup [R, \infty) \cup \{\pm s_{j,\nu} : j = 1, 2 \dots\}).$$

Besides, the points $\pm s_{j,\nu}$ are simple poles. So, we have

$$\int_{\mathbf{C}} H_{r,y}^{\delta,\nu}(z)\, dz = 0, \tag{26}$$

where $\mathbf{C}$ is the integration contour given by $\mathbf{S}_\varepsilon \cup \mathbf{I}$, being $\mathbf{I}$ the interval $[-R, R]$ warped with arcs of radius ε centered at the origin, at the zeros $\pm s_{j,\nu}$ of $J_\nu(z)$ and at $\pm R$, see Figure 2.

The existence of the integral is clear for the path $\mathbf{I}$; for $\mathbf{S}_\varepsilon$ this fact can be verified by using (20), (21) and (23). Indeed, on $\mathbf{S}_\varepsilon$, we obtain that $\left|\frac{H_\nu^{(1)}(z)}{J_\nu(z)}\right| \le Ce^{-2\,\mathrm{Im}(z)}$ (in (21) we can consider $\theta \le \arctan(\varepsilon/R)$, in this way the asymptotic can be used for the complete path $\mathbf{S}_\varepsilon$). Similarly, one has on $\mathbf{S}_\varepsilon$

$$|(ry)^{1/2} z J_\nu(zr) J_\nu(zy)| \le Ce^{\mathrm{Im}(z)(r+y)} h_{r,y}(|z|)$$

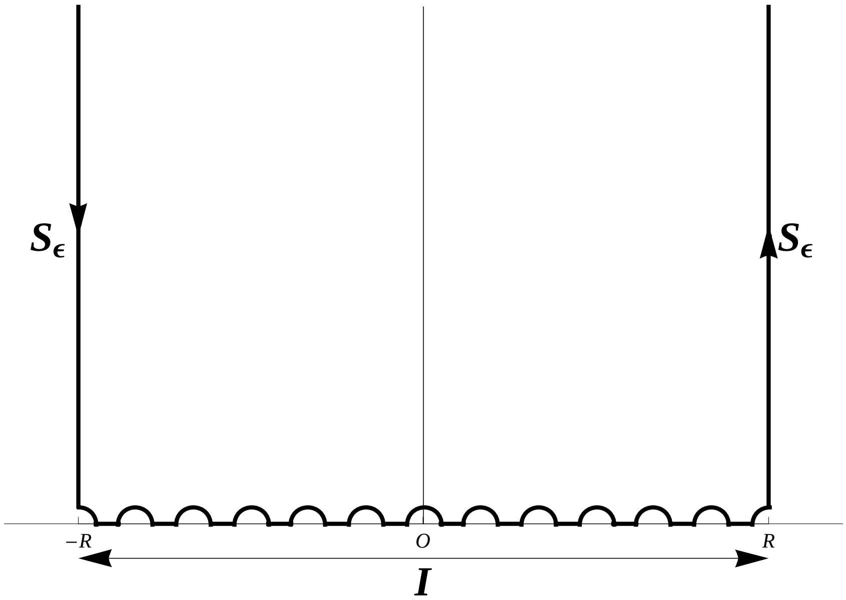

FIGURE 2. The integration contour **C**.

where

$$h_{r,y}(|z|) = \begin{cases} (ry)^{\nu+1/2}|z|^{2\nu+1}, & \text{for } -1 < \nu < -1/2, \\ 1, & \text{for } \nu \geq -1/2. \end{cases}$$

Thus $|H_{r,y}^{\delta,\nu}(z)| \leq C h_{r,y}(|z|) e^{-\operatorname{Im}(z)(2-r-y)}$, and the integral on $\mathbf{S}_\varepsilon$ is convergent.

Now it is not difficult to see that when $\varepsilon \to 0$,

$$\int_{\mathbf{I}} H_{r,y}^{\delta,\nu}(z)\, dz \to \\ -4(ry)^{1/2} \sum_{j\geq 1} \left(1 - \frac{(s_{j,\nu})^2}{R^2}\right)_+^\delta \frac{J_\nu(s_{j,\nu} r) J_\nu(s_{j,\nu} y)}{(J'_\nu(s_{j,\nu}))^2} + \int_{-R}^{R} H_{r,y}^{\delta,\nu}(z)\, dz, \tag{27}$$

using the fact that

$$\begin{aligned} \operatorname{Res}(H_{r,y}^{\delta,\nu}(z), s_{j,\nu}) &= \operatorname{Res}(H_{r,y}^{\delta,\nu}(z), -s_{j,\nu}) \\ &= -\frac{2i(ry)^{1/2}}{\pi} \left(1 - \frac{(s_{j,\nu})^2}{R^2}\right)^\delta \frac{J_\nu(s_{j,\nu} r) J_\nu(s_{j,\nu} y)}{(J_{\nu+1}(s_{j,\nu}))^2}. \end{aligned} \tag{28}$$

The first identity in (28) is a consequence of

$$J_\nu(-z) = e^{\nu\pi i} J_\nu(z).$$

To complete the proof of (28) we first recall the identities

$$-J'_\nu(z)H^{(1)}_\nu(z) + J_\nu(z)(H^{(1)}_\nu)'(z) = \frac{2i}{\pi z}$$

(see [**61**, p. 76]), and

$$zJ'_\nu(z) - \nu J_\nu(z) = -zJ_{\nu+1}(z).$$

If we insert $z = s_{j,\nu}$ in these equations, we have

$$\begin{aligned}
\operatorname{Res}(H^{\delta,\nu}_{r,y}(z), s_{j,\nu}) &= \lim_{z\to s_{j,\nu}} (z - s_{j,\nu})H^{\delta,\nu}_{r,y}(z) \\
&= \sqrt{ry}\left(1 - \frac{(s_{j,\nu})^2}{R^2}\right)^\delta \frac{s_{j,\nu}H^{(1)}_\nu(s_{j,\nu})J_\nu(s_{j,\nu}r)J_\nu(s_{j,\nu}y)}{J'_\nu(s_{j,\nu})} \\
&= -\frac{2i\sqrt{ry}}{\pi}\left(1 - \frac{(s_{j,\nu})^2}{R^2}\right)^\delta \frac{J_\nu(s_{j,\nu}r)J_\nu(s_{j,\nu}y)}{(J'_\nu(s_{j,\nu}))^2} \\
&= -\frac{2i\sqrt{ry}}{\pi}\left(1 - \frac{(s_{j,\nu})^2}{R^2}\right)^\delta \frac{J_\nu(s_{j,\nu}r)J_\nu(s_{j,\nu}y)}{(J_{\nu+1}(s_{j,\nu}))^2}.
\end{aligned}$$

Now, using the definition of $H^{(1)}_\nu$ in terms of J_ν y $J_{-\nu}$, we have

$$\begin{aligned}
&\int_{-R}^R H^{\delta,\nu}_{r,y}(z)\,dz \\
&\quad= \int_{-R}^R \left(1 - \frac{z^2}{R^2}\right)^\delta \frac{z(-J_\nu(z)e^{-i\nu\pi} + J_{-\nu}(z))J_\nu(zr)J_\nu(zy)}{i\sin\nu\pi J_\nu(z)}\,dz \\
&\quad= \frac{ie^{-i\nu\pi}}{\sin\nu\pi}\int_{-R}^R z\left(1 - \frac{z^2}{R^2}\right)^\delta J_\nu(zr)J_\nu(zy)\,dz \\
&\qquad - \frac{ie^{-i\nu\pi}}{\sin\nu\pi}\int_{-R}^R z\left(1 - \frac{z^2}{R^2}\right)^\delta \frac{J_{-\nu}(z)}{J_\nu(z)}e^{i\nu\pi}J_\nu(zr)J_\nu(zy)\,dz.
\end{aligned}$$

Note that, in the subtrahend, it is verified

$$\frac{J_{-\nu}(-z)}{J_\nu(-z)}J_\nu(-zr)J_\nu(-zy) = \frac{J_{-\nu}(z)}{J_\nu(z)}J_\nu(zr)J_\nu(zy).$$

In this way,

$$z\left(1 - \frac{z^2}{R^2}\right)^\delta \frac{J_{-\nu}(z)}{J_\nu(z)}e^{i\nu\pi}J_\nu(zr)J_\nu(zy)$$

is an odd function, and we are integrating in a symmetric interval, therefore the subtrahend vanishes. On the other hand, since

$$J_\nu(-zr)J_\nu(-zy) = e^{2i\nu\pi}J_\nu(zr)J_\nu(zy),$$

we have

$$\begin{aligned}\int_{-R}^{R} H_{r,y}^{\delta,\nu}(z)\,dz &= \frac{ie^{-i\nu\pi}(ry)^{1/2}}{\sin\nu\pi}\int_{-R}^{R} z\left(1-\frac{z^2}{R^2}\right)^{\delta} J_\nu(zr)J_\nu(zy)\,dz\\
&= \frac{ie^{-i\nu\pi}(ry)^{1/2}}{\sin\nu\pi}\left(\int_{0}^{R} z\left(1-\frac{z^2}{R^2}\right)^{\delta} J_\nu(zr)J_\nu(zy)\,dz\right.\\
&\qquad \left. -e^{2i\pi\nu}(ry)^{1/2}\int_{0}^{R} z\left(1-\frac{z^2}{R^2}\right)^{\delta} J_\nu(zr)J_\nu(zy)\,dz\right)\\
&= 2(ry)^{1/2}\int_{0}^{R} z\left(1-\frac{z^2}{R^2}\right)^{\delta} J_\nu(zr)J_\nu(zy)\,dz,\end{aligned}$$

thus

$$\int_{-R}^{R} H_{r,y}^{\delta,\nu}(z)\,dz = 2(ry)^{1/2}\int_{0}^{R} z\left(1-\frac{z^2}{R^2}\right)^{\delta} J_\nu(zr)J_\nu(zy)\,dz. \tag{29}$$

So, from (26), (27) and (29) the proof is complete. □

First, consider the study of the integral $I_{R,1}^{\delta,\nu}(r,y)$. We are going to give a bound in the region $(r,y)\in(0,1)\times(0,1)$ off the diagonal. The main goal is the proof of the following lemma.

LEMMA 2.3. *For $\nu>-1$, $R>0$, $\max\{r,y\}>4/R$ and $|r-y|>\frac{4}{3R}$, the following holds:*

$$|I_{R,1}^{\delta,\nu}(r,y)| \le \frac{C_\nu}{R^\delta}\frac{\Psi_\nu(Rr)\Psi_\nu(Ry)}{|r-y|^{\delta+1}}.$$

Let us define

$$N^\delta(a,b)=\sqrt{ab}\int_0^1 s(1-s^2)^\delta J_\nu(as)J_\nu(bs)\,ds.$$

The proof of Lemma 2.3 will follow from the estimate

$$\left|N^\delta(a,b)\right| \le C_\nu \frac{\Psi_\nu(a)\Psi_\nu(b)}{|a-b|^{\delta+1}}, \tag{30}$$

for $\nu>-1$ and $a,b>0$, which will be proved further on. (30) implies the statement of Lemma 2.3 since the change of variable $z=Rs$ in $I_{R,1}^{\delta,\nu}$ gives

$$I_{R,1}^{\delta,\nu}(r,y)=RN^\delta(Rr,Ry).$$

Estimate (30) for $\max\{a,b\}>4$ and $|a-b|>4/3$ generalizes a similar one obtained in [**11**] for $\nu\ge 0$. Moreover, our method of achieving is completely different: we will show an explicit expression for N^δ with $\delta=m\in\mathbb{N}$ and this will lead us to the estimate for N^δ in the integer case. Finally, the result will be completed using an identity in which N^δ, with a general δ, is related to the previously analyzed integer cases. Some of the ideas in this proof have been taken from [**59**].

The next four lemmas contain the technical tools used to prove the estimate (30). Let us introduce some notations. We define the functions

$$\begin{aligned} F_1(a,b) &= \sqrt{ab}J_\nu(a)J_\nu(b), & F_2(a,b) &= \sqrt{ab}J_{\nu+1}(a)J_\nu(b),\\ F_3(a,b) &= \sqrt{ab}J_\nu(a)J_{\nu+1}(b), & F_4(a,b) &= \sqrt{ab}J_{\nu+1}(a)J_{\nu+1}(b), \end{aligned}$$

and the operator

$$\mathcal{D} = \frac{1}{a^2-b^2}\left(b\frac{\partial}{\partial b} - a\frac{\partial}{\partial a}\right).$$

LEMMA 2.4. *Let $u = a^2 - b^2$. Supposed that $a \neq b$, the following equalities*

$$\begin{aligned} \mathcal{D}(F_1(a,b)) &= \frac{1}{u}(aF_2(a,b) - bF_3(a,b)),\\ \mathcal{D}(F_2(a,b)) &= \frac{-1}{u}(aF_1(a,b) - (2\nu+1)F_2(a,b) + bF_4(a,b)),\\ \mathcal{D}(F_3(a,b)) &= \frac{1}{u}(bF_1(a,b) - (2\nu+1)F_3(a,b) + aF_4(a,b)) \end{aligned}$$

and

$$\mathcal{D}(F_4(a,b)) = \frac{1}{u}(bF_2(a,b) - aF_3(a,b))$$

hold.

PROOF. It suffices to apply the identities

$$tJ_\nu'(t) + \nu J_\nu(t) = tJ_{\nu-1}(t) \quad \text{and} \quad tJ_\nu'(t) - \nu J_\nu(t) = -tJ_{\nu+1}(t),$$

$$\begin{aligned} \mathcal{D}(F_1(a,b)) &= \mathcal{D}(\sqrt{ab}J_\nu(a)J_\nu(b))\\ &= \frac{\sqrt{ab}}{u}\Big(J_\nu(a)(\nu J_\nu(b) - bJ_{\nu+1}(b)) - J_\nu(b)(\nu J_\nu(a) - aJ_{\nu+1}(a))\Big)\\ &= \frac{1}{u}(aF_2(a,b) - bF_3(a,b)), \end{aligned}$$

$$\begin{aligned} \mathcal{D}(F_2(a,b)) = \mathcal{D}(\sqrt{ab}J_{\nu+1}(a)J_\nu(b))&\\ &= \frac{\sqrt{ab}}{u}\Big(J_{\nu+1}(a)(\nu J_\nu(b) - bJ_{\nu+1}(b))\\ &\qquad - J_\nu(b)(aJ_\nu(a) - (\nu+1)J_{\nu+1}(a))\Big)\\ &= \frac{-1}{u}(aF_1(a,b) - (2\nu+1)F_2(a,b) + bF_4(a,b)), \end{aligned}$$

$$\begin{aligned}\mathcal{D}(F_3(a,b)) = \mathcal{D}(\sqrt{ab}J_{\nu+1}(b)J_\nu(a))&\\ &= \frac{\sqrt{ab}}{u}\Big(J_\nu(a)(bJ_\nu(b)-(\nu+1)J_{\nu+1}(b))\\ &\qquad - J_{\nu+1}(b)(\nu J_\nu(a)-aJ_{\nu+1}(a))\Big)\\ &= \frac{1}{u}(bF_1(a,b)-(2\nu+1)F_3(a,b)+aF_4(a,b)),\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}(F_4(a,b)) = \mathcal{D}(\sqrt{ab}J_{\nu+1}(a)J_{\nu+1}(b))&\\ &= \frac{\sqrt{ab}}{u}\Big(J_{\nu+1}(a)(bJ_\nu(b)-(\nu+1)J_{\nu+1}(b))\\ &\qquad - J_{\nu+1}(b)(aJ_\nu(a)-(\nu+1)J_{\nu+1}(a))\Big)\\ &= \frac{1}{u}(bF_2(a,b)-aF_3(a,b)).\end{aligned}$$

□

Lemma 2.5. *Let $\nu > -1$, $m \in \mathbb{N}\cup\{0\}$, $u = a^2-b^2$ and $v = a^2+b^2$. Then, there exist polynomials A_m, B_m, C_m, D_m (in the variables a and b) such that, for $a \neq b$,*

$$\begin{aligned}N^m(a,b) &= \frac{2^m m!}{u^{2m}}\\ &\times\left(A_mF_1(a,b)+B_m\frac{a}{u}F_2(a,b)-C_m\frac{b}{u}F_3(a,b)-D_m abF_4(a,b)\right).\end{aligned}\tag{31}$$

Moreover, $A_0 = D_0 = 0$, $B_0 = C_0 = 1$ and A_m, B_m, C_m, D_m satisfy the recurrence relation

$$\begin{pmatrix}A_{m+1}\\B_{m+1}\\C_{m+1}\\D_{m+1}\end{pmatrix} = \begin{pmatrix}u^2\mathcal{D}(A_m)\\u^2\mathcal{D}(B_m)\\u^2\mathcal{D}(C_m)\\u^2\mathcal{D}(D_m)\end{pmatrix}$$
$$+\begin{pmatrix}4mv & -a^2 & -b^2 & 0\\ u^2 & 2(2m+1)v+2u\nu & 0 & -u^2b^2\\ u^2 & 0 & 2(2m+1)v-2u\nu & -u^2a^2\\ 0 & 1 & 1 & 4mv\end{pmatrix}\begin{pmatrix}A_m\\B_m\\C_m\\D_m\end{pmatrix}.$$

Proof. We will argue by induction over m. Recall Lommel's formula (see [**58**, Ch. 5, p. 134]):

$$\int_0^z sJ_\nu(as)J_\nu(bs)\,ds = z\frac{aJ_{\nu+1}(za)J_\nu(zb)-bJ_\nu(za)J_{\nu+1}(zb)}{a^2-b^2}.$$

With $z = 1$ we have the result for $m = 0$. To complete the induction, we shall consider the identity

$$N^{m+1}(a,b) = 2(m+1)\mathcal{D}(N^m(a,b)), \qquad m = 0,1,2,\ldots,$$

which is based on integration by parts and Lommel's formula. Indeed,

$$\begin{aligned}
N^{m+1}(a,b) &= \sqrt{ab}\int_0^1 s(1-s^2)^{m+1} J_\nu(as)J_\nu(bs)\,ds \\
&= 2(m+1)\sqrt{ab}\int_0^1 s^2(1-s^2)^m \frac{aJ_{\nu+1}(sa)J_\nu(sb) - bJ_\nu(sa)J_{\nu+1}(sb)}{a^2-b^2}\,ds \\
&= 2(m+1)\sqrt{ab}\int_0^1 s(1-s^2)^m \mathcal{D}(J_\nu(sa)J_\nu(sb))\,ds \\
&= 2(m+1)\mathcal{D}(N^m(a,b)).
\end{aligned}$$

So,

$$\begin{aligned}
N^{m+1}(a,b) &= 2^{m+1}(m+1)! \\
&\quad \times \mathcal{D}\left(u^{-2m}\left(A_m F_1 + B_m \frac{a}{u}F_2 - C_m\frac{b}{u}F_3 - D_m abF_4\right)\right).
\end{aligned}$$

Now, using Lemma 2.4, it follows that

$$\begin{aligned}
\mathcal{D}\left(u^{-2m}A_mF_1\right) &= \mathcal{D}(u^{-2m})A_mF_1 + u^{-2m}\mathcal{D}(A_m)F_1 + u^{-2m}A_m\mathcal{D}(F_1) \\
&= u^{-2(m+1)}\left((4mvA_m + u^2\mathcal{D}(A_m))F_1 + u^2A_m\frac{a}{u}F_2 - u^2A_m\frac{b}{u}F_3\right),
\end{aligned}$$

$$\begin{aligned}
&\mathcal{D}\big(u^{-2m}B_m\frac{a}{u}F_2\big) \\
&\quad = \mathcal{D}(u^{-2m-1})B_maF_2 + u^{-2m-1}\mathcal{D}(B_m)aF_2 + u^{-2m-1}B_m\mathcal{D}(aF_2) \\
&\quad = u^{-2(m+1)}\big(- a^2B_mF_1 + ((2(2m+1)v + 2u\nu)B_m \\
&\qquad\qquad + u^2\mathcal{D}(B_m)\frac{a}{u})F_2 - B_m abF_4\big),
\end{aligned}$$

$$\begin{aligned}
&\mathcal{D}\big(u^{-2m}C_m\frac{b}{u}F_3\big) \\
&\quad = \mathcal{D}(u^{-2m-1})C_mbF_3 + u^{-2m-1}\mathcal{D}(C_m)bF_3 + u^{-2m-1}C_m\mathcal{D}(bF_3) \\
&\quad = u^{-2(m+1)}\big(b^2C_mF_1 + ((2(2m+1)v - 2u\nu)C_m \\
&\qquad\qquad + u^2\mathcal{D}(C_m))\frac{b}{u}F_3 + C_m abF_4\big),
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{D}\big(u^{-2m} D_m abF_4\big)\\
&\qquad = \mathcal{D}(u^{-2m}) D_m abF_4 + u^{-2m}\mathcal{D}(D_m) abF_4 + u^{-2m} D_m \mathcal{D}(abF_4)\\
&\qquad = u^{-2(m+1)}\big(u^2 b^2 D_m \frac{a}{u} F_2 - u^2 a^2 D_m \frac{b}{u} F_3\\
&\qquad\qquad\qquad + (4mvD_m + u^2\mathcal{D}(D_m)) abF_4\big).
\end{aligned}$$

The last equations show that N^{m+1} is of the form (31). $\square$

To estimate the polynomials in (31), we need to know the behavior of the operator $\mathcal{D}$ acting on them. Our polynomials have a unique expression of the form

$$\sum_{j,k\geq 0} c_{j,k} u^k v^j,$$

and the following lemma analyzes the action of $\mathcal{D}$ on this kind of polynomials.

Lemma 2.6. *Let $u = a^2 - b^2$, $v = a^2 + b^2$ and*

$$P = P(u,v) = \sum_{j,k} c_{j,k} u^k v^j.$$

Defining

$$P^{\#} = P^{\#}(u,v) = \sum_{j,k} |c_{j,k}| |u|^k v^j,$$

the estimate

$$(u^2\mathcal{D}(P))^{\#} \leq C v P^{\#}$$

holds with a constant C depending on the degree of P.

Proof. It is clear that

$$(P+Q)^{\#} \leq P^{\#} + Q^{\#} \qquad \text{and} \qquad (cP)^{\#} = |c| P^{\#},$$

so we can consider $P = u^k v^j$. Furthermore, we have

$$b\frac{\partial}{\partial b} - a\frac{\partial}{\partial a} = -\Big(2u\frac{\partial}{\partial v} + 2v\frac{\partial}{\partial u}\Big).$$

Therefore,

$$u^2\mathcal{D}(P) = -2u\Big(u\frac{\partial}{\partial v} + v\frac{\partial}{\partial u}\Big)P = -2u\Big(uju^k v^{j-1} + vku^{k-1}v^j\Big).$$

Then, with the obvious estimate $|u| \leq v$, we arrive at

$$\begin{aligned}
(u^2\mathcal{D}(P))^{\#} &\leq C\Big(|u|^2\big(|u|^k v^{j-1}\big) + |u| v\big(|u|^{k-1} v^j\big)\Big)\\
&\leq C\left(|u|^2 \frac{P^{\#}}{v} + |u| v \frac{P^{\#}}{|u|}\right) \leq C(|u|+v)P^{\#} \leq CvP^{\#}.
\end{aligned}$$

$\square$

In the next lemma we relate $N^{[\delta]}$ to N^{δ}, for $\delta > 0$, where $[\cdot]$ denotes the integer part function.

LEMMA 2.7. *Let $\nu > -1$, $\delta > 0$ and $\delta \notin \mathbb{N}$. Then*

$$N^{\delta}(a,b) = C(\delta) \int_0^1 (1-s^2)^{\delta-[\delta]-1} s^{2[\delta]+2} N^{[\delta]}(as,bs)\, ds \tag{32}$$

where

$$C(\delta) = 2\frac{\delta(\delta-1)\cdots(\delta-[\delta])}{[\delta]!}.$$

PROOF. Let us suppose that $0 < \delta < 1$. Then, the use of the identity

$$\sqrt{ab}\int_0^s tJ_\nu(at)J_\nu(bt)\, dt = sN^0(as,bs)$$

and integration by parts yield

$$N^{\delta}(a,b) = 2\delta \int_0^1 (1-s^2)^{\delta-1} s^2 N^0(as,bs)\, ds.$$

Now, for $m < \delta < m+1$, applying integration by parts $m+1$ times and using the identity

$$\int_0^s t^{2m+2} N^m(at,bt)\, dt = \frac{s^{2m+3}}{2(m+1)} N^{m+1}(as,bs), \quad m = 0,1,\dots, \tag{33}$$

we obtain (32). □

PROOF OF LEMMA 2.3. As we observed above, it is enough to show (30) to complete the proof.

It is easy to see that (30) holds for the cases $\max\{a,b\} \le 4$, and $\max\{a,b\} > 4$ with $|a-b| \le 4/3$. For the first case, $\max\{a,b\} \le 4$, (30) holds from

$$|N^{\delta}(a,b)| \le C(ab)^{\nu+1/2} \tag{34}$$

and for the second one, $\max\{a,b\} > 4$ with $|a-b| \le 4/3$, the inequality

$$|N^{\delta}(a,b)| \le C \tag{35}$$

yields (30). Estimate (34) is a consequence of (22): it is enough to observe that, for $\max\{a,b\} \le 4$,

$$|N^{\delta}(a,b)| \le C(ab)^{\nu+1/2} \int_0^1 s^{2\nu+1}(1-s^2)^{\delta}\, ds$$

and the last integral is convergent. To show (35), using Schwartz's inequality, we have to show that for $\alpha > 0$

$$\int_0^1 (1-s^2)^{\delta} \alpha s J_\nu^2(\alpha s)\, ds \le C,$$

which is obtained by using (22), and take $\alpha = a, b$.

We continue by proving (30) for $\delta = m$ a positive integer and for $\max\{a,b\} > 4$ with $|a-b| > 4/3$. From (31), with $u = a^2 - b^2$, it follows that

$$|N^m(a,b)| \le \frac{2^m m!}{|u|^{2m}} \left(|A_m F_1| + \frac{a}{|u|}|B_m F_2| + \frac{b}{|u|}|C_m F_3| + ab|D_m F_4| \right).$$

Clearly,

$$|F_i(a,b)| \le C_\nu \Psi_\nu(a)\Psi_\nu(b), \quad i = 1, \dots, 4.$$

So

$$|N^m(a,b)| \le \frac{C_\nu}{|u|^{2m}} \Psi_\nu(a)\Psi_\nu(b) \left(|A_m| + \frac{a}{|u|}|B_m| + \frac{b}{|u|}|C_m| + ab|D_m| \right).$$

Now, as $|P| \le P^{\#}$, it yields

$$\begin{aligned} |N^m(a,b)| &\le C_\nu \frac{\Psi_\nu(a)\Psi_\nu(b)}{|u|^{2m}} \left(A_m^{\#} + \frac{a}{|u|} B_m^{\#} + \frac{b}{|u|} C_m^{\#} + ab D_m^{\#} \right) \\ &\le C_\nu \frac{\Psi_\nu(a)\Psi_\nu(b)}{|a-b|^{m+1}} \frac{|a-b|^{m+1}}{|u|^{2m}} \left(A_m^{\#} + \frac{a+b}{|u|}(B_m^{\#} + C_m^{\#}) + ab D_m^{\#} \right). \end{aligned}$$

Comparing this with (30) and using the inequality $ab \le (a^2+b^2)/2 \sim (a+b)^2$, it is clear that it suffices to show that

$$\frac{1}{|a-b|^{m-1}(a+b)^{2m}} \left(A_m^{\#} + \frac{1}{|a-b|}(B_m^{\#} + C_m^{\#}) + (a+b)^2 D_m^{\#} \right) \le C.$$

Recall that $A_0^{\#} = D_0^{\#} = 0$ and $B_0^{\#} = C_0^{\#} = 1$ and $|a-b| > 4/3$, so the case $m = 0$ is settled. For $m \ge 1$ it is enough to prove

$$\begin{aligned} A_m^{\#} &\le C|a-b|^{m-1}(a+b)^{2m}, \\ B_m^{\#} + C_m^{\#} &\le C|a-b|^{m}(a+b)^{2m}, \\ D_m^{\#} &\le C|a-b|^{m-1}(a+b)^{2m-2}. \end{aligned} \tag{36}$$

Considering the recurrence relation for the polynomials A_{m+1}, B_{m+1}, C_{m+1} and D_{m+1} in Lemma 2.5 and the estimate in Lemma 2.6, we obtain that

$$\begin{aligned} A_{m+1}^{\#} &= \left(u^2 \mathcal{D}(A_m) + 4mvA_m - \left(\frac{u+v}{2}\right) B_m - \left(\frac{v-u}{2}\right) C_m \right)^{\#} \\ &\le C\Big((u^2\mathcal{D}(A_m))^{\#} + v(A_m^{\#} + B_m^{\#} + C_m^{\#}) \Big) \\ &\le Cv\Big(2A_m^{\#} + B_m^{\#} + C_m^{\#} \Big), \end{aligned}$$

and, in a similar way,

$$\begin{aligned} B_{m+1}^{\#} &\le C\Big(|u|^2 A_m^{\#} + 2vB_m^{\#} + |u|^2 v D_m^{\#} \Big), \\ C_{m+1}^{\#} &\le C\Big(|u|^2 A_m^{\#} + 2vC_m^{\#} + |u|^2 v D_m^{\#} \Big), \\ D_{m+1}^{\#} &\le C\Big(B_m^{\#} + C_m^{\#} + 2vD_m^{\#} \Big). \end{aligned}$$

With this, using that $|a-b|>4/3$ and induction over m, the proof of (36) is completed. Hence, we have proved (30) for the case $m\in\mathbb{N}$.

Let us continue with the case $\delta\notin\mathbb{N}$ with $|a-b|>4/3$ and $\max\{a,b\}>4$. By the identity (32) we can write $N^\delta(a,b)$, except for the constant $C(\delta)$, as the sum of the integrals

$$I_1=\int_{1-\frac{1}{|a-b|}}^{1}(1-s^2)^{\delta-[\delta]-1}s^{2[\delta]+2}N^{[\delta]}(sa,sb)\,ds$$

and

$$I_2=\int_0^{1-\frac{1}{|a-b|}}(1-s^2)^{\delta-[\delta]-1}s^{2[\delta]+2}N^{[\delta]}(sa,sb)\,ds.$$

Integrating by parts I_2 and using (33), we have

$$I_2=\frac{1}{2([\delta]+1)}\left((1-s^2)^{\delta-[\delta]-1}s^{2[\delta]+3}N^{[\delta]+1}(sa,sb)\Big|_{s=0}^{1-\frac{1}{|a-b|}}\right.$$
$$\left.+2\int_0^{1-\frac{1}{|a-b|}}(1-s^2)^{\delta-[\delta]-2}s^{2[\delta]+4}N^{[\delta]+1}(sa,sb)\,ds\right).$$

For $s=0$ the first summand is zero; so, we only have to analyze its behavior for $s=1-\frac{1}{|a-b|}$. By using (30) for $N^{[\delta]+1}$ with $a,b>0$ we obtain that

$$(1-s^2)^{\delta-[\delta]-1}s^{2[\delta]+3}|N^{[\delta]+1}(sa,sb)|\le C(1-s)^{\delta-[\delta]-1}\frac{\Psi_\nu(sa)\Psi_\nu(sb)}{|a-b|^{[\delta]+2}}s^{[\delta]+1}.$$

From this fact, using that $\frac{1}{2}<1-\frac{1}{|a-b|}<1$, and

$$\Psi_\nu(sr)\le Cs^{-1/2}\Psi_\nu(r), \tag{37}$$

which holds for $\nu>-1$, $0<s\le1$ and $r>1$, we obtain that the first summand in I_2 is bounded by a constant multiple of $\Psi_\nu(a)\Psi_\nu(b)|a-b|^{-(\delta+1)}$. Inequality (37) follows by separately considering the cases $1<r\le1/s$ and $1/s<r$ with $0<s\le1$.

Let us analyze the second summand in I_2. To this end, we observe that

$$\left|\int_0^{1-\frac{1}{|a-b|}}(1-s^2)^{\delta-[\delta]-2}s^{2[\delta]+4}N^{[\delta]+1}(sa,sb)\,ds\right|$$
$$\le C\int_0^{1-\frac{1}{|a-b|}}(1-s^2)^{\delta-[\delta]-2}s^{2[\delta]+4}\frac{\Psi_\nu(sa)\Psi_\nu(sb)}{|sa-sb|^{[\delta]+2}}\,ds$$
$$=\frac{C}{|a-b|^{[\delta]+2}}\int_0^{1-\frac{1}{|a-b|}}(1-s^2)^{\delta-[\delta]-2}s^{[\delta]+2}\Psi_\nu(sa)\Psi_\nu(sb)\,ds.$$

To complete the estimate for I_2, it is enough to show that

$$\int_0^{1-\frac{1}{a-b}}(1-s^2)^{\delta-[\delta]-2}s^{[\delta]+2}\Psi_\nu(sa)\Psi_\nu(sb)\,ds\le C\frac{\Psi_\nu(b)}{(a-b)^{\delta-[\delta]-1}} \tag{38}$$

for $b < a$, $a - b > 4/3$ and $a > 4$. Now, (38) is obtained from (37) by using that

$$\int_0^{1-\frac{1}{a-b}} (1-s^2)^{\delta-[\delta]-2} s^{[\delta]+1}\, ds \le C \int_0^{1-\frac{1}{a-b}} (1-s)^{\delta-[\delta]-2}\, ds$$
$$= C((a-b)^{-\delta+[\delta]+1} - 1) \le C(a-b)^{-\delta+[\delta]+1}.$$

Finally, let us estimate the integral I_1. We can use (30) for $N^{[\delta]}$ since $[\delta]$ is an integer. From the definition of Ψ_ν we see that $\Psi_\nu(sa) \le C$ in this situation. Moreover, $\Psi_\nu(sb) \le C(sb)^{\nu+1/2}$, for $0 < b < 4/3$, and $\Psi_\nu(sb) \le C$, for $b \ge 4/3$. With this, it is clear that

$$|I_1| \le C \frac{\Psi_\nu(b)}{|a-b|^{[\delta]+1}} \int_{1-\frac{1}{|a-b|}}^1 (1-s)^{\delta-[\delta]-1}\, ds = C \frac{\Psi_\nu(b)}{|a-b|^{\delta+1}}.$$

This completes the proof of (30) in all the cases. □

Now, we shall show that the integral $I_{R,2}^{\delta,\nu}$ can be controlled by the same function that was a bound for $I_{R,1}^{\delta,\nu}$.

LEMMA 2.8. *For $\nu > -1$, $\delta > 0$, $R > 8$, and $(r,y) \in A_3 \cup A_4 \cup A_5$, the following holds*

$$|I_{R,2}^{\delta,\nu}(r,y)| \le \frac{C_\nu}{R^\delta} \frac{\Psi_\nu(Rr)\Psi_\nu(Ry)}{|r-y|^{\delta+1}}.$$

PROOF. We will show that

$$|I_{R,2}^{\delta,\nu}(r,y)| \le C \left(\frac{\Psi_\nu(Ry)}{R^\delta (2-r-y)^{\delta+1}} + \frac{\Psi_\nu(Ry)}{R^{2\delta}(2-r-y)^{2\delta+1}} \right) \tag{39}$$

for

$$(r,y) \in \left\{ (r,y) : 4/R < r < 1, 0 < y < r - \frac{4}{3R} \right\}.$$

In the case

$$(r,y) \in \left\{ (r,y) : 4/R < y < 1, 0 < r < y - \frac{4}{3R} \right\}$$

we can obtain a similar estimate and the proof is analogous. So, the required bound for $|I_{R,2}^{\delta,\nu}(r,y)|$ will follow from the fact that

$$|r-y| \le 2-r-y \quad \text{and} \quad |r-y| > \frac{4}{3R}.$$

To obtain (39), we use the asymptotic expansions given in (23) and (21) for $H_\nu^{(1)}(z)$ and $J_\nu(z)$. Let $\mathbf{S}_\varepsilon$ be the path of integration described in Lemma 2.2. For $t = \operatorname{Im}(z)$ the estimate

$$\left| \frac{H_\nu^{(1)}(z)}{J_\nu(z)} \right| \le C e^{-2t}$$

holds for $\varepsilon < t < \infty$, for each $\varepsilon > 0$ (as in the proof of Lemma 2.2 we have to consider $\theta \le \arctan(\varepsilon/R)$ in (21)). Now, from (20) and (21), it is clear that

$$|z\sqrt{ry}J_\nu(zr)J_\nu(zy)| \le C\Psi_\nu((R+t)y)e^{(r+y)t},$$

and so

$$|I^{\delta,\nu}_{R,2}(r,y)| \le C\int_0^\infty \Psi_\nu((R+t)y)\left|\frac{t^2}{R^2}-\frac{2it}{R}\right|^\delta e^{-(2-r-y)t}\,dt.$$

If either $y > 1/R$ or $\nu + 1/2 \le 0$, (39) follows immediately by using the inequalities

$$\Psi_\nu((R+t)y) \le C\Psi_\nu(Ry)$$

and

$$\left|\frac{t^2}{R^2}-\frac{2it}{R}\right|^\delta \le C\left(\frac{t^{2\delta}}{R^{2\delta}}+\frac{t^\delta}{R^\delta}\right),$$

and integrating. If $y \le 1/R$ and $\nu + 1/2 > 0$, we obtain (39) with the estimate

$$\Psi_\nu((R+t)y) \le C(\Psi_\nu(Ry) + (ty)^{\nu+1/2}).$$

Indeed,

$$\begin{aligned}|I^{\delta,\nu}_{R,2}(r,y)| \le C\Bigg(&\int_0^\infty \Psi_\nu(Ry)\left|\frac{t^2}{R^2}-\frac{2it}{R}\right|^\delta e^{-(2-r-y)t}\,dt\\ &+\int_0^\infty (ty)^{\nu+1/2}\left|\frac{t^2}{R^2}-\frac{2it}{R}\right|^\delta e^{-(2-r-y)t}\,dt\Bigg).\end{aligned}$$

The first integral gives the required bound for $|I^{\delta,\nu}_{R,2}(r,y)|$ as in the previous case. For the second one we have

$$\begin{aligned}&y^{\nu+1/2}\int_0^\infty \left|\frac{t^2}{R^2}-\frac{2it}{R}\right|^\delta t^{\nu+1/2}e^{-(2-r-y)t}\,dt\\ &\quad\le Cy^{\nu+1/2}\left(\frac{1}{R^\delta(2-r-y)^{\delta+\nu+3/2}}+\frac{1}{R^{2\delta}(2-r-y)^{2\delta+\nu+3/2}}\right)\\ &\quad\le C(Ry)^{\nu+1/2}\left(\frac{1}{R^\delta(2-r-y)^{\delta+1}}+\frac{1}{R^{2\delta}(2-r-y)^{2\delta+1}}\right),\end{aligned}$$

where in the last step we have used the fact that $2 - r - y > C/R$ for (r, y) in this region. □

We have just concluded the proof of Lemma 2.3 and therefore, the proof of Lemma 2.1 is completed.

3. Proof of the Main Theorem

Assume $u(r)$ satisfies the assumptions of the theorem. In these conditions, the operator $B_R^{\delta,\nu}$ is well defined for each $f \in L^p((0,1), u^p(r)\,dr)$ because the integrals defining the coefficients exist. Indeed, using that $|\varphi_j^\nu(y)| \le C\Psi_\nu(s_{j,\nu}y)$, we have

$$\left|\int_0^1 f(y)\varphi_j^\nu(y)\,dy\right| \le C\left(\int_0^1 \left(\frac{\Psi_\nu(s_{j,\nu}y)}{u(y)}\right)^{p'} dy\right)^{1/p'} \|f\|_{L^p((0,1),u^p(r)\,dr)}$$

and the convergence of the integral is clear by using (11).

For the kernel associated with the operator $B_R^{\delta,\nu}$ we have that

$$|K_R^{\delta,\nu}(r,y)| = \sum_{k=1}^5 |K_R^{\delta,\nu}(r,y)|\chi_{A_k}(r,y),$$

and

$$|B_R^{\delta,\nu} f| \le \sum_{k=1}^5 T_k f,$$

where we define

$$T_k(f,r) = \int_0^1 |f(y)||K_R^{\delta,\nu}(r,y)|\chi_{A_k}(r,y)\,dy.$$

We will use tacitly and repeatedly Lemma 2.1 to get the boundedness of the different T_k, $k = 1, \dots, 5$.

Boundedness of T_1. Using Hölder's inequality, we have then

$$\begin{aligned}\int_0^1 |u(r)T_1(f,r)|^p\,dr &\le C\int_0^{\frac{4}{R}}\left(u(r)\int_0^{\frac{4}{R}} (ry)^{\nu+1/2}R^{2(\nu+1)}|f(y)|\,dy\right)^p dr\\
&\le CR^{2p(\nu+1)}\left(\int_0^{\frac{4}{R}} (u(r)r^{\nu+1/2})^p\,dr\right)\left(\int_0^{\frac{4}{R}}\left(\frac{r^{\nu+1/2}}{u(r)}\right)^{p'} dr\right)^{\frac{p}{p'}}\\
&\qquad\times\int_0^1 |u(r)f(r)|^p\,dr\\
&\le C\int_0^1 |u(r)f(r)|^p\,dr,\end{aligned}$$

where in the last step we have used (14).

Boundedness of T_2 and T_5. The corresponding inequality for $T_2(f,r)$ and $T_5(f,r)$ is a consequence of the estimate

$$|T_2(f,r)| + |T_5(f,r)| \le CM(f,r). \tag{40}$$

We conclude the bound for these two parts using (13) and (17). To demonstrate (40) with T_2 it is enough to observe that

$$T_2(f,r) \le CR\int_{r-\frac{4}{3R}}^{\min\left\{r+\frac{4}{3R},1\right\}} |f(y)|\,dy \le CM(f,r).$$

To analyze (40) for the operator T_5 we use the decomposition

$$A_5 = \bigcup_{k=1}^{m} (A_5 \cap \{(r,y) : 2^k < R|r-y| \le 2^{k+1}\}),$$

with $m = [\log_2 R] - 1$. In this manner

$$\begin{aligned} T_5(f,r) &\le C \sum_{k=1}^{m} 2^{-k(\delta+1)} R \int_{\{y: 2^k < R|r-y| \le 2^{k+1}\}} |f(y)|\,dy \\ &\le C \sum_{k=1}^{m} 2^{-k\delta} M(f,x) \le C M(f,r). \end{aligned}$$

Boundedness of T_3. In this situation we have

$$\begin{aligned} &\int_0^1 |u(r) T_3(f,r)|^p\,dr \\ &\quad \le C \int_0^1 \left(u(r)\chi_{[4/R,1)}(r) \int_0^{r/2} \frac{\Psi_\nu(Ry)}{R^\delta |r-y|^{\delta+1}} |f(y)|\,dy \right)^p dr \\ &\quad \le C \int_0^1 \left(\frac{u(r)\chi_{[4/R,1)}(r)}{R^\delta r^{\delta+1}} \int_0^{r/2} \Psi_\nu(Ry) |f(y)|\,dy \right)^p dr. \end{aligned}$$

Now, taking into account the uniform boundedness for the operator H with the weights

$$U_R(r) = u(r) \frac{\chi_{[4/R,1)}(r)}{R^\delta r^{\delta+1}} \quad \text{and} \quad V_R(r) = \frac{u(r)}{\Psi_\nu(Rr)},$$

and the condition (11) we conclude.

Boundedness of T_4. For this last operator it is verified that

$$\begin{aligned} &\int_0^1 |u(r) T_4(f,r)|^p\,dr \\ &\quad \le C \int_0^1 \left(u(r)\Psi_\nu(Rr) \int_{3r/2}^1 \frac{\chi_{[4/R,1)}(y)}{R^\delta |r-y|^{\delta+1}} |f(y)|\,dy \right)^p dr \\ &\quad \le C \int_0^1 \left(u(r)\Psi_\nu(Rr) \int_{3r/2}^1 \frac{\chi_{[4/R,1)}(y)}{R^\delta y^{\delta+1}} |f(y)|\,dy \right)^p dr. \end{aligned}$$

We finish using the uniform boundedness of the operator H^* with the weights

$$U_R(r) = u(r)\Psi_\nu(Rr) \quad \text{and} \quad V_R(r) = \chi_{[4/R,1)}(r) u(r) R^\delta r^{\delta+1}$$

and the condition (12).

4. Weighted inequalities with power weights: proof of the Main Corollary

In this section, we are going to show the sufficiency and necessity of c_p conditions to get inequality (19).

To prove the sufficiency of the conditions c_p in the corollary we have to check that they imply (11)–(14).

The inequalities in c_p give that

$$\sup_{0<R}\sup_{0<s<1}\left(\int_s^1\left(\frac{r^{a-\delta-1}\chi_{[4/R,1)}(r)}{R^\delta}\right)^p dr\right)^{\frac{1}{p}} \times\left(\int_0^s\left(\frac{\Psi_\nu(Rr)}{r^a}\right)^{p'} dr\right)^{\frac{1}{p'}}<\infty,$$

and

$$\sup_{0<R}\sup_{0<s<1}\left(\int_0^s (r^a\Psi_\nu(Rr))^p\,dr\right)^{\frac{1}{p}}\left(\int_s^1\left(\frac{\chi_{[4/R,1)}(r)}{R^\delta r^{a+\delta+1}}\right)^{p'} dr\right)^{\frac{1}{p'}}<\infty.$$

The inequalities

$$-\frac{1}{p}-\left(\nu+\frac{1}{2}\right)<a<1-\frac{1}{p}+\left(\nu+\frac{1}{2}\right),$$

imply

$$\sup_{0<R} R^{2(\nu+1)}\left(\int_0^{4/R} r^{(\nu+1/2+a)p}\,dr\right)^{1/p}\left(\int_0^{4/R} r^{(\nu+1/2-a)p'}\,dr\right)^{1/p'}<\infty.$$

Finally the condition

$$\sup_{0<w<v<\min\{1,2w\}}\frac{1}{v-w}\left(\int_w^v r^{ap}\,dr\right)^{1/p}\left(\int_w^v r^{-ap'}\,dx\right)^{1/p'}<\infty,$$

holds for any $a\in\mathbb{R}$.

The necessity of the conditions

$$-\frac{1}{p}-\left(\nu+\frac{1}{2}\right)<a<1-\frac{1}{p}+\left(\nu+\frac{1}{2}\right).$$

follows from the fact that $r^a\varphi_j^\nu(r)$ has to belong to $L^p((0,1),dr)$ and $r^{-a}\varphi_j^\nu(r)$ to $L^q((0,1),dr)$. Indeed, using (20) and (21), we can show that

$$r^\gamma\varphi_j^\nu(r)\in L^p((0,1),dr)\iff\gamma>-\frac{1}{p}-\left(\nu+\frac{1}{2}\right).$$

We still have to prove the necessity of

$$-\delta-\frac{1}{p}<a<\delta+1-\frac{1}{p}.$$

From (19) and taking into account Theorem 2.1 in [**9**], we deduce that

$$\left\| r^a \mathbf{B}_R^\delta (f,r) \right\|_{L^p((0,\infty),dr)} \le C \left\| r^a f(r) \right\|_{L^p((0,\infty),dr)},$$

where

$$\mathbf{B}_R^{\delta,\nu} f = \mathcal{H}_\nu \left(\left(1 - \frac{r^2}{R^2}\right)_+^\delta \mathcal{H}_\nu f \right),$$

with $\mathcal{H}_\nu$ denoting the Hankel transform of order ν

$$\mathcal{H}_\nu(f,r) = \int_0^\infty (ry)^{1/2} J_\nu(ry) f(y)\, dy, \qquad r > 0.$$

Consider the space $\mathcal{S}_\nu = r^{\nu+1/2}\mathcal{S}_{\mathrm{e}}(\mathbb{R})$, where $\mathcal{S}_{\mathrm{e}}(\mathbb{R})$ is the space of all even Schwartz functions on $\mathbb{R}$. On this space, the Hankel transform is well-defined and an isomorphism. Let $f \in \mathcal{S}_\nu$ such that $\mathcal{H}_\nu(f,r) = r^{\nu+1/2}$ for $r \in (0,1)$, we conclude, by Sonine's integral [**58**, p. 373], that

$$\mathbf{B}_1^{\delta,\nu}(f,r) = A(\delta)\frac{J_{\nu+\delta+1}(r)}{r^{\delta+1/2}}.$$

In this manner we have

$$r^a \mathbf{B}_1^{\delta,\nu}(f,r) \in L^p((0,\infty),dr) \Rightarrow r^a \frac{J_{\nu+\delta+1}(r)}{r^{\delta+1/2}} \in L^p((1,\infty),dr)$$

and

$$r^a \frac{J_{\nu+\delta+1}(r)}{r^{\delta+1/2}} \in L^p((1,\infty),dr) \iff a < \delta + 1 - 1/p.$$

The necessity of the inequality $-\delta - \frac{1}{p} < a$ can be deduced by a duality argument.

5. Consequences of the Main Corollary

The first consequence we obtain from the Main Corollary in this chapter is the convergence of the Bochner-Riesz means. It follows in a standard manner by the density of our orthonormal system.

Corollary 2.7. *Let $\nu > -1$, $\delta > 0$ and $1 < p < \infty$. Then, for every $f \in L^p((0,1), r^{ap}\,dr)$,*

$$\lim_{R\to\infty} B_R^{\delta,\nu} f = f \qquad \text{in} \quad L^p((0,1), r^{ap}\,dr)$$

if and only if (a,ν,δ) satisfy the c_p conditions.

As we stated at the beginning of this chapter, the weighted convergence of Bochner-Riesz means for Fourier-Bessel expansions is directly related to the convergence of Bochner-Riesz means for the multidimensional Fourier-Bessel series for radial functions. In this way, the result of convergence obtained in Corollary 2.7 gives us the convergence of $\mathcal{B}_R^{\delta,d} f$ in $L^p_{\mathrm{rad}}(B^d, dx)$ if and only if (8) is satisfied.

Partial sums and Bochner-Riesz means are particular cases of multipliers for the Fourier-Bessel series. Let $m(r)$ be a bounded function in $(s_{1,\nu}, \infty)$; we define the multiplier for the Fourier-Bessel function associated with m by means of the identity

$$T_m(f,r) = \sum_{j=1}^{\infty} m(s_{j,\nu}) a_j(f) \varphi_j^{\nu}(r).$$

For instance, taking

$$m_{R,\delta}(r) = \left(1 - \frac{r^2}{R^2}\right)_+^{\delta},$$

we have

$$\mathcal{S}_n^{\nu} = T_{m_{s_n^{\nu},0}} \text{ and } B_R^{\delta,\nu} = T_{m_{R,\delta}}, \text{ for } \delta > 0.$$

With the function $m_\delta(r) = r^{-\delta}$, $\delta > 0$, we define the fractional integrals. They are given by

$$F_\delta(f,r) = \sum_{j=1}^{\infty} \frac{a_j(f)}{s_{j,\nu}^{\delta}} \varphi_j^{\nu}(r), \qquad t > 0.$$

With the function $m_t(r) = e^{-tr^2}$, we obtain the Gauss-Weierstrass multiplier

$$H_t(f,r) = \sum_{j=1}^{\infty} e^{-ts_{j,\nu}^2} a_j(f) \varphi_j^{\nu}(r), \qquad t > 0. \tag{41}$$

Finally, from here, we define the Poisson semigroup as

$$P_t(f,r) = \sum_{j=1}^{\infty} e^{-ts_{j,\nu}} a_j(f) \varphi_j^{\nu}(r), \quad t > 0. \tag{42}$$

The following lemma shows the relations among these operators. From these relations and using the result in the Main Corollary (Corollary (2.6)), we will obtain L^p inequalities for these multipliers. Expressions (43) and (45) relate the heat semigroup and fractional integrals to Bochner-Riesz means. Identity (44) allows to express the Poisson semigroup in terms of heat semigroup. This kind of relations are usually called *subordination formulas.*

LEMMA 2.9. *Let $\nu > -1$ and $\delta > 0$. Then, for $f \in L^2((0,1), dr)$, we have*

$$H_t(f,r) = \frac{2t^{\delta+1}}{\Gamma(\delta+1)} \int_0^{\infty} e^{-tR^2} R^{2\delta+1} B_R^{\delta,\nu}(f,r)\, dR, \tag{43}$$

$$P_t(f,r) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-u}}{\sqrt{u}} H_{t^2/(4u)}(f,r)\, du, \tag{44}$$

and

$$F_\delta(f,r) = \frac{2\Gamma\left(\frac{3\delta}{2}+1\right)}{\Gamma(\delta+1)\Gamma\left(\frac{\delta}{2}\right)} \int_0^{\infty} B_R^{\delta,\nu}(f,r) \frac{dR}{R^{\delta+1}}. \tag{45}$$

PROOF. To obtain (43) it is enough to show that

$$\sum_{j=1}^{\infty} e^{-t(s_{j,\nu})^2} a_j(f)\varphi_j^{\nu}(r) = \frac{2t^{(\delta+1)}}{\Gamma(\delta+1)} \int_0^{\infty} e^{-tR^2} R^{2\delta+1} B_R^{\delta,\nu}(f,r)\, dR \tag{46}$$

and this follows immediately using the fact that

$$B_R^{\delta,\nu}(f,r) = \sum_{j=1}^{n} \left(1 - \frac{(s_{j,\nu})^2}{R^2}\right)_+^{\delta} a_j(f)\varphi_j^{\nu}(r),$$

for $s_{n,\nu} < R \le s_{n+1,\nu}$, $n \in \mathbb{N}$, and $B_R^{\delta,\nu}(f,r) = 0$ for $0 < R \le s_{1,\nu}$. Indeed,

$$\begin{aligned}
&\int_0^{\infty} e^{-tR^2} R^{2\delta+1} B_R^{\delta,\nu}(f,r)\, dR \\
&\quad = \sum_{k=1}^{\infty} \int_{s_{k,\nu}}^{s_{k+1,\nu}} e^{-tR^2} R^{2\delta+1} \sum_{j=1}^{k} \left(1 - \frac{(s_{j,\nu})^2}{R^2}\right)_+^{\delta} a_j(f)\varphi_j^{\nu}(r)\, dR \\
&\quad = \sum_{j=1}^{\infty} a_j(f)\varphi_j^{\nu}(r) \sum_{k=j}^{\infty} \int_{s_{k,\nu}}^{s_{k+1,\nu}} e^{-tR^2} R^{2\delta+1} \left(1 - \frac{(s_{j,\nu})^2}{R^2}\right)_+^{\delta} dR;
\end{aligned}$$

with the change $R^2 = z$ and taking into account the definition of Gamma function, the last identity equals

$$\begin{aligned}
\frac{1}{2} \sum_{j=1}^{\infty} a_j(f)\varphi_j^{\nu}(r) \int_{s_{j,\nu}^2}^{\infty} e^{-tz} z^{\delta} \left(1 - \frac{(s_{j,\nu})^2}{z}\right)_+^{\delta} dz& \\
= \frac{1}{2} \sum_{j=1}^{\infty} a_j(f)\varphi_j^{\nu}(r)\Gamma(\delta+1) t^{-\delta-1} e^{-(s_{j,\nu})^2 t}&,
\end{aligned}$$

and this yields (43).

In order to prove (44), we will use the formula

$$e^{-t} = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-u}}{\sqrt{u}} e^{-\frac{t^2}{4u}}\, du,$$

(for the proof of this equality, see [**52**, p. 6-7]). From here, and taking into account the definitions of Poisson semigroup (42) and heat semigroup (41) it is easy to deduce the subordination formula (44).

Finally, identity (45) is obtained by multiplying (46) by $t^{\delta/2-1}$ and integrating on $(0,\infty)$ with respect to t. Then we have

$$\begin{aligned}
&\int_0^{\infty} \sum_{j=1}^{\infty} t^{\delta/2-1} e^{-t(s_{j,\nu})^2} a_j(f)\varphi_j^{\nu}(x)\, dt \\
&\qquad = \int_0^{\infty} \frac{2t^{\frac{3\delta}{2}}}{\Gamma(\delta+1)} \int_0^{\infty} e^{-tR^2} R^{2\delta+1} B_R^{\delta,\nu}(f,r)\, dR\, dt.
\end{aligned} \tag{47}$$

The left-hand side of (47) equals

$$\sum_{j=1}^{\infty} a_j(f)\varphi_j^\nu(x)\int_0^\infty t^{\delta/2-1}e^{-t(s_{j,\nu})^2}\,dt = \Gamma\left(\frac{\delta}{2}\right)\sum_{j=1}^{\infty}\frac{a_j(f)\varphi_j^\nu(x)}{(s_{j,\nu})^\delta}.$$

Applying Fubini's theorem to the right-hand side of (47) gives

$$\begin{aligned}\frac{2}{\Gamma(\delta+1)}\int_0^\infty R^{2\delta+1}B_R^{\delta,\nu}(f,r)\int_0^\infty e^{-tR^2}t^{\frac{3\delta}{2}}\,dt\,dR\\ = \frac{2\Gamma(\frac{3\delta}{2}+1)}{\Gamma(\delta+1)}\int_0^\infty B_R^{\delta,\nu}(f,r)\frac{dR}{R^{\delta+1}}.\end{aligned}$$

Now, with the two previous identities, (46) is obtained. □

We can state, for the fractional integrals, the following corollary:

Corollary 2.8. *Let $\nu > -1$, $\delta > 0$ and $1 < p < \infty$. Then, if (a,ν,δ) satisfy the c_p conditions, the inequality*

$$\|r^a F_\delta(f,r)\|_{L^p((0,1),dr)} \le C\,\|r^a f(r)\|_{L^p((0,1),dr)}$$

holds, with a constant C independent of f.

Proof. In order to obtain the proof of this corollary, we use (45), Minkowski's inequality and Corollary 2.6. In this way, taking into account that $B_R^{\delta,\nu}(f,r) = 0$ when $0 < R < s_{1,\nu}$,

$$\begin{aligned}\|r^a F_\delta(f,x)\|_{L^p((0,1),dr)} &\le C\|r^a B_R^{\delta,\nu}(f,r)\|_{L^p((0,1),dr)}\int_{s_{1,\nu}}^\infty \frac{dR}{R^{\delta+1}}\\ &\le C\int_{s_{1,\nu}}^\infty \frac{dR}{R^{\delta+1}}\,\|r^a f(r)\|_{L^p((0,1),dr)} \le C\,\|r^a f(r)\|_{L^p((0,1),dr)}.\end{aligned}$$

□

The heat semigroup for the Fourier-Bessel series can be also treated as follows:

Corollary 2.9. *Let $\nu > -1$ and $1 < p < \infty$. Then*

$$\|r^a H_t(f,r)\|_{L^p((0,1),dr)} \le C\,\|r^a f(r)\|_{L^p((0,1),dr)}, \tag{48}$$

with a constant C independent of t and f, if and only if the conditions

$$-\frac{1}{p} - \left(\nu + \frac{1}{2}\right) < a < 1 - \frac{1}{p} + \left(\nu + \frac{1}{2}\right) \tag{49}$$

hold.

Proof. The proof of this result is obtained by using Corollary 2.6 and identity (43). For each p, let us take a value for δ such that (a,ν,δ)

verify the conditions c_p. So,

$$\begin{aligned}
&\|r^a H_t(f,r)\|_{L^p((0,1),dr)} \\
&\quad \leq \left(\frac{2t^{\delta+1}}{\Gamma(\delta+1)}\right) \int_0^\infty e^{-tR^2} R^{2\delta+1} \left\| r^a B_R^{\delta,\nu}(f,r) \right\|_{L^p((0,1),dr)} dR \\
&\quad \leq C \left(\frac{2t^{\delta+1}}{\Gamma(\delta+1)}\right) \|r^a f(r)\|_{L^p((0,1),dr)} \int_0^\infty e^{-tR^2} R^{(2\delta+1)}\, dR \\
&\quad \leq C\, \|r^a f(r)\|_{L^p((0,1),dr)}
\end{aligned}$$

where we have used Minkowski's inequality and (19), which follows from (49) and our choice of δ. Necessity of conditions is obtained as in Corollary 2.6. □

In the end, we have the following result concerning the Poisson semigroup:

COROLLARY 2.10. *Let $\nu > -1$ and $1 < p < \infty$. Then*

$$\|r^a P_t(f,r)\|_{L^p((0,1),dr)} \leq C\, \|r^a f(r)\|_{L^p((0,1),dr)}\,, \tag{50}$$

with a constant C independent of t and f, if and only if

$$-\frac{1}{p} - \left(\nu + \frac{1}{2}\right) < a < 1 - \frac{1}{p} + \left(\nu + \frac{1}{2}\right) \tag{51}$$

hold.

PROOF. To prove the result, we take the subordination formula (44) and make the change of variable $\frac{t^2}{4u} = z$, so

$$P_t(f,r) = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/(4z)}}{z^{3/2}} H_z(f,r)\, dz.$$

Now, using the definition of Gamma function and (48), (50) follows from conditions (51). The necessity of these conditions follows again as in Corollary 2.6. □

6. Weak type inequalities for $p = 1$

Our estimate for the kernel $K_R^{\delta,\nu}$ can be used to obtain more inequalities for the Bochner-Riesz means. For example, we can show weak type inequalities for $p = 1$.

Now, for a nonnegative weight u, we consider the following set of conditions:

(52)

$$\sup_{0<R}\ \sup_{0<s<1} \left(\int_s^1 \left(\frac{s}{r}\right)^\alpha \frac{u(r)\chi_{[4/R,1)}(r)}{R^\delta r^{\delta+1}}\, dr \right) \left(\operatorname*{ess\,sup}_{r\in(0,s)} \frac{\Psi_\nu(Rr)}{u(r)} \right) < \infty,$$

$$\sup_{0<R}\sup_{0<s<1} s^{\nu+1/2}\left(\int_0^s u(r)\chi_{(0,4/R)}(r)\,dr\right) \times\left(\operatorname*{ess\,sup}_{r\in(s,1)}\frac{R^{\nu+1/2-\delta}\chi_{[4/R,1)}(r)}{r^{\delta+1}u(r)}\right)<\infty, \tag{53}$$

$$\sup_{0<R}\sup_{0<s<1}\left(\int_0^s\left(\frac{r}{s}\right)^{\alpha}u(r)r^{\nu+1/2}\chi_{(0,4/R)}(r)\,dr\right) \times\left(\operatorname*{ess\,sup}_{r\in(s,1)}\frac{R^{\nu+1/2-\delta}\chi_{[4/R,1)}(r)}{r^{\delta+1}u(r)}\right)<\infty, \tag{54}$$

$$\sup_{0<R}\sup_{0<s<1}\left(\int_0^s u(r)\chi_{[4/R,1)}(r)\,dr\right)\left(\operatorname*{ess\,sup}_{r\in(r,1)}\frac{\chi_{[4/R,1)}(r)}{R^{\delta}r^{\delta+1}u(r)}\right)<\infty, \tag{55}$$

(56)
$$\sup_{0<R}R^{2(\nu+1)}\|r^{\nu+1/2}\chi_{(0,4/R)}(r)\|_{L^{1,\infty}((0,1),u^p(r)\,dr)}\sup_{0<r<4/R}\frac{r^{\nu+1/2}}{u(r)}<\infty.$$

In (52) and (54) we suppose that there exists a positive α such that the quantity is finite.

This time, we will take into account some results about the Hardy operators H_η and H_η^* that were defined in (15) and (16). The three following results are the corresponding ones to Theorems 2, 4 and 5 from [**1**]. Note also that the interval considered therein is $(0,\infty)$, but we use the interval $(0,1)$ without affecting the result.

THEOREM 2.11. *Let $1\le p\le q<\infty$ and $\eta\ge 0$. There is a finite C for which, for all $\lambda>0$,*

$$\left(\int_{\{0<r<1:|H_\eta(f,r)|>\lambda\}}U(r)\,dr\right)^{1/q}\le\frac{C}{\lambda}\left(\int_0^1|f(r)|^pV(r)\,dr\right)^{1/p}$$

is true if and only if

$$\sup_{0<s<1}\left(\int_s^1\left(\frac{s}{r}\right)^{\alpha}\frac{U(r)}{r^{\eta q}}\,dr\right)^{\frac1q}\left(\int_0^s V(r)^{-\frac{1}{p-1}}\,dr\right)^{\frac{1}{p'}}<\infty,$$

for some $\alpha>0$.

THEOREM 2.12. *Let $1\le p\le q<\infty$ and $\eta\ge 0$. There is a finite C for which, for all $\lambda>0$,*

$$\left(\int_{\{0<r<1:|H_\eta^*(f,r)|>\lambda\}}U(r)\,dr\right)^{1/q}\le\frac{C}{\lambda}\left(\int_0^1|f(r)|^pV(r)\,dr\right)^{1/p}$$

is true if and only if

$$\sup_{0<s<1}s^{-\eta}\left(\int_0^s U(r)\,dr\right)^{\frac1q}\left(\int_s^1 V(r)^{-\frac{1}{p-1}}\,dr\right)^{\frac{1}{p'}}<\infty.$$

THEOREM 2.13. *Let $1 \le p \le q < \infty$ and $\eta < 0$. There is a finite C for which, for all $\lambda > 0$,*

$$\left(\int_{\{0<r<1:|H_\eta^*(f,r)|>\lambda\}} U(r)\,dr\right)^{1/q} \le \frac{C}{\lambda}\left(\int_0^1 |f(r)|^p V(r)\,dr\right)^{1/p}$$

is true if and only if

$$\sup_{0<s<1}\left(\int_0^s \left(\frac{r}{s}\right)^\alpha \frac{U(r)}{r^{\eta q}}\,dr\right)^{\frac{1}{q}}\left(\int_s^1 V(r)^{-\frac{1}{p-1}}\,dr\right)^{\frac{1}{p'}} < \infty,$$

for some $\alpha > 0$.

Once again, we can reach uniform weighted weak type inequalities by including a proper supremum in the conditions given in the results above. With a slight modification of the proofs of Theorems 2, 4 and 5 in [**1**], we can show the following:

PROPOSITION 2.14. *If $1 \le p \le q < \infty$, $\eta \ge 0$ and*

$$\sup_{R>0}\sup_{0<s<1}\left(\int_s^1 \left(\frac{s}{r}\right)^\alpha \frac{U_R(r)}{r^{\eta q}}\,dr\right)^{\frac{1}{q}}\left(\int_0^s V_R(r)^{-\frac{1}{p-1}}\,dr\right)^{\frac{1}{p'}} < \infty,$$

for some $\alpha > 0$, then the inequality

$$\left(\int_{\{0<r<1:|H_\eta(f,r)|>\lambda\}} U_R(r)\,dr\right)^{1/q} \le \frac{C}{\lambda}\left(\int_0^1 |f(r)|^p V_R(r)\,dr\right)^{1/p}$$

holds for all $\lambda > 0$, with a constant C independent of R.

PROPOSITION 2.15. *If $1 \le p \le q < \infty$, $\eta \ge 0$ and*

$$\sup_{R>0}\sup_{0<s<1} s^{-\eta}\left(\int_0^s U_R(r)\,dr\right)^{\frac{1}{q}}\left(\int_s^1 V_R(r)^{-\frac{1}{p-1}}\,dr\right)^{\frac{1}{p'}} < \infty,$$

then the inequality

$$\left(\int_{\{0<r<1:|H_\eta^*(f,r)|>\lambda\}} U_R(r)\,dr\right)^{1/q} \le \frac{C}{\lambda}\left(\int_0^1 |f(r)|^p V_R(r)\,dr\right)^{1/p}$$

holds for all $\lambda > 0$, with a constant C independent of R.

PROPOSITION 2.16. *If $1 \le p \le q < \infty$, $\eta < 0$ and*

$$\sup_{R>0}\sup_{0<s<1}\left(\int_0^s \left(\frac{r}{s}\right)^\alpha \frac{U_R(r)}{r^{\eta q}}\,dr\right)^{\frac{1}{q}}\left(\int_s^1 V_R(r)^{-\frac{1}{p-1}}\,dr\right)^{\frac{1}{p'}} < \infty,$$

for some $\alpha > 0$, then the inequality

$$\left(\int_{\{0<r<1:|H_\eta^*(f,r)|>\lambda\}} U_R(r)\,dr\right)^{1/q} \le \frac{C}{\lambda}\left(\int_0^1 |f(r)|^p V_R(r)\,dr\right)^{1/p}$$

holds for all $\lambda > 0$, with a constant C independent of R.

From these statements, we have that condition (52) is sufficient for the inequality

$$\int_{\{0<r<1:|H_{\delta+1}(f,r)|>\lambda\}} u(r)\chi_{[4/R,1)}(r)\,dr \le \frac{C}{\lambda}\int_0^1 |f(r)|\frac{R^\delta u(r)}{\Psi_\nu(Rr)}\,dr, \quad \lambda>0 \tag{57}$$

to hold with C independent of R; this follows from Proposition 2.14 taken with $p=q=1$, $\eta=\delta+1>0$, $U_R(r)=u(r)\chi_{[4/R,1)}(r)$ and $V_R(r)=R^\delta\Psi_\nu(Rr)^{-1}u(r)$ for $r\in(0,1)$. The conditions (53) when $\nu\in(-1,-1/2]$ or (54) when $\nu\in(-1/2,\infty)$ are sufficient for the inequality

$$\int_{\{0<r<1:|H^*_{-(\nu+1/2)}(f,r)|>\lambda\}} u(r)\chi_{(0,4/R)}(r)\,dr \le \frac{C}{\lambda}\int_0^1 |f(r)|R^{\delta-(\nu+1/2)}r^{\delta+1}u(r)\chi_{[4/R,1)}(r)\,dr, \quad \lambda>0 \tag{58}$$

to hold, with C independent of R. In this case this follows from Proposition 2.15 and Proposition 2.16 taken with $p=q=1$, $\eta=-(\nu+1/2)$, $U_R(r)=u(r)\chi_{(0,4/R)}(r)$ and $V_R(r)=R^{\delta-(\nu+1/2)}r^{\delta+1}u(r)\chi_{[4/R,1)}(r)$ for $r\in(0,1)$. The condition (55) is sufficient for the inequality

$$\int_{\{0\le r<1:|H^*(f,r)|>\lambda\}} u(r)\chi_{[4/R,1)}(r)\,dr \le \frac{C}{\lambda}\int_0^1 |f(r)|R^\delta r^{\delta+1}u(r)\chi_{[4/R,1)}(r)\,dr, \quad \lambda>0 \tag{59}$$

to hold with C independent of R; this follows from Proposition 2.15 taken with $p=q=1$, $\eta=0$, $U_R(r)=u(r)\chi_{[4/R,1)}(r)$ and $V_R(r)=R^\delta r^{\delta+1}u(r)\chi_{[4/R,1)}(r)$ for $r\in(0,1)$. Finally, the condition (56) will be used to treat the operator in the region A_1.

In this case of $p=1$, condition (13) is written as

$$\sup_{0<w<v<\min\{1,2w\}} \frac{1}{v-w}\left(\int_w^v u(r)\,dr\right)\operatorname*{ess\,sup}_{r\in(w,v)} u(r)^{-1}<\infty, \tag{60}$$

and it is necessary and sufficient for the weighted weak type $(1,1)$ inequality

$$\int_{\{0<r<1:|M(f,r)|>\lambda\}} u(r) \le \frac{C}{\lambda}\int_0^1 |f(r)|u(r)\,dr, \qquad \lambda>0 \tag{61}$$

to hold, see [**42**, Section 6].

THEOREM 2.17. *Let $\nu>-1$, $\delta>0$, and $u(r)$ be a weight on $(0,1)$ that satisfies the conditions* (52)*,* (55)*,* (56)*,* (60)*, and* (53) *for $\nu\in$*

$(-1,-1/2]$ *or* (54) *for* $\nu \in (-1/2,\infty)$. *Then*

$$\int_{\{0<r<1:|B_R^{\delta,\nu}(f,r)|>\lambda\}} u(r)\,dr \le \frac{C}{\lambda}\int_0^1 |f(r)|u(r)\,dr, \quad \lambda > 0,$$

for all $f \in L^1((0,1),u(r)\,dr)$, *with a constant* C *independent of* R *and* f.

As it happens with the Main Theorem (see Theorem 2.5 and its Main Corollary, 2.6), we can also obtain the corresponding result for power weights.

Corollary 2.18. *Let* $\nu > -1$, $\delta > 0$ *and* $R > 0$. *If*

$$-\delta - 1 \le a \le \delta, \tag{62}$$

and

$$-1-(\nu+1/2) \le a \le \nu + 1/2, \tag{63}$$

for $\nu \neq -1/2$, *or*

$$-1 < a \le 0, \tag{64}$$

for $\nu = -1/2$, *then*

$$\int_{\{0<r<1:|B_R^{\delta,\nu}(f,r)|>\lambda\}} r^a\,dr \le \frac{C}{\lambda}\int_0^1 |f(r)|r^a\,dr, \quad \lambda > 0.$$

Proof of Theorem 2.17. Assume $u(r)$ satisfies the assumptions of the theorem. As in Theorem 2.5, the operator $B_R^{\delta,\nu}$ exists for $f \in L^1((0,1),u(r)\,dr)$ because the coefficients are well defined. In this case this fact is a consequence of condition (52).

Then, by Lemma 2.1, we obtain

$$\begin{aligned}
&\int_{\{0<r<1:\ |T_3(f,r)|>\lambda\}} u(r)\,dr \\
&\qquad \le C\int_{\{0<r<1:\ r^{-(\delta+1)}\int_0^{r/2}\Psi_\nu(Ry)R^{-\delta}|f(y)|\,dy>\lambda\}} u(r)\chi_{[4/R,1)}(r)\,dr \\
&\qquad \le C\int_{\{0<r<1:\ H_{\delta+1}(\Psi_\nu(Ry)R^{-\delta}|f(y)|,r)>\lambda\}} u(r)\chi_{[4/R,1)}(r)\,dr.
\end{aligned}$$

So, using (57), which is possible by condition (52), we get

$$\int_{\{0<r<1:\ |T_3(f,r)|>\lambda\}} u(r)\,dr \le \frac{C}{\lambda}\int_0^1 |f(r)|u(r)\,dr.$$

The inequality for T_4 required a thorough analysis. From Lemma 2.1, it is obtained that

$$
\begin{aligned}
&\int_{\{0<r<1:\ |T_4(f,r)|>\lambda\}} u(r)\,dr \\
&\quad\le C\int_{\{0<r<1:\ r^{\nu+1/2}\int_r^1 R^{\nu+1/2-\delta}|f(y)|y^{-(\delta+1)}\,dy>\lambda\}} \chi_{(0,4/R)}(r)u(r)\,dr \\
&\qquad + C\int_{\{0<r<1:\ \int_r^1 R^{-\delta}y^{-(\delta+1)}|f(y)|\,dy>\lambda\}} \chi_{[4/R,1)}(r)u(r)\,dr \\
&\quad\le C\int_{\{0<r<1:\ H^*_{-(\nu+1/2)}(R^{\nu+1/2-\delta}y^{-(\delta+1)}|f(y)|,r)>\lambda\}} \chi_{(0,4/R)}(r)u(r)\,dr \\
&\qquad + C\int_{\{0<r<1:\ H^*(R^{-\delta}y^{-(\delta+1)}|f(y)|,r)>\lambda\}} \chi_{[4/R,1)}(r)u(r)\,dr.
\end{aligned}
$$

Now, with (53) for $\nu\in(-1,-1/2]$ and (54) for $\nu\in(-1/2,\infty)$, we can apply (58) to have

$$
\int_{\{0<r<1:\ H^*_{-(\nu+1/2)}(R^{\nu+1/2-\delta}y^{-(\delta+1)}|f(y)|,r)>\lambda\}} \chi_{(0,4/R)}(r)u(r)\,dr \le \frac{C}{\lambda}\int_0^1 |f(r)|u(r)\,dr.
$$

To complete the estimate for T_4, we consider (59), which follows from (55), to conclude that

$$
\int_{\{0<r<1:\ H^*(R^{-\delta}y^{-(\delta+1)}|f(y)|,r)>\lambda\}} \chi_{[4/R,1)}(r)u(r)\,dr \le \frac{C}{\lambda}\int_0^1 |f(r)|u(r)\,dr.
$$

The corresponding inequality for $T_2(f,r)$ and $T_5(f,r)$ follows from (60) as in the proof of Theorem 2.5 by taking into account (61). Finally, for the case of $T_1(f,r)$, by using Hölder's inequality,

$$
\int_0^{\frac{4}{R}} f(y)(ry)^{\nu+\frac12}R^{2(\nu+1)}\,dy \le Cr^{\nu+\frac12}R^{2(\nu+1)}\|u(y)f(y)\|_{L^1((0,1),\,dy)} \sup_{0<y<4/R}\frac{y^{\nu+\frac12}}{u(y)}
$$

then

$$
\begin{aligned}
&\int_{\{0<r<1:\ |T_1(f,r)|>\lambda\}} u(r)\,dr \\
&\le C\lambda^{-1}R^{2(\nu+1)}\|r^{\nu+\frac12}\chi_{(0,4/R)}(r)\|_{L^{1,\infty}((0,1),u(r)\,dr)} \\
&\times \|u(y)f(y)\|_{L^1((0,1),\,dy)} \sup_{0<y<4/R}\frac{y^{\nu+\frac12}}{u(y)} \le C\lambda^{-1}\|u(y)f(y)\|_{L^1((0,1),dy)},
\end{aligned}
$$

where we have applied (56). $\square$

Proof of Corollary 2.18. It can be checked easily that (62) and (63), for $\nu \neq -1/2$, and (62) and (64), for $\nu = -1/2$, imply conditions (52), (53), (54), (55) and (56). The condition (60) holds for any $a \in \mathbb{R}$. $\square$

7. Almost everywhere convergence

Another interesting question to be dealt with is the almost everywhere convergence of the Bochner-Riesz means and Gauss-Weierstrass summability method for functions in $L^p((0,1),dr)$.

To treat this question we need to analyze $L^p((0,1),dr)$ inequalities for the maximal operator

$$\mathrm{B}^{\delta,\nu}(f,r) = \sup_{R>0} \left| B_R^{\delta,\nu}(f,r) \right|.$$

The ultimate purpose is to prove that, for $\delta > 0$

$$\lim_{R\to\infty} B_R^{\delta,\nu}(f,r) = f(r)$$

for almost every $r > 0$ and $f \in L^p((0,1),dr)$.

First, we will prove an inequality of the form

$$\left\| r^a \mathrm{B}^{\delta,\nu}(f,r) \right\|_{L^p((0,1),dr)} \leq C \left\| r^a f(r) \right\|_{L^p((0,1),dr)} \tag{65}$$

for $\delta > 0$, $1 < p \leq \infty$. Besides, a weak type result for $|\mathrm{B}^{\delta,\nu}(f,r)|$ will be proved for $p = 1$. As an immediate consequence, we obtain a result of almost everywhere convergence for our means.

Muckenhoupt and Webb, in [**39**], give inequalities for Cesàro means of Laguerre polynomial series and for the supremum of these means with certain parameters and $1 < p \leq \infty$. For $p = 1$, they prove a weak type result too. They also obtain similar estimates for Cesàro means of Hermite polynomial series and for the supremum of those means in [**40**]. In both of them, an almost everywhere convergence result is obtained as a corollary. The result about Laguerre polynomials is an extension of a previous result in [**53**]. This kind of matters have been also studied by Ciaurri and Varona in [**21**] for the Cesàro means of generalized Hermite expansions.

In order to obtain our result about almost everywhere convergence, we will recall the c_p conditions, but this time we need some extra conditions in the end points. For each $\nu > -1, \delta > 0$ and $1 \leq p \leq \infty$, the parameters (a, ν, δ) satisfy the c_p conditions if

$$a > -1/p - (\nu + 1/2) \;\; (\geq \text{ if } p = \infty), \tag{66}$$

$$a < 1 - 1/p + (\nu + 1/2) \;\; (\leq \text{ if } p = 1), \tag{67}$$

$$a > -\delta - 1/p \;\; (\geq \text{ if } p = \infty), \tag{68}$$

$$a < 1 + \delta - 1/p \;\; (\leq \text{ if } p = 1). \tag{69}$$

The main results in this section are the following:

THEOREM 2.19. *Let $\nu > -1$, $\delta > 0$ and $1 < p \le \infty$. Then*

$$\left\| r^a \mathrm{B}^{\delta,\nu}(f,r) \right\|_{L^p((0,1),dr)} \le C \left\| r^a f(r) \right\|_{L^p((0,1),dr)}, \tag{70}$$

with a constant C independent of f, if the parameters (a, ν, δ) satisfy the c_p conditions.

Note that, in the case of $1 < p < \infty$, the c_p conditions are also necessary. The necessity in Theorem 2.19 is immediate, since (70) implies (19), and the conditions c_p are necessary for (19), as can be seen in Section 4.

THEOREM 2.20. *If $\nu > -1$, $\delta > 0$, (a, ν, δ) satisfy the c_1 conditions and*

$$E_\lambda = \left\{ r \in (0,1) : r^a |\mathrm{B}^{\delta,\nu}(f,r)| > \lambda \right\},$$

then

$$\int_{E_\lambda} dr \le \frac{C}{\lambda} \left\| r^a f(r) \right\|_{L^1((0,1),dr)}$$

holds with C independent of f and λ.

As an immediate consequence of Theorems 2.19 and 2.20 we have the following:

COROLLARY 2.21. *If $1 \le p < \infty$, $\nu > -1$, $\delta > 0$, (a, ν, δ) satisfy the c_p conditions and*

$$\left\| r^a f(r) \right\|_{L^p((0,1),dr)} < \infty,$$

then

$$\lim_{R \to \infty} B_R^{\delta,\nu}(f,r) = f(r)$$

for almost every $r > 0$.

Theorem 2.20 and the sufficiency of Theorem 2.19 will follow by showing that, if $1 \le p \le \infty$,

$$\left\| \sup_{R>0} \int_0^1 y^{-a} r^a |K_R^{\delta,\nu}(r,y)| |f(y)| \chi_{A_j} \, dy \right\|_{L^p((0,1),dr)} \le C \left\| f(r) \right\|_{L^p((0,1),dr)} \tag{71}$$

holds for $j = 1, 3, 4$, and that

$$\int_0^1 y^{-a} r^a |K_R^{\delta,\nu}(r,y)| |f(y)| \chi_{A_j} \, dy \le C \mathcal{M}(f,r), \tag{72}$$

for $j = 2, 5$, where $\mathcal{M}$ is the classical Hardy-Littlewood maximal function of f,

$$\mathcal{M}(f,r) = \sup_{y \ne r} \frac{1}{y-r} \int_r^y |f(t)| \, dt,$$

and C is independent of R and f. These results and the fact that $\mathcal{M}$ is $(1,1)$-weak and (p,p)-strong if $1 < p \le \infty$ complete the proofs of Theorem 2.20 and Theorem 2.19.

The proof of (72) follows from the given estimate for the kernel $K_R^{\delta,\nu}(r,y)$ and $y^{-a}r^a \sim C$ in $A_2 \cup A_5$. In the case of A_2, from $|K_R^{\delta,\nu}(r,y)| \le CR$ we deduce easily the required inequality. For A_5 the result is a consequence of $\Psi_\nu(Rr)\Psi_\nu(Ry) \le C$ and of a decomposition of the region in strips such that $R|r-y| \sim 2^k$, with $k = 0, \dots, [\log_2 R]-1$, as was done in the proof of Theorem 2.5.

In this manner, to complete the proof of Theorem 2.19 and 2.20 we only have to show (71) for $j = 1, 3, 4$ in the conditions c_p for $1 \le p \le \infty$.

To prove (71) for $j = 1, 3, 4$ we will use an interpolation argument based on six lemmas. These are stated below. They are small modifications of the six lemmas contained in Section 3 of [**39**] where a sketch of their proofs can be found.

LEMMA 2.10. *Let $\xi_0 > 0$. If $w < -1$, $w+t \le -1$ and $w+s+t \le -1$, then, for $p = 1$,*

$$\left\| r^w \chi_{[1,\infty)}(r) \sup_{\xi_0 \le \xi \le r} \xi^s \int_\xi^r y^t |f(y)|\, dy \right\|_{L^p((0,\infty),dr)} \le C\, \|f(r)\|_{L^p((0,\infty),dr)}$$

with C independent of f. If $w \le 0$, $w+t \le -1$ and $w+s+t \le -1$ with equality holding in at most one of the first two inequalities, this holds for $p = \infty$.

LEMMA 2.11. *Let $\xi_0 > 0$. If $t \le 0$, $w+t \le -1$ and $w+s+t \le -1$, with strict inequality in the last two in case of equality in the first, then, for $p = 1$,*

$$\left\| r^w \chi_{[1,\infty)}(r) \sup_{\xi_0 \le \xi \le r} \xi^s \int_r^\infty y^t |f(y)|\, dy \right\|_{L^p((0,\infty),dr)} \le C\, \|f(r)\|_{L^p((0,\infty),dr)}$$

with C independent of f. If $t < -1$, $w+t \le -1$ and $w+s+t \le -1$, this holds for $p = \infty$.

LEMMA 2.12. *If $s < 0$, $s+t \le 0$ and $w+s+t \le -1$, with equality holding in at most one of the last two inequalities, then, for $p = 1$,*

$$\left\| r^w \chi_{[1,\infty)}(r) \sup_{\xi \ge r} \xi^s \int_r^\xi y^t |f(y)|\, dy \right\|_{L^p((0,\infty),dr)} \le C\, \|f(r)\|_{L^p((0,\infty),dr)}$$

with C independent of f. If $s \le 0$, $s+t \le -1$ and $w+s+t \le -1$, this holds for $p = \infty$.

LEMMA 2.13. *If $t \le 0$, $s+t \le 0$ and $w+s+t \le -1$, with strict inequality holding in the first two in case the third is an equality, then, for $p = 1$,*

$$\left\| r^w \chi_{[1,\infty)}(r) \sup_{\xi \ge r} \xi^s \int_\xi^\infty y^t |f(y)|\, dy \right\|_{L^p((0,\infty),dr)} \le C\, \|f(r)\|_{L^p((0,\infty),dr)}$$

with C independent of f. If $t < -1$, $s+t \le -1$ and $w+s+t \le -1$, this holds for $p = \infty$.

Lemma 2.14. *If $s<0$, $w+s<-1$ and $w+s+t\le -1$, then, for $p=1$,*

$$\left\| r^w \chi_{[1,\infty)}(r) \sup_{\xi\ge r} \xi^s \int_1^r y^t |f(y)|\,dy \right\|_{L^p((0,\infty),dr)} \le C\,\|f(r)\|_{L^p((0,\infty),dr)}$$

with C independent of f. If $s<0$, $w+s\le 0$ and $w+s+t\le -1$, with equality holding in at most one of the last two inequalities, this holds for $p=\infty$.

Lemma 2.15. *If $w<-1$, $w+s<-1$ and $w+s+t\le -1$, then, for $p=1$,*

$$\left\| r^w \chi_{[1,\infty)}(r) \sup_{1\le\xi\le r} \xi^s \int_1^\xi y^t |f(y)|\,dy \right\|_{L^p((0,\infty),dr)} \le C\,\|f(r)\|_{L^p((0,\infty),dr)}$$

with C independent of f. If $w\le 0$, $w+s\le 0$ and $w+s+t\le -1$, with equality in at most one of the last two inequalities, this holds for $p=\infty$.

We are going to proceed with the proofs of the inequality (71) for regions A_1, A_3 and A_4. The results we will prove are included in the following:

Lemma 2.16. *If $\nu>-1$, $\delta>0$, $R>0$, $j=1,3,4$ and (a,ν,δ) satisfy the c_1 conditions, then (71) holds for $p=1$ with C independent of f.*

Lemma 2.17. *If $\nu>-1$, $\delta>0$, $R>0$, $j=1,3,4$ and (a,ν,δ) satisfy the c_∞ conditions, then (71) holds for $p=\infty$ with C independent of f.*

Corollary 2.22. *If $1\le p\le\infty$, $\nu>-1$, $\delta>0$, $R>0$, (a,ν,δ) satisfy the c_p conditions and $j=1,3,4$, then (71) holds with C independent of f.*

Proof of Corollary 2.22. It is enough to observe that if $1<p<\infty$ and (a,ν,δ) satisfy the c_p conditions, then $(a-1+1/p,\nu,\delta)$ satisfy the c_1 conditions. So, by Lemma 2.16

$$\left\| \sup_{R>0} \int_0^1 y^{-a+1-1/p} r^{a-1+1/p} |K_R^{\delta,\nu}(r,y)| \chi_{A_j}(r,y) |f(y)|\,dy \right\|_{L^1((0,1),dr)} \le C\,\|f(r)\|_{L^1((0,1),dr)},$$

and this is equivalent to

$$\int_0^1 r^{a+1/p} \sup_{R>0} \int_0^1 |K_R^{\delta,\nu}(r,y)| \chi_{A_j}(r,y) |f(y)|\,dy \frac{dr}{r} \le C \int_0^1 r^{a+1/p} |f(r)| \frac{dr}{r},$$

where $j = 1, 3, 4$. Similarly, $(a + 1/p, \nu, \delta)$ satisfy the c_∞ conditions. Hence, by Lemma 2.17,

$$\left\| r^{a+1/p} \sup_{R>0} \int_0^1 |K_R^{\delta,\nu}(r,y)| \chi_{A_j}(r,y) |f(y)|\, dy \right\|_{L^\infty((0,1),dr)} \\ \leq C \left\| r^{a+1/p} f(r) \right\|_{L^\infty((0,1),dr)}.$$

Now, we can use the Marcinkiewicz interpolation theorem to obtain the inequality

$$\int_0^1 \left(r^{a+1/p} \sup_{R>0} \int_0^1 |K_R^{\delta,\nu}(r,y)| \chi_{A_j}(r,y) |f(y)|\, dy \right)^p \frac{dr}{r} \\ \leq C \int_0^1 \left(r^{a+1/p} |f(r)| \right)^p \frac{dr}{r},$$

for $1 < p < \infty$ and the proof is finished. □

We will prove Lemmas 2.16 and 2.17 for A_j, $j = 1, 3$ and 4, separately.

Proof of Lemma 2.16 and Lemma 2.17 for A_1. First of all, we have to note that $B_R^{\delta,\nu}(f, r) = 0$ for in $0 < R < s_{1,\nu}$, where $s_{1,\nu}$ is the first positive zero of J_ν. In this manner, using the estimate (24), the left side of (71), in this case, is bounded by

$$C \Bigg\| r^{a+\nu+1/2} \chi_{[0,1]}(r) \\ \times \sup_{s_{1,\nu} < R \leq 4/r} R^{2(\nu+1)} \int_0^{4/R} y^{-a+\nu+1/2} |f(y)|\, dy \Bigg\|_{L^p((0,1),\, dr)}.$$

Making the change of variables $r = 4/u$ and $y = 4/v$, we have

$$C \Bigg\| u^{-a-\nu-\frac{1}{2}-\frac{2}{p}} \chi_{[4,\infty)}(u) \\ \times \sup_{s_{1,\nu} \leq R \leq u} R^{2(\nu+1)} \int_R^\infty v^{a-(\nu+\frac{1}{2})-2+\frac{2}{p}} g(v)\, dv \Bigg\|_{L^p((0,\infty),\, du)},$$

where $\|h\|_{L^p((0,\infty),du)}$ denotes the L^p norm in the variable u, and

$$g(v) = v^{-2/p} |f(v^{-1})|.$$

Note that function $g(v)$ is supported in $(1, \infty)$ and $\|g\|_{L^p((0,\infty),du)} = \|f\|_{L^p((0,1),dr)}$. This notation will be used through the section. Now, splitting the inner integral at u, we obtain the sum of

$$C\left\| u^{-a-\nu-\frac{1}{2}-\frac{2}{p}}\chi_{[4,\infty)}(u) \times \sup_{s_{1,\nu}\le R\le u} R^{2(\nu+1)}\int_R^u v^{a-(\nu+\frac{1}{2})-2+\frac{2}{p}}g(v)\,dv\right\|_{L^p((0,\infty),\,du)} \tag{73}$$

and

$$C\left\| u^{-a-\nu-\frac{1}{2}-\frac{2}{p}}\chi_{[4,\infty)}(u) \times \sup_{s_{1,\nu}\le R\le u} R^{2(\nu+1)}\int_u^\infty v^{a-(\nu+\frac{1}{2})-2+\frac{2}{p}}g(v)\,dv\right\|_{L^p((0,\infty),\,du)}. \tag{74}$$

From Lemma 2.10 we get the required estimate for (73), using condition (66); for $p=1$, this must be strict. Lemma 2.11 is applied to estimate (74), there we need condition (67), and it must be strict for $p=\infty$. This completes the proof of Lemmas 2.16 and 2.17 for $j=1$. □

Proof of Lemma 2.16 and Lemma 2.17 for A_3. Clearly, the left side of (71) is bounded by

$$C\left\| r^a\chi_{[4/R,1]}(r)\sup_{4/r\le R}\int_0^{r/2} y^{-a}|K_R^{\delta,\nu}(r,y)||f(y)|\,dy\right\|_{L^p((0,1),\,dr)}.$$

Splitting the inner integral at $2/R$, using the bound for the kernel given in (24) and the definition of Ψ_ν, we have this expression majorized by the sum of

$$\left\| r^a\chi_{[0,1]}(r)\sup_{4/r\le R}\int_0^{2/R}|f(y)|\frac{(Ry)^{\nu+1/2}y^{-a}}{R^\delta|r-y|^{\delta+1}}\,dy\right\|_{L^p((0,1),\,dr)} \tag{75}$$

and

$$\left\| r^a\chi_{[0,1]}(r)\sup_{4/r\le R}\int_{2/R}^{r/2}\frac{|f(y)|y^{-a}}{R^\delta|r-y|^{\delta+1}}\,dy\right\|_{L^p((0,1),\,dr)}. \tag{76}$$

For (75), taking into account that $|r-y|\sim r$ in A_3, the changes of variables $r=4/u$, $y=2/v$ give us

$$C\left\| u^{-a+(\delta+1)-\frac{2}{p}}\chi_{[4,\infty)}(u) \times \sup_{u\le R} R^{-\delta+(\nu+1/2)}\int_R^\infty v^{-(\nu+1/2)+a+\frac{2}{p}-2}g(v)\,dv\right\|_{L^p((0,\infty),\,du)}.$$

Lemma 2.13 can be used on here. The required conditions for $p = 1$ are (67) and (69). For $p = \infty$ the same inequalities are needed but they must be strict.

On the other hand, in (76), using again that $|r-y| \sim r$, by changing of variables $r = 4/u$ and $y = 2/v$ we have

$$C \left\| u^{-a+(\delta+1)-\frac{2}{p}} \chi_{[4,\infty)}(u) \sup_{u\le R} R^{-\delta} \int_{2u}^{R} v^{a+\frac{2}{p}-2} g(v)\, dv \right\|_{L^p((0,\infty),\, du)}$$
$$\le C \left\| u^{-a+(\delta+1)-\frac{2}{p}} \chi_{[4,\infty)}(u) \sup_{u\le R} R^{-\delta} \int_{u}^{R} v^{a+\frac{2}{p}-2} g(v)\, dv \right\|_{L^p((0,\infty),\, du)}.$$

Lemma 2.12 can then be applied. For $p = 1$, we need $\delta > 0$, which is an hypothesis, and (69). For $p = \infty$ the inequalities are the same, with the requirement that (69) is strict. This completes the proof of Lemmas 2.16 and 2.17 for $j = 3$. □

Proof of Lemma 2.16 and Lemma 2.17 for A_4. In this case, the left side of (71) is estimated by

$$C \left\| r^a \chi_{[0,1/2]}(r) \sup_{R>4} \int_{\max(4/R,2r)}^{1} y^{-a} |K_R^{\delta,\nu}(r,y)||f(y)|\, dy \right\|_{L^p((0,1),\, dr)}.$$

To majorize this, we decompose the R-range in two regions: $4 < R \le 2/r$ and $R \ge 2/r$. In this manner, with the bound for the kernel given in (24) and the definition of Ψ_ν, the previous norm is controlled by the sum of

$$C \left\| r^a \chi_{[0,1/2]}(r) \sup_{4<R\le 2/r} \int_{4/R}^{1} |f(y)| \frac{(Rr)^{\nu+1/2} y^{-a}}{R^\delta |r-y|^{\delta+1}}\, dy \right\|_{L^p((0,1),\, dr)}$$

and

$$C \left\| r^a \chi_{[0,1/2]}(r) \sup_{R\ge 2/r} \int_{2r}^{1} \frac{|f(y)| y^{-a}}{R^\delta |r-y|^{\delta+1}}\, dy \right\|_{L^p((0,1),\, dr)}.$$

Next, using that $|r-y| \sim y$ in A_4, the changes of variables $r = 2/u$ and $y = 1/v$ control the previous norms by

$$C \left\| u^{-a-\frac{2}{p}-(\nu+\frac{1}{2})} \chi_{[4,\infty)}(u) \times \sup_{4<R\le u} R^{-\delta+(\nu+\frac{1}{2})} \int_{1}^{R/4} v^{a+\frac{2}{p}-2+(\delta+1)} g(v)\, dv \right\|_{L^p((0,\infty),\, du)} \tag{77}$$

and

$$C \left\| u^{-a-\frac{2}{p}} \chi_{[4,\infty)}(u) \sup_{R\ge u} R^{-\delta} \int_{1}^{u/4} v^{a+\frac{2}{p}-2+(\delta+1)} g(v)\, dv \right\|_{L^p((0,\infty),\, du)}. \tag{78}$$

In (77), we use Lemma 2.15; for $p = 1$, conditions (66) and (68) are needed, they must be strict; we need the same for $p = \infty$ without those restrictions. For (78), Lemma 2.14 requires the hypothesis $\delta > 0$ and condition (68) for $p = 1$ and the same for $p = \infty$, with strict inequality for (68) in the case $p = 1$. This proves Lemmas 2.16 and 2.17 for $j = 4$. $\square$

With this, we finish completely the proof of Lemmas 2.16 and 2.17.

Recall now that the heat semigroup for Fourier-Bessel expansions is defined, for $t > 0$, by

$$H_t(f, r) = \sum_{j=1}^{\infty} e^{-ts_{j,\nu}^2} \int_0^1 f(y)\varphi_j^\nu(r)\varphi_j^\nu(y)\, dy.$$

We define the maximal operator by

$$\mathrm{H}(f, r) = \sup_{t>0} |H_t(f, r)|\,.$$

Weighted inequalities for H are consequences of the results obtained for $\mathrm{B}^{\delta,\nu}$ because, using the subordination formula in Lemma 2.9, we have

$$|\mathrm{H}(f, r)| \le C \left|\mathrm{B}^{\delta,\nu}(f, r)\right|.$$

From Theorems 2.19 and 2.20 we will show some estimates for the supremum of H_t. An immediate consequence of these inequalities will be an almost everywhere result for this operator.

Corollary 2.23. *Let $\nu > -1$ and $1 < p \le \infty$. Then*

$$\|r^a \mathrm{H}(f, r)\|_{L^p((0,1),\,dr)} \le C\, \|r^a f(r)\|_{L^p((0,1),\,dr)}\,,$$

with a constant C independent of t and f, if the conditions (66) *and* (67) *are satisfied.*

Corollary 2.24. *If $\nu > -1$, the conditions* (66) *and* (67) *are satisfied, for $p = 1$, and*

$$N_\mu = \{r \in (0, 1) : r^a|\mathrm{H}(f, r)| > \mu\},$$

then

$$\int_{N_\mu} dr \le \frac{C}{\mu}\, \|r^a f(r)\|_{L^1((0,1),\,dr)}$$

holds with C independent of f and μ.

Corollary 2.25. *If $1 \le p < \infty$, $\nu > -1$,* (66) *and* (67) *are satisfied and*

$$\|r^a f(r)\|_{L^p((0,1),\,dr)} < \infty,$$

then

$$\lim_{t\to 0} H_t(f, r) = f(r)$$

for almost every $r > 0$.

In previous Section 5 it was shown that, for $1 < p < \infty$,

$$\|r^a H_t(f,r)\|_{L^p((0,1),\,dr)} \le C\,\|r^a f(r)\|_{L^p((0,1),dr)}\,,$$

if and only if the conditions (66) and (67) are satisfied. In this manner, we have that the conditions are also necessary in the case $1 < p < \infty$ for the inequality in Corollary 2.23.

The proofs of Corollary 2.23 and Corollary 2.24 are deduced from the identity (43), shown in Section 5,

$$H_t(f,r) = \frac{2t^{\delta+1}}{\Gamma(\delta+1)} \int_0^\infty e^{-tR^2} R^{2\delta+1} B_R^\delta(f,r)\,dR.$$

From this fact we obtain that

$$\sup_{t>0} |H_t(f,r)| \le C \sup_{R>0} |B_R^{\delta,\nu}(f,r)|$$

for any $\delta > 0$. Now, to finish we choose a δ such that the conditions (68) and (69) are satisfied and we use Theorems 2.19 and 2.20.

Corollary 2.25 follows from Corollaries 2.23 and 2.24 in the same way that Corollary 2.21 was obtained from Theorems 2.19 and 2.20.

CHAPTER 3

Littlewood-Paley-Stein g_k-functions for Fourier-Bessel expansions

1. Introduction

In this chapter, we define and study g_k-functions related to the Poisson semigroup of Fourier-Bessel expansions, for each $k \geq 1$. Mainly, it will be proved that these g_k-functions are Calderón-Zygmund operators in the sense of the associated space of homogeneous type, hence their mapping properties follow from the general theory.

For our convenience, we are going to introduce a new orthonormal system. We will take the functions

$$\phi_j^\nu(r) = d_{j,\nu} s_{j,\nu}^{1/2} J_\nu(s_{j,\nu} r) r^{-\nu}, \quad n = 1, 2, \dots,$$

with $d_{j,\nu} = \frac{\sqrt{2}}{|s_{j,\nu}^{1/2} J_{\nu+1}(s_{j,\nu})|}$. These functions are a slight modification of the functions that we have been using up to now but they are more appropriate to obtain our target in this chapter. Moreover, this system is directly related to the multidimensional Fourier-Bessel system when reduced to radial functions. The system $\{\phi_j^\nu\}_{j\geq 1}$ is a complete orthonormal basis in $L^2((0,1), d\mu_\nu)$ where

$$d\mu_\nu(r) = r^{2\nu+1} dr.$$

Furthermore, the functions $\{\phi_j^\nu\}_{j\geq 1}$ are eigenfunctions of the second order differential operator

$$L_\nu = -\frac{d^2}{dr^2} - \frac{2\nu+1}{r}\frac{d}{dr}$$

with the corresponding eigenvalue $s_{j,\nu}^2$. The Fourier-Bessel expansion of a function f is

$$f = \sum_{j=1}^{\infty} a_j(f) \phi_j^\nu,$$

where $a_j(f) = \int_0^1 f(y)\phi_j^\nu(y)\, d\mu_\nu(y)$ provided the integrals exist.

For $t > 0$, the corresponding Poisson semigroup, $P_t^\nu f$ is defined by

$$P_t^\nu f = \sum_{j=1}^{\infty} e^{-t s_{j,\nu}} a_j(f) \phi_j^\nu, \qquad f \in L^2((0,1), d\mu_\nu). \tag{79}$$

We can write (79) in the form

$$P_t^\nu f(r) = \int_0^1 P_t^\nu(r,y) f(y) d\mu_\nu(y)$$

where the kernel $P_t^\nu(r,y)$ is given by

$$P_t^\nu(r,y) = \sum_{j=1}^\infty e^{-s_{j,\nu}t} \phi_j^\nu(r)\phi_j^\nu(y).$$

In order to simplify our computations, we make a change of variable $e^{-t} = u$ and, using that $s_{j,\nu} \sim j$, from now on we will take $e^{-ts_{j,\nu}}$ as u^j, $0 < u < 1$; in this way, we write (79) in the form

$$\mathcal{P}_u^\nu f(r) = \int_0^1 \mathcal{P}_u^\nu(r,y) f(y) d\mu_\nu(y)$$

where the kernel $\mathcal{P}_u^\nu(r,y)$ is given by

$$\mathcal{P}_u^\nu(r,y) = \sum_{j=1}^\infty u^j \phi_j^\nu(r)\phi_j^\nu(y).$$

We now define the g_k functions, for each integer $k \geq 1$, by

$$(g_k(f,r))^2 = \int_0^1 \left| \left(u\frac{\partial}{\partial u}\right)^k \mathcal{P}_u^\nu f(r)\right|^2 \left(\log\frac{1}{u}\right)^{2k-1} \frac{du}{u},$$

where $\left(u\frac{\partial}{\partial u}\right)^k$ means that we are applying k times the operator $u\frac{\partial}{\partial u}$. Note that our definition of g_k-function is inspired by the definition of Thangavelu in [**55**, Chapter 4] after the change of variable $e^{-t} = u$ (to be precise, Thangavelu works with the heat semigroup for Hermite expansions).

We denote by $A_p^\nu = A_p^\nu((0,1), d\mu_\nu)$ the Muckenhoupt class of A_p weights on the space $((0,1), d\mu_\nu, |\cdot|)$, where $|\cdot|$ denotes the Euclidean metric. More precisely, A_p^ν is the class of all nonnegative functions $w \in L^1_{\text{loc}}((0,1), d\mu_\nu)$ such that $w^{-p'/p} \in L^1_{\text{loc}}((0,1), d\mu_\nu)$ and

$$\sup_{I\in\mathcal{I}} \left(\frac{1}{\mu_\nu(I)}\int_I w(r)\, d\mu_\nu(r)\right)\left(\frac{1}{\mu_\nu(I)}\int_I w(r)^{-p'/p}\, d\mu_\nu(r)\right)^{p/p'} < \infty$$

where $\mathcal{I}$ is the class of all intervals in $((0,1), |\cdot|)$.

With the previous notation, the target of this chapter is the proof of the following theorem:

THEOREM 3.1. *Let $w \in A_p^\nu$, $1 < p < \infty$. There exist two constants C_1 and C_2 such that, for all $f \in L^p((0,1), w\, d\mu_\nu)$, the following inequalities are valid for each $k \geq 1$:*

$$C_1 \|f\|_{L^p((0,1),w\,d\mu_\nu)} \leq \|g_k(f)\|_{L^p((0,1),w\,d\mu_\nu)} \leq C_2 \|f\|_{L^p((0,1),w\,d\mu_\nu)}. \tag{80}$$

2. The g_k-functions as vector-valued Calderón-Zygmund operators

The proof of Theorem 3.1 is an application of the theory of vector-valued Calderón-Zygmund operators defined on spaces of homogeneous type. Now, let us introduce some concepts to fix the setting in which we are going to work.

Following [**13**], a space of homogeneous type (X, μ, ρ) is a set X together with a quasimetric ρ and a positive measure μ on X such that for every $x \in X$ and $r > 0$, $\mu(B(x,r)) < \infty$, and such that there exists $0 < C < \infty$ verifying that $\mu((B(x,2r)) \leq C\mu(B(x,r))$; i.e., μ is a doubling measure.

Given $\nu > -1$, we shall work on the space $(0,1)$, equipped with the measure $d\mu_\nu$ and with the Euclidean distance $|\cdot|$. Since $d\mu_\nu$ possesses the doubling property, the triple $((0,1), d\mu_\nu, |\cdot|)$ forms a space of homogeneous type.

For $\mathbb{B}$ a Banach space, we say that a kernel $K : X \times X \setminus \{(r,r)\} \to \mathbb{B}$ is a standard kernel if there exist $\varepsilon > 0$ and $C < \infty$ such that for all $r, y, z \in X$ $(r \neq y)$, with $\rho(r,z) \leq \varepsilon\rho(r,y)$, then

$$\|K(r,y)\|_{\mathbb{B}} \leq \frac{C}{\mu(B(r,\rho(r,y)))} \tag{81}$$

and

$$\|K(r,y) - K(z,y)\|_{\mathbb{B}} + \|K(y,r) - K(y,z)\|_{\mathbb{B}} \leq C\frac{\rho(r,z)}{\rho(r,y)\mu(B(r,\rho(r,y)))} \tag{82}$$

hold. Thus, a vector-valued Calderón-Zygmund operator with associated kernel K is a linear operator T bounded from $L^2(X, d\mu)$ into $L^2_{\mathbb{B}}(X, d\mu)$ such that, for every $f \in L^2(X, d\mu)$ and r outside the support of f,

$$Tf(r) = \int_X K(r,y)f(y)\,d\mu.$$

It is known that any vector-valued Calderón-Zygmund operator as above is bounded from $L^p(X, w\,d\mu)$ into $L^p_{\mathbb{B}}(X, w\,d\mu)$, for $1 < p < \infty$ and any weight w in the Muckenhoupt type class $A_p(X, d\mu)$, see [**50**]. The weights in $A_p(X, d\mu)$ are nonnegative functions $w \in L^1_{\text{loc}}(X, d\mu)$ such that $w \in L^{p'/p}_{\text{loc}}(X, d\mu)$ and

$$\sup_{B\in\mathcal{B}} \left(\frac{1}{\mu(B)}\int_B w(r)\,d\mu(r)\right)\left(\frac{1}{\mu(B)}\int_B w(r)^{-p'/p}\,d\mu(r)\right)^{p/p'} < \infty,$$

where $\mathcal{B}$ is the class of all the balls in (X, ρ).

The g_k-functions can be seen as vector-valued operators taking values in a Banach space. For each integer $k \geq 1$, we consider the space

$$\mathbb{B}_k = L^2\left((0,1), \left(\log\frac{1}{u}\right)^{2k-1} \frac{du}{u}\right).$$

and the vector-valued kernel

$$G_{u,k}(r,y) = \left(u\frac{\partial}{\partial u}\right)^k \mathcal{P}_u^\nu(r,y).$$

Defining

$$G_{u,k}f(r) = \int_0^1 G_{u,k}(r,y) f(y)\, d\mu_\nu(r),$$

clearly, the identity

$$g_k(f,r) = \|G_{u,k}f(r)\|_{\mathbb{B}_k}$$

holds. To prove that operator $G_{u,k}f(r)$ is bounded from $L^2((0,1),d\mu_\nu)$ into the space $L^2_{\mathbb{B}_k}((0,1),d\mu_\nu)$ it is enough the result contained in the following lemma where it is established that g_k-functions are isometries in $L^2((0,1),d\mu_\nu)$.

Lemma 3.1. *For each $k \geq 1$ and $f \in L^2((0,1),d\mu_\nu)$ one has*

$$\|g_k(f)\|^2_{L^2((0,1),d\mu_\nu)} = 2^{-2k}\Gamma(2k)\, \|f\|^2_{L^2((0,1),d\mu_\nu)}.$$

Proof. The proof works the same as in [**55**, Theorem 4.1.1] after a change of variable. □

In this way, showing the inequalities

$$\|G_{u,k}(r,y)\|_{\mathbb{B}_k} \leq \frac{C}{\mu_\nu(B(r,|r-y|))} \tag{83}$$

and

$$\|\nabla_{r,y} G_{u,k}(r,y)\|_{\mathbb{B}_k} \leq \frac{C}{|r-y|\mu_\nu(B(r,|r-y|))} \tag{84}$$

(note that (83) and (84) imply (81) and (82)), we have the following proposition:

Proposition 3.2. *For each $k \geq 1$, $G_{u,k}f(r)$ is a vector-valued Calderón-Zygmund operator taking values in $\mathbb{B}_k$.*

Proof of Theorem 3.1. *The direct inequality.* Observe that

$$g_k(f,r) = \|G_{u,k}f(r)\|_{\mathbb{B}_k}.$$

Therefore, the boundedness of $g_k(f,r)$ in $L^p((0,1),w\,d\mu_\nu)$ is equivalent to the boundedness of the operator $G_{u,k}$ from $L^p((0,1),w\,d\mu_\nu)$ into $L^p_{\mathbb{B}_k}((0,1),w\,d\mu_\nu)$. By Proposition 3.2, $G_{u,k}$ is a vector-valued Calderón-Zygmund operator. Thus, by the general theory, for any $1<p<\infty$ and $w \in A_p^\nu$, $G_{u,k}$ is bounded from $L^p((0,1),w\,d\mu_\nu)$ into $L^p_{\mathbb{B}_k}((0,1),w\,d\mu_\nu)$.

The reverse inequality. By polarizing the isometry

$$\|g_k(f)\|^2_{L^2((0,1),d\mu_\nu)} = 2^{-2k}\Gamma(2k)\,\|f\|^2_{L^2((0,1),d\mu_\nu)}$$

of Lemma 3.1, we obtain

$$\left|\int_0^1 f_1(r)f_2(r)\,d\mu_\nu(r)\right| = \frac{2^{2k}}{\Gamma(2k)}\int_0^1\int_0^1\left(\log\frac{1}{u}\right)^{2k-1} \times \left(\left(u\frac{\partial}{\partial u}\right)^k \mathcal{P}^\nu_u f_1(r)\right)\left(\left(u\frac{\partial}{\partial u}\right)^k \mathcal{P}^\nu_u f_2(r)\right)\frac{du}{u}\,d\mu_\nu(r),$$

which leads to the inequality

$$\left|\int_0^1 f_1(r)f_2(r)\,d\mu_\nu(r)\right| \le \frac{2^{2k}}{\Gamma(2k)}\int_0^1 g_k(f_1,r)g_k(f_2,r)\,d\mu_\nu(r).$$

Taking $h(r) = w(r)^{1/p}f_2(r)$ we get

$$\left|\int_0^1 f_1(r)w(r)^{1/p}f_2(r)\,d\mu_\nu(r)\right| \le \frac{2^{2k}}{\Gamma(2k)}\int_0^1 g_k(f_1,r)w(r)^{1/p}w(r)^{-1/p}g_k(h,r)\,d\mu_\nu(r).$$

By applying Holder's inequality then

$$\left|\int_0^1 f_1(r)w(r)^{1/p}f_2(r)\,d\mu_\nu(r)\right| \le C\,\|g_k(f_1)\|_{L^p((0,1),w\,d\mu_\nu)}\,\|g_k(h)\|_{L^{p'}((0,1),w'\,d\mu_\nu)},$$

with $w' = w^{\frac{-1}{p-1}}$. Since if $w \in A^\nu_p$ then $w^{\frac{-1}{p-1}} \in A^\nu_{p'}$ and by the direct part of (80), we have

$$\|g_k(h)\|_{L^{p'}((0,1),w'\,d\mu_\nu)} \le C\,\|h\|_{L^{p'}((0,1),w'\,d\mu_\nu)} = C\,\|f_2\|_{L^{p'}((0,1),d\mu_\nu)}.$$

Therefore, we have the inequality

$$\left|\int_0^1 f_1(r)w(r)^{1/p}f_2(r)\,d\mu_\nu(r)\right| \le C\,\|g_k(f_1)\|_{L^p((0,1),w\,d\mu_\nu)}\,\|f_2\|_{L^{p'}((0,1),d\mu_\nu)}.$$

Taking supremum over all f_2 with $\|f_2\|_{L^{p'}((0,1),d\mu_\nu)} \le 1$, then

$$\|f_1\|_{L^p((0,1),w\,d\mu_\nu)} = \sup\left|\int_0^1 f_1(r)w(r)^{1/p}f_2(r)\,d\mu_\nu(r)\right|$$

and we get

$$\|f_1\|_{L^p((0,1),w\,d\mu_\nu)} \le C\,\|g_k(f_1)\|_{L^p((0,1),w\,d\mu_\nu)}.$$

$\square$

3. Technical lemmas

In this section we will develop several technical results that will be useful to get the proofs of the estimates (83) and (84) in Sections 4 and 5. Many of the ideas and procedures in these three sections are taken from [**17**].

Recall that the Bessel function J_ν satisfies

$$J'_\nu(t) = \frac{\nu}{t} J_\nu(t) - J_{\nu+1}(t). \tag{85}$$

We will use the fact that

$$s_{j,\nu} = O(j), \qquad d_{j,\nu} = O(1). \tag{86}$$

The following asymptotics will be used (see [**31**, p. 122]):

$$\sqrt{z} J_\nu(z) = \sum_{n=0}^{M} \left(\frac{A_{\nu,n}}{z^n} \sin z + \frac{B_{\nu,n}}{z^n} \cos z \right) + H_M(z), \tag{87}$$

where $M = 0, 1, \dots$ and $|H_M(z)| \le C z^{-(M+1)}$, $z \to \infty$. On the other hand,

$$J_\nu(z) = O(z^\nu), \qquad z \to 0^+. \tag{88}$$

Poisson's integral formula

$$J_\nu(z) = C_\nu z^\nu \int_0^1 (1 - t^2)^{\nu - 1/2} \cos(zt)\, dt, \quad \nu > -1/2, \tag{89}$$

will also be helpful.

LEMMA 3.2. *Let $\nu > -1$, ℓ be a nonnegative integer and γ be a real number. Then each of the four functions*

$$d_{j,\nu}^2 s_{j,\nu}^\gamma \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \big(s_{j,\nu}(r \pm y)\big), \qquad j = 1, 2, \dots,$$

can be written as the sum of terms of the form

$$j^\gamma \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\pi j (r \pm y)) E_{\gamma,\ell}(j, r, y),$$

where

$$E_{\gamma,\ell}(j, r, y) = \sum_{k=0}^{\ell} \frac{A_k(r, y)}{j^k} + q_j^{(\ell)}(r, y),$$

and $A_k(r, y)$, $k = 0, 1, \dots, \ell$, and $q_j^{(\ell)}(r, y)$, $j = 1, 2, \dots$, are functions such that $|A_k(r, y)| \le C$, $|q_j^{(\ell)}(r, y)| \le C j^{-\ell-1}$, $0 < r, y < 1$, with a constant $C = C_{\nu,\ell,\gamma}$.

The lemma follows by taking $\mu = \nu$, $m = j = 0$ in [**17**, Lemmas 4.1 and 4.2] (the functions $A_k(r, y)$ now incorporate some bounded functions that appear in those lemmas).

The following estimate will be used from now on with no additional comments

$$u\left(\log\frac{1}{u}\right)^{2k-1}\le C\begin{cases}1, & 0<u<1/2,\\ (1-u)^{2k-1}, & 1/2\le u<1.\end{cases}\tag{90}$$

By P_u and Q_u, $0<u<1$, we denote the usual Poisson and conjugate Poisson kernels,

$$P_u(r)=\frac{1}{2}+\sum_{j=1}^{\infty}u^j\cos(jr)=\frac{1-u^2}{2(1-2u\cos r+u^2)},$$

$$Q_u(r)=\sum_{j=1}^{\infty}u^j\sin(jr)=\frac{u\sin r}{1-2u\cos r+u^2}.$$

Notice that for $r\neq 2k\pi$, $k\in\mathbf{Z}$, $\lim_{u\to 1^-}P_u(r)=\frac{1}{2}$, $\lim_{u\to 1^-}Q_u(r)=\frac{1}{2}\cot(\frac{\pi}{2}r)$.

Lemma 3.3. *For $j=1,2,\dots$, $0<u<1$ and $0<|r|<3\pi/2$, we have*

$$\left|\left(u\frac{\partial}{\partial u}\right)^j P_u(r)\right|\le C_j u\frac{(1-u)^{j+1}+|\sin\frac{r}{2}|^{j+1}}{((1-u)^2+4u\sin^2\frac{r}{2})^{j+1}},$$

$$\left|\left(u\frac{\partial}{\partial u}\right)^j Q_u(r)\right|\le C_j u\frac{(1-u)^{j+1}+|\sin\frac{r}{2}|^{j+1}}{((1-u)^2+4u\sin^2\frac{r}{2})^{j+1}}.$$

Proof. We have that $\left|\left(u\frac{\partial}{\partial u}\right)^j P_u(r)\right|$ or $\left|\left(u\frac{\partial}{\partial u}\right)^j Q_u(r)\right|$ has the form

$$\frac{uS_j(u,r)}{((1-u)^2+4u\sin^2\frac{r}{2})^{j+1}},$$

where

$$\begin{aligned}S_{j+1}(u,r)&=u\left((1-u)^2+4u\sin^2\frac{r}{2}\right)\frac{dS_j(u,r)}{du}\\&\quad+\left((1-u)^2+4u\sin^2\frac{r}{2}-2(j+1)u(2\sin^2\frac{r}{2}-(1-u))\right)S_j(u,r)\end{aligned}$$

and $S_0(u,r)=\frac{1}{2}(1-u^2)$ in the case of P_u or $S_0(u,r)=u\sin r$ in the case Q_u. It follows inductively that $S_j(u,r)$ can be written as the sum of terms of the form $f(u,r)(1-u)^m\sin^k\frac{r}{2}$, where $f(u,r)$ is a bounded function and $m+k\ge j+1$. From this, $|S_j(u,r)|\le C\left((1-u)^{j+1}+|\sin\frac{r}{2}|^{j+1}\right)$. $\square$

Remark 3.1. *The result in the previous lemma allows us to obtain estimates for $\left|\left(u\frac{\partial}{\partial u}\right)^j P_u(r)\right|$ and $\left|\left(u\frac{\partial}{\partial u}\right)^j Q_u(r)\right|$ depending on u or r separately. The exact bounds are*

$$\left|\left(u\frac{\partial}{\partial u}\right)^j P_u(r)\right|\le C_j|r|^{-(j+1)},\qquad \left|\left(u\frac{\partial}{\partial u}\right)^j Q_u(r)\right|\le C_j|r|^{-(j+1)},$$

and

$$\left|\left(u\frac{\partial}{\partial u}\right)^j P_u(r)\right| \le C_j u(1-u)^{-(j+1)},$$

$$\left|\left(u\frac{\partial}{\partial u}\right)^j Q_u(r)\right| \le C_j u(1-u)^{-(j+1)}.$$

These results follow immediately using the previous lemma and the estimates

$$\frac{(1-u)^{j+1} + |\sin\frac{r}{2}|^{j+1}}{((1-u)^2 + 4u\sin^2\frac{r}{2})^{j+1}} \le C|r|^{-(j+1)},$$

for $1-u \le |r| < 3\pi/2$, *and*

$$\frac{(1-u)^{j+1} + |\sin\frac{r}{2}|^{j+1}}{((1-u)^2 + 4u\sin^2\frac{r}{2})^{j+1}} \le C(1-u)^{-(j+1)},$$

for $0 < |r| < 1-u$.

LEMMA 3.4. *For* $k \ge 1$, $m = 1, 2, \dots$, *and* $0 < |r| < 3\pi/2$, *we have*

$$\int_0^1 \left(\log\frac{1}{u}\right)^{2k-1} \left(\sum_{j=1}^{\infty} j^{k+m-1} u^j \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(jr)\right)^2 \frac{du}{u} \le Cr^{-2m}$$

with a constant C *independent of* r.

PROOF. Our task reduces to proving that

$$\int_0^1 \left(\log\frac{1}{u}\right)^{2k-1} \left(\left(u\frac{\partial}{\partial u}\right)^{k+m-1} P_u(r)\right)^2 \frac{du}{u} \le Cr^{-2m},$$

and the analogous for $\left(u\frac{\partial}{\partial u}\right)^{k+m-1} Q_u(r)$. We show the proof for P_u, the other one is obtained following the same reasoning. By Lemma 3.3, the left hand side of the previous inequality is bounded by

$$\int_0^1 \left(\log\frac{1}{u}\right)^{2k-1} \left(u\frac{(1-u)^{k+m} + |\sin\frac{r}{2}|^{k+m}}{((1-u)^2 + 4u\sin^2\frac{r}{2})^{k+m}}\right)^2 \frac{du}{u}. \tag{91}$$

To estimate the integral (91) we decompose it into the intervals $0 < u \le 1/2$ and $1/2 < u < 1$. For the first one, using (90) we obtain that

$$\int_0^{1/2} \left(\log\frac{1}{u}\right)^{2k-1} \left(u\frac{(1-u)^{k+m} + |\sin\frac{r}{2}|^{k+m}}{((1-u)^2 + 4u\sin^2\frac{r}{2})^{k+m}}\right)^2 \frac{du}{u}$$
$$\le C\int_0^{1/2} \frac{C}{(1-u)^{2(k+m)}}\, du \le C.$$

For the interval $1/2 < u < 1$, using (90), taking into account that $\sin r/2 \sim r/2$, and applying the change of variable $(1-u)/|r| = w$, we have

$$\int_{1/2}^{1}\left(\log\frac{1}{u}\right)^{2k-1}\left(u\frac{(1-u)^{k+m}+|\sin\frac{r}{2}|^{k+m}}{((1-u)^2+4u\sin^2\frac{r}{2})^{k+m}}\right)^2\frac{du}{u}$$

$$\leq C\int_{1/2}^{1}(1-u)^{2k-1}\left(\frac{(1-u)^{k+m}+|\sin\frac{r}{2}|^{k+m}}{((1-u)^2+4u\sin^2\frac{r}{2})^{k+m}}\right)^2 du$$

$$\sim\int_{1/2}^{1}(1-u)^{2k-1}\left(\frac{(1-u)^{k+m}+|r|^{k+m}}{((1-u)^2+r^2)^{k+m}}\right)^2 du$$

$$= C\frac{1}{|r|^{2m}}\int_0^{1/(2|r|)} w^{2k-1}\left(\frac{w^{k+m}+1}{(w^2+1)^{k+m}}\right)^2 dw \leq C|r|^{-2m}$$

where in the last step we use that $\int_0^\infty w^{2k-1}\left(\frac{w^{k+m}+1}{(w^2+1)^{k+m}}\right)^2 dw < \infty$. $\square$

Lemma 3.5. *Let $0<\gamma<1$ and f be a 2π-periodic function such that $|f(r)|\leq C|r|^{-\gamma}$ for $0<|r|\leq\pi$. Then, the estimate*

$$\left|\left(P_u-\frac{1}{2}\right)*f(r)\right|\leq Cu\frac{(1-u)^\gamma+|\sin\frac{r}{2}|^\gamma}{((1-u)^2+4u\sin^2\frac{r}{2})^\gamma}$$

holds, with $C=C_\gamma$ independent of $0<u<1$ and $0<|r|<3\pi/2$.

Proof. It is sufficient to check that

$$|(P_u-\frac{1}{2})*f(r)|\leq Cr\frac{(1-u)^\gamma+|\sin\frac{r}{2}|^\gamma}{((1-u)^2+4u\sin^2\frac{r}{2})^\gamma}$$

for $0<|r|<\pi/2$; and $|(P_u-\frac{1}{2})*f(r)|\leq Cu$ for $\pi/2\leq|r|\leq3\pi/2$, with constants independent of u and r. Since the periodicity of f allows the hypothesized estimate $|f(r)|\leq C|r|^{-\gamma}$ to hold for $0<|r|\leq3\pi/2$, the proof of the first bound reduces to showing that

$$\int_0^\pi\left|P_u(y)-\frac{1}{2}\right||r\pm y|^{-\gamma}\,dy \leq Cu\frac{(1-u)^\gamma+(\sin\frac{r}{2})^\gamma}{((1-u)^2+4u\sin^2\frac{r}{2})^\gamma},\quad 0<r<\pi/2. \tag{92}$$

To show (92) we split the region of integration $(0,\pi)$ into three parts, $A=(0,r/2)$, $B=(2r,\pi)$ and $D=(r/2,2r)$ and we take into account that

$$\left|P_u(r)-\frac{1}{2}\right|\leq Cu\frac{1-u}{(1-u)^2+4u\sin^2\frac{r}{2}}.$$

For the region A, we separately consider $1-u\leq r$ and $0<r<1-u$. For the first case, the fact that

$$\int_{-\pi}^{\pi}\frac{u(1-u)}{(1-u)^2+4u\sin^2\frac{y}{2}}\,dy=\frac{2\pi u}{1+u}$$

easily gives

$$\int_A \left| P_u(y) - \frac{1}{2} \right| |r \pm y|^{-\gamma}\, dy \le Cur^{-\gamma} \le Cu \frac{(1-u)^\gamma + (\sin\frac{r}{2})^\gamma}{((1-u)^2 + 4u\sin^2\frac{r}{2})^\gamma},$$

where we have used that $\frac{(1-u)^\gamma+(\sin\frac{r}{2})^\gamma}{((1-u)^2+4u\sin^2\frac{r}{2})^\gamma} \sim \left(\frac{1}{1-u+r}\right)^\gamma$, and $1-u \le r$ implies that $\left(\frac{1}{1-u+r}\right)^{-\gamma} \le Cr^\gamma$. For the second case, $0 < r < 1-u$, we have that $r/2 < (1-u+r)/4$, hence we enlarge the region of integration to get

$$\begin{aligned}\int_A \left| P_u(y) - \frac{1}{2} \right| |r \pm y|^{-\gamma}\, dy &\le \int_0^{\frac{1-u+r}{4}} \left| P_u(y) - \frac{1}{2} \right| |r \pm y|^{-\gamma}\, dy \\ &\le \frac{u}{1-u} \int_0^{\frac{1-u+r}{4}} |r \pm y|^{-\gamma}\, dy \le C\frac{u}{1-u}(1-u+r)^{1-\gamma} \\ &\le C\frac{u}{(1-u+r)^\gamma},\end{aligned}$$

and the last expression is equivalent to $Cu\frac{(1-u)^\gamma+(\sin\frac{r}{2})^\gamma}{((1-u)^2+4u\sin^2\frac{r}{2})^\gamma}$.

For the region B we also consider separately the cases $1-u \le r$ and $0 < r < 1-u$. For the first one, we make the change of variable $t = \pi - y$ to obtain the integral

$$\int_0^{\pi-2r} u \frac{1-u}{(1-u)^2 + 4u\cos^2\frac{t}{2}} |r \pm (\pi - t)|^{-\gamma}\, dt$$

and the bound is obtained as in the case $1-u \le r$ in the region A, since

$$\int_{-\pi}^{\pi} \frac{u(1-u)}{(1-u)^2 + 4u\cos^2\frac{t}{2}}\, dt = \frac{2\pi u}{1+u}.$$

For the case $0 < r < 1-u$, we split the integral into two parts

$$\begin{aligned}&\int_B \left| P_u(y) - \frac{1}{2} \right| |r \pm y|^{-\gamma}\, dy \\ &\qquad = \int_{2r}^{r+(1-u)} \left| P_u(y) - \frac{1}{2} \right| |r \pm y|^{-\gamma}\, dy \\ &\qquad\qquad + \int_{r+(1-u)}^{\pi} \left| P_u(y) - \frac{1}{2} \right| |r \pm y|^{-\gamma}\, dy =: J_1 + J_2.\end{aligned}$$

First, we have

$$\begin{aligned}J_1 &\le \frac{u}{1-u} \int_{2r}^{1-u+r} |r \pm y|^{-\gamma}\, dy \\ &\qquad \le C\frac{u}{1-u}(1-u+r)^{1-\gamma} \le C\frac{u}{(1-u+r)^\gamma}.\end{aligned}$$

On the other hand,

$$J_2 \le Cu \int_{r+(1-u)}^{\pi} |r \pm y|^{-\gamma}\, dy \le Cu(1-u+r)^{-\gamma},$$

and the result is proved for the entire region B because $(1-u+r)^{-\gamma} \sim \frac{(1-u)^\gamma + (\sin\frac{r}{2})^\gamma}{((1-u)^2+4u\sin^2\frac{r}{2})^\gamma}$.

In the region D, we have

$$\begin{aligned}\int_D \left|P_u(y) - \frac{1}{2}\right| |r\pm y|^{-\gamma}\, dy &\le \int_D u\frac{1-u}{(1-u)^2+4u\sin^2\frac{y}{2}}|r\pm y|^{-\gamma}\, dy\\ &\le Cu\frac{1-u}{(1-u)^2+4u\sin^2\frac{r}{2}}\int_D |r\pm y|^{-\gamma}\, dy\\ &\le Cu\frac{1-u}{(1-u)^2+4u\sin^2\frac{r}{2}} r^{1-\gamma}\\ &\le Cu\left(\frac{1-u}{(1-u)^2+4u\sin^2\frac{r}{2}}\right)^\gamma,\end{aligned}$$

where we have used that $\frac{1-u}{(1-u)^2+4u\sin^2\frac{r}{2}} \le Cr^{-1}$. This bound for the integral in D is enough for our purpose.

The proof for the region $\pi/2 \le |r| \le 3\pi/2$ reduces to showing that

$$\int_0^\pi y^{-\gamma}\left|P_u(y\pm r) - \frac{1}{2}\right| dy \le Cu, \quad \pi/2 \le r \le 3\pi/2.$$

The observation

$$\left|P_u(r) - \frac{1}{2}\right| \le Cu\frac{1-u}{(1-u)^2+4u\sin^2\frac{r}{2}} = C\frac{2u}{1+u}P_u(r)$$

implies that

$$\int_0^\pi y^{-\gamma}\left|P_u(y\pm r) - \frac{1}{2}\right| dy \le Cu\int_0^\pi y^{-\gamma}P_u(y\pm r)\, dy,$$

and the estimate follows since this last integral is bounded by a constant, as was seen in [**17**, Lemma 3.2]. □

Lemma 3.6. *For $k \ge 1$, $\gamma > 0$, and $0 < |r| < 3\pi/2$, we have*

$$\int_0^1 \left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=1}^\infty j^{k+\gamma-1}u^j \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(jr)\right)^2 \frac{du}{u} \le Cr^{-2\gamma} \tag{93}$$

with a constant C independent of r.

Proof. For $\gamma = m$, $m = 1, 2, \dots$, the result is contained in Lemma 3.4. To prove (93) for other values of γ we are going to prove that for each $\beta > 1$, the expression

$$\left|\sum_{j=1}^\infty j^{\beta-1}u^j \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(jr)\right|$$

can be written as a finite sum of terms of the kind

$$u\frac{(1-u)^\delta + |\sin\frac{r}{2}|^\delta}{((1-u)^2+4u\sin^2\frac{r}{2})^\delta}, \qquad \delta \le \beta.$$

With this estimate, proceeding as in Lemma 3.4, the result follows.

The following estimate is stated in [**63**, (13), p. 70, Vol. I]: for any $0<\beta<1$ and $0<|r|<3\pi/2$

$$\left|\sum_{j=1}^\infty j^{\beta-1}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(jr)\right| \le C_\gamma |r|^{-\beta}.$$

This, together with Lemma 3.5, shows that

$$\left|\sum_{j=1}^\infty j^{\beta-1}u^j\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(jr)\right| \le Cu\frac{(1-u)^\beta + |\sin\frac{r}{2}|^\beta}{((1-u)^2+4u\sin^2\frac{r}{2})^\beta} \tag{94}$$

for $0<\beta<1$.

Suppose now that $m<\beta<m+1$ with $m\ge 1$. To simplify the notation write

$$S_\beta(r) = S_\beta(u,r) = \sum_{j=1}^\infty u^j j^{\beta-1}\sin(jr),$$

and

$$C_\beta(r) = C_\beta(u,r) = \sum_{j=1}^\infty u^j j^{\beta-1}\cos(jr).$$

In [**17**, (3.3), p. 4449] we can find the identity

$$\begin{aligned} S_\beta(r) = S_m(r) &+ \sum_{s=1}^m\sum_{p=s}^m a_{s,p,\beta}\left(S_s(r)C_{\beta-p}(r) + C_s(r)S_{\beta-p}(r)\right) \\ &+ \sum_{s=1}^m \left(S_s(r)A_{s,\beta}(u,r) + C_s(r)B_{s,\beta}(u,r)\right), \end{aligned} \tag{95}$$

where $a_{s,p,\beta}$ are constants and $A_{s,\beta}(u,r)$, $B_{s,\beta}(u,r)$ are bounded functions. An analogous formula holds for C_β (to be precise, on the right side of (95), S_m, S_s, C_s and the second plus sign have to be replaced by C_m, C_s, S_s and the minus sign). So, with (95), we finish by using the bounds in Lemma 3.3 and (94). □

LEMMA 3.7. *For $k\ge 1$, $0<r<1$, and $J=[1/r]$, we have*

$$\int_0^1 \left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=1}^J j^{k+\beta}u^j\right)^2\frac{du}{u} \le Cr^{-(2\beta+2)}, \tag{96}$$

for $\beta>-1$, and

$$\int_0^1 \left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=J+1}^\infty j^{k+\beta}u^j\right)^2\frac{du}{u} \le Cr^{-(2\beta+2)}, \tag{97}$$

for $\beta < -1$.

Proof. We will prove (96), the proof of (97) is analogous and it will be omitted. First, we will think of the case $k = 1$; we are going to estimate

$$\int_0^1 \log\frac{1}{u}\left(\sum_{j=1}^{J} j^{1+\beta}u^j\right)^2 \frac{du}{u}.$$

Integrating by parts, this expression equals

$$\begin{aligned}
&\int_0^1 \left(\int_0^u \left(\sum_{j=1}^{J} s^{j-1/2} j^{1+\beta}\right)^2 ds\right)\frac{du}{u}\\
&= \int_0^1 \left(\int_0^u \left(\sum_{j,n=1}^{J} s^{j+n-1}(jn)^{1+\beta}\right) ds\right)\frac{du}{u}\\
&= \int_0^1 \left(\sum_{j,n=1}^{J} \frac{u^{j+n}}{j+n}(jn)^{1+\beta}\right)\frac{du}{u} \le C\int_0^1 \left(\sum_{j,n=1}^{J} \frac{u^{j+n}}{\sqrt{jn}}(jn)^{1+\beta}\right)\frac{du}{u}\\
&= C\int_0^1 \sum_{j,n=1}^{J} u^{j+n-1}(jn)^{1/2+\beta}\,du \le C\sum_{j,n=1}^{J}\frac{(jn)^{1/2+\beta}}{\sqrt{jn}}\\
&= C\left(\sum_{j=1}^{J} j^{\beta}\right)^2 \le CJ^{2\beta+2} \le Cr^{-(2\beta+2)}.
\end{aligned}$$

For $k > 1$, we integrate by parts $2k-2$ times to obtain

$$\int_0^1 \left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=1}^{J} j^{k+\beta}u^j\right)^2\frac{du}{u} \le C\int_0^1 \log\frac{1}{u}\left(\sum_{j=1}^{J} j^{\beta+1}u^j\right)^2\frac{du}{u}$$

and then apply the result for $k = 1$.

□

The next lemma analyzes a situation similar to the case $\gamma = 0$ in Lemma 3.6. This case has to be investigated separately because in this situation the proof of Lemma 3.4 does not work.

Lemma 3.8. *For* $k \ge 1$, $0 < |r| < 3\pi/2$, *and* $J = [1/|r|]$, *we have*

$$\int_0^1 \left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=J}^{\infty} j^{k-1}u^j \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi jr)\right)^2\frac{du}{u} \le C, \tag{98}$$

with a constant C *independent of* r.

Proof. We will prove the result for the case of sines. The other case is analogous. Applying the summation by parts formula,

$$\sum_{j=J}^{\infty} a_j(b_{j+1}-b_j) = -a_J b_J - \sum_{j=J}^{\infty} b_{j+1}(a_{j+1}-a_j),$$

with the sequences $a_j = j$ and $b_j = -\sum_{n=j}^{\infty} n^{k-2} u^n \sin(\pi n r)$, we have

$$\left|\sum_{j=J}^{\infty} j^{k-1} u^j \sin(\pi j r)\right| \\ \leq J \left|\sum_{n=J}^{\infty} n^{k-2} u^n \sin(\pi n r)\right| + \left|\sum_{j=J}^{\infty} \sum_{n=j+1}^{\infty} n^{k-2} u^n \sin(\pi n r)\right|.$$

The last sum, after a translation $n - j = m$ and using the formula for $\sin(a + b)$, equals

$$\sum_{s=0}^{k-2} \binom{k-2}{s} \left(\sum_{j=J}^{\infty} j^{k-s-2} u^r \cos(\pi j r) \sum_{m=1}^{\infty} m^s u^m \sin(\pi m r) \right. \\ \left. + \sum_{j=J}^{\infty} j^{k-s-2} u^j \sin(\pi j r) \sum_{m=1}^{\infty} m^s u^m \cos(\pi m r) \right),$$

hence

$$\left|\sum_{j=J}^{\infty} j^{k-1} u^j \sin(\pi j r)\right| \leq \left| J \sum_{n=J}^{\infty} n^{k-2} u^n \sin(\pi n r)\right| \\ + \left| \sum_{s=0}^{k-2} \binom{k-2}{s} \left(\sum_{j=J}^{\infty} j^{k-s-2} u^j \cos(\pi j r) \sum_{m=1}^{\infty} m^s u^m \sin(\pi m r) \right. \right. \\ \left. \left. + \sum_{j=J}^{\infty} j^{k-s-2} u^j \sin(\pi j r) \sum_{m=1}^{\infty} m^s u^m \cos(\pi m r) \right) \right|.$$

Concerning the first summand, using (97) with $\beta = -2$, the expression

$$\int_0^1 \left(\log \frac{1}{u}\right)^{2k-1} \left(\sum_{j=J}^{\infty} j^{k-2} u^j \sin(\pi j r) \right)^2 \frac{du}{u}$$

is bounded by r^2, so the result follows, since $J^2 \sim r^{-2}$. To estimate the second summand, by Lemma 3.3 in [**17**], we have that

$$\sum_{m=1}^{\infty} m^s u^m \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\pi m r) \leq C r^{-2(s+1)},$$

and, on the other hand,

$$\int_0^1 \left(\log \frac{1}{u}\right)^{2k-1} \left(\sum_{j=J}^{\infty} j^{k-s-2} u^j \right)^2 \frac{du}{u}$$

is bounded by $r^{2(s+1)}$, after applying (97) with $\beta = -s - 2$, which finishes the proof. □

In the following lemma, we give estimates for the measure of the balls $B(r,|r-y|)$ in the space $((0,1),d\mu_\nu)$.

Lemma 3.9. *For $\nu>-1$ and $r\neq y$, the inequality*

(99)
$$\mu_\nu(B(r,|r-y|))\le C\begin{cases} r^{2\nu+2}, & 0<y\le r/2,\\ (ry)^{\nu+1/2}|r-y|, & r/2<y<\min\{1,3r/2\},\\ y^{2\nu+2}, & \min\{1,3r/2\}\le y<1,\end{cases}$$

holds.

Proof. (99) follows easily by studying separately the three considered regions. Thus, for the case $0<y\le r/2$,

$$\begin{aligned}\mu_\nu(B(r,|r-y|))&=\int_{B(r,|r-y|)}d\mu_\nu(t)=\int_y^{2r-y}t^{2\nu+1}\,dt\\ &=C((2r-y)^{2\nu+2}-y^{2\nu+2})\le Cr^{2\nu+2}.\end{aligned}$$

The case $\min\{1,3r/2\}\le y<1$ follows analogously. Concerning the case $r/2<y<\min\{1,3r/2\}$ we distinguish between $\nu\ge -1/2$ and $-1<\nu<-1/2$. For the first one, we obtain, taking into account that $r\sim y$,

$$\begin{aligned}\mu_\nu(B(r,|r-y|))&=\int_{B(r,|r-y|)}d\mu_\nu(t)=\int_{r-|r-y|}^{r+|r-y|}t^{2\nu+1}\,dt\\ &\le 2C|r+|r-y||^{2\nu+1}|r-y|\le C(ry)^{\nu+1/2}|r-y|.\end{aligned}$$

For the second case, we need to consider two regions separately. For the points (r,y) such that $r/2<y<r$, we have

$$\begin{aligned}\mu_\nu(B(r,|r-y|))&=\int_{B(r,|r-y|)}d\mu_\nu(t)=\int_y^{2r-y}t^{2\nu+1}\,dt\\ &\le Cr^{2\nu+1}|r-y|\le C(ry)^{\nu+1/2}|r-y|.\end{aligned}$$

In the case in which $r<y<\min\{1,3r/2\}$,

$$\begin{aligned}\mu_\nu(B(r,|r-y|))&=\int_{B(r,|r-y|)}d\mu_\nu(t)=\int_{2r-y}^{y}t^{2\nu+1}\,dt\\ &\le C(2r-y)^{2\nu+1}|r-y|\le Cy^{2\nu+1}|r-y|\\ &\le C(ry)^{\nu+1/2}|r-y|.\end{aligned}$$

□

4. Proof of the estimate (83)

The estimate (83) is an immediate consequence of Lemma 3.9 and the following proposition.

PROPOSITION 3.3. *Let* $\nu > -1$. *For* $k \geq 1$,

$$\int_0^1 \left| \left(u \frac{\partial}{\partial u} \right)^k \mathcal{P}_u^\nu(r,y) \right|^2 \left(\log \frac{1}{u} \right)^{2k-1} \frac{du}{u} \tag{100}$$
$$\leq C \begin{cases} r^{-4(\nu+1)}, & 0 < y \leq r/2, \\ (ry)^{-2(\nu+1/2)} |r-y|^{-2}, & r/2 < y < \min\{1, 3r/2\}, \\ y^{-4(\nu+1)}, & \min\{1, 3r/2\} \leq y < 1, \end{cases}$$

with C *independent of* $0 < u < 1$, r *and* y.

PROOF. *Case 1*: $0 < y \leq r/2$.

We split the series defining $\left(u \frac{\partial}{\partial u}\right)^k \mathcal{P}_u^\nu(r,y)$ into

$$\begin{aligned} A &= \sum_{j=1}^{J-1} j^k u^j \phi_j^\nu(r) \phi_j^\nu(y) \\ &= \sum_{j=1}^{J-1} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} (s_{j,\nu} r)^{1/2} J_\nu(s_{j,\nu} r) \cdot (s_{j,\nu} y)^{1/2} J_\nu(s_{j,\nu} y) \end{aligned}$$

and

$$\begin{aligned} B &= \sum_{j=J}^{\infty} j^k u^j \phi_j^\nu(r) \phi_j^\nu(y) \\ &= \sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} (s_{j,\nu} r)^{1/2} J_\nu(s_{j,\nu} r) \cdot (s_{j,\nu} y)^{1/2} J_\nu(s_{j,\nu} y), \end{aligned}$$

where $J = [1/r]$. Using (86) and (88) we write

$$\begin{aligned} |A| &\leq \sum_{j=1}^{J-1} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} |(s_{j,\nu} r)^{1/2} J_\nu(s_{j,\nu} r)| |(s_{j,\nu} y)^{1/2} J_\nu(s_{j,\nu} y)| \\ &\leq C (ry)^{-\nu} \sum_{j=1}^{J-1} j^{k+1} u^j |J_\nu(s_{j,\nu} r)| |J_\nu(s_{j,\nu} y)| \\ &\leq C \sum_{j=1}^{J-1} j^{2\nu+k+1} u^j. \end{aligned}$$

Now, using (96) with $\beta = 2\nu + 1$ we obtain

$$\int_0^1 \left(\log \frac{1}{u} \right)^{2k-1} \left| \sum_{j=1}^{J-1} j^{2\nu+k+1} u^j \right|^2 \frac{du}{u} \leq C r^{-4(\nu+1)}.$$

To get the same estimate for $|B|$ it is enough to show that for $0 < u < 1$, $0 < r < 1$, $0 < y \leq r/2$ and $\nu > -1/2$

(101)
$$\int_0^1 \left(\log \frac{1}{u}\right)^{2k-1} \left(\sum_{j=J}^{\infty} j^k u^j \phi_j^{\nu}(r) d_{j,\nu} s_{j,\nu}^{\nu+1/2} \cos(s_{j,\nu} y)\right)^2 \frac{du}{u} \leq C r^{-4(\nu+1)},$$

and the analogous estimate with the exponents $\nu + 1/2$ and $-4(\nu + 1)$ replaced by $(\nu + 2) + 1/2$ and $-4((\nu + 2) + 1)$ correspondingly (this is needed in the case $-1 < \nu \leq -1/2$ only). Indeed, using (101), Poisson's integral formula (89) applied to $J_\nu(s_{j,\nu} y)$, and Minkowsky's inequality give, for $\nu > -1/2$,

$$\begin{aligned}
&\int_0^1 \left(\log\frac{1}{u}\right)^{2k-1} \left(\sum_{j=J}^{\infty} j^k u^j \phi_j^{\nu}(r) d_{j,\nu} s_{j,\nu}^{\nu+1/2}\right. \\
&\qquad \left.\times \int_0^1 (1-t^2)^{\nu-1/2} \cos(s_{j,\nu} y t)\, dt\right)^2 \frac{du}{u} \\
&\qquad \leq C \int_0^1 \left(\log \frac{1}{u}\right)^{2k-1} \left(\int_0^1 (1-t^2)^{\nu-1/2}\right. \\
&\qquad \left.\times \left|\sum_{j=J}^{\infty} j^k u^j \phi_j^{\nu}(r) d_{j,\nu} s_{j,\nu}^{\nu+1/2} \cos(s_{j,\nu} y t)\right| dt\right)^2 \frac{du}{u} \\
&\qquad \leq C \left(\int_0^1 (1-t^2)^{\nu-1/2} \left(\int_0^1 \left(\log \frac{1}{u}\right)^{2k-1}\right.\right. \\
&\qquad \left.\left.\times \left(\sum_{j=J}^{\infty} j^k u^j \phi_j^{\nu}(r) d_{j,\nu} s_{j,\nu}^{\nu+1/2} \cos(s_{j,\nu} y t)\right)^2 \frac{du}{u}\right)^{1/2} dt\right)^2 \\
&\qquad \leq C r^{-4(\nu+1)}.
\end{aligned}$$

In the case $-1 < \nu \leq -1/2$, applying the identity

$$J_\nu(z) = -J_{\nu+2}(z) + \frac{2(\nu+1)}{z} J_{\nu+1}(z),$$

gives

$$\begin{aligned}
B = &-\sum_{j=J}^{\infty} j^k u^j \phi_j^{\nu}(r) d_{j,\nu} (s_{j,\nu} y)^{1/2} y^{-(\nu+1/2)} J_{\nu+2}(s_{j,\nu} y) \\
&+ 2(\nu+1) \sum_{j=J}^{\infty} j^k u^j \phi_j^{\nu}(r) d_{j,\nu} (s_{j,\nu} y)^{-1/2} y^{-(\nu+1/2)} J_{\nu+1}(s_{j,\nu} y).
\end{aligned}$$

Now, using Poisson's formula (89) for $J_{\nu+1}(s_{j,\nu} y)$ and $J_{\nu+2}(s_{j,\nu} y)$ (together with the assumption $y \leq r/2$ in the first summand) and applying (101) we obtain the result.

Proving (101) (the proof of its counterpart with aforementioned replacements in exponents is completely analogous hence we do not treat it separately) we use (87) to expand $(s_{j,\nu}r)^{1/2}J_\nu(s_{j,\nu}r)$ and choose M to be the unique nonnegative integer satisfying $M-1 \le \nu+1/2 < M$. It is then clear that

$$\left|\sum_{j=J}^{\infty} j^k u^j \phi_j^\nu(r) d_{j,\nu} s_{j,\nu}^{\nu+1/2}\cos(s_{j,\nu}y)\right| \le C\sum_{n=0}^{M} r^{-n-(\nu+1/2)}(|\mathcal{C}_n|+|\mathcal{S}_n|)+G_M, \tag{102}$$

where

$$\begin{Bmatrix}\mathcal{S}_n\\ \mathcal{C}_n\end{Bmatrix} = \sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 s_{j,\nu}^{-n+\nu+1/2}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(s_{j,\nu}(r\pm y)),$$

$n=0,1,\dots,M$, and

$$G_M = r^{-(\nu+1/2)}\sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 |H_M(s_{j,\nu}r)| s_{j,\nu}^{\nu+1/2}.$$

Then, using (86),

$$G_M \le Cr^{-(\nu+M+3/2)}\sum_{j=J}^{\infty} j^{k+\nu-M-1/2}u^j.$$

Since $M>\nu+1/2$, we use (97) with $\beta=\nu-M-1/2$ to obtain

$$r^{-2(\nu+M+3/2)}\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\left|\sum_{j=J}^{\infty} j^{k+\nu-M-1/2}u^j\right|^2\frac{du}{u} \le Cr^{-2(\nu+M+3/2)}r^{-2(\nu-M+1/2)} \le Cr^{-4(\nu+1)}.$$

We now take into account (102), to finish the proof of (101). It follows from Lemma 3.2 that for given $n=0,1,\dots,M$, $\mathcal{S}_n$ and $\mathcal{C}_n$ are sums of series of the form

$$\sum_{j=J}^{\infty} j^{k-n+\nu+1/2}u^j E_{k-n+\nu+1/2,M-n}(j,r,y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi j(r\pm y)). \tag{103}$$

It is therefore clear that our task is reduced to dealing with the absolute value of each of the series in (103). Given $n=0,\dots,M$, we use the expression for $E_{k-n+\nu+1/2,M-n}(j,r,y)$ from Lemma 3.2 to estimate (103) with the sum of the absolute value of

$$R_{n,m} = \sum_{j=J}^{\infty} j^{k-n-m+\nu+1/2}u^j\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi j(r\pm y)), \tag{104}$$

for $m = 0, \dots, M-n$, and

$$\left| \sum_{j=J}^{\infty} j^{k-n+\nu+1/2} u^j q_j^{(M-n)}(r,y) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\pi j (r \pm y)) \right|.$$

For the term involving $q_j^{(M-n)}(r,y)$, it is verified that

$$\left| \sum_{j=J}^{\infty} j^{k-n+\nu+1/2} u^j q_j^{(M-n)}(r,y) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\pi j (r \pm y)) \right| \\ \leq C \sum_{j=J}^{\infty} j^{k+\nu-M-1/2} u^j.$$

Then, using that $-M - 1/2 + \nu < -1$, we can apply (97) with $\beta = \nu - M - 1/2$ to obtain

$$r^{-2n-2(\nu+1/2)} \int_0^1 \left(\log \frac{1}{u} \right)^{2k-1} \left(\sum_{j=J}^{\infty} j^{k+\nu-M-1/2} u^j \right)^2 \frac{du}{u} \leq C r^{-4(\nu+1)}.$$

The hypothesis made on M shows that $\nu + 1/2 - n - m > -1$ for $n = 0, \dots, M$ and $m = 0, \dots, M-n$ when $M - 1 < \nu + 1/2$ and the same is true for $n = 0, \dots, M-1$ and $m = 0, \dots, M-n-1$ when $M - 1 = \nu + 1/2$. Hence, in these cases,

$$\left| \sum_{j=1}^{J-1} j^{k+\nu+1/2-n-m+} u^j \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\pi j (r \pm y)) \right| \leq C \sum_{j=1}^{J-1} j^{k+\nu+1/2-n-m} u^j,$$

and using (96) with $\beta = \nu + 1/2 - n - m$,

$$r^{-2n-2(\nu+1/2)} \int_0^1 \left(\log \frac{1}{u} \right)^{2k-1} \left(\sum_{j=1}^{J-1} j^{k+\nu+1/2-n-m} u^j \right)^2 \frac{du}{u} \\ \leq C r^{-4(\nu+1)+2m} \leq C r^{-4(\nu+1)}.$$

In consequence, in (104) we can extend the sum to start from $j = 1$ and then use Lemma 3.6 to get

$$r^{-2n-2(\nu+1/2)} \int_0^1 \left(\log \frac{1}{u} \right)^{2k-1} \\ \times \left(\sum_{j=1}^{\infty} j^{k+\nu+1/2-n-m} u^j \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\pi j (r \pm y)) \right)^2 \frac{du}{u} \\ \leq C r^{-4(\nu+1)+2m}.$$

This inequality completes the estimate involving the terms $R_{n,m}$, $m = 0, 1, \dots, M-n$, except for the cases of $R_{n,M-n}$ when $M - 1 = \nu + 1/2$

for $n = 0, \dots, M$. In these exceptional cases we have to show that

$$r^{-2n-2(\nu+1/2)} \int_0^1 \left(\log \frac{1}{u}\right)^{2k-1} (R_{n,M-n})^2 \, \frac{du}{u} \le C r^{-2M-2(\nu+1/2)}.$$

Since $R_{n,M-n}$ takes the form of the series in (98), then Lemma 3.8 and the fact that $J = [1/r] \sim [1/r+y] \sim [1/|r-y|]$ give the desired bound.

Case 2: $r/2 < y < \min\{1, 3r/2\}$.

We use (87) with $M = 1$ to expand the functions $(s_{j,\nu}r)^{1/2} J_\nu(s_{j,\nu}r)$ and $(s_{j,\nu}y)^{1/2} J_\nu(s_{j,\nu}y)$. Then, taking $J = [1/r] \sim [1/y]$, we will write $\left(u\frac{\partial}{\partial u}\right)^k \mathcal{P}_u^\nu(r,y)$ as the sum

$$F(u,r,y) + \sum_{n,l=0}^{1} r^{-n} y^{-l} O_{n,l}(u,r,y) + J_1(u,r,y) + J_2(u,r,y) + G_1(u,r,y),$$

where

$$F(u,r,y) = \sum_{j=1}^{J-1} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} (s_{j,\nu}r)^{1/2} J_\nu(s_{j,\nu}r) \cdot (s_{j,\nu}y)^{1/2} J_\nu(s_{j,\nu}y),$$

and, for the remainder sum that starts from $j = J$, the $O_{n,l}$ terms capture the part that comes from the main parts of the aforementioned expansions and are sums of terms of the form

$$D_{n,l} \sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} s_{j,\nu}^{-n-l} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (s_{j,\nu}r) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (s_{j,\nu}y)$$

($D_{n,l}$ is a product of $A_{\nu+1,n}$ or $B_{\nu+1,n}$ and $A_{\nu,l}$ or $B_{\nu,l}$ depending on the choice of the sine or cosine), J_1 gathers the part that comes from the main parts of the second expansion and the remainder of the first one, hence its absolute value is bounded by

$$\begin{aligned} |J_1(u,r,y)| \le C &\left| \sum_1^2 \sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} H_1(s_{j,\nu}r) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (s_{j,\nu}y) \right| \\ &+ C y^{-1} \left| \sum_1^2 \sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} s_{j,\nu}^{-1} H_1(s_{j,\nu}r) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (s_{j,\nu}y) \right| \end{aligned}$$

(the sign $\sum_1^2$ indicates that we add two series, one for the choice of the sine and another for the cosine), J_2 acts as J_1 but with the position of the both expansions switched, and its absolute value is controlled by

$$
\begin{aligned}
|J_2(u,r,y)| \le C & \left| \sum_1^2 \sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (s_{j,\nu} r) H_1(s_{j,\nu} y) \right| \\
& + Cr^{-1} \left| \sum_1^2 \sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} s_{j,\nu}^{-1} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (s_{j,\nu} r) H_1(s_{j,\nu} y) \right|
\end{aligned}
$$

and, eventually, G_1 captures the part that comes from the remainders,

$$
G_1(u,r,y) = \sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} H_1(s_{j,\nu} r) H_1(s_{j,\nu} y).
$$

For $F(u,r,y)$, using (88) and (86) we have

$$
|F(u,r,y)| \le C \sum_{j=1}^{J-1} j^{k+2\nu+1} u^j,
$$

then, applying (96) with $\beta = 2\nu+1$, we obtain the estimate

$$
\int_0^1 \left(\log \frac{1}{u} \right)^{2k-1} (F(u,r,y))^2 \frac{du}{u} \le Cr^{-4(\nu+1)} \le C(ry)^{-2(\nu+1/2)} |r-y|^{-2}.
$$

For $J_1(u,r,y)$ (and the same reasoning works for $J_2(u,r,y)$), using $H_1(z) = O(z^{-2})$, $z \ge 1$, and again (88) and (86), shows that

$$
\begin{aligned}
&|J_1(u,r,y)| \\
&\quad \le Cr^{-2} \left(\sum_{j=J}^{\infty} j^{k-2} u^j (ry)^{-(\nu+1/2)} + y^{-1} \sum_{j=J}^{\infty} j^{k-3} u^j (ry)^{-(\nu+1/2)} \right).
\end{aligned}
$$

Then, the required bound in this case boils down to estimating

$$
r^{-4} (ry)^{-2(\nu+1/2)} \int_0^1 \left(\log \frac{1}{u} \right)^{2k-1} \left(\sum_{j=J}^{\infty} j^{k-2} u^j \right)^2 \frac{du}{u}
$$

and

$$
r^{-4} y^{-2} (ry)^{-2(\nu+1/2)} \int_0^1 \left(\log \frac{1}{u} \right)^{2k-1} \left(\sum_{j=J}^{\infty} j^{k-3} u^j \right)^2 \frac{du}{u}.
$$

Applying (97) with $\beta = -2$ and $\beta = -3$, these expressions are bounded by a constant times $r^{-2}(ry)^{-2(\nu+1/2)}$ and $y^{-2}(ry)^{-2(\nu+1/2)}$, respectively, and the task is done.

We show, in the same manner, that

$$
|G_1(u,r,y)| \le C(ry)^{-2} (ry)^{-(\nu+1/2)} \sum_{j=J}^{\infty} j^{k-4} u^j
$$

and

$$(ry)^{-4}(ry)^{-2(\nu+1/2)}\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=J}^{\infty}j^{k-4}u^j\right)^2\frac{du}{u}$$
$$\le Cr^{-2}(ry)^{-2(\nu+1/2)},$$

by (97) with $\beta=-4$. The remainder part of the proof is concerned with a more delicate analysis of the $r^{-n}y^{-l}O_{n,l}(u,r,y)$ terms. We start with the $r^{-1}y^{-1}O_{1,1}(u,r,y)$ term. It is clear that

$$|r^{-1}y^{-1}O_{1,1}(u,r,y)|\le Cr^{-2}\sum_{j=J}^{\infty}j^{k-2}u^j(ry)^{-(\nu+1/2)},$$

and, using (97) with $\beta=-2$,

$$r^{-4}(ry)^{-2(\nu+1/2)}\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=J}^{\infty}j^{k-2}u^j\right)^2\frac{du}{u}\le r^{-2}(ry)^{-2(\nu+1/2)}.$$

Lemma 3.2 with $\gamma=-1$ and $\ell=0$ yields

$$\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}|r^{-1}O_{1,0}(u,r,y)|^2\frac{du}{u}\le Cr^{-2}(ry)^{-2(\nu+1/2)},$$

once we show that

$$\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\left|\sum_{j=J}^{\infty}j^{k-1}u^jE_{-1,0}(j,r,y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi j(r\pm y))\right|^2\frac{du}{u}\le C.$$

The form of $E_{-1,0}$ reduces the task to showing the estimates

$$\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=J}^{\infty}j^{k-1}u^j\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi j(r\pm y))\right)^2\frac{du}{u}\le C \tag{105}$$

and

$$\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=J}^{\infty}j^{k-1}u^jq_j^{(0)}(r,y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi j(r\pm y))\right)^2\frac{du}{u}\le C, \tag{106}$$

where $|q_j^{(0)}(r,y)|\le Cj^{-1}$. (105) follows immediately from Lemma 3.8. The estimate (106) is proved taking into account that the inner series is bounded by $\sum_{j=J}^{\infty}j^{k-2}u^j$. So, using (97) with $\beta=-2$, the left term in (106) is bounded by r^2, therefore controlled by a constant. The estimate for $y^{-1}O_{0,1}(u,r,y)$ follows analogously.

It remains to consider the case of $O_{0,0}(u,r,y)$. Using Lemma 3.2 with $\gamma=0$ and $\ell=1$ shows that each of the four terms of $O_{0,0}(u,r,y)$

is a sum of terms of the form

$$(ry)^{-(\nu+1/2)}\sum_{j=J}^{\infty} j^k u^j\Big(A_0+\frac{A_1(r,y)}{j}+q_j^{(1)}(r,y)\Big)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi j(r\pm y)).$$

The estimate for the remainder term is immediate since, using

$$|q_j^{(1)}(r,y)|\le Cj^{-2}\quad \text{for}\quad 0<r,y<1,$$

gives

$$\left|\sum_{j=J}^{\infty} j^k u^j q_j^{(1)}(r,y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi j(r\pm y))\right|\le C\sum_{j=J}^{\infty} j^{k-2}u^j.$$

Then, using (97) with $\beta=-2$,

$$\begin{aligned}(ry)^{-2(\nu+1/2)}&\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\\ &\times\left(\sum_{j=J}^{\infty} j^k u^j q_j^{(1)}(r,y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi j(r\pm y))\right)^2\frac{du}{u}\\ &\le C(ry)^{-2(\nu+1/2)}\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\\ &\times\left(\sum_{j=J}^{\infty} j^{k-2}u^j\right)^2\frac{du}{u}\le C(ry)^{-2(\nu+1/2)}.\end{aligned}$$

Concerning the term involving $A_1(r,y)j^{-1}$, observe that $A_1(r,y)$ is a bounded function on $0<r,y<1$, hence our task reduces to estimating (105). Finally, consider the term involving A_0. It is possible to extend the summation of the series involving this term from $j=1$ since

$$\left|\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=1}^{J-1} j^k u^j\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi j(r\pm y))\right)^2\frac{du}{u}\right|\le Cr^{-2},$$

after using (96) with $\beta=0$. Then, we have to prove that

$$\left|\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=1}^{\infty} j^k u^j\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi j(r\pm y))\right)^2\frac{du}{u}\right|\le C|r-y|^{-2}. \tag{107}$$

With this, we can apply Lemma 3.4 with $m=1$ in the case $|r-y|\le 3/2$, to obtain (107) for the minus sign. When $r+y\le 3/2$ the result follows in the same manner taking into account that $(r+y)^{-1}\le|r-y|^{-1}$. For $r+y>3/2$ we need an extra argument. Points (r,y) such that $r+y>3/2$ are contained in the region where $3/8<r,y<1$. Then, writing $r = 1 - z$ and $y = 1 - v$, we have $z + v \le 3/2$ and

$$\sum_{j=1}^{\infty} j^k u^j \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\pi j(r+y)) = \sum_{j=1}^{\infty} j^k u^j \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\pi j(z+v)).$$

In this way, for $r + y > 3/2$, by Lemma 3.4 with $m = 1$, it is verified that

$$\left| \int_0^1 \left(\log \frac{1}{u}\right)^{2k-1} \left(\sum_{j=1}^{\infty} j^k u^j \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\pi j(r \pm y)) \right)^2 \frac{du}{u} \right| \le C(z+v)^{-2}$$
$$= C(2-r-y)^{-2} \le C|r-y|^{-2},$$

and the proof of (107) is completed.

Case 3: $\min\{1, 3r/2\} \le y < 1$.

This case is completely analogous to the Case 1 so we omit the proof. □

5. Proof of the estimate (84)

The result in the following proposition (where $\frac{\partial}{\partial r,y}$ means the partial derivative against either r or y) and the Lemma 3.9 allow us to complete the proof of (84).

Proposition 3.4. *Let $\nu > -1$ and $k \ge 1$. Then*

$$\int_0^1 \left(\log \frac{1}{u}\right)^{2k-1} \left| \frac{\partial}{\partial r, y} \left(u \frac{\partial}{\partial u}\right)^k \mathcal{P}_u^\nu(r,y) \right|^2 \frac{du}{u}$$
$$\le C \begin{cases} r^{-(4\nu+6)}, & 0 < y \le r/2, \\ (ry)^{-2(\nu+1/2)}|r-y|^{-4}, & r/2 < y < \min\{1, 3r/2\}, \\ y^{-(4\nu+6)}, & \min\{1, 3r/2\} \le y < 1. \end{cases}$$

with C independent of $0 < u < 1$, r and y.

Proof. We use (85) to find that

$$\frac{d\phi_j^\nu(r)}{dr} = -s_{j,\nu} d_{j,\nu} s_{j,\nu}^{1/2} J_{\nu+1}(s_{j,\nu} r) r^{-\nu}.$$

In this way (exchanging summation with differentiation is easily seen to be possible)

$$\frac{\partial}{\partial r} \left(u \frac{\partial}{\partial u}\right)^k \mathcal{P}_u^\nu(r,y) = -\sum_{j=1}^{\infty} j^k u^j s_{j,\nu} d_{j,\nu} s_{j,\nu}^{1/2} J_{\nu+1}(s_{j,\nu} r) r^{-\nu} \phi_j^\nu(y)$$

and

$$\frac{\partial}{\partial y} \left(u \frac{\partial}{\partial u}\right)^k \mathcal{P}_u^\nu(r,y) = -\sum_{j=1}^{\infty} j^k u^j s_{j,\nu} \phi_j^\nu(r) d_{j,\nu} s_{j,\nu}^{1/2} J_{\nu+1}(s_{j,\nu} y) y^{-\nu}.$$

We shall consider the case $\frac{\partial}{\partial r}\left(u\frac{\partial}{\partial u}\right)^k \mathcal{P}_u^\nu(r,y)$ only. Note that treating $\frac{\partial}{\partial y}\left(u\frac{\partial}{\partial u}\right)^k \mathcal{P}_u^\nu(r,y)$ is completely analogous.

Case 1: $0 < y \le r/2$.

This case is proved analogously to the case $\min\{1, 3r/2\} \le y < 1$ that will be proved later.

Case 2: $r/2 < y < \min\{1, 3r/2\}$.

We use the asymptotic expansion (87) with $M = 2$, to expand the functions $(s_{j,\nu}r)^{1/2}J_{\nu+1}(s_{j,\nu}r)$ and $(s_{j,\nu}y)^{1/2}J_\nu(s_{j,\nu}y)$ and take $J = \left[\frac{1}{r}\right] \sim \left[\frac{1}{y}\right]$ to write $\frac{\partial}{\partial r}\left(u\frac{\partial}{\partial u}\right)^k \mathcal{P}_u^\nu(r,y)$ as the sum

$$F(u,r,y) + \sum_{n,l=0}^{2} r^{-n}y^{-l}O_{n,l}(u,r,y) + J_1(u,r,y) + J_2(u,r,y) + G_2(u,r,y).$$

Here

$$F(u,r,y) = \sum_{j=1}^{J-1} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} s_{j,\nu}(s_{j,\nu}r)^{1/2}J_{\nu+1}(s_{j,\nu}r)\cdot(s_{j,\nu}y)^{1/2}J_\nu(s_{j,\nu}y),$$

and, for the remainder sum that starts from $j = J$, the $O_{n,l}$ terms capture the part that comes from the main parts of the aforementioned expansions and are the sums of terms of the form

$$D_{n,l}\sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} s_{j,\nu}^{-n-l+1} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(s_{j,\nu}(r\pm y)),$$

J_1 gathers the part that comes from the main parts of the second expansion and the remainder of the first one, hence its absolute value is bounded by

$$\begin{aligned}|J_1(u,r,y)| \le C&\left|\sum_{1}^{2}\sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} s_{j,\nu} H_2(s_{j,\nu}r)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(s_{j,\nu}y)\right|\\
&+ Cy^{-1}\left|\sum_{1}^{2}\sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} H_2(s_{j,\nu}r)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(s_{j,\nu}y)\right|\\
&+ Cy^{-2}\left|\sum_{1}^{2}\sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} s_{j,\nu}^{-1} H_2(s_{j,\nu}r)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(s_{j,\nu}y)\right|.\end{aligned}$$

J_2 acts as J_1 but with the position of the both expansions switched, and its absolute value is controlled by

$$|J_2(u,r,y)| \le C\left|\sum_{1}^{2}\sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} s_{j,\nu}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(s_{j,\nu}r)H_2(s_{j,\nu}y)\right|$$
$$+ Cr^{-1}\left|\sum_{1}^{2}\sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(s_{j,\nu}r)H_2(s_{j,\nu}y)\right|$$
$$+ Cr^{-2}\left|\sum_{1}^{2}\sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} s_{j,\nu}^{-1}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(s_{j,\nu}r)H_2(s_{j,\nu}y)\right|$$

and, eventually, G_2 captures the part that comes from the remainders,

$$G_2(u,r,y) = \sum_{j=J}^{\infty} j^k u^j d_{j,\nu}^2 (ry)^{-(\nu+1/2)} s_{j,\nu} H_2(s_{j,\nu}r)H_2(s_{j,\nu}y).$$

We will now analyze separately each of the summands in the above decomposition of $\frac{\partial}{\partial r}\left(u\frac{\partial}{\partial u}\right)^k \mathcal{P}_u^\nu(r,y)$ and bound them by $C(ry)^{-2(\nu+1/2)}|r-y|^{-4}$.

For $F(u,r,y)$, using (88) and (86), we have

$$|F(u,r,y)| \le Cr\sum_{j=1}^{J-1} j^{2\nu+k+3}u^j.$$

Then, using (96) with $\beta = 2\nu+3$, we obtain that

$$\int_0^1 \left(\log\frac{1}{u}\right)^{2k-1} (F(u,r,y))^2\,\frac{du}{u}$$
$$\le Cr^2\int_0^1 \left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=1}^{J-1} j^{2\nu+k+3}u^j\right)^2\frac{du}{u}$$
$$\le Cr^{-2(2\nu+3)} \le C(ry)^{-2(\nu+1/2)}|r-y|^{-4}.$$

For $J_1(u,r,y)$ (and the same reasoning works for $J_2(u,r,y)$), using $H_2(z) = O(z^{-3})$, $z \ge 1$, and again (88) and (86), shows that

$$|J_1(u,r,y)| \le Cr^{-3}\left(\sum_{j=J}^{\infty} j^{k-2}u^j(ry)^{-(\nu+1/2)}\right.$$
$$\left. + y^{-1}\sum_{j=J}^{\infty} j^{k-3}u^j(ry)^{-(\nu+1/2)} + y^{-2}\sum_{j=J}^{\infty} j^{k-4}u^j(ry)^{-(\nu+1/2)}\right).$$

Then, the required bound comes down to estimating

$$r^{-6}(ry)^{-2(\nu+1/2)}\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=J}^{\infty} j^{k-2}u^j\right)^2\frac{du}{u},$$

$$r^{-6}y^{-2}(ry)^{-2(\nu+1/2)}\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=J}^{\infty}j^{k-3}u^j\right)^2\frac{du}{u},$$

and

$$r^{-6}y^{-4}(ry)^{-2(\nu+1/2)}\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=J}^{\infty}j^{k-4}u^j\right)^2\frac{du}{u}.$$

Applying (97) with $\beta=-2$, -3 and -4, these expressions are bounded by a constant times

$$r^{-4}(ry)^{-2(\nu+1/2)},\quad r^{-2}y^{-2}(ry)^{-2(\nu+1/2)}\quad\text{and}\quad y^{-4}(ry)^{-2(\nu+1/2)},$$

respectively, and the task is done. In a similar way we show that

$$|G_2(u,r,y)|\le C(ry)^{-3}(ry)^{-(\nu+1/2)}\sum_{j=J}^{\infty}j^{k-5}u^j$$

and

$$(ry)^{-6}(ry)^{-2(\nu+1/2)}\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=J}^{\infty}j^{k-5}u^j\right)^2\frac{du}{u} \le Cr^{-4}(ry)^{-2(\nu+1/2)},$$

by (97) with $\beta=-5$. The remainder part of the proof is concerned with a more delicate analysis of the $r^{-n}y^{-l}O_{n,l}(u,r,y)$ terms. We start with the $r^{-2}y^{-2}O_{2,2}(u,r,y)$ term. It is clear that

$$|r^{-2}y^{-2}O_{2,2}(u,r,y)|\le Cr^{-4}\sum_{j=J}^{\infty}j^{k-3}u^j(ry)^{-(\nu+1/2)},$$

and, using (97) with $\beta=-3$,

$$r^{-8}(ry)^{-2(\nu+1/2)}\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=J}^{\infty}j^{k-3}u^j\right)^2\frac{du}{u}\le r^{-4}(ry)^{-2(\nu+1/2)}.$$

Similarly, for $|r^{-2}y^{-1}O_{2,1}(u,r,y)|$,

$$|r^{-2}y^{-1}O_{2,1}(u,r,y)|\le Cr^{-2}y^{-1}\sum_{j=J}^{\infty}j^{k-2}u^j(ry)^{-(\nu+1/2)}$$

holds, and

$$r^{-4}y^{-2}(ry)^{-2(\nu+1/2)}\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=J}^{\infty}j^{k-2}u^j\right)^2\frac{du}{u} \le r^{-2}y^{-2}(ry)^{-2(\nu+1/2)}$$

by using (97) with $\beta=-2$. In a similar way, we can obtain the same bound for $|r^{-1}y^{-2}O_{1,2}(u,r,y)|$.

The estimate of $|r^{-2}O_{2,0}(u,r,y)|$ by $(ry)^{-2(\nu+1/2)}C|r-y|^{-4}$ uses Lemma 3.2 with $\gamma=-1$ and $\ell=0$, and essentially is contained in the estimate of $|r^{-1}O_{1,0}(u,r,y)|$ already discussed when proving (100) in the region $r/2<y<\min\{1,3r/2\}$. The estimate of $|y^{-2}O_{0,2}(u,r,y)|$ as well as $|r^{-1}y^{-1}O_{1,1}(u,r,y)|$ follows analogously.

The estimate of $|r^{-1}O_{1,0}(u,r,y)|$ by $C(ry)^{-2(\nu+1/2)}|r-y|^{-4}$ uses Lemma 3.2 with $\gamma=0$ and $\ell=1$, and essentially is contained in the estimate of $|O_{0,0}(u,r,y)|$ already discussed when proving (100) in the considered region. The estimate for the term with $|y^{-1}O_{0,1}(u,r,y)|$ follows analogously.

It remains to consider the case of $O_{0,0}(u,r,y)$. We use Lemma 3.2 with $\gamma=1$ and $\ell=2$, to conclude that each of the terms of $O_{0,0}(u,r,y)$ is a sum of series of the form

$$\sum_{j=J}^{\infty} j^{k+1}u^j\Big(A_0+\frac{A_1(r,y)}{j}+\frac{A_2(r,y)}{j^2}+q_j^{(2)}(r,y)\Big)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi j(r\pm y)).$$

The estimate for the remainder follows from the bound $|q_j^{(2)}(r,y)|\le Cj^{-3}$ for $0<r,y<1$. Indeed, in this case we have, using (97) with $\beta=-2$,

$$\begin{aligned}(ry)^{-2(\nu+1/2)}&\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\\ &\times\left(\sum_{j=J}^{\infty}j^k u^j q_j^{(2)}(r,y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi j(r\pm y))\right)^2\frac{du}{u}\\ &\qquad\le C(ry)^{-2(\nu+1/2)}\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\\ &\qquad\qquad\times\left(\sum_{j=J}^{\infty}j^{k-2}u^j\right)^2\frac{du}{u}\le C(ry)^{-2(\nu+1/2)}\end{aligned}$$

which is enough for our purpose. The series resulting from taking into account either A_1 or A_2 were already discussed in Case 2 of Proposition 3.3 and are bounded by $C(ry)^{-2(\nu+1/2)}|r-y|^{-4}$ in the considered region. We are left with the series A_0. Note that

$$\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=1}^{J-1}j^{k+1}u^j\right)^2\frac{du}{u}\le r^{-4}$$

so it is possible to extend the summation in the series from $j=1$. Now, we have to show

$$\left|\int_0^1\left(\log\frac{1}{u}\right)^{2k-1}\left(\sum_{j=1}^{\infty}j^{k+1}u^j\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi j(r\pm y))\right)^2\frac{du}{u}\right|\le C|r-y|^{-4}.$$

In the case of the minus sign the estimate is a consequence of Lemma 3.4 with $m = 2$. For the plus sign we have to consider separately the cases $r + y \leq 3/2$ and $r + y > 3/2$ and this can be done as in the previous proposition.

Case 3: $\min\{1, 3r/2\} \leq y < 1$.

We split the series defining $\frac{\partial}{\partial r}\left(u\frac{\partial}{\partial u}\right)^k \mathcal{P}_u^\nu(r, y)$ into A and B, being

$$\begin{aligned} A &= \sum_{j=1}^{J-1} j^k u^j s_{j,\nu} d_{j,\nu} s_{j,\nu}^{1/2} J_{\nu+1}(s_{j,\nu} r) r^{-\nu} \phi_j^\nu(y) \\ &= \sum_{j=1}^{J-1} j^k u^j s_{j,\nu} d_{j,\nu}^2 (ry)^{-(\nu+1/2)} (s_{j,\nu} r)^{1/2} J_{\nu+1}(s_{j,\nu} r) \cdot (s_{j,\nu} y)^{1/2} J_\nu(s_{j,\nu} y) \end{aligned}$$

and

$$\begin{aligned} B &= \sum_{j=J}^{\infty} j^k u^j s_{j,\nu} d_{j,\nu} s_{j,\nu}^{1/2} J_{\nu+1}(s_{j,\nu} r) r^{-\nu} \phi_j^\nu(y) \\ &= \sum_{j=J}^{\infty} j^k u^j s_{j,\nu} d_{j,\nu}^2 (ry)^{-(\nu+1/2)} (s_{j,\nu} r)^{1/2} J_{\nu+1}(s_{j,\nu} r) \cdot (s_{j,\nu} y)^{1/2} J_\nu(s_{j,\nu} y) \end{aligned}$$

with $J = [1/y]$. Using (86) and (88) we get

$$\begin{aligned} |A| &\leq \sum_{j=1}^{J-1} j^k u^j s_{j,\nu}^2 d_{j,\nu}^2 (ry)^{-\nu} |J_{\nu+1}(s_{j,\nu} r)| |J_\nu(s_{j,\nu} y)| \\ &\leq C(ry)^{-\nu} \sum_{j=1}^{J-1} j^{k+2} u^j |J_{\nu+1}(s_{j,\nu} r)| |J_\nu(s_{j,\nu} y)| \\ &\leq Cr \sum_{j=1}^{J-1} j^{2\nu+k+3} u^j. \end{aligned}$$

Then, using (96) with $\beta = 2\nu + 3$, we obtain

$$r^2 \int_0^1 \left(\log \frac{1}{u}\right)^{2k-1} \left| \sum_{j=1}^{J-1} j^{2\nu+k+3} u^j \right|^2 \frac{du}{u} \leq Cr^2 y^{-(4\nu+8)} \leq Cy^{-(4\nu+6)}.$$

To get the analogous estimate for $|B|$ it is enough to show that for $0 < u < 1$, $0 < r \leq 2y/3$, $0 < y < 1$ and $\nu > -1$

$$r^2 \int_0^1 \left(\log \frac{1}{u}\right)^{2k-1} \left(\sum_{j=J}^{\infty} j^k u^j s_{j,\nu} s_{j,\nu}^{\nu+3/2} d_{j,\nu} \cos(s_{j,\nu} r) \phi_j^\nu(y) \right)^2 \frac{du}{u} \leq Cr^2 y^{-(4\nu+8)}. \tag{108}$$

Indeed, using (108), Minkowski's inequality and Poisson's integral formula (89) applied to $J_\nu(s_{j,\nu} y)$, for $\nu > -1$, we obtain the result as in Case 1 in the proof of Proposition 3.3. To check (108) we can also proceed as in the proof of (101) in Proposition 3.3. □

Conclusions and further work

This work was born with the spirit and intention of researching into the convergence in L^p norm of the operator of the Bochner-Riesz means for the multidimensional Fourier-Bessel expansions. Up to now, the achievements we have obtained concerning this matter in the case of radial functions are shown in the second chapter of this memory, and they cover an interesting hole in the study of summability methods in harmonic analysis. At least, this is another small brick in the contribution to this vast field of mathematical analysis.

There is a wide open door to continue investigating more topics directly related to this research, such as the study of the convergence of the Bochner-Riesz means for multidimensional Fourier-Bessel series, in the general case. This involves the analysis of vectorial boundedness of the type

$$\left\| \left(\sum_{k=0}^{\infty} \left| B_R^{\delta,\nu+k} f_k \right|^2 \right)^{1/2} \right\|_{L^p((0,1),dr)} \le C \left\| \left(\sum_{k=0}^{\infty} |f_k|^2 \right)^{1/2} \right\|_{L^p((0,1),dr)}.$$

As a generalization of the inequality above, it could be of great interest the study of vectorial estimates for the Bochner-Riesz means of the kind

$$\left\| \left(\sum_{k=0}^{\infty} \left| B_R^{\delta,\nu+k} f_k \right|^q \right)^{1/q} \right\|_{L^p((0,1),dr)} \le C \left\| \left(\sum_{k=0}^{\infty} |f_k|^q \right)^{1/q} \right\|_{L^p((0,1),dr)}.$$

While developing the initial work, we had to cope with several problems that required more sophisticated techniques and we broadened our first target to the study of other kind of operators related to the Fourier-Bessel series. This led us to define and study the g_k-functions related to our expansions. We show our result about this topic in the third chapter of the dissertation. This research is based on very sharp and not straightforward computations. Borne in mind there stays a continuation of this work, and the next step should be the proper definition of the so-called g_k^*-functions as well as the study of its weighted boundedness. Both tools, g_k and g_k^*-functions are essential to obtain a theorem about multipliers for the Fourier-Bessel expansions satisfying Hörmander's conditions. With a result of this nature, we could deduce, among other things, the boundedness of the Cesàro means associated to the Fourier-Bessel series from our work about the Bochner-Riesz means, and using a subordination formula that relates both types of means.

In our future work, we also consider the study of other operators related to Fourier-Bessel series. In fact, we are developing research about higher order Riesz transforms for Fourier-Bessel expansions, which are defined as follows: for each $d \geq 1$,

$$\mathcal{R}_\nu^d(f,r) = \sum_{j=1}^{\infty} s_{j,\nu}^{-d} \langle f, \varphi_j^\nu \rangle \delta_\nu^d \varphi_j^\nu(r),$$

where δ_ν^d is a first order differential operator suitably defined. This investigation extends the results for the conjugate operator associated to the Fourier-Bessel series (which is the Riesz transform of order one) given in [**18**].

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Acotaciones con pesos de operadores relacionados con las series de Fourier-Bessel

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Introducción

Sea $(\Omega, \mathcal{M}, d\mu)$ un espacio de medida ($\mu \geq 0$). Consideramos

$$L^p(\Omega, d\mu) = \{f : \Omega \to \mathbb{C},\ f \text{ medible } : \|f\|_{L^p(\Omega,d\mu)} < \infty\},$$

donde

$$\|f\|_{L^p(\Omega,d\mu)} = \left(\int_\Omega |f|^p \, d\mu\right)^{1/p},$$

para $1 \leq p < \infty$, y

$$\|f\|_{L^\infty(\Omega,d\mu)} = \sup \operatorname{esn}\{|f(x)| : x \in \Omega\}.$$

Sea $\{\psi_j\}_{j\in\Lambda}$ una sucesión de funciones, donde Λ es un conjunto de índices que está contenido en $\mathbb{Z}$ (o en $\mathbb{Z}^d$), ortonormales en $L^2(\Omega, d\mu)$, es decir,

$$\int_\Omega \psi_j \overline{\psi_m} \, d\mu = \delta_{j,m}.$$

La serie de Fourier de una función apropiada f asociada al sistema $\{\psi_j\}_{j\in\Lambda}$ viene dada por

$$f \sim \sum_{j\in\Lambda} a_j(f)\psi_j,$$

donde

$$a_j(f) = \int_\Omega f\overline{\psi_j} \, d\mu.$$

Las propiedades de convergencia de estas series en espacios $L^p(\Omega, d\mu)$ han sido estudiadas con profundidad para conjuntos particulares de $\{\psi_j\}_{j\in\Lambda}$. Si definimos las sumas parciales asociadas a las series de Fourier para cada $f \in L^2(\Omega, d\mu)$ como

$$S_n f = \sum_{\substack{j\in\Lambda \\ |j|\leq n}} a_j(f)\psi_j,$$

entonces, la teoría de espacios de Hilbert asegura que

$$\|S_n f\|_{L^2(\Omega,d\mu)} = \left(\sum_{\substack{j\in\Lambda \\ |j|\leq n}} |a_j(f)|^2\right)^{1/2} \leq \|f\|_{L^2(\Omega,d\mu)}$$

de lo cual se deduce que

$$\lim_{n\to\infty} \int_\Omega |f - S_n f|^2 \, d\mu = 0,$$

para cada $f \in L^2(\Omega, d\mu)$, siempre y cuando el sistema $\{\psi_j\}_{j\in\Lambda}$ sea completo en dicho espacio. La primera cuestión que se plantea inmediatamente es el estudio de lo que se denomina la convergencia en media, es decir, para qué valores de $p, 1 \leq p < \infty$, se tiene, para cada $f \in L^p(\Omega, d\mu)$, que

$$\lim_{n\to\infty} \int_\Omega |f - S_n f|^p \, d\mu = 0.$$

Cuando esto último se verifica, se dice que la sucesión $\{\psi_j\}_{j\in\Lambda}$ forma una base para el espacio de estas funciones. Para resolver el problema sólo es necesario estudiar la acotación uniforme del operador S_n.

Problemas de esta naturaleza fueron inicialmente estudiados por Riesz [**49**], que analizó la convergencia de la serie de Fourier clásica, asociada al sistema $\{e^{ikx}\}_{k\in\mathbb{Z}}$. Otros sistemas ortonormales han sido estudiados en la literatura matemática a lo largo de los años.

Por ejemplo, Pollard [**45, 46, 47, 48**], Muckenhoupt [**33**], Badkov [**3**], Pérez [**43**], Varona [**56**] y Guadalupe-Pérez-Varona [**26**] se han planteado la investigación de la convergencia de las series de Fourier de polinomios de Jacobi, ortonormales en $L^2([-1,1],(1-x)^\alpha(1+x)^\beta\, dx)$.

Para resultados similares sobre polinomios y funciones de Laguerre, ortonormales en $L^2((0,\infty), e^{-x}x^\alpha\, dx)$ y en $L^2((0,\infty), dx)$ respectivamente, se pueden ver los trabajos de Askey-Wainger [**2**] y Muckenhoupt [**34, 35**]. Estas mismas referencias pueden ser consultadas para los polinomios y las funciones de Hermite, ortonormales en $L^2(\mathbb{R}, e^{-x^2}\, dx)$ y en $L^2(\mathbb{R}, dx)$ respectivamente.

Sobre el sistema de Fourier-Bessel, constituido por funciones de Bessel y ortonormal en $L^2([0,1], x\, dx)$, pueden examinarse los trabajos de Wing [**62**], Benedek-Panzone [**7, 8**], Pérez [**43**], Varona [**56**] y Guadalupe-Pérez-Ruiz-Varona [**25**].

Por último, se pueden consultar los trabajos de Benedek-Panzone [**6**], Barceló-Córdoba [**5**], Ciaurri [**14**], Ciaurri-Guadalupe-Pérez-Varona [**15**] y Generozov [**24**] sobre ciertos sistemas ortonormales formados por autofunciones de operadores diferenciales de segundo orden.

Cuando la convergencia de los operadores suma parcial falla es habitual considerar otros métodos de sumación para las series de Fourier. Podemos encontrar resultados muy interesantes que consideran las medias de Cesàro con pesos potenciales, obtenidos por Muckenhoupt-Webb [**39, 40**] en el contexto de las series de Fourier-Laguerre y Fourier-Hermite, respectivamente. Estos resultados extienden un trabajo previo desarrollado por Poiani [**44**]. Recientemente, Ciaurri-Varona han trabajado sobre la misma cuestión en [**21**] en el contexto de las series de Fourier de funciones de Hermite generalizadas, extendiendo los resultados de [**40**]. En [**12**] se ha llevado a cabo un trabajo exhaustivo para las medias de Cesàro de las series de Fourier-Jacobi, en particular se analiza el comportamiento débil de dichas medias.

Las medias de Bochner-Riesz son otro importante método de sumación utilizado frecuentemente en análisis armónico. La sumabilidad Bochner-Riesz se ha tratado, por ejemplo, para la transformada de Fourier y la transformada de Hankel. En el caso de la transformada de Fourier multidimensional, el estudio de este problema sólo ha sido resuelto completamente para dimensión dos, [**22**], y para funciones radiales, [**60**]. El caso general sigue siendo un problema abierto. La transformada de Fourier para funciones radiales multidimensionales se convierte en la transformada de Hankel de un orden fijo. El estudio de las medias de Bochner-Riesz para la transformada de Hankel ha sido analizado, para valores cualesquiera del orden, en [**20, 19**].

Para $\delta > 0$, definimos las medias de Bochner-Riesz asociadas a un sistema ortonormal $\{\psi_j\}_{j\in\Lambda}$ mediante la identidad

$$B_R^\delta(f,x) = \sum_{j\in\Lambda} \left(1 - \frac{r_j^2}{R^2}\right)_+^\delta a_j(f)\psi_j(x),$$

donde $R > 0$, $(1-s^2)_+ = \text{máx}\{1-s^2, 0\}$ y r_j es una sucesión convenientemente elegida. Como en el caso de los operadores de suma parcial, es fácil ver que la convergencia de este método de sumación se sigue a partir de la acotación uniforme para B_R^δ, si tenemos densidad del sistema $\{\psi_j\}_{j\in\Lambda}$.

El primer objetivo de esta memoria es analizar la convergencia, para funciones radiales, de las medias de Bochner-Riesz de las series de Fourier-Bessel multidimensionales. En este caso radial, y sin más comentarios, nos referiremos a la serie como serie de Fourier-Bessel. Este sistema ortonormal está constituido por las autofunciones del operador de Laplace en la bola unidad d-dimesional. El análisis de las sumas parciales relacionadas con este sistema ortonormal se ha hecho en [**4**] para espacios de Lebesgue dotados de una norma mixta.

Hay otro tipo de convergencia que ha sido frecuentemente analizada en la literatura, la convergencia en casi todo punto. El origen de esta cuestión se puede encontrar en la conjetura de Lusin (1915). Afirma que la serie de Fourier clásica converge en casi todo punto a f, para cada $f \in L^2$. Como es conocido, la demostración de esta conjetura se debe a Carleson [**10**], y la extensión a L^p, $1 < p < \infty$, a Hunt [**30**]. En general, si el sistema ortonormal $\{\psi_j\}_{j\in\Lambda}$ es denso, la acotación del correspondiente operador maximal de las sumas parciales de la serie de Fourier, en el espacio $L^p(\Omega, d\mu)$ implica la convergencia en casi todo punto de la serie de Fourier.

Nuestro segundo objetivo será el análisis del supremo de las medias de Bochner-Riesz para la serie de Fourier-Bessel. Estudiaremos desigualdades con pesos en la norma L^p para este operador cuando $1 < p \leq \infty$.

Las sumas parciales y las medias de Bochner-Riesz pueden verse como casos particulares de multiplicadores para la serie de Fourier-Bessel. Existe una herramienta de gran ayuda que se aplica frecuentemente en el estudio de multiplicadores, las funciones cuadrado o g_k-funciones. La teoría de g_k-funciones fue en principio desarrollada a lo largo de los años treinta del pasado siglo por Littlewood y Paley, Zygmund y Marcinkiewicz. En estos primeros enfoques, las g_k-funciones se definieron para la serie de Fourier clásica. Son operadores no-lineales que nos permiten dar una caracterización muy útil de la norma L^p de una función en términos del comportamiento de su integral de Poisson. Entre otras aplicaciones, las g_k-funciones se usan para obtener resultados para multiplicadores con condiciones de tipo Hörmander y estimaciones para la transformada de Riesz. Se han probado teoremas para multiplicadores usando g_k-funciones en [**51**] para la transformada de Fourier d-dimensional, en [**38**] para las series ultraesféricas, en [**54**] para series de Hermite, en [**32**] para semigrupos generales, y en [**23**] para series de Laguerre e incluyendo pesos potenciales. Las g_k-funciones aparecen en resultados relacionados con la transformada de Riesz en [**27**] para el semigrupo de Ornstein-Uhlenbeck, en [**28**] para el semigrupo de Hermite, y en [**41**] para las series de Laguerre.

Como último objetivo, se definirán las g_k-funciones relacionadas con el semigrupo de Poisson de las series de Fourier-Bessel para cada $k \geq 1$. Probaremos que estas g_k-funciones son operadores de Calderón-Zygmund cuyo espacio asociado es de tipo homogéneo, y después deduciremos propiedades funcionales a partir de la teoría general. Este último resultado es el primer paso hacia un teorema muy general sobre multiplicadores para las series de Fourier-Bessel, y por lo tanto queda abierto un problema para su futura investigación.

La memoria se divide en tres capítulos. El primero es de naturaleza introductoria y describimos allí el sistema de funciones ortogonales que será analizado en este trabajo. Además, en este capítulo trataremos varios aspectos sobre el operador suma parcial. Se presentarán resultados conocidos para las sumas parciales relacionadas con las series de Fourier-Bessel multidimensionales, en el caso general y radial.

El Capítulo 2 está dedicado al estudio de la convergencia, para funciones radiales, de las medias de Bochner-Riesz de la serie de Fourier-Bessel multidimensional. En la primera sección comenzamos comprobando que este hecho es equivalente a la acotación con pesos de las medias de Bochner-Riesz para la serie de Fourier-Bessel. Antes de introducir los resultados principales, necesitaremos mostrar nuestras condiciones sobre los pesos, y varias definiciones. Procederemos enunciando el resultado principal, una desigualdad con pesos generales, y su corolario principal, un resultado similar con pesos potenciales. En la segunda sección obtendremos una estimación puntual para el núcleo de las medias de Bochner-Riesz de las series de Fourier-Bessel. La Sección 3 contendrá la demostración del Teorema Principal. En la cuarta sección se prueba el Corolario Principal. En la Sección 5 se obtienen algunas consecuencias del Corolario Principal, tales como la convergencia de las medias de Bochner-Riesz y la acotación de otros operadores relacionados con la serie de Fourier-Bessel. Probamos desigualdades de tipo débil para $p = 1$ en la Sección 6 y, en la última sección, estudiamos la convergencia en casi todo punto.

En el tercer capítulo llevamos a cabo un estudio de las g_k-funciones para las series de Fourier-Bessel. El principal resultado es una caracterización de la norma L^p de una función en términos del comportamiento de la integral de Poisson correspondiente a la serie de Fourier-Bessel, incluyendo pesos. Antes de enunciar nuestro resultado sobre g_k-funciones, introducimos algunos conceptos concernientes a estos operadores y, por otro lado, sobre la teoría Calderón-Zygmund. En la segunda sección, afirmaremos que las g_k-funciones pueden tratarse como operadores Calderón-Zygmund vector-valorados y de este hecho concluiremos la acotación buscada. El resto del capítulo está dedicado a demostrar este hecho, mostrando varios lemas técnicos y proposiciones necesarias para comprobar que el núcleo asociado a las g_k-funciones es un núcleo estándar en el espacio de Banach apropiado.

Resumen de los capítulos

Presentamos a continuación un breve resumen de cada uno de los capítulos de esta memoria.

1. El sistema de Fourier-Bessel

En el primer capítulo de la memoria describimos el sistema de Fourier-Bessel multidimensional. Aunque los resultados principales se centran en una generalización del caso radial, presentamos la familia completa de funciones. Este sistema ortonormal está constituido por las autofunciones del operador de Laplace sobre la bola unidad d-dimensional, con $d \geq 2$. A continuación, tratamos diversos aspectos de los operadores suma parcial respecto de este sistema. Para ello, presentamos varios resultados conocidos para las sumas parciales asociadas a la serie de Fourier-Bessel multidimensional, tanto en el caso general (ver [**4**]), como en el caso radial (ver [**62, 7, 25**]).

2. Convergencia con pesos de las medias de Bochner-Riesz para las series de Fourier-Bessel

El Capítulo 2 está dedicado al estudio de las medias de Bochner-Riesz de las series de Fourier-Bessel. En primer lugar, definimos dichas medias y mostramos el resultado de acotación con pesos relacionado con ellas. Este resultado se basa en una estimación puntual apropiada para el núcleo de las medias.

Como corolario, obtenemos que las condiciones impuestas sobre nuestros pesos son necesarias cuando estos pesos son potenciales. A partir de aquí deducimos otros resultados, como la convergencia en L^p y la acotación de otros operadores relacionados con las medias de Bochner-Riesz, como son el semigrupo del calor, el semigrupo de Poisson y las integrales fraccionarias.

De la acotación puntual para el núcleo de las medias de Bochner-Riesz obtenemos también desigualdades de tipo débil para $p = 1$ y acotaciones para el supremo de las medias de Bochner-Riesz.

3. g_k-funciones de Littlewood-Paley-Stein para las series de Fourier-Bessel

En el último capítulo de la memoria definimos y estudiamos, para $k \geq 1$, las g_k-funciones relacionadas con el semigrupo de Poisson de las series de Fourier-Bessel. Probamos que estos operadores son operadores de Calderón-Zygmund cuyo espacio asociado es de tipo homogéneo, por lo tanto, sus propiedades funcionales se derivan de la teoría general. Para ello, necesitamos demostrar una serie de resultados técnicos muy precisos, que utilizaremos para probar condiciones que aseguran que el núcleo asociado a los operadores de las g_k-funciones es un núcleo estándar en el espacio de Banach adecuado.

Conclusiones y trabajo futuro

Este trabajo nació con el espíritu y la intención de investigar la convergencia en la norma L^p del operador de las medias de Bochner-Riesz para las series de Fourier-Bessel multidimensionales. Hasta ahora, los logros que hemos obtenido en relación a este hecho en el caso de funciones radiales se muestran en el segundo capítulo de esta memoria, y cubren un interesante hueco en el estudio de métodos de sumación en el análisis armónico. Al menos, este es otro pequeño ladrillo en la construcción del vasto campo del análisis matemático.

Existe una puerta abierta para continuar con el examen de más cuestiones relacionadas directamente con esta investigación, tales como el estudio de la convergencia de las medias de Bochner-Riesz para las series de Fourier-Bessel multidimensionales en el caso general. Esto implica el análisis de acotaciones vectoriales del tipo

$$\left\| \left(\sum_{k=0}^{\infty} \left| B_R^{\delta,\nu+k} f_k \right|^2 \right)^{1/2} \right\|_{L^p((0,1),dr)} \leq C \left\| \left(\sum_{k=0}^{\infty} |f_k|^2 \right)^{1/2} \right\|_{L^p((0,1),dr)}.$$

Como generalización de la desigualdad de arriba, podría ser de gran interés el estudio de estimaciones vectoriales para las medias de Bochner-Riesz del tipo

$$\left\| \left(\sum_{k=0}^{\infty} \left| B_R^{\delta,\nu+k} f_k \right|^q \right)^{1/q} \right\|_{L^p((0,1),dr)} \leq C \left\| \left(\sum_{k=0}^{\infty} |f_k|^q \right)^{1/q} \right\|_{L^p((0,1),dr)}.$$

Mientras desarrollábamos el trabajo inicial, tuvimos que superar diversos problemas que requerían técnicas más sofisticadas y ampliamos nuestro primer objetivo al estudio de otro tipo de operadores relacionados con las series de Fourier-Bessel. Esto nos condujo a la definición y estudio de las g_k-funciones relacionadas con nuestras series. Mostramos nuestro resultado sobre esta cuestión en el tercer capítulo de la tesis. Esta investigación se basa en cálculos muy finos y nada triviales. Tenemos presente la continuación de esta linea de trabajo, y el siguiente paso debería ser la obtención de la definición apropiada de las denominadas g_k^*-funciones, así como el estudio de su acotación con pesos. Ambas herramientas, g_k y g_k^*-funciones son esenciales para obtener un teorema sobre multiplicadores para las series de Fourier-Bessel satisfaciendo condiciones de tipo Hörmander. Con un resultado de esta naturaleza podríamos deducir, entre otras cosas, la acotación de las medias de Cèsaro asociadas a las series de Fourier-Bessel, a partir de nuestro trabajo sobre las medias de Bochner-Riesz, y usando una fórmula de subordinación que relaciona ambos tipos de medias.

En nuestro trabajo futuro también consideramos el estudio de otros operadores relacionados con las series de Fourier-Bessel. De hecho, estamos desarrollando la investigación sobre las transformadas de Riesz de orden superior para las series de Fourier-Bessel, que se definen como sigue: para cada $d \geq 1$,

$$\mathcal{R}_\nu^d(f,r) = \sum_{j=1}^{\infty} s_{j,\nu}^{-d} \langle f, \varphi_j^\nu \rangle \delta_\nu^d \varphi_j^\nu(r),$$

donde δ_ν^d es un operador diferencial de primer orden definido de manera adecuada. Esta investigación extiende los resultados para el operador conjugado relacionado con las series de Fourier-Bessel (que es la transformada de Riesz de orden uno) dados en [**18**].

Publicaciones

Incluimos a continuación los resúmenes de las publicaciones y prepublicaciones a las que ha dado lugar el trabajo presentado en esta memoria.

- **Óscar Ciaurri y Luz Roncal, The Bochner-Riesz means for Fourier-Bessel expansions, *Journal of Functional Analysis* 228 (2005), 89–113.**

  En este trabajo se analizan las medias de Bochner-Riesz para las series de Fourier-Bessel. Demostramos una acotación uniforme con dos pesos potenciales para dichas medias. El resultado nos proporciona condiciones necesarias y suficientes para la acotación. Además, obtenemos algunos corolarios acerca de la convergencia de estas medias y la acotación de otros operadores relacionados con las series de Fourier-Bessel.

- **Óscar Ciaurri y Luz Roncal, Weighted inequalities for the Bochner-Riesz means related to the Fourier-Bessel expansions, *Journal of Mathematical Analysis and Applications* 329 (2007), 1170–1180.**

  En este artículo probamos desigualdades para las medias de Bochner-Riesz de las series de Fourier-Bessel con pesos más generales $u(r)$ que los pesos potenciales previamente considerados. Estas estimaciones vienen dadas mediante la utilización de la teoría A_p local y las desigualdades de Hardy con pesos. Además, también obtenemos desigualdades de tipo $(1,1)$ débil. La estimación en el caso $u(r) = r^a$ se obtiene como corolario.

- **Óscar Ciaurri y Luz Roncal, Bochner-Riesz means for Fourier-Bessel expansions: almost everywhere convergence, *Preprint.***

  En esta ocasión analizamos el supremo de las medias de Bochner-Riesz para las series de Fourier-Bessel. Estudiamos una desigualdad en la norma L^p con pesos potenciales para este operador cuando $1 < p \leq \infty$; las condiciones de las hipótesis resultan ser necesarias y suficientes para la desigualdad. Aparte de esto, obtenemos una desigualdad de tipo débil para $p = 1$. A partir de estos hechos obtenemos un resultado de convergencia en casi todo punto para estas medias. Finalmente, a través de una fórmula de subordinación, estudiamos el semigrupo del calor para las series de Fourier-Bessel.

- **Óscar Ciaurri y Luz Roncal, Littlewood-Paley-Stein g_k-functions for Fourier-Bessel expansions, *Preprint.***

  Definimos las g_k-funciones relacionadas con el semigrupo de Poisson de las series de Fourier-Bessel, para cada $k \geq 1$. Se demuestra que estas g_k-funciones son operadores de Calderón-Zygmund cuyo espacio asociado es de tipo homogéneo y, por lo tanto, sus propiedades funcionales se deducen de la teoría general.