NUMERICAL REPRESENTATION OF CHOICE FUNCTIONS*

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ABSTRACT

Numerical representations of choice functions allow the expression of a problem of choice as a problem of finding maxima of real valued functions, which requires less information to be defined and which is easier to work with. In this paper, the existence of numerical representations of choice functions by imposing assumptions directly on the choice function, is obtained. In particular, it is proved that (IIA) is a necessary and sufficient condition for a choice function to be representable, if the set of alternatives is countable, while the conjunction of (IIA) and a "continuity condition" is sufficient to ensure it in separable topological spaces.

Key words: Choice function, numerical representation.

1. INTRODUCTION

In Social Sciences, the most common description of individual choice consists of assuming that the agent has an "a priori" ordering, or a ranking over the different alternatives, that is, the agent knows his preference relation. Then, "rational" behavior requires choosing the "best" elements, according to this criterion, in every feasible set presented for choice (i.e., to choose the *maximal elements*).

A different approach is given by removing the assumption that the agent knows "a priori" his preference relation. In this case, the way of analyzing the *rationality* of the choice function consists of observing the different choices individuals make when different subsets of alternatives are presented for choice, and comparing them. Thus, rationality is based on the analysis of some coherent properties between the different choices individuals make when the feasible set changes.

The relationship between these two ways of analyzing the choice problem has been one of the most important subjects in the theory of choice. Thus, although it is clear that from a preference relation we can define a choice function by maximizing it in each feasible set of alternatives, the converse (that is, given a choice function to find a preference relation whose maximal elements define the choice set) is not always so obvious. This is the problem of the *rationality of choice functions* (a revision of the most important results on this topic can be found in [4,5,8]).

However, there are some economic models of choice in which the existence of additional information about what the choice, the preferences or the feasible sets are like, allow the problem to be analyzed in a different way. This is the case, for instance, of models of applied mathematics or control theory where the best options are chosen by means of an optimization process of a particular real function (vectorial or scalar); or the usual economic models of consumer and demand theory, where the decision of the agents is based on the maximization of a utility function on the budget sets. In all of these choice problems, it is assumed that the choice of the best alternatives is achived by the *maximization of a real valued function*. This approach is known in the literature as the problem of *numerical representation of a choice function*.

The interest of numerical representations of choice functions is clear. On the one hand, they transform a choice problem into a problem of the existence of maxima of a real valued function, which is, in general, easier to formulate and solve, due to the mathematical tools which can be applied to them. On the other hand, the use of numerical representations has the additional advantage of requiring less information to be defined.

The analysis of numerical representations of choice functions is closely related to another independent field of the literature, devoted to the analysis of binary relations: *numerical representations of binary relations*.

Results on numerical representations of binary relations, together with those on the rationality of choice functions, provide an indirect way of analyzing the existence of numerical representations of choice functions. Thus, by imposing conditions on a choice function its rationality can be obtained and, once we know it is rationalized by a binary relation, the existence of a numerical representation of this relation will guarantee the existence of a numerical representation of the choice function. In this line, we have to mention the work of Deb [3], who analyzes the existence of numerical representations of choice functions by means of this indirect way, in contexts with finite or countable sets of alternatives.

However, in order to obtain this kind of result in non-countable sets, it is necessary to impose additional assumptions of separability, continuity, etc., on both the relation which rationalizes the choice function, as well as on the set where it is defined, but not on our initial "input": the choice function. So it seems interesting to analyze directly (without making use of rationality results) the existence of numerical representations of choice functions. This kind of analysis is not usual in the literature and has been only analyzed in some particular contexts (see for instance [6,7], where the numerical representation of bargaining solutions, taken as choice functions, is analyzed).

The aim of this work is, therefore, to state sufficient conditions for a choice function to be numerically representable in general contexts (countable and non-countable ones) only by means of assumptions directly imposed on the choice function (without considering any kind of conditions over possible rationalizations of the choice function).

2. PRELIMINARIES

Let X be the set of alternatives and $P(X)$ the family of non-empty subsets of X. A choice function $F:D(X) \longrightarrow X$, $D(X) \subseteq P(X)$, is a functional relationship that associates a non-empty subset of A, F(A), to each choice situation $A \in D(X)$. As is usual in the literature, it is assumed throughout the paper that $D(X)$ contains all finite subsets of X (in many contexts it is assumed that $D(X) = P(X)$). Let us denote by $\mathscr{R}(X)$ the set of complete binary relations defined over X.

Given a complete binary relation $R \in \mathscr{R}(X)$ we denote by M(A,R) the set of maximal elements over $A \in P(X)$,

$$M(A,R) = \{a \in A \mid a \; R \; x, \; \forall x \in A\}.$$

In a similar way, given a real valued function $u:X \longrightarrow \mathbb{R}$, we denote by M(A,u) the set of maxima of the function over $A \in P(X)$, that is,

$$M(A,u) = \{a \in A \mid u(a) \geq u(x), \; \forall x \in A\}.$$

A choice function F is *rational* if there exists a binary relation $R \in \mathscr{R}(X)$ whose maximal elements define the choice set in each feasible set of alternatives, that is

$$\exists \; R \in \mathscr{R}(X) \text{ such that } \forall A \in D(X), \; F(A) = M(A,R)$$

In this case, R is called a *rationalization* of the choice function.

The most well-known kind of rationality of choice functions is that of rationality by means of a preorder (that is a complete, reflexive and transitive binary relation).

In the literature about rationality of choice functions, many papers have dealt with the problem of characterizing it by means of axioms that explain the choice function behavior when the set of alternatives changes. The following assumption is one of the most usual in these results,

(IIA) Independence of Irrelevant Alternatives (Arrow, [1])

If $A \subseteq B$, $F(B) \cap A \neq \emptyset$, then $F(A) = F(B) \cap A$

In other words, if the set of feasible alternatives contracts, but some of the chosen alternatives remain in the new feasible set, then these alternatives are the only chosen ones in the new set. This assumption is a necessary and sufficient condition for the rationality of a choice function by means of a preorder.

Theorem 1. (Arrow, [1])

Let $F:P(X) \longrightarrow X$ be a choice function. Then

F is rationalized by a preorder $\Leftrightarrow$ F satisfies **(IIA)**

In order to prove the existence of numerical representations of a choice function, one possibility consists of making use of the results concerning the numerical representation of preference relations. In this context, a real valued function $u:X \longrightarrow \mathbb{R}$ is called a *utility function* which represents the binary relation R, if for every $x,y \in X$, it is satisfied that

$$x \; R \; y \quad \Leftrightarrow \quad u(x) \geq u(y).$$

In a similar way, real valued functions which characterize choice functions are known in the literature as *numerical representations of choice functions* (see [3]).

Definition 1.

A choice function $F:\mathcal{D}(X) \longrightarrow X$ is said to be *representable* iff there exists a real valued function $u:X \longrightarrow \mathbb{R}$ such that for all $A \in \mathcal{D}(X)$ it is satisfied that

$$\varnothing \neq M(A,u) = F(A)$$

Then u is called a *representation* of F.

Remark: It is easy to prove that, in contexts in which the domain of the choice function is $\mathcal{P}(X)$, or at least contains all finite subsets of X, numerical representations of a choice function are unique, unless they are monotone transformations (in the same way as happens with utility functions). In more general domains, it is only satisfied that any monotone transformation of a numerical representation of a choice function is also a numerical representation.

Deb [3] obtains some results about representations of choice functions by analyzing the relationship between the representation of a choice function and its rationality. Thus, he proves that the existence of the representation of a choice function implies its rationality by a preorder and that the converse is also true in the case of considering countable sets of alternatives, (in this case any preorder has an associated utility function, see [2]). Therefore, all of the results known in the literature about rationality by a preorder of a choice function can be presented as results about representability of choice functions in this particular context.

Proposition 1. (Deb,[3])

Let $F:\mathcal{D}(X) \longrightarrow X$ be a choice function.

i). If $u:X \longrightarrow \mathbb{R}$ is a representation of F, then F is rationalized by a preorder and u is a utility function of this preorder.

ii). If X is countable, then F is representable if and only if it is rationalized by a preorder.

As a consequence of this result and Theorem 1, Deb [3] obtains that the representability of a choice function is provided by **(IIA)** when X is a countable set of alternatives. However, in non-countable contexts, this result is not true as the following example shows.

Example 1.

Consider $X = \mathbb{R}\times\{0,1\}$ and the choice function $F:\mathcal{D}(X) \longrightarrow X$ defined as follows:

$$F(A) = \{(a,i) \in A \mid a \geq b \quad \forall(b,j) \in A,\ i \geq j \quad \forall(a,j) \in A\}$$

where the domain of the choice function $D(X)$ consists of all $D \subseteq X$ such that $F(D) \neq \varnothing$. On the one hand, it is clear that F satisfies **(IIA)**. But, on the other hand, if there exists a representation of this choice function, $u: X \longrightarrow \mathbb{R}$, it has to satisfy

$$u(b,0) < u(b,1) \qquad\qquad \forall b \in \mathbb{R}$$

since if we consider $A = \{(b,0),(b,1)\}$, then $F(A) = \{(b,1)\}$. So, the open interval of real numbers $(u(b,0),u(b,1))$ is non-empty and we can define a function $g: \mathbb{R} \longrightarrow \mathbb{Q}$ as follows,

$$g(b) = q \in \mathbb{Q} \cap (u(b,0),u(b,1))$$

This is a one-to-one function, since if $a > b$, then $F(\{(a,0),(b,1)\} = \{(a,0)\}$, so $g(b) < u(b,1) < u(a,0) < g(a)$. But it implies that $\mathbb{R}$ is countable, a contradiction. Therefore we can conclude that there does not exist a representation of the choice function.

3. RESULTS

This section is devoted to proving the existence of numerical representations of choice functions directly, without making use of rationality results. It allows us to observe explicitly the way of defining the real valued function (the representation) from the choice function. We present results about the existence of representation, for both the case of countable sets of alternatives, as well as for the case of separable topological spaces.

The first result we present assumes that the set of alternatives is finite or countable, $X = \{x_i, i \in I \subseteq \mathbb{N}\}$, and that the choice function is defined in a domain $D(X)$ which contains all finite subsets of X (in fact, to obtain our results, it is sufficient to consider that each subset of X with three or less alternatives belongs to $D(X)$). In this context the equivalence between Arrow's axiom **(IIA)** and the representability of a choice function is proved.

Theorem 2.

Let X be a countable set of alternatives and $F:D(X) \longrightarrow X$ a choice function. Then, F is representable $\Leftrightarrow$ F satisfies **(IIA)**.

Proof.

It is clear that the existence of numerical representation of a choice function implies that it satisfies **(IIA)** due to the properties of the maxima of a real valued function.

Conversely, assume that F satisfies **(IIA)**. In order to define its representation, we associate the following set with each alternative $x \in X$,

$$A(x) = \{\ i \in I \mid \{x\} = F(\{x,x_i\})\ \}$$

where $X = \{x_i,\ i \in I \subseteq \mathbb{N}\}$ and then, we define the following function:

$$u(x) = \sum_{i \in A(x)} (1/2)^i$$

We are going to prove that this function is a representation of F.

In order to prove that $F(A) \subseteq M(A,u)$ it is sufficient to show that if $x \in F(A)$ then $A(y) \subseteq A(x)$, $\forall\ y \in A$. Consider $x \in F(A)$ and $y \in A$. By applying **(IIA)** we can ensure that $x \in F(\{x,y\})$, since

$$F(A) \cap \{x,y\} = F(\{x,y\})$$

Let $i \in A(y)$, that is $\{y\} = F(\{y,x_i\})$. If we consider the subset $\{x,y,x_i\}$, then we can ensure that $x \in F(\{x,y,x_i\})$; if not, since $\{x_i,y\} \subseteq \{x,y,x_i\}$ and $F(\{x,y,x_i\}) \cap \{x_i,y\} \neq \emptyset$, by **(IIA)**

$$F(\{x,y,x_i\}) \cap \{x_i,y\} = F(\{y,x_i\})$$

which implies $\{y\} = F(\{x,y,x_i\})$. But then, by applying **(IIA)**, to the subsets $\{x,y\}$, $\{x,y,x_i\}$ we obtain

$$F(\{x,y,x_i\}) \cap \{x,y\} = F(\{x,y\})$$

and then $x \notin F(\{x,y\})$, a contradiction. Therefore $x \in F(\{x,y,x_i\})$ and, by applying **(IIA)** to the subsets $\{x,x_i\}$, $\{x,y,x_i\}$,

$$F(\{x,y,x_i\}) \cap \{x,x_i\} = F(\{x,x_i\}),$$

that is $x \in F(\{x,x_i\})$; but since $x_i \notin F(\{y,x_i\})$, **(IIA)** implies $x_i \notin F(\{x,y,x_i\})$, so $\{x\} = F(x,x_i)$, which implies $i \in A(x)$.

Conversely, consider $x \in M(A,u)$ and assume that $x \notin F(A)$. Since $F(A) \neq \emptyset$, let $y \in F(A)$; then, by reasoning as above, we know that $u(y) \geq u(x)$. Now, let $k \in I$ be such that $y = x_k$, so $k \in A(y)$. Suppose that $k \in A(x)$, that is $\{x\} = F(\{x,x_k\})$; since $F(A) \cap \{x,x_k\} \neq \emptyset$ we can apply **(IIA)** and obtain that

$$F(A) \cap \{x,x_k\} = F(\{x,x_k\})$$

which implies $x \in F(A)$, a contradiction. Then $k \notin A(x)$ and $u(y) > u(x)$, which is not possible, since $x \in M(A,u)$.■

Now, we analyze the representability of a choice function in the case of considering the set of alternatives as being a separable topological space, and that the choice function is defined on a domain *D(X)* which contains all finite subsets of X. Unlike the discrete case, as Example 1 shows, **(IIA)** is not a sufficient condition (although it is a necessary one) to ensure the existence of a representation. In order to know the role played by each of the used assumptions, the proof of the existence of numerical representation is given by means of some previous lemmas.

Lemma 1.

Let X be a separable topological space and $F:\mathcal{D}(X) \longrightarrow X$ a choice function. If F satisfies **(IIA)**, then there exists a real valued function $u:X \longrightarrow \mathbb{R}$ such that

$$F(\{x,y\}) = \{y\} \quad \Rightarrow \quad u(y) \geq u(x)$$

Proof.

Let $Q = \{q_k,\ k \in \mathbb{N}\}$ a countable and dense subset of X (provided by the separability of X). For each $x \in X$, we define $A(x) = \{k \in \mathbb{N} \mid \{x\} = F(\{q_k,x\})\}$ and the following function:

$$u(x) = \sum_{i \in A(x)} (1/2)^i$$

In order to show that this function satisfies the required property, consider $x,y \in X$ such that $F(\{x,y\}) = \{y\}$ and $j \in A(x)$, that is $\{x\} = F(\{q_j,x\})$. Then, by applying **(IIA)** we know that $x \notin F(\{q_j,x,y\})$ and $q_j \notin F(\{q_j,x,y\})$, therefore $F(\{q_j,x,y\}) = \{y\}$, and, by **(IIA)**, we obtain $F(\{q_j,y\}) = \{y\}$, that is $j \in A(y)$. So we can conclude that $u(x) \leq u(y)$.■

In order to obtain the discrimination between u(x) and u(y) whenever $F(\{x,y\}) = \{y\}$, we need to introduce a continuity assumption.

(C1) If $F(\{x,y\}) = \{y\}$, then there exists a nonempty open set $A \subseteq X$ such that for all $z \in A$

$$x \notin F(\{x,z\}) \quad \text{and} \quad z \notin F(\{y,z\})$$

In other words, if x is rejected in the presence of y, then it is possible ("by continuity") to find an open set where x is rejected and y is chosen in front of any other alternative of such an open set.

Lemma 2.

Let X be a separable topological space and $F:\mathcal{D}(X) \longrightarrow X$ a choice function satisfying **(IIA)** and **(C1)**. Then there exists a real valued function $u:X \longrightarrow \mathbb{R}$ such that

$$F(\{x,y\}) = \{y\} \quad \Rightarrow \quad u(y) > u(x)$$

Proof.

Let $x,y \in X$ such that $F(\{x,y\}) = \{y\}$. By applying **(C1)**, there exists an open set A such that $z \notin F(\{z,y\})$, $x \notin F(\{x,z\})$ for all $z \in A$. Let $Q = \{q_k, k \in \mathbb{N}\}$ a countable and dense subset of X. Then, since Q is dense and A open, $Q \cap A \neq \emptyset$, so there is some k such that $q_k \in A$ and then $k \in A(y)$, $k \notin A(x)$, where the sets A(.) are defined as in Lemma 1. By considering now the function defined in the proof of Lemma 1, $u(x) < u(y)$ is obtained.∎

The next result shows that the maxima of the real valued function "determine the choice" in the case of binary sets.

Lemma 3.

Let X be a separable topological space and $F:\mathcal{D}(X) \longrightarrow X$ a choice function satisfying **(IIA)**. Then, there exists a real valued function $u:X \longrightarrow \mathbb{R}$ such that

$$u(y) > u(x) \quad \Rightarrow \quad F(\{x,y\}) = \{y\}$$

Proof.

Let u(x) be the function defined in Lemma 1 and $x,y \in X$ such that $u(x) < u(y)$. By the way of contradiction, assume that $x \in F(\{x,y\})$. Then if $j \in A(y)$,

$$x \in F(\{q_j,x,y\})$$

if not, by applying **(IIA)** we know that $y \in F(\{q_j,x,y\})$, and then

$$F(\{x,y\}) = F(\{q_j,x,y\}) \cap \{x,y\} = \{y\}$$

a contradiction. But then, by applying **(IIA)** again, $\{x\} = F(\{q_j,x\})$, so $j \in A(x)$ and we obtain $u(x) \geq u(y)$, also a contradiction. Therefore $x \notin F(\{x,y\})$, that is $F(\{x,y\}) = \{y\}$.■

The following proposition summarizes the results proved in the previous lemmas and proves the existence of representation of choice functions for binary sets:

Proposition 1.

Let X be a separable topological space and $F{:}\mathcal{D}(X) \longrightarrow X$ a choice function satisfying **(IIA)** and **(C1)**. Then, there exists a function $u{:}X \longrightarrow \mathbb{R}$ such that

$$F(\{x,y\}) = \max \{u(x),u(y)\}$$

Finally, the following theorem extends the previous result to non-binary sets.

Theorem 3.

Let $F:D(X) \longrightarrow X$ be a choice function satisfying **(IIA)**. If a topology on X can be defined such that the space is separable and F satisfies **(C1)**, then there exists a function $u:X \longrightarrow \mathbb{R}$ such that

$$F(B) = M(B,u) \qquad\qquad \forall B \in D(X)$$

Proof.

Let u(x) be the function defined in Lemma 1, $B \in D(X)$ and $a \in F(B)$. If there exists $x \in B$ such that $u(a) < u(x)$, then Lemma 3 implies that $F(\{x,a\}) = \{x\}$, a contradiction with **(IIA)**. Therefore, $F(B) \subseteq M(B,u)$.

Consider $a \in M(B,u)$. If $a \notin F(B)$ and $z \in F(B)$, by applying **(IIA)**, $F(\{a,z\}) = \{z\}$, so by Lemma 2 $u(a) < u(z)$, a contradiction. Therefore $M(B,u) \subseteq F(B)$.∎

As we have mentioned, **(IIA)** is also a necessary condition for the existence of the representation of a choice function; however **(C1)** is not a necessary condition as the following example shows.

Example 2.

Consider $X = [0,1]\cup[2,3]$, endowed with an arbitrary topology, and the choice function defined as follows:

$$F(D) = \text{argmax } \{x,\ x \in D\}$$

where the domain of the choice function $\mathcal{D}(X)$ consists of all $D \subseteq X$ such that $F(D) \neq \varnothing$. In the way that it has been defined, function $u(x) = x$ is a representation of the choice function; however **(C1)** is not satisfied since $F(\{1,2\}) = \{2\}$ and we can not find a nonempty open set $A \subseteq X$ satisfying that for every $z \in A$,

$$1 \notin F(\{z,1\}) \quad \text{and} \quad z \notin F(\{z,2\}).$$

4. FINAL COMMENTS

Throughout this work, the existence of representations of choice functions has been analyzed. This problem consists of finding a real valued function which totally characterizes the choice function, in the sense that the maxima of the function coincide with the choice set in each particular situation considered. However, just as in the case of representations of binary relations, there exist other kinds of representations of choice functions (weaker than the one considered in this work) which, on the one hand, provide less information about the choice function but, on the other hand, exist in more general contexts. In particular, we have to mention the notion of *weakly representable choice function* introduced in [3] which consists of finding a real valued function whose maxima are a selection of the choice set (u:X $\longrightarrow \mathbb{R}$ such that $\emptyset \neq$ M(A,u) $\subseteq$ F(A) $\forall$A $\in$ D(X)). The interpretation given to this kind of representation is clear: by means of the real valued function we have a quick process for selecting some of the alternatives chosen by the choice function. In [3] some results about the existence of weak representations for choice functions are proved in the particular case of considering finite and countable sets of alternatives. It would be interesting to obtain the same kind of results on the existence of weak representations of choice functions in non-countable sets of alternatives and, as we have done for the existence of representations, without making use of the results about the rationality of choice functions.

R E F E R E N C E S

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