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One-dimensional conservative coupled discretization of the shallow-water with scalar transport equations

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Abstract

The quality of the solutions obtained by numerical methods in shallow-water equations is strongly dependent on the discretization used. This is the greatest difficulty in shallow-water simulations. In this work we show that only a coupled discretization of the whole system of equations and a careful definition of the flux limiter function in second order TVD schemes are necessary in order to preserve uniform solute profiles in situations of 1D unsteady subcritical flow. An ideal dam-break with analytical solution and two practical applications on river flow and furrow irrigation are presented.

Keywords: One dimensional flow, advection, dispersion, coupled discretization, upwind schemes, conservative formulation, shallow water, river flow, furrow irrigation.

1 Introduction

A one-dimensional shallow-water model including solute transport, both forming a coupled or a decoupled system of equations, can be assumed to govern longitudinal mixing processes. To obtain an accurate solution of the advective part of the transport process, an option is to use Eulerian schemes of the appropriate order for the separate system of equations [9, 14]. Furthermore, Eulerian schemes can also be applied to the coupled set of equations [13, 3]. It is necessary as a first requirement to evaluate to what extent numerical schemes are able to preserve uniform initial solute profiles in irregular geometries or unsteady flow conditions. As a second goal, a suitable conservative scheme must be able to ensure bounded concentration values. It is not a trivial task since the solute concentration is not one of the conserved variables in our equations system. Two upwind finite volume techniques are presented and a few options considered for their numerical resolution.

The ideal dambreak unsteady flow with uniform solute concentration is used to evaluate the ability of the methods to preserve good properties in the solute distributions. Two practical applications of solute transport and unsteady flow on pollutant spill in a river and on furrow irrigation are finally presented.

2 The equations

The one-dimensional system formed by the cross sectional averaged liquid mass conservation, momentum balance in main stream direction and solute transport can be expressed in conservation form as:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}^c}{\partial x} + \frac{\partial \mathbf{D}}{\partial x} = \mathbf{S}^c, \tag{1}$$

where $\mathbf{U}$ is the vector of conserved variables, $\mathbf{F}^c$ the flux vector, $\mathbf{S}^c$ the source term vector, and $\mathbf{D}$ stands for diffusion:

$$\mathbf{U} = \begin{pmatrix} A \\ Q \\ As \end{pmatrix}, \quad \mathbf{F}^c = \begin{pmatrix} Q \\ gI_1 + \frac{\beta Q^2}{A} \\ Qs \end{pmatrix}, \quad \mathbf{S}^c = \begin{pmatrix} 0 \\ g[I_2 + A(S_0 - S_f)] \\ 0 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 0 \\ 0 \\ -KA\frac{\partial s}{\partial x} \end{pmatrix}. \tag{2}$$

with A the wetted cross section, Q the discharge, s the cross sectional average solute concentration, g the gravity constant, S_0 the longitudinal bottom slope, S_f the longitudinal friction slope, K the diffusion coefficient, I_1 and I_2 represent pressure forces:

$$I_1 = \int_0^h (h - z) b \, dz, \quad I_2 = \int_0^h (h - z) \frac{\partial b}{\partial x} \, dz, \tag{3}$$

with h the water depth, b the cross section width, and β a coefficient resulting from the cross sectional averaging of the velocity:

$$\beta = \frac{A}{Q^2} \int_A v_x^2 \, dA, \tag{4}$$

with v_x the longitudinal component of the velocity at any point of the cross section. Different models of β can be seen in [4]. The friction slope is widely modelled by means of the Gauckler-Manning law [6, 12]:

$$S_f = \frac{n^2 Q|Q| P^{\frac{4}{3}}}{A^{\frac{10}{3}}}, \tag{5}$$

with P the cross sectional wetted perimeter. The diffusion coefficient contains all the information related to molecular or viscous diffusion, turbulent diffusion and dispersion derived from the averaging process. In this paper, the model proposed by Rutherford [16] will be used:

$$K = 10\sqrt{gPA|S_f|}. \tag{6}$$

The system of equations can be expressed in non-conservative form taking into account:

$$\frac{d\mathbf{F}^c(x,\mathbf{U})}{dx} = \frac{\partial \mathbf{F}^c}{\partial x} + \mathbf{J}\frac{\partial \mathbf{U}}{\partial x}, \tag{7}$$

with $\mathbf{J}$ the flux Jacobian:

$$\mathbf{J} = \frac{\partial \mathbf{F}^c}{\partial \mathbf{U}} = \begin{pmatrix} 0 & 1 & 0 \\ c^2 - \beta u^2 & 2\beta u & 0 \\ -us & s & u \end{pmatrix}, \tag{8}$$

where $u = Q/A$ is the cross sectional average velocity, $c = \sqrt{gA/B}$ is the velocity of the infinitesimal waves and B is the cross section top width. Inserting in (1):

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{J}\frac{\partial \mathbf{U}}{\partial x} + \frac{\partial \mathbf{D}}{\partial x} = \mathbf{S}^{nc}, \tag{9}$$

with $\mathbf{S}^{nc}$ the non-conservative source term:

$$\mathbf{S}^{nc} = \mathbf{S}^c - \frac{\partial \mathbf{F}^c}{\partial x} = \begin{pmatrix} 0 \\ c^2\frac{\partial A}{\partial x} - Au^2\frac{\partial \beta}{\partial x} - gA\left(\frac{\partial z_s}{\partial x} + S_f\right) \\ 0 \end{pmatrix}, \tag{10}$$

where z_s is the water surface level.

The Jacobian matrix can be made diagonal:

$$\mathbf{J} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}, \quad \mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & 0 \\ s & s & 1 \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{11}$$

with $\mathbf{\Lambda}$ the eigenvalues diagonal matrix, $\mathbf{P}$ the diagonalizer matrix and λ_i the Jacobian eigenvalues corresponding to the characteristic propagation celerities:

$$\lambda_1 = \beta u + \sqrt{c^2 + (\beta^2 - \beta)u^2}, \quad \lambda_2 = \beta u - \sqrt{c^2 + (\beta^2 - \beta)u^2}, \quad \lambda_3 = u. \tag{12}$$

The eigenvalues are related to the flow regime:

- $\beta u^2 > c^2 \Rightarrow$ Supercritical flow.
- $\beta u^2 < c^2 \Rightarrow$ Subcritical flow.

By defining the differential characteristic variables $d\mathbf{W}$ as:

$$d\mathbf{W} = \mathbf{P}^{-1}\, d\mathbf{U}, \tag{13}$$

and left-multiplying the non-conservative equation (9) by $\mathbf{P}^{-1}$ the characteristic differential equations are obtained:

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{\Lambda}\frac{\partial \mathbf{W}}{\partial x} + \mathbf{P}^{-1}\frac{\partial \mathbf{D}}{\partial x} = \mathbf{P}^{-1}\mathbf{S}^{nc}. \tag{14}$$

3 Separate discretization of the solute transport equation

The simplest and most common form to solve the system of equations (2) is to discretize the mass and momentum flow equations separately, in each time step, from the solute transport equation. Letting aside the method applied to the flow equations, let us consider the conservative form of the transport equation:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0, \tag{15}$$

being $U = As$ the conserved variable and $F = uAs - KA\frac{\partial s}{\partial x}$ the flux. This flux can be decomposed into a flux T due to transport and another flux D due to diffusion. In this case:

$$F = T + D, \quad T = uAs, \quad D = -KA\frac{\partial s}{\partial x}, \tag{16}$$

and we shall next concentrate on the description and discussion of different numerical methods suitable for the discretization of this equation alone.

3.1 First order upwind scheme with implicit diffusion

Upwind schemes are based on a spatial discretization according to the sign of the characteristic celerities of propagation in the system. Hence, spatial derivatives are evaluated at every point using a computational cell larger than the correct region of influence of that point. Combining the first order explicit upwind scheme applied to the advection and the centred implicit scheme to solve the diffusion, both in conservative form, the following scheme is obtained:

$$\Delta U_i^n = -\frac{\Delta t}{\delta x}\left[\left(\delta T^+\right)_{i-(1/2)}^n + \left(\delta T^-\right)_{i+(1/2)}^n + D_{i+(1/2)}^{n+\theta} - D_{i-(1/2)}^{n+\theta}\right], \tag{17}$$

with δT^+ and δT^- associated to propagation velocities positive and negative respectively:

$$u^\pm = \frac{1}{2}(u \pm |u|), \quad \delta T^\pm = \frac{u^\pm}{u}\delta T, \tag{18}$$

where the notation $f^{n+\theta} = \theta f^{n+1} + (1-\theta)f^n$ has been used. This scheme is TVD for [3]:

$$0 \le \theta \le 1, \quad \Delta t \le \frac{\delta x^2}{|u|\delta x + (1-\theta)2K}. \tag{19}$$

3.2 Second order in space and time TVD scheme with implicit diffusion

By combining the Sweby second order in space and time TVD explicit scheme [18] applied to the advection and the centred implicit scheme to solve the diffusion, both in conservative form, the following second order in space TVD semi-implicit scheme is obtained:

$$\Delta U_i^n = -\frac{\Delta t}{\delta x}\left\{\left(\delta T^+\right)_{i-(1/2)}^n + \left(\delta T^-\right)_{i+(1/2)}^n + D_{i+(1/2)}^{n+\theta} - D_{i-(1/2)}^{n+\theta} + \frac{1}{2}\left[\left(\Psi^+\delta E^+\right)_{i-(1/2)}^n - \right.\right.$$

$$ - \left(\Psi^+ \delta E^+\right)^n_{i-(3/2)} + \left(\Psi^- \delta E^-\right)^n_{i+(1/2)} - \left(\Psi^- \delta E^-\right)^n_{i+(3/2)}\Big]\Big\}, \tag{20} $$

with:

$$ \delta E^{\pm} = (1 \mp \sigma)\delta T^{\pm}, \quad \sigma = u\frac{\Delta t}{\delta x}, \quad \sigma^{\pm} = \frac{1}{2}(\sigma \pm |\sigma|). \tag{21} $$

It is worth signalling that, although this scheme is named second order in space and time, this is not strictly true since it is not second order in time for the diffusion term.

The dependence of the flux limiter functions is defined as:

$$ \Psi^+_{i+(1/2)} = \Psi\left(\frac{(\delta E^+)^n_{i+(3/2)}}{(\delta E^+)^n_{i+(1/2)}}\right), \quad \Psi^-_{i+(1/2)} = \Psi\left(\frac{(\delta E^-)^n_{i-(1/2)}}{(\delta E^-)^n_{i+(1/2)}}\right), \tag{22} $$

so that:

- $\Psi(r) = 1$: Warming-Beam second order upwind scheme [19].
- $\Psi(r) = r$: Lax-Wendroff second order centred scheme [10].

Infinite second order schemes can be built for intermediate values between $\Psi(r) = 1$ and $\Psi(r) = r$. Applying the TVD conditions, the flux limiter will be a positive function so that [8]

$$ \Psi(r) = 0,\ \forall r < 0; \quad \Psi(r) \leq 2r,\ \forall r > 0; \quad \Psi(r) \leq 2,\ \forall r \tag{23} $$

leading to the following stability conditions [8]:

$$ 0 \leq \theta \leq 1, \quad \Delta t \leq \frac{\delta x^2}{|u|\delta x + (1-\theta)2K}. \tag{24} $$

Many particular flux limiter functions are defined in previous works [15, 1, 11]. We use the extreme value:

- *superbee* [15]: $\Psi(r) = \max\{0, \min(1, 2r), \min(2, r)\}$

4 Coupled discretization of the system

In the rest of this paper, our interest will be focused in the analysis of the discretization of the coupled system of equations using the first order upwind and the second order TVD schemes. Despite the apparently unnecessary extra complexity of this approach, it will prove the only way to improve the quality of the numerical solution in many cases, as previously reported [13, 3].

4.1 First order upwind scheme with implicit diffusion

The following decomposition matrices are defined:

$$ \mathbf{\Phi}^{\pm} = \begin{pmatrix} \phi_1^{\pm} & 0 & 0 \\ 0 & \phi_2^{\pm} & 0 \\ 0 & 0 & \phi_3^{\pm} \end{pmatrix}, \tag{25} $$

and, at the same time, the upwind matrices associated to the propagation directions:

$$\phi_i^{\pm} = \frac{1}{2}(1 \pm \operatorname{sign}(\lambda_i)), \quad \mathbf{\Omega}^{\pm} = \mathbf{P}\mathbf{\Phi}^{\pm}\mathbf{P}^{-1}, \quad \mathbf{G}^{\pm} = \mathbf{\Omega}^{\pm}\mathbf{G}, \quad \mathbf{G}^n_{i+(1/2)} = \left(\mathbf{S}^c - \frac{\partial \mathbf{F}^c}{\partial x}\right)_{i+(1/2)}. \tag{26}$$

In order to deal with transcritical problems of the type subcritical to supercritical flow, the introduction of an artificial viscosity like the one proposed by Harten-Hyman [7] is necessary. Then, the first order upwind scheme with implicit diffusion is defined as:

$$\Delta \mathbf{U}_i^n = \Delta t \left[\left(\mathbf{G}^+ - \nu \frac{\delta \mathbf{U}}{\delta x}\right)^n_{i-(1/2)} + \left(\mathbf{G}^- + \nu \frac{\delta \mathbf{U}}{\delta x}\right)^n_{i+(1/2)} - \frac{1}{\delta x}\left(\mathbf{D}^{n+\theta}_{i+(1/2)} - \mathbf{D}^{n+\theta}_{i-(1/2)}\right)\right], \tag{27}$$

with ν an artificial viscosity coefficient defined as [2]:

$$\nu^n_{i+(1/2)} = \max_k \begin{cases} \frac{1}{4}\left[\delta(\lambda_k) - 2|\lambda_k|\right]^n_{i+(1/2)}, & \text{if } (\lambda_k)^n_i < 0 \text{ and } (\lambda_k)^n_{i+1} > 0 \\ 0, & \text{otherwise} \end{cases} \tag{28}$$

Note that, for supercritical flow, $\mathbf{\Omega}^+ = \mathbf{1}$, $\mathbf{\Omega}^- = \mathbf{0}$ and this discretization is identical to (17). The same is not true for subcritical flow. We shall postulate that the TVD condition for this combined scheme is governed by the most restrictive among the different eigenvalues, that is:

$$0 \le \theta \le 1, \quad \Delta t \le \frac{\delta x}{\beta|u| + \sqrt{(\beta^2 - \beta)u^2 + c^2}}, \quad \Delta t \le \frac{\delta x^2}{|u|\delta x + (1-\theta)2K}. \tag{29}$$

4.2 Second order in space and time TVD scheme with implicit diffusion

The simplest form to extend the described scalar second order in space and time TVD scheme (20) to the coupled system of equations is:

$$\Delta \mathbf{U}_i^n = \Delta t \left\{\left(\mathbf{G}^+ - \nu \frac{\delta \mathbf{U}}{\delta x}\right)^n_{i-(1/2)} + \left(\mathbf{G}^- + \nu \frac{\delta \mathbf{U}}{\delta x}\right)^n_{i+(1/2)} - \frac{1}{\delta x}\left(\mathbf{D}^{n+\theta}_{i+(1/2)} - \mathbf{D}^{n+\theta}_{i-(1/2)}\right) + \right.$$
$$\left. + \frac{1}{2}\left[\left(\mathbf{\Psi}^+\mathbf{E}^+\right)^n_{i-(1/2)} - \left(\mathbf{\Psi}^+\mathbf{E}^+\right)^n_{i-(3/2)} + \left(\mathbf{\Psi}^-\mathbf{E}^-\right)^n_{i+(1/2)} - \left(\mathbf{\Psi}^-\mathbf{E}^-\right)^n_{i+(3/2)}\right]\right\}, \tag{30}$$

with the second order vectors:

$$\mathbf{E}^{\pm} = \left(\mathbf{1} \mp \mathbf{J}\frac{\Delta t}{\delta x}\right)\mathbf{G}^{\pm}. \tag{31}$$

The flux limiting matrices are defined as:

$$\mathbf{\Psi}^{\pm}_{i+(1/2)} = \begin{pmatrix} \Psi\left(\frac{(\mathbf{E}^{\pm})^1_{i+(1/2)\pm 1}}{(\mathbf{E}^{\pm})^1_{i+(1/2)}}\right) & 0 & 0 \\ 0 & \Psi\left(\frac{(\mathbf{E}^{\pm})^2_{i+(1/2)\pm 1}}{(\mathbf{E}^{\pm})^2_{i+(1/2)}}\right) & 0 \\ 0 & 0 & \Psi\left(\frac{(\mathbf{E}^{\pm})^3_{i+(1/2)\pm 1}}{(\mathbf{E}^{\pm})^3_{i+(1/2)}}\right) \end{pmatrix}, \tag{32}$$

with $(\mathbf{E}^{\pm})^i$ the i component of vector $\mathbf{E}^{\pm}$. This new form of defining the flux limiting matrices, based on the components of the second order vector, will be called vectorial limiting discretization.

Another alternative is to define the second order vectors as:

$$\mathbf{L}^{\pm} = \left(\mathbf{1} \mp \mathbf{\Lambda}^{\pm}\frac{\Delta t}{\delta x}\right)\mathbf{P}^{-1}\mathbf{G}^{\pm} \tag{33}$$

EqxtTVDSecChar Then, the second order in space and time TVD scheme is written as:

$$\Delta\mathbf{U}_i^n = \Delta t\left\{\left(\mathbf{G}^+ - \nu\frac{\delta\mathbf{U}}{\delta x}\right)^n_{i-(1/2)} + \left(\mathbf{G}^- + \nu\frac{\delta\mathbf{U}}{\delta x}\right)^n_{i+(1/2)} - \frac{1}{\delta x}\left(\mathbf{D}^{n+\theta}_{i+(1/2)} - \mathbf{D}^{n+\theta}_{i-(1/2)}\right) + \right.$$

$$\left. +\frac{1}{2}\left[\left(\mathbf{P\Psi^+L^+}\right)^n_{i-(1/2)} - \left(\mathbf{P\Psi^+L^+}\right)^n_{i-(3/2)} + \left(\mathbf{P\Psi^-L^-}\right)^n_{i+(1/2)} - \left(\mathbf{P\Psi^-L^-}\right)^n_{i+(3/2)}\right]\right\}, \tag{34}$$

and the flux limiting matrices are:

$$\mathbf{\Psi}^{\pm}_{i+(1/2)} = \begin{pmatrix} \Psi\left(\frac{(\mathbf{L}^{\pm})^1_{i+(1/2)\pm 1}}{(\mathbf{L}^{\pm})^1_{i+(1/2)}}\right) & 0 & 0 \\ 0 & \Psi\left(\frac{(\mathbf{L}^{\pm})^2_{i+(1/2)\pm 1}}{(\mathbf{L}^{\pm})^2_{i+(1/2)}}\right) & 0 \\ 0 & 0 & \Psi\left(\frac{(\mathbf{L}^{\pm})^3_{i+(1/2)\pm 1}}{(\mathbf{L}^{\pm})^3_{i+(1/2)}}\right) \end{pmatrix}. \tag{35}$$

This second form of defining the flux limiting matrices, more in the line of the characteristic form of the scheme, will be named characteristic limiting discretization. Using that in the scalar case the TVD conditions for this scheme are identical to those for the first order scheme, we shall postulate that this scheme is TVD whenever (29) holds.

5 Practical applications

5.1 *Ideal dambreak with uniform solute concentration*

When the initial concentration as well as the boundary conditions are uniform, $\left(\frac{\partial s}{\partial x} = 0\right)$, the third of the conservation equations (1) becomes:

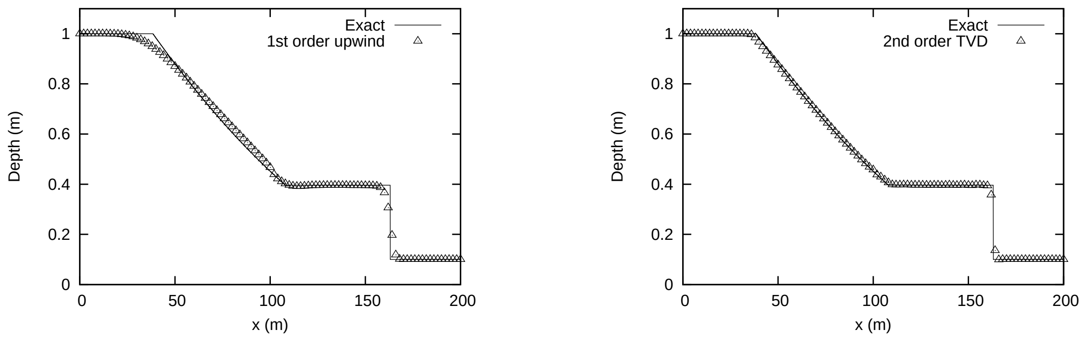
$$\frac{\partial(As)}{\partial t} + \frac{\partial(Qs)}{\partial x} = \frac{\partial}{\partial x}\left(KA\frac{\partial s}{\partial x}\right) = 0 \tag{36}$$

EqSolConstCons By developing the derivatives and using the mass conservation equation:

$$A\frac{\partial s}{\partial t} + s\left(\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x}\right) = 0 \Rightarrow \quad \frac{\partial s}{\partial t} = 0, \tag{37}$$

indicating that, under these conditions, the concentration must stay constant in time whatever the flow conditions. A numerical scheme unable to reproduce this important property will be unacceptable.

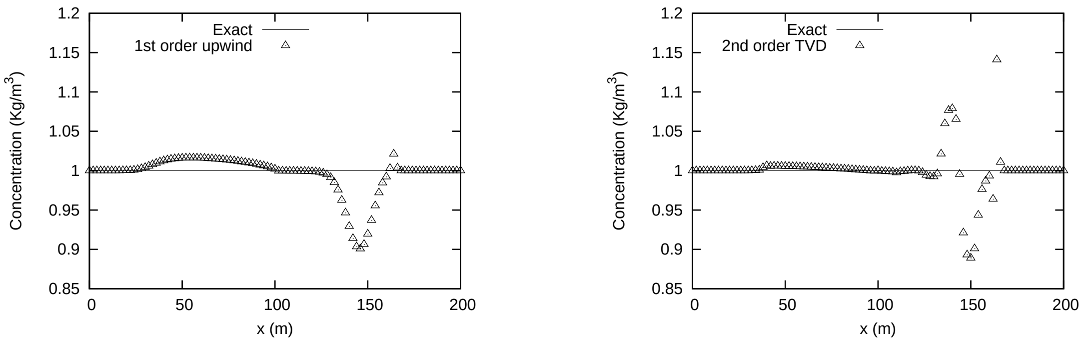
The ideal dambreak problem is one of the classical examples used as test cases for unsteady shallow water flow simulations. The reason is that for flat and frictionless bottom, rectangular cross section and no diffusion, the problem defined by zero initial velocity and initial discontinuities in the water depth and solute concentration has an exact solution easy to compute [17]. A rectangular channel $200m$ long and $1m$ wide has been considered with an initial depth ratio $1m : 0.1m$ together with a uniform initial solute concentration of $1kg/m^3$. A grid spacing of $\delta x = 2m$ and $CFL = 0.9$ was used for all the simulations. The plots in Fig. 1 show the numerical solution for the water depth of the three schemes and the exact solution at $t = 20s$. Fig. 2 displays the concentration



(a) (b)

Figure 1.— Ideal dambreak depth with the schemes of (a) 1st order upwind and (b) 2nd order TVD.

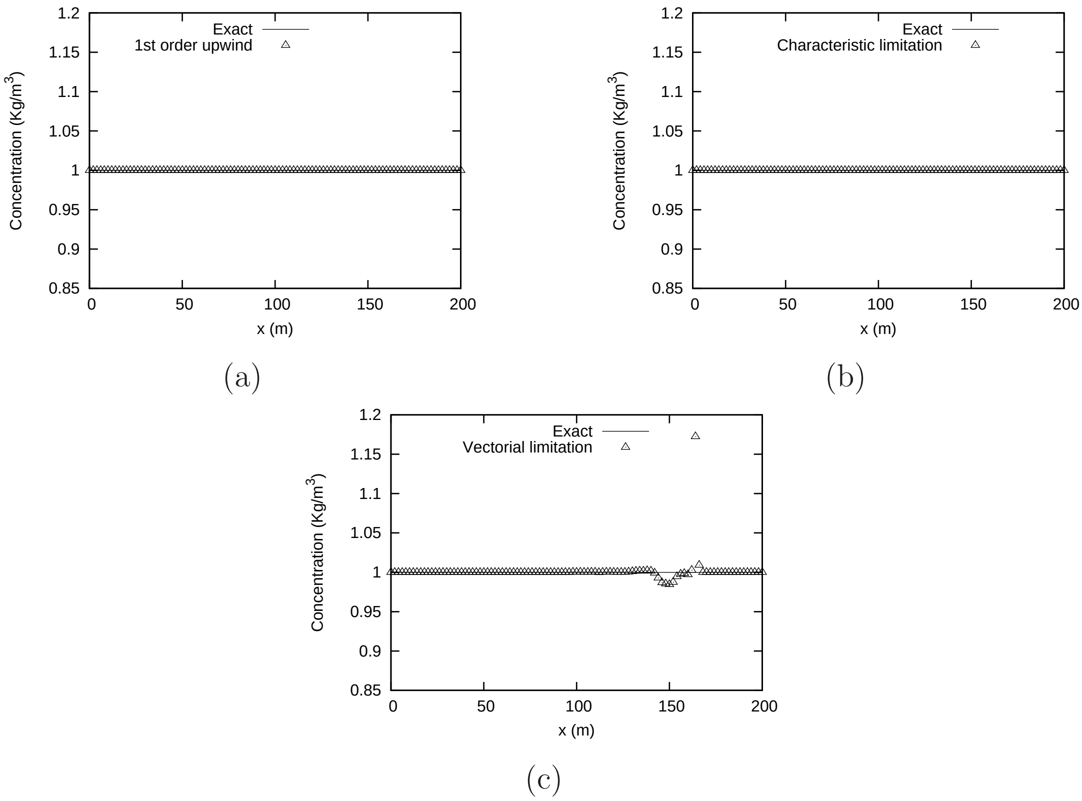
results at $t = 20s$ using the separated discretization. None of the numerical schemes is able to keep the concentration uniform as time progresses. Fig. 3 shows the results



(a) (b)

Figure 2.— Ideal dambreak concentration with the separated discretization and the schemes of (a) 1st order upwind and (b) 2nd order TVD.

obtained with the coupled discretization for the same test case. The first order upwind scheme preserves the uniform concentration as well as the second order TVD scheme with different flux limiters if the characteristic limiting formulation is used. When the vectorial limiting discretization is used for the limiters, the numerical solution is not free from oscillations.



(a) (b) (c)

Figure 3.— Ideal dambreak concentration with the coupled discretization and the schemes of (a) 1st order upwind, (b) and (c) 2nd order TVD with (b) characteristic and (c) vectorial limiting discretization.

5.2 Pollutant spill in a river

In order to show the practical application of the model in a river flow context, a hazardous and instantaneous pollutant spill of $20T$ of petrol, located at a point at $700m$ of the upstream end, in a $11.4km$ reach of the Ebro River will be simulated. The solubility of the petrol at the typical temperature of the river water was estimated as $0.03kg/m^3$. For higher concentrations, the petrol was assumed to precipitate to the bottom remaining there. The steady annual base river discharge of $200m^3/s$ was assumed. In a first run, the steady state water surface profile corresponding to that discharge in the river reach was calculated. Fig. 4a represents the bed and surface levels at steady state. Fig. 4b shows two concentration longitudinal profiles at $1h$ and $2h$ of the spill. The Spanish law establishes that $9.5mg/m^3$ is the limit of tolerance for the pernicious influence of petrol concentration in riverine ecological systems. Fig. 4c represents the time evolution of borders of the contaminant cloud with a concentration exceeding the dangerous limit.

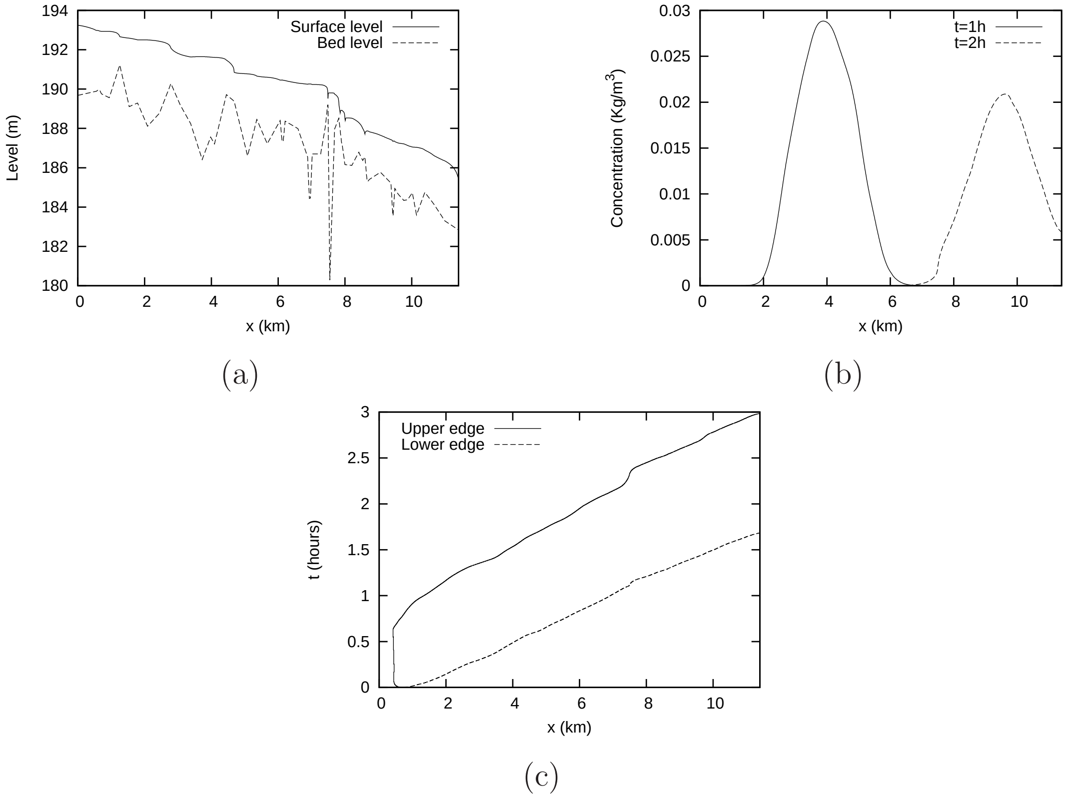


Figure 4.— Petrol spill in Ebro river: (a) longitudinal bed and water surface profiles, (b) longitudinal concentration profiles at $1h$ and $2h$ after the spill, (c) time evolution of the plume of concentration exceeding the dangerous threshold.

5.3 Fertigation in furrows

A field experiment was conducted during the summer of 2003 at the research farm of the Agricultural and Technological Research Center of the Government of Aragón in Zaragoza, Spain [5]. Four isolated furrows were built using the same field machinery (Fig. 5). The cross-sectional average furrow dimensions were: base width $0.14m$, top width $0.80m$ and furrow depth $0.27m$. All furrows showed zero slope.

Water was diverted to each furrow using a pump. A volumetric water meter was installed downstream from the pump in order to verify that a constant discharge was applied to each furrow. An irrigation evaluation, under free draining downstream conditions, was performed at each furrow. Evaluations were characterized by different irrigation discharges: 1, 2, 3 and $4L/s$.

The monitored furrow length was $100m$. The furrow spacing was $1m$. Stations were marked along each furrow every $10m$. All stations were used to monitor the advance phase (advance stations). Five of them, every $20m$, were additionally used to monitor fertilizer hydrodynamics (fertilizer stations). Fertilizer was applied to the irrigation water in all evaluations. The initial fertilizer concentration (approximately $10.6g/L$) was kept constant for all furrow irrigation events. In Fig. 6 the time evolution of measured

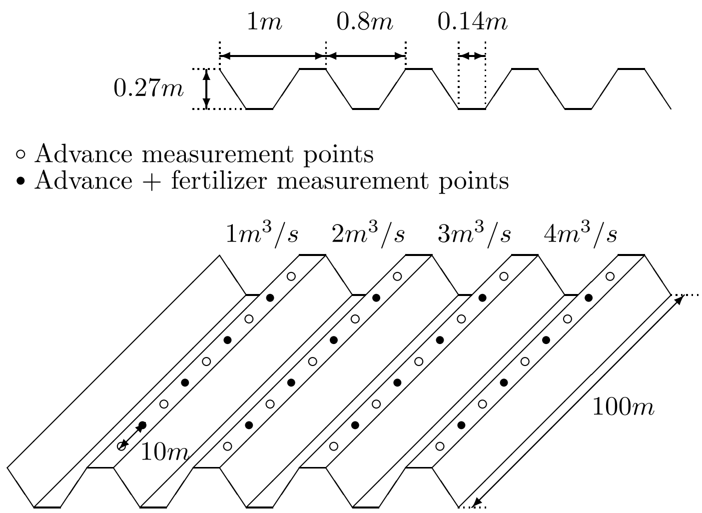


Figure 5.— Experimental set-up used for furrow fertigation evaluation.

and simulated fertilizer concentration is presented at four gauge locations for the four experiments.

6 Conclusions

A conservative formulation of the system of equations governing the water flow and the solute transport has been adopted as the basis of our study. The formulation of two finite volume conservative upwind schemes well suited for the numerical simulation of one-dimensional shallow-water flow and solute transport has been provided. Two possibilities have been identified, separate or coupled discretization, leading to different degree of influence of the flow processes to the solute transport at the discrete level.

It has been proved that well balanced conservative upwind schemes based on a separate discretization of the scalar solute transport from the shallow water equations are not able to preserve uniform solute profiles in situations of unsteady subcritical flow even when using first order methods. However, the coupled formulation and discretization of the system is proved to lead to the correct solution in first order approximations.

When seeking more accuracy second order TVD schemes can be applied. It has been shown that a careful definition of the flux limiter function is required in order to preserve uniform solute profiles in the solute distribution function in cases of unsteady subcritical flow.

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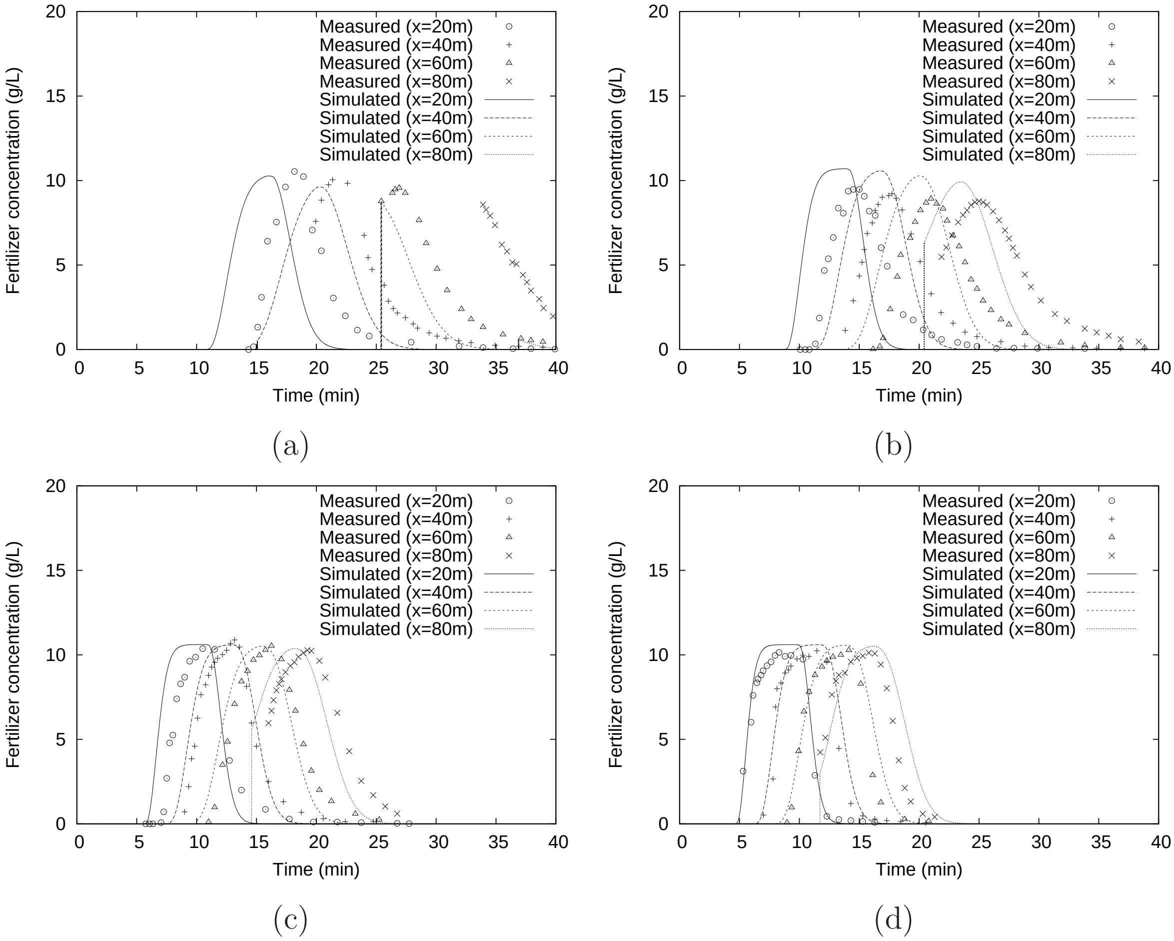

Figure 6.— Fertilizer concentration measured and simulated at four locations for discharges of (a) $1m^3/s$, (b) $2m^3/s$, (c) $3m^3/s$ and (d) $4m^3/s$.

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