EXISTENCE AND NASH IMPLEMENTATION OF EFFICIENT SHARING RULES FOR A COMMONLY OWNED TECHNOLOGY

Luis C. Corchón and M. Socorro Puy

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Correspondence to Luis C. Corchón: Universidad Carlos III. Departamento de Economía. C/ Madrid 126. 28093 Getafe. Spain e-mail: lcorchon@eco.uc3m.es.

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** L.C. Corchón: Universidad Carlos III. M. Socorro: Universidad de Málaga.

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ABSTRACT

Suppose that a group of individuals owns collectively a technology which produces a consumption good by means of a (possibly heterogeneous) input. A sharing rule associates input contributions with a vector of consumptions that are technologically feasible. We show that the set of allocations obtained by any continuous sharing rule contains a subselection that is Pareto efficient. We also present a mechanism that implements in Nash equilibrium the Pareto efficient allocations contained in an arbitrary sharing rule.

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1. Introduction

Consider a group of people owning a technology which transforms a possibly heterogeneous input (labor) in an homogeneous output (consumption). Inputs are also provided by the owners. Different proposals on how to distribute the output can be found in the literature.

From the point of view of fairness some authors have translated philosophical criteria into solution concepts in the class of environments in which the input is homogenous: Roemer and Silvestre (1988) proposed the Proportional Solution and the Equal Benefit Solution, Mas-Colell (1980) proposed the Constant Returns Equivalent Solution. Several characterizations of these solutions are provided in Moulin (1990), Moulin and Roemer (1989), and Maniquet (1996). When heterogenous inputs are considered, other solutions have been proposed: Equal Sharing, Marginal Cost Rule, Aumann-Shapley Prices, Reference Welfare Equivalent Budget, etc. (see Aumann and Shapley (1974), Billera and Heath (1982), Moulin (1987), Tauman (1988), Pfingsten (1991) and Fleurbaey and Maniquet (1996)).

In this note we approach this problem from a different angle. We focus our attention on contracts that are offered to the owners of the inputs. We assume that the quantity of inputs is contractible but preferences are not. These contracts, that we will call Sharing Rules, are a function which specifies the list of consumptions depending on input contributions. The sharing rule together with the quantity of inputs determine income distribution inside the firm. It is worth to notice that in spite of the different motivations, all solutions mentioned above qualify as Sharing Rules except the constant return equivalent solution (the idea of expressing the share of output in terms of the inputs appeared in Moulin (1990), p.445 for the proportional and the equal benefit solutions). Our analytical task consists in checking which Sharing Rules satisfy two basic requirements: Efficiency and Implementability.

Firstly, we focus attention on those Sharing Rules which are compatible with Pareto efficiency, which we call Efficient Sharing Rules. It is well known that the proportional and the equal benefit solutions are Efficient Sharing Rules. We generalize these results by showing that *any* continuous sharing rule is an efficient sharing rule in the set of classical economies (continuous and convex preferences). Our proof is inspired by the proof of Negishi (1960) of the existence of a Walrasian Equilibrium.

Secondly, we consider the incentive properties of efficient sharing rules. A sharing rule that gives incentives to distort preferences or productivities can not be regarded as satisfactory. Roemer (1988), Gevers (1986) and Maniquet and Fleurbaey (1996) showed respectively that the proportional, the equal benefit solution and the reference welfare equivalent budget are Nash implementable[1], i.e. there is a mechanism whose Nash equilibrium strategies generate the desired allocations. Suh (1995) introduced a mechanism that implements the proportional solution in Nash, undominated Nash and Strong equilibria. Shin and Suh (1997) provide a simple mechanism which doubly implements a class of solutions in Nash and strong equilibrium. In this paper we provide a simple mechanism that implements in Nash equilibrium *every* efficient sharing rule in the set of classical economies when there are at least three individuals[2]. We assume that the planner knows the sharing rule but not the preferences of the agents that determine the set of Pareto efficient allocations for each economy. Our procedure has the advantage over Shin and Suh's that our conditions on the solutions to be implemented are easy to check and they include economies with heterogeneous inputs, but the disadvantage that we only implement in Nash equilibrium.

Our mechanism has been inspired by the canonical mechanism used in Nash implementation. People are arranged in a circle and each agent proposes the amount of input supplied by him and the agent next to him. Three cases are then identified:

First, when the amount of input proposed by each agent coincides with the amount suggested by her monitor. In this case the mechanism distributes the output according to the sharing rule.

Second, when there are, at most, two consecutive agents whose proposals differ from what was proposed for them. Then, the agent with the lowest index (the dissident) has the right to choose an allocation in a certain budget set that is only profitable if he has deviated from a non efficient allocation. Since a deviation can only happen if the monitor of the dissident has tried to fool the mechanism, then the monitor is severely fined: he gets zero consumption and has to contribute up the maximum amount of labor. All other agents obtain some arbitrary bundle.

Third, for any other message it is not possible to identify the dissident. Thus things are very muddy and the mechanism has to reflect that. In this case, the mechanism divides the agents into two groups: the ones that consume but do not work, and the ones that work but do not consume, in such a way that the agents

[1]Implementation in dominant strategies is usually impossible. However see Schmeidler and Tauman (1989) for a case in which it is possible.

[2]We consider adverse selection problems only. Moral hazard in teams is considered in Holmstrom (1982).

of the second group always have incentive to deviate the mechanism to the second case. Notice that, contrarily to what happens in the canonical mechanism, agents do not play integer games.

Our construction avoids some of the criticism made by Jackson (1992) to Maskin-type mechanisms: Message spaces are bounded and thus, there is always a best reply for each agent.

2. The Model and the Main Results

There is one consumption good produced from a vector of possibly heterogenous inputs using a publicly owned technology.
There are n individuals indexed by i. Let $N = \{1, ..., n\}$. They are endowed with $\bar{\ell} \in \mathbb{R}_+$, units of input[3]. Each individual consumption set is defined by:

$$\mathbb{X} = \left\{ (x_i, \ell_i) : x_i \in \mathbb{R}_+, \ell_i \in \left[0, \overline{\ell}\right] \right\}$$

where x_i is agent i's consumption and ℓ_i is input contribution. Each agent has preferences defined on $\mathbb{X}$, that can be represented by a utility function

$$u_i : \mathbb{X} \longrightarrow \mathbb{R}$$

The utility function is assumed to be continuous, concave, strictly increasing in x_i and strictly decreasing in ℓ_i. Thus

$$\underset{(x_i, \ell_i) \in \mathbb{X}}{\arg\min}\, u_i (x_i, \ell_i) = \left(0, \overline{\ell}\right) \forall i \in N.$$

The technology is represented by a production function

$$f : \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+$$

The function f is continuous, increasing in each component, concave, continuously differentiable in each component and with $f(0, ..., 0) = 0$.
We define the **feasible set,** denoted by $\mathcal{X}$ as follows

$$\mathcal{X} = \left\{ (x_1, \ell_1, ..., x_n, \ell_n) \in \mathbb{X}^n : \sum_{i \in N} x_i \leq f\ (\ell_1, ..., \ell_n) \right\}$$

[3] When inputs are heterogenous, we shall consider that each agent is endowed with $\bar{\ell}_i \in \mathbb{R}_+$ units of labor, where each $\bar{\ell}_i$ is common knowledge.

A feasible allocation is denoted by $(x, \ell) \in \mathcal{X}$ where $x = (x_1, .., x_n)$ and $\ell = (\ell_1, ..., \ell_n)$.
We assume that $\mathcal{X}$ is fixed and utility functions vary. Thus, an **economy,** denoted by $u = (u_1, ..., u_n)$, is a list of utility functions satisfying the assumptions listed above. The set of admissible economies is denoted by $\mathcal{E}$.

The **Pareto efficient solution** $\varphi^E : \mathcal{E} \longrightarrow \mathcal{X}$ associates to each economy in the domain the set of Pareto Efficient allocations for this economy. Formally,

$$\varphi^E(u) = \left\{ \begin{array}{c} (x, \ell) \in \mathcal{X} : \nexists \left(x', \ell'\right) \in \mathcal{X} \ / \ u_h(x'_h, \ell'_h) \geq u_h(x_h, \ell_h) \ \forall h \in N \\ \text{and } u_j\left(x'_j, \ell'_j\right) > u_j(x_j, \ell_j) \text{ for at least one } j \in N. \end{array} \right\}$$

We next introduce the concept of a sharing rule. A sharing rule is a contract that specifies the consumptions as a function of input´s contributions.
A **Sharing Rule** $P = (P_1, ..., P_n)$ is a collection of functions such that $P_i : \left[0, \overline{\ell}\right]^n \longrightarrow \mathbb{R}_+ \ \forall i \in N$ with $\sum_{i \in N} P_i(\ell) = f(\ell) \ \forall \ell \in \left[0, \overline{\ell}\right]^n$.

Each P_i yields the consumption of i as a function of ℓ. Moreover, P distributes the total output. Some examples of solutions that can be expressed as sharing rules are the following:
The Proportional Solution in which the sharing rule is:

$$P_i(\ell) = \frac{f(\ell)}{\sum_{i \in N} \ell_i} \ell_i \text{ for } \forall i \in N$$

where the amount of output consumed by an agent is proportional to the amount of input that she contributes.
The Equal Benefit Solution, in which the sharing rule is:

$$P_i(\ell) = \frac{\partial f(\ell)}{\partial \ell_i} \ell_i + \frac{1}{n} \left[f(\ell) - \sum_{i \in N} \left(\frac{\partial f(\ell)}{\partial \ell_i} \ell_i \right) \right] \text{ for } \forall i \in N$$

where each agent consumes according to the budget constraint in the Walrasian equilibrium with equal profits. Clearly, other rules of profit distribution also qualify as sharing rules.
The Equal Sharing Solution, in which the sharing rule is:

$$P_i(\ell) = \frac{f(\ell)}{n} \text{ for } \forall i \in N$$

where each agent consumes an equal part of the total output.
The Aumann-Shapley prices, in which denoting by $\ell \in \left[0, \bar{\ell}\right]^n$ a vector of input contributions, the sharing rule is:

$$P_i(\ell) = \int_0^1 \frac{\partial f(t\ell)}{\partial \ell_i} dt \ \ell_i \text{ for } \forall i \in N$$

where each agent consumes proportionally to the contribution of her input to the total production. When the input is homogenous it coincides with the proportional sharing rule.

Furthermore, the family of methods proposed by Moulin (1987) also qualify as sharing rules, indeed, all these methods are compromise between the equal and the proportional sharing rules:
>From Moulin (1987) Theorem 1, where for each $\mu \in \mathbb{R}_+$,

$$P_i(\ell) = \begin{array}{l} \frac{\ell_i}{\sum\limits_{i \in N} \ell_i} f(\ell) \text{ if } \frac{f(\ell)}{\sum\limits_{i \in N} \ell_i} \leq \mu \\ \ell_i \mu + \frac{1}{n}\left(f(\ell) - \mu \sum\limits_{i \in N} \ell_i\right) \text{ if } \frac{f(\ell)}{\sum\limits_{i \in N} \ell_i} \geq \mu \text{ for } \forall i \in N. \end{array}$$

>From Moulin (1987) Theorem 4, where for each $\lambda \in [0,1]$,

$$P_i(\ell) = \frac{f(\ell)}{n} + \left(\ell_i - \frac{1}{n} \sum_{i \in N} \ell_i\right) \left[\left(1 + \frac{f(\ell)}{\sum_{i \in N} \ell_i}\right)^{\lambda} - 1\right] \text{ for } \forall i \in N.$$

>From Moulin (1987) Theorem 5, where for each $\lambda \in [0,1]$,

$$P_i(\ell) = \frac{\ell_i^{\lambda}}{\sum\limits_{i \in N} \ell_i^{\lambda}} f(\ell) \text{ for } \forall i \in N.$$

>From Moulin (1987) Theorem 6, where for each $\lambda \in [0,1)$

$$\begin{aligned} P_i(\ell) &= \left(\ell_i^{1-\lambda} + \alpha\right)^{\frac{1}{1-\lambda}} - \ell_i \text{ and } \alpha \text{ is unique} \\ \text{solution of } &: \sum_{i \in N} \left(\ell_i^{1-\lambda} + \alpha\right)^{\frac{1}{1-\lambda}} = \sum_{i \in N} \ell_i + f(\ell) \text{ for } \forall i \in N. \end{aligned}$$

Finally, note that every convex combination of the mentioned solutions, is also a sharing rule, see for instance Corchón and Puy (1998).

We assume that every sharing rule verifies that if $\ell_i = \bar{\ell}$, $P_i\left(\ell_1, .., \bar{\ell}, \ell_{i+1} .., \ell_n\right) \neq 0$.

Pareto Efficiency and Sharing Rules

In the sequel we will be interested in the intersection of the Pareto efficient solutions φ^E and a given sharing rule P, that we denote Efficient Sharing Rule φ^{PE} and is defined by

$$\varphi^{PE}(u, P) = \left\{(x, \ell) \in \varphi^E(u) : x_i = P_i(\ell), \ \forall i \in N\right\}.$$

We now prove that φ^{PE} exists provided that the sharing rule is continuous[4].

Theorem 1: Given $u \in \mathcal{E}$ and a continuous sharing rule P, then $\varphi^{PE}(u, P) \neq \emptyset$.

Proof: Let $\alpha \in \Delta^{n-1}$ where $\Delta^{n-1} = \left\{\alpha \in \mathbb{R}_+^n : \sum\limits_{i \in N} \alpha_i = 1\right\}$.
Consider the problem:

$$\max_{(x,\ell) \in \mathcal{X}} \sum_{i \in N} \alpha_i u_i(x_i, \ell_i)$$

by continuity of u_i and compactness of $\mathcal{X}$, there always exists a solution to this problem, which is by definition Pareto efficient.
This maximization defines a correspondence denoted by ϕ, such that

$$\phi : \Delta^{n-1} \longrightarrow \mathcal{X}$$

By concavity of u and convexity of $\mathcal{X}$, ϕ is convex valued. By Berge's Maximum Theorem, ϕ is upper hemicontinuous.
We define $D_i : \mathbb{R}_+ \times \left[0, \bar{\ell}\right]^n \longrightarrow \mathbb{R}$ for each $i \in N$ as follows:

$$D_i(x_i, \ell) = P_i(\ell) - x_i$$

Note that for every i, D_i is continuous by continuity of P_i.
Fix an allocation $\left(\hat{x}, \hat{\ell}\right)$ and consider the following maximization program:

$$\max_{\alpha \in \Delta^{n-1}} \sum_{i \in N} \alpha_i D_i\left(\hat{x}_i, \hat{\ell}\right)$$

[4] It can be easily checked that all the rules mentioned before are continuous (for Aumann-Shapley prices, see Mirman and Tauman (1980)).

by continuity of each D_i and compactness of Δ^{n-1}, there always exists a solution to this problem. This maximization defines a correspondence denoted by Φ, such that

$$\Phi : \mathcal{X} \longrightarrow \Delta^{n-1}$$

where Φ is convex valued and upper hemicontinuous by Berge's Maximum Theorem.

Now consider the following mapping

$$\Phi \times \phi : \Delta^{n-1} \times \mathcal{X} \longrightarrow \Delta^{n-1} \times \mathcal{X}$$

This is a upper hemicontinuous mapping from a compact convex set into itself, with non empty and convex values. By Kakutani's fixed point theorem, there exist a fixed point (α^*, x^*, ℓ^*).

Notice that it is impossible to have (x_i^*, ℓ^*) such that $D_i(x_i^*, \ell^*) < 0 \ \forall i \in N$ or $D_i(x_i^*, \ell^*) > 0 \ \forall i \in N$, because the sharing rules verify that $\sum\limits_{i \in N} x_i = f(\ell)$. Thus if $D_i(x_i^*, \ell^*) > 0$, then $\exists j \in N : D_j\left(x_j^*, \ell^*\right) < 0$ and so $\alpha_j^* = 0$. But this implies that ϕ will assign $\left(x_j^*, \ell_j^*\right) = \left(0, \bar{\ell}\right)$ and so $D_j\left(x_j^*, \ell^*\right) \geq 0$, a contradiction. Therefore the fixed point verifies $D_i(x_i^*, \ell^*) = 0 \ \forall i \in N$ and this implies that $x_i^* = P_i(\ell) \ \forall i \in N$. **Q.E.D.**

Remark 1. *We shall point out that to guarantee the continuity of these sharing rules, the continuity of f, in some cases, and the continuous differentiability of f are crucial. As a first approach we do not deal with non-continuous or/and non-differentiable technologies.*

Remark 2. *Note that all the examples of sharing rules that we have already mentioned as well as all the convex combination of these sharing rules are continuous sharing rules so that they are Efficient Sharing Rules.*

Implementation of Efficient Sharing Rules

A **mechanism** Γ is a list $\left\{(S_i)_{i \in N}, g\right\}$ where S_i is the strategy space for agent i and g is the outcome function, mapping each strategy profile into an element of the feasible set :

$$g : \prod_{i \in N} S_i \longrightarrow \mathcal{X}$$

The outcome received by each agent is $g_i(s) = (x_i, \ell_i)$.
Let s_{-i} the list of strategies for all the agents except for i, then, the set of **Nash equilibria** of the game (Γ, u) is denoted by NE(Γ, u).

$$NE(\Gamma, u) = \left\{ s \in \prod_{i\in N} S_i \ : \ u_i(g_i(s)) \geq u_i(g_i(s'_i, s_{-i}) \ \forall i \in N \ , \forall s'_i \in S_i) \right\}$$

We say that a mechanism implements φ^{PE} in Nash equilibrium when it verifies that:

$$NE \ \ (\Gamma, u) \neq \emptyset$$

$$\varphi^{PE}(u, P) = g(NE \ \ (\Gamma, u)), \ \forall u \in \mathcal{E}.$$

Let us introduce the mechanism $\Gamma(P)$ which implements the efficient sharing rule P.
Each individual **strategy space** is defined by $S_i = \left[0, \bar{\ell}\right]^2 \subset \mathbb{R}^2_+$.
A strategy for i is a pair, $s_i = \left(\ell^i_i, \ell^{i+1}_i\right)$, whenever $i = n$ we define $i + 1 = 1$ and when $i = 1$, $i - 1 = n$. Each individual strategy may be interpreted as a proposed labor allocation for herself $\left(\ell^i_i\right)$ and the individual next to her $\left(\ell^{i+1}_i\right)$. This is a particular instance of a "Tweed Ring" mechanism[5].

The **outcome function** is divided in three cases which we denote as rules:
Rule 1 (Unanimity):
If $\forall \ j \in N : \ell^j_j = \ell^j_{j-1}$ then $g_j(s) = \left(P_j(\ell), \ell^j_j\right)$ where $\ell = (\ell^1_1, ..., \ell^n_n)$.
Rule 2 (Dissident right):
If $\forall j \in N - \{i\} : \ell^j_j = \ell^j_{j-1}$ and for $i : \ell^i_i \neq \ell^i_{i-1}$,
or if $\forall j \in N - \{i, i+1\} : \ell^j_j = \ell^j_{j-1}$ and for i and $i+1 : \ell^i_i \neq \ell^i_{i-1}$ and $\ell^{i+1}_i \neq \ell^{i+1}_{i+1}$, agent i is called the dissident, agent $i - 1$ is the punished agent and the rest are denoted by k, then,

$$g_i(s) = \left(x_i, \ell^i_i\right), \ g_{i-1}(s) = \left(0, \bar{\ell}\right), \ g_k(s) = \left(\frac{f\left(\ell^1_1, ., \ell^{i-2}_{i-2}, \bar{\ell}, \ell^i_i, ., \ell^n_n\right) - x_i}{n-2}, \ell^k_k\right)$$

where $x_i = P_i(\ell') + \frac{\partial f(\ell')}{\partial \ell_i}\left(\ell^i_i - \ell^i_{i-1}\right)$ with $\ell' = \left(\ell^1_1, .., \ell^{i-1}_{i-1}, \ell^i_{i-1}, \ell^{i+1}_{i+1}, .., \ell^n_n\right)$.
And only in the case that from this rule $x_i > f\left(\ell^1_1, ., \ell^{i-2}_{i-2}, \bar{\ell}, \ell^i_i, ., \ell^n_n\right)$ then,

$$g_i(s) = \left(f\left(\ell^1_1, .., \ell^{i-2}_{i-2}, \bar{\ell}, \ell^i_i, ., \ell^n_n\right), \ell^i_i\right), \ g_{i-1}(s) = \left(0, \bar{\ell}\right), \ g_k(s) = \left(0, \ell^k_k\right).$$

[5] Another example of a "Tweed Ring" mechanism is due to M. Walker (1981).

<u>Rule 3</u>:
It applies when we are not in rule 1 nor in 2. Let $M = \{i \in N : s_i = (0,0)\}$, then, for $\forall i \in M : g_i(s) = \left(\frac{f(\ell)}{\#M}, 0\right)$ and for those $j \in N - M : g_j(s) = \left(0, \max\left\{\frac{\bar{\ell}}{\beta}, \min\left\{\beta\ell_j^j, \bar{\ell}\right\}, \min\left\{\beta\ell_j^{j+1}, \bar{\ell}\right\}\right\}\right)$ with $\beta > 1$.

The interpretation of this mechanism is the following:
In **Rule 1** each individual is given the share of the total output according to the sharing rule P.
Rule 2 applies when one individual deviates. Notice that the maximization problem for the dissident agent is

$$\max_{\ell_i^i \in [0,\bar{\ell}]} u_i\left(P_i(\ell') + \frac{\partial f(\ell')}{\partial \ell_i}\left(\ell_i^i - \ell_{i-1}^i\right), \ell_i^i\right)$$

considering interior solutions, the first order condition to this problem is a necessary condition for an efficient allocation:

$$\frac{\partial u_i / \partial x_i}{\partial u_i / \partial \ell_i} = \frac{-1}{\partial f(\ell') / \partial \ell_i}$$

and it is sufficient if this equality is verified for every agent. Thus, if the announcement of the rest of agents does not lead to a Pareto efficient allocation, one individual have incentive to deviate[6]. Also, Rule 2 punishes the individual who did not monitor adequately. The punished individual is given her worst allocation. If the consumption proposed by the dissident is not feasible, we give him all the output.
In **Rule 3**, the mechanism divides the agents into two groups: the ones that consume and do not work, which announce $s_i = (0,0)$ and the ones that work but do not consume, which announce $s_i \neq (0,0)$. As we next show, there is always an agent of this second group with incentive to deviate the mechanism to Rule 2.

Theorem 2. If $n \geq 3$,the mechanism $\Gamma(P)$ implements $\varphi^{PE}(u,P)$ in Nash equilibrium.

Proof : <u>First</u>, let us show that $\varphi^{PE}(u,P) \subseteq g(NE(\Gamma(P), u))\ \forall u \in \mathcal{E}$.

[6] It can be shown that for non interior solutions, rule 2 also gives incentives to deviate from a non optimal allocation.

Let $(x,\ell) \in \varphi^{PE}(u,P)$ for some $u \in \mathcal{E}$ and some P, let $s \in \prod_{i \in N} S_i$ be a strategy profile defined by $s = (s_1, ..., s_n)$ and $s_i = \left(\ell_i^i, \ell_i^{i+1}\right)$ such that $\ell_i^i = \ell_{i-1}^i \; \forall i \in N$. Then, the outcome induced by s is $g(s) = (x,\ell)$. We verify that $s \in NE(\Gamma(P), u)$.
If $(x,\ell) \in \varphi^E(u)$ is an interior solution, then $\forall i$, ℓ_i^i verifies first order optimallity conditions:

$$\frac{\partial u_i / \partial x_i}{\partial u_i / \partial \ell_i} = \frac{-1}{\partial f(\ell') / \partial \ell_i} \; \forall i \in N$$

and so, if one agent deviates, rule 2 applies and the dissident agent can not get anything preferred to (x_i, ℓ_i^i). Even if $(x,\ell) \in \varphi^E(u)$ is a non interior solution, there is no dissident agent which can get anything preferred to (x_i, ℓ_i^i). Therefore, $s \in NE(\Gamma(P), u)$.
<u>Second</u>, let us show that $g(NE(\Gamma(P), u)) \subseteq \varphi^{PE}(u,P) \; \forall u \in \mathcal{E}$.
Let $s \in NE(\Gamma(P), u)$ and $g(s) = (x,\ell)$.
Case 1: when (x,ℓ) is in rule 1 but $(x,\ell) \notin \varphi^E(u)$. Then, $\exists \left(x', \ell'\right) \in \mathcal{X}$ such that at least one agent strictly improves. If this agent deviates in the announcement of the components of the strategy space, rule 2 applies. Let $\ell' = \left(\ell_1^1, .., \ell_{i-1}^{i-1}, \ell_{i-1}^i, \ell_{i+1}^{i+1}, .., \ell_n^n\right)$ then, the attainable set for this agent $\mathbb{A}_i$ is as follows

$$\mathbb{A}_i = \left\{(x_i, \ell_i) \in \mathbb{X} : \; x_i = P_i(\ell') + \tfrac{\partial f(\ell')}{\partial \ell_i}\left(\ell_i - \ell_{i-1}^i\right)\right\}$$

Since (x,ℓ) is not efficient, there is at least one agent for whom the marginal rate of substitution is greater (respectively lower) than the marginal rate of transformation, then by announcing a lower (respectively higher) input contribution, he improves. This contradicts that $s \in NE(\Gamma(P), u)$.
Case 2: (x,ℓ) comes from rule 2. The punished individual $i-1$ receives $g_{i-1}(s) = \left(0, \overline{\ell}\right)$ and then, any deviation in the announcement of $i-1$, will improve her:
- When $\ell_j^j = \ell_{j-1}^j \; \forall j \in N - \{i\}$, then a deviation of agent $i-1$ announcing $\ell_{i-1}'^i = \ell_i^i$ leads to the unanimity rule.
- When $\ell_j^j = \ell_{j-1}^j \; \forall j \in N - \{i, i+1\}$, a deviation of agent $i-1$ in the announcement of ℓ_{i-1}^{i-1} will lead to the third rule.
Case 3: (x,ℓ) comes from rule 3. Therefore, one of the following two cases must occur:
Case 3.1: there are at least three successive disagreements, i.e., at least three successive agents for whom: $\ell_j^j \neq \ell_{j-1}^j$.
Case 3.2: there are at least two non-successive disagreements.

In both cases, for all $j \in N - M$, $s_j = \left(\ell_j^j, \ell_j^{j+1}\right)$ where $\ell_j^j > \frac{\bar{\ell}}{\beta^2}$ or $\ell_j^{j+1} > \frac{\bar{\ell}}{\beta^2}$, can not be a strategy of a Nash equilibrium.
Suppose first that s is a Nash equilibrium satisfying **case 3.1**:

If $n = 3$, we have that there are three disagreements. Therefore, there are at least two agents with payoff $\left(0, \frac{\bar{\ell}}{\beta}\right)$. One of these agents, however, have incentive to move the mechanism to rule 2 by means of announcing $s_j' = \left(\ell_{j-1}^j, \ell_{j+1}^{j+1}\right)$, since then, he obtains the payoff (x_j, ℓ_j) where $x_j \geq 0$ and $\ell_j \leq \frac{\bar{\ell}}{\beta^2}$.Thus, s can not be a Nash equilibrium.

If $n > 3$, suppose first that there are three or more successive agents, that we denote by $\{1, 2, 3, ...\}$, such that $s_j \neq (0, 0)$ which obtain $g_j(s) = \left(0, \frac{\bar{\ell}}{\beta}\right)$. We have then that agent 2 can deviate to $s_2' = (0, 0)$ so that the mechanism does not move from rule 3 (there are at least two non-successive disagreements or three successive disagreements) and agent 2 improves since he obtains the payoff $(x_2, 0)$ where $x_2 > 0$. Therefore, if there is a Nash equilibrium satisfying Case 3.1, it can not consist of more than two successive agents, denoted by $\{1, 2\}$, who announce $s_j \neq (0, 0)$. We then have that, $s_1 = (\ell_1^1, \ell_1^2)$, $s_2 = (\ell_2^2, \ell_2^3)$ and $s_i = (0, 0)$ for $\forall i \in N - \{1, 2\}$ where $\ell_1^1 \neq 0$, $\ell_1^2 \neq \ell_2^2$ and $\ell_2^3 \neq 0$. And so, $g(s) = \left(\left(0, \frac{\bar{\ell}}{\beta}\right), \left(0, \frac{\bar{\ell}}{\beta}\right), (x_3, 0), ..., (x_n, 0)\right)$. However, agent 1 can improve deviating the mechanism to rule 2 by means of announcing $s_1' = (0, \ell_2^2)$, since then he obtains the payoff $(x_1, 0)$ with $x_1 \geq 0$. We therefore conclude that there is no Nash equilibrium which satisfies case 3.1.
Suppose second that s is a Nash equilibrium satisfying **case 3.2**:

Consider first that there are more than two non-successive disagreements. Then, there must be at least three agents such that $s_j \neq (0, 0)$, which obtain a payoff of $\left(0, \frac{\bar{\ell}}{\beta}\right)$. Clearly, one of these agents have incentive to deviate to $s_j' = (0, 0)$ since rule 3 also applies, but then, he obtains a payoff of $(x_j, 0)$ where $x_j > 0$. Therefore, if there is a Nash equilibrium satisfying case 3.2, it can not consist of more than two non-successive agents announcing $s_j \neq (0, 0)$. Thus, it shall consist of two non-successive agents $\{j, k\}$ announcing $s_j \neq (0, 0)$, $s_k \neq (0, 0)$ and for $\forall i \in N - \{j, k\}$, $s_i = (0, 0)$. However, it can not be a Nash equilibrium since agent j improves moving the mechanism to rule 2 by means of announcing $s_j' = (0, 0)$. He then obtains the payoff $(x_j, 0)$ where $x_j \geq 0$.

We therefore conclude that there is no Nash equilibrium in rule 3. **Q.E.D.**

Remark 3. *The proposed mechanism satisfies some nice properties. Indeed, we propose a natural message space where the announcements are the amounts of* *input contributions. Furthermore, participants contribute, in equilibrium, with the amount they announce (a similar property is called Forthrightness by Saijo, Tatamitani and Yamato (1996)). Finally, the proposed mechanism is bounded (see Jackson (1992)), as we have already shown, there is always a best reply for each agent.*

Remark 4. *Our result is related to that of Shin and Suh (1997). They succeed in showing a class of interior and efficient solutions which are double implementable in Nash and Strong Nash equilibria. Although we do not show doubly implementation, the class of efficient solutions that we implement, also includes efficient solutions for economies with heterogeneous inputs.*

It is also interesting, that our class of efficient solutions, are characterized by a novel property (continuity of the sharing rule), since it is not only an easy-checking property but also a meaningful property (in the sense that it is contractible). We consider that it establish some links between Implementation Theory and Contract Theory within the commons problem, with worthy implications for future research.

3. Final Comments

In this note we have shown that any continuous sharing rule is compatible with efficiency and incentives. In this sense, our results suggest the existence of a large degree of freedom concerning income distribution within the firm, unless other consideration are introduced. See Corchón and Puy (1998) for a study of sharing rules yielding individually rational allocations and sharing rules that arise from voting inside the firm.

4. References

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