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Semiflujos discretos exteriores

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Contents

Introduction

A background on semi-flows and previous contributions from other authors

In the first part of this introduction, we deal with other authors' contributions made previously on discrete semi-flows and their connections with other fields: dynamical systems, differential equations, fractals, iterative processes and exterior spaces. This will help us establish the current framework in which our thesis takes place.

Continuous and discrete semi-flows and differential equations

Some of the origins of dynamical systems and flow theory can be traced back to Poincaré's pioneer work [70, 71] on the topological properties of the solutions of autonomous differential equations, in the late 19th century. It should also be noted A. M. Lyapunov's work [56], in which he developed his theory of stability of solutions of a system of first-order ordinary differential equations. While most of the research done by Poincaré analyzed global properties of dynamical systems, that of Lyapunov was more focused on the consideration of their local stability. The theories associated with dynamical systems reached a high level of evolution due to the studies of G. D. Birkhoff [9], who could be considered as one of the founders of this theory.

The development of dynamical systems has been quite ample, has been tackled from many points of view and has had several applications in diverse fields. A study of their basic properties can be seen in [9, 8].

One of the methods that Poincaré introduced for the study of properties of autonomous differential equations was the first return map: if we consider a periodic solution (a compact closed curve) of a differential equation in $\mathbb{R}^2$ and x_0 is a point in it, we can take a perpendicular line T in x_0 and, given a point $x \in T$, consider the unique solution with initial point x and look for the first return $f(x)$ of the trajectory to the perpendicular. In this way, a discretization process is obtained, which associates the continuous flow with the discrete flow determined by the first return map at the perpendicular. The technique of Poincaré's return and other discretization methods induce the corresponding discrete flows, and some inverse methods such as suspension can construct an associated continuous dynamical system from a discrete one.

Let us notice that there are many other discretization methods that are not as complicated as Poincaré's; without going any further, a discrete flow can be associated with each continuous flow when a discrete time scale is chosen. This process is usually carried out as a first step toward making these models suitable for numerical evaluation and implementation on digital computers.

For an autonomous Lipschitzian differential equation system, one always has an induced local or global flow. As this sort of differential equations, the autonomous ones induced by continuous vector fields usually appear in numerous scientific contexts, too; nevertheless, given an initial condition, there are existence theorems for these equations but, in general, the solution uniqueness may not hold. For continuous vector fields, the solutions of an equation do not have, in general, the structure of a continuous flow; an interesting particular case arises when, for some initial condition, we can only assure the existence and uniqueness of solutions at a future time, obtaining in this case a continuous semi-flow. For a continuous semi-flow, we can consider discretization processes and the corresponding discrete semi-flows, which are precisely the cornerstone of our doctoral thesis.

Fractals and flows induced by iterative processes

A fractal is a geometric shape whose basic structure, either fragmented or rough, is repeated at different scales. This expression was first proposed by B. B. Mandelbrot [59] in 1975 and it comes from Latin *fractus*, which means to be "broken", "fractured", or "irregular". A lot of natural structures are of fractal type. Although the term "fractal" is recent, the objects nowadays named fractals had been well-known since the early 20th century.

A possible definition of fractal, given in [61], is the following: "A fractal is a set whose Hausdorff-Besicovitch dimension is strictly greater than its topological dimension." A more recent definition was given by K. Falconer [31] in 1990: "A fractal structure is one that satisfies some of the following properties: it cannot be described in traditional Euclidean geometric language, both locally and globally; it has a fine or detailed structure at arbitrarily small scales; it is self-similar; its fractal dimension is greater than its topological dimension; and it has a simple and perhaps recursive definition."

To find the first cases of fractals, we have to go back to the late 19th century when, in 1872, Weierstrass function, whose graph is considered as a fractal now, was first introduced as an example of a continuous function which is not differentiable anywhere. Later, some more examples with similar properties appeared, but they had a more geometric definition. These examples of fractals could be made from an initial pattern to which a series of simple geometric constructions were applied. The family of figures obtained approached a limit figure that corresponded to what today we call fractal set. In this way, in 1904, H. V. Koch defined a curve with similar properties to that of Weierstrass: the Koch snowflake. In 1915, W. Sierpinski constructed the Sierpinski triangle and, one year later, the Sierpinski carpet.

By 1918, two French mathematicians, P. J. L. Fatou [32] and G. M. Julia [52], though working independently, arrived essentially simultaneously at results describing what are now seen as fractal behavior associated with mapping complex numbers and iterative functions and leading to further ideas about attractors and repellers (i.e., points that attract or repel other points), which have become very important in the study of fractals. They were among the first mathematicians who studied the complex dynamics generated by iterative processes. Their works were popularized later on by B. B. Mandelbrot [60, 61] who, combining his geometric vision and his programming abilities, gave an essential boost for the creation of a new mathematical field, sometimes named fractal geometry or fractal dynamics. Some works regarding research in the fields related to the complex dynamics, the chaos theory, the fractal notion and the connection between Julia and Mandelbrot sets can be found in [69, 6, 16, 1]. In several of these works, one can usually observe numerous illustrations that reflect the beauty of the fractals created from the referred iterative processes.

The Julia sets appear as a result of the iteration of analytic functions $z \mapsto f(z) \mapsto f(f(z)) \mapsto f(f(f(z))) \mapsto \dots$ In the particular case of polynomial functions f of degree greater than 1, it is very possible that, when iterating them, the result tends to infinity. The set of values $z \in \mathbb{C}$ that do not escape to infinity by means of this procedure is called filled-in Julia set, and its boundary $J(f)$ is just called Julia set. The family of Julia sets $\{J(f_c)\}$ associated with the iteration of functions of the form $f_c(z) = z^2 + c$ has a surprising variety of sets; the set of complex numbers $c \in \mathbb{C}$ such that the Julia set associated with $f_c(z) = z^2 + c$ is connected is called Mandelbrot set.

Another kind of fractal related to some iterative process is the Newton fractal. Newton's method transforms a polynomial $p(z)$ into a rational (meromorphic) function $N_p(z) = z - \dfrac{p(z)}{p'(z)}$. The iteration of N_p induces the basins of attraction of the roots of $p(z) = 0$, and the boundary of those basins has a fractal structure.

One of the objectives of this doctoral thesis is to study the iteration of a continuous map f in a topological space; in other words, a discrete semi-flow. An interesting particular case is the iteration of an analytic function defined on a Riemann surface: when considering this case, the semi-flow theory has many relations to fractal geometry.

Exterior space theory

Over the 20th century, a large number of techniques were developed by many mathematical research groups with the aim of studying non-compact spaces; now, we shall mention some of those related to the lines of research carried out in this area.

In 1923, B. Kerékjártó [53] found a classification of non-compact surfaces. In order to do so, he defined an "ideal point" as a principal invariant. After that, in 1931, H. Freudenthal [33] defined the notion of "end point" of a topological space, which in the case of surfaces coincides with the concept of Kerékjártó's "ideal point". These were the first invariants of what would be named proper homotopy theory afterwards, and one of its first applications in the classification of open manifolds.

In 1965, L. C. Siebenmann [78] gave necessary and sufficient conditions in order for a differentiable n-manifold to be the interior of a compact manifold with boundary, when $n \geq 6$. In the referred work, he associated new invariants with the semi-stable ends of Freudenthal by using the fundamental group and inverse limits of groups. Later on, in his work [79] in 1970, Siebenmann suggested that, when considering non-compact spaces, the homotopy hypothesis should be given in the category of proper maps (i.e., continuous functions with inverse images of compact closed subsets being compact) rather than in the category of continuous functions. This proposal was subsequently considered by numerous research groups and several works with significant contributions in this field were developed until the late twentieth century –for a general view about proper homotopy, see [25, 73]. However, one of the problems concerning the proper category is that it lacks enough limits and colimits in order to make the most usual homotopy theory constructions.

A solution to this problem has been the introduction of the category of exterior spaces [34, 35, 27] and the study of its properties, carried out over the last years of the 20th century and until nowadays in the 21st century. The notion of exterior space was made with the aim of having a topological-type model at one's disposal to study non-compact spaces and the shape of compact T_2 spaces. An exterior space consists of a topological space provided with a "system of exterior open subsets" (that is to say, a non-empty quasi-filter of open subsets) which play the role of a family of open neighborhoods at infinity when considering the proper homotopy.

More specifically, the concept of exterior space was introduced in the work [34], where a full and faithful natural transformation from the category $\mathbf{P}$ of topological spaces and proper maps to the category of exterior spaces $\mathbf{E}$ was given. In addition, it was proved that the exterior category $\mathbf{E}$ admits a Quillen model structure [75]. This technique has permitted the use of various homotopic constructions that cannot be made in proper homotopy since, as mentioned above, the proper category does not have enough limits and colimits. The study of the homotopy theory of exterior spaces has been extensively developed in several works, such as [36], [27], [15] and [24].

As we will see in the next subsection of this introduction, the exterior space theory can be applied to the study of dynamical systems. It has been observed that a dynamical system (either flow or semi-flow, continuous or discrete) can be provided with diverse exterior space structures, so that all the constructions and properties developed for exterior spaces have a natural application to the classification and the study of properties of dynamical systems. The introduction of the exterior dynamical systems [28, 38, 39] and their development and applications is being one of the ongoing research lines on which our research team is working at the moment.

Exterior flows

The theory of Differential Equations and the Invariant theory of Algebraic Topology are two important mathematical techniques whose applications to scientific-technical fields have been a constant source of progress and innovation. The research work on exterior spaces and exterior flows has unified some aspects regarding these methods and has designed computational tools which permit a more effective transfer of these advances to science and technology.

The potential of these new techniques has recently been analyzed, together with the use of exterior spaces to study dynamical systems: an original procedure, consisting of providing flows and semi-flows with an additional structure of exterior space, has been developed, originating a hybrid theory of exterior dynamical systems. In this way, the study of dynamical systems can be carried out through the consideration of exterior spaces. There have been some recent works on the study of continuous flows through the use of the theory of exterior spaces, such as [28], [38], [39] and [40]. For more advances achieved in these fields, see [29].

Author's contributions and structure of the thesis

The second part of this introduction is devoted to describe the advances evolved in the works in which the author considered some of the topics developed in this thesis during his time as a doctoral student in association with his supervisors and some of their close research collaborators. Examples of these topics include the analysis of exterior discrete flows and the applications of flow theory to the field of Numerical Analysis. All these aspects are broadened and covered in depth along the different chapters of this dissertation, whose structure will be shown at the end.

Exterior discrete flows

Bearing in mind the results on the connection between exterior spaces and continuous dynamical systems pointed out at the end of the previous section, it has been contemplated to find out the connections between exterior spaces and dynamics of discrete flows. It is known that some discretization (such as the first return map) and anti-discretization (such as suspension) processes determine an interdependence between the properties of discrete dynamical systems and those of continuous ones. In consequence, many of the properties, results and applications that figure in [38, 39] should be matched with notions and results concerning discrete dynamical systems. Precisely, some of the connections between exterior spaces and discrete semi-flows are analyzed in this thesis, and the notion of exterior discrete semi-flow emerges as a result of them. The obtained results, and some additional ones, led to a joint work with J. M. García-Calcines [37].

Some differences between continuous flows and discrete semi-flows are essential for their technical handling. For example, a semi-flow consists in a semigroup of continuous maps instead of a group of homeomorphisms, so the construction and properties of left and right omega-limits are very different. A continuous flow has the property that all the points in a trajectory belong to the same connected component; nevertheless, this cannot be ensured for discrete semi-flows. These differences must be taken into account when analyzing the interrelations between the exterior space theory and the discrete semi-flow theory. As a matter of fact, we also have in [37] some similarities with the results and tools given in [38], but new (non-analogous) techniques had to be developed for a better analysis of discrete semi-flows.

With the aim of studying exterior discrete semi-flows, some concepts such as region of exterior attraction of an externology, limit and bar-limit of an externology and different notions of end points, which also appear in [37], will be analyzed in depth throughout this dissertation; at least, one can consider three different kinds of end points given by the analogues of the 0-dimensional homotopy invariants of the Borsuk-Čech $\check{\pi}_0$, Steenrod π_0^{S} and Brown-Grossman π_0^{BG} type.

What is more, in discrete dynamical systems, it is common to consider the notion of region of attraction of a right-invariant subset, the omega-limit of a point, periodic points, basins of n-cycles, et cetera; here, in determined parts of this work, we will go into detail about the analysis and use of certain techniques that provide a connection between the notions associated with an exterior space and dynamical notions associated with a discrete semi-flow, and we will see that the regions of exterior attraction of an externology are related to the regions of attraction of an adequate invariant subset, the concept of limit is related to that of subset of periodic points, the notion of bar-limit is connected to the notion of omega-limit, the basin of an end point of Borsuk-Čech type is related to the basin of a fixed point and the basin of an end point of Brown-Grossman type is related to the basin of a periodic point and the basin of an n-cycle.

Some applications of flow theory to iterative methods

One of the utilities of the exterior discrete semi-flows lies in their application to the dynamics of numerical iterative methods, such as those of Newton-Raphson or Tchebychev, when finding roots of complex polynomials. Some examples of works that deal with this topic can be found in [30, 82], and a general study on the iteration of rational functions of complex variable can be seen in [7].

In order to study basins of end points related to the dynamics of certain iterative processes and as part of the results achieved in this doctoral thesis, diverse algorithms were implemented in *Sage* [48] and *Mathematica* [62] by the author and his supervisors to visualize basins of attraction of end points of a discrete semi-flow associated with a rational function defined on the Riemann sphere by using its geometry and complex structure. These algorithms give a global view of the basins, in contrast to other usual implementations which provide partial visualizations in rectangles. For each integer $p \geq 1$, with these implementations one obtains a decomposition of the sphere into the union of basins of p-cyclic points and their complementary, and, moreover, the areas of these basins can be calculated. The visualization and area-computation algorithms are based on procedures of consecutive subdivisions starting from a cubic structure on the 2-sphere. The research presented in the mentioned publications is reflected along the last chapters of this document, in which we show, in addition, a rigorous study that allows us to demonstrate that the method we use for estimating the area of the basins by such iterated subdivisions is appropriate.

Some applications of the referred algorithms for the graphical visualization and area measure of basins are exhibited in chapter 7 of this thesis in the analysis of the influence of the multiplicity of complex polynomial roots on the area of the corresponding basins obtained by means of Newton's method. This work includes a comparison between two subdivision methods considered on the Riemann sphere to see which one is better to obtain more efficient computations, and has resulted in a joint publication [44] together with the author's supervisors and J. M. Gutiérrez.

Structure of the document

The aim of this doctoral thesis is to profoundly study the notion of exterior discrete semi-flow and to apply it to the analysis of iterative processes induced by some numerical methods. It is organized as follows.

Chapter 0 is the preliminary chapter and it is devoted to a review of some fundamental concepts on which this work is based, such as topological spaces, categories, exterior spaces, discrete semi-flows, measures and regular CW-complexes. In turn, we set the notations that will appear in it and introduce the terminology that will be used.

The three chapters following chapter 0 are chiefly inspired by the theorem that we are going to recall in the next lines, since along them we intend as a main goal to establish connections among three of the most usual end sets of an exterior space and their analogues in the category of exterior discrete semi-flows: those of Brown-Grossman $\pi_0^{\mathrm{BG}}(X)$, Steenrod $\pi_0^{\mathrm{S}}(X)$ and Borsuk-Čech $\check{\pi}_0(X)$. That theorem, which can be found in [49] (oriented to pro-spaces) and [41] (in the context of exterior spaces), claims that, if X is an exterior space, then there is an exact sequence that provides a nice connection between the Brown-Grossman and Steenrod homotopy groups given by the first row of the following diagram:

$$\begin{tikzcd}
\pi_q^{\mathrm{S}}(X) \arrow[rr] \arrow[dr, two heads] & & \pi_q^{\mathrm{BG}}(X) \arrow[r, "\mathrm{Id}_X - \mathrm{Sh}_X"] & \pi_q^{\mathrm{BG}}(X) \arrow[r] & \pi_{q-1}^{\mathrm{S}}(X) \arrow[r] & \dots \\
& \ker(\mathrm{Id}_X - \mathrm{Sh}_X) \cong \check{\pi}_q(X) \arrow[ur, tail] & & & &
\end{tikzcd}$$

This long exact sequence is actually an analogue for exterior spaces of the exact sequence given by Quigley [74] in shape theory or by Porter [72] in proper homotopy theory, and in the context of pro-spaces it is also considered or used by Hernández-Paricio in [47] and by him and Porter in [50, 51]. Furthermore, under the condition of first countable at infinity, the kernel of the map $\mathrm{Id}_X - \mathrm{Sh}_X$ from the q-th Brown-Grossman group to itself is just the q-th Borsuk-Čech group, with $q \geq 1$.

The problem arises precisely when $q = 0$, since these sets have not got a group structure in that case and, therefore, defining the map $\mathrm{Id}_X - \mathrm{Sh}_X$ is senseless as, given $a \in \pi_0^{\mathrm{BG}}(X)$, the element $\mathrm{Sh}_X(a) \in \pi_0^{\mathrm{BG}}(X)$ would not have an inverse $-\mathrm{Sh}_X(a) \in \pi_0^{\mathrm{BG}}(X)$. Nevertheless, the kernel of the map $\mathrm{Id}_X - \mathrm{Sh}_X$ is the same as the equalizer of Id_X and Sh_X if $q \geq 1$ and this equalizer makes sense for the sets concerned when $q = 0$, "fixing" our problem.

Considering the foregoing, chapter 1 shows the connections among the end sets $\pi_0^{\mathrm{BG}}(X)$, $\pi_0^{\mathrm{S}}(X)$ and $\check{\pi}_0(X)$ when X is a first-countable at infinity exterior space. Not only do we intend to complete the study in depth of the diagram above in the lowest dimension $q = 0$, but we also try to relate some of these end sets to limit spaces of the exterior space, and these limit spaces and end sets amongst each other. The strategy that we follow to do so lies in the definition of natural transformations between these functorial constructions.

The main result of chapter 1 that provides a detailed description of the above-mentioned exact sequence in dimension zero is Theorem 1.5.1, which also states that, if the exterior space X is first-countable at infinity, then the set of end points $\check{\pi}_0(X)$ is the equalizer of the identity map and the shift operator of $\pi_0^{\mathrm{BG}}(X)$. The principal difficulty is to prove the existence of the natural transformation from $\check{\pi}_0$ to π_0^{BG}, assuming the exterior space to be first-countable at infinity.

The central concept of this work, which is that of exterior discrete semi-flow, is thoroughly considered along chapter 2. An exterior discrete semi-flow can be seen as a discrete semi-flow whose action is exterior. In addition, it is **d**-exterior if the orbit of each point in the exterior space reaches an end point; that is to say, it is eventually contained in an exterior open subset, for a large enough time. To construct end sets of an exterior discrete semi-flow X, it is desirable to define a **d**-exterior subspace, which is called the region of exterior attraction of the exterior discrete semi-flow and denoted by $D(X)$. In Theorem 2.2.1, we show that the functor which carries a discrete exterior semi-flow X to a discrete **d**-exterior semi-flow $D(X)$ is a right adjoint. The existence of an action of $\mathbb{N}$ on X will allow us to consider end points that can be represented by the orbit of a point in $D(X)$. This fact makes it possible for us to introduce the notion of ω-end point and to define new sets of ω-end points ${}^{\omega}\pi_0^{\mathrm{BG}}(X)$, ${}^{\omega}\pi_0^{\mathrm{S}}(X)$ and ${}^{\omega}\check{\pi}_0(X)$ in a natural way as subfunctors of $\pi_0^{\mathrm{BG}}(X)$, $\pi_0^{\mathrm{S}}(X)$ and $\check{\pi}_0(X)$, respectively. We also prove in Theorem 2.3.1 an analogue of Theorem 1.5.1 for ω-end points. Another important fact is the partition of the region of exterior attraction $D(X)$ into a disjoint union of basins of end points of ${}^{\omega}\pi_0^{\mathrm{BG}}(X)$ –see Corollary 2.3.1. A particular case of this kind of partitions will be considered in the last chapters for discrete semi-flows induced on the 2-sphere by the iteration of a rational map, so long as it is different from the identity.

In chapter 3, we take a step further and create a new kind of Brown-Grossman and Steenrod end sets of exterior discrete semi-flows, called Ω-end sets, based on the previous ones but such that the paths that define the homotopy between exterior sequences or exterior semi-rays, as appropriate, are constructed successively as images of the very first path, at time zero, of that homotopy under the action of the discrete semi-flow. In order to relate these end sets to that corresponding to Borsuk-Čech type, it is necessary to define the notion of intrinsic path, which is a path whose image, for a big enough time, is eventually contained in every exterior open subset. Intrinsic paths are associated with the intrinsic topology, generated by the union of the open subsets and the right-invariant subsets of the exterior open subsets taken as a subbasis: by way of example, a path is intrinsic if and only if it is continuous in the intrinsic topology and its image is completely contained in the region of exterior attraction. From this comes naturally the notion of intrinsic path component: two points in the region of exterior attraction belong to the same intrinsic path component if and only if there is an intrinsic path connecting them. This notion permits the construction of Ω-end sets of Borsuk-Čech type from inverse limits of intrinsic path components –instead from inverse limits of simple path components, as initially. The new functorial Ω-end points can be related to each other using natural transformations, following again the same sketch above.

Within chapter 3, we prove in Proposition 3.2.2 that, for a given exterior discrete semi-flow X, one has that there is a bijection between ${}^{\Omega}\pi_0^{\mathrm{BG}}(X)$ and ${}^{\Omega}\pi_0^{\mathrm{BG}}(D)$ and, moreover, ${}^{\Omega}\pi_0^{\mathrm{BG}}(D)$ is isomorphic to the set of intrinsic path components $\pi_0^{\mathrm{int}}(D)$ and to the set of ω-representable end points ${}^{\omega}\pi_0^{\mathrm{BG}}(D^{\mathrm{int}})$, where $D = D(X)$ is provided with the intrinsic topology. The functor ${}^{\Omega}\check{\pi}_0$ has similar properties, since ${}^{\Omega}\check{\pi}_0(X) \cong {}^{\Omega}\check{\pi}_0(D) \cong {}^{\omega}\check{\pi}_0(D^{\mathrm{int}})$. However, the new functor ${}^{\Omega}\pi_0^{\mathrm{S}}$ does not satisfy, in general, this kind of properties. Again, for the new functors ${}^{\Omega}\pi_0^{\mathrm{BG}}$, ${}^{\Omega}\pi_0^{\mathrm{S}}$ and ${}^{\Omega}\check{\pi}_0$, we also have the exact sequence given in Theorem 3.2.1 analogous to Theorem 1.5.1 and Theorem 2.3.1.

With a view to make a study about basins of attraction of fixed and, in general, m-periodic points associated with a rational map defined on the surface of the sphere S^2, we consider in chapter 4 externologies given by families of the open subsets that contain determined right-invariant subsets. In section 4.1, under certain nice conditions, we compare pairs of externologies of the described type when they each are induced by right-invariant subsets such that one is contained in the other; the main result of the section, which is none other than Theorem 4.1.1, claims that, given a pair of such subsets and an ω-representable end point when considering the externology induced by the smaller subset, the basin of this end point in the region of exterior attraction induced by the larger subset is just the same as the basin in the region of exterior attraction induced by the smaller one. In section 4.2, the concept of asymptotically stable cycle comes into play (for a rough definition of this notion, see the opening paragraph of chapter 4), and there we prove several interesting results when the appropriate conditions exist, namely:

- Every ω-representable end point can be given by an asymptotically stable periodic point –see Corollary 4.2.2.

- The region of exterior attraction of an asymptotically stable l-cycle of a continuous map f can be decomposed into a disjoint union of regions of exterior attraction of asymptotically stable fixed points of f^l –see Proposition 4.2.3.

- The regions of exterior attraction of the sets of asymptotically stable fixed points, n-cyclic points and m-periodic points of a continuous map f can be decomposed into the disjoint unions of the regions of exterior attraction of the corresponding asymptotically stable fixed points, n-cyclic points and m-periodic points belonging to them. Furthermore, the region of exterior attraction of the set of asymptotically stable m-periodic points of f can be decomposed into the disjoint union of the regions of exterior attraction of the sets of asymptotically stable n-cyclic points such that n divides m –see Theorem 4.2.2.

- Consequently, under the same conditions, if m and n are coprime integers different from one another such that the sets of asymptotically stable m-periodic and n-periodic points are finite, then the intersection of their regions of exterior attraction equals the region of exterior attraction of the set of asymptotically stable fixed points –see Corollary 4.2.3.

In the last section, we apply the results seen to the particular case in which the subjacent space is the sphere S^2 and the action of the exterior discrete semi-flow is induced by a continuous map such that the set of m-periodic points is finite, for all natural m. Special relevance has Theorem 4.3.1(ii), which claims that the basin of an ω-representable end point associated with the externology induced by the set of m-periodic points and the basin of the same end point seen as related to the externology induced by the countable set of all periodic points are exactly the same; this result is based on Lemma 4.3.2.

Ultimately, along chapter 4 we are actually connecting some of the novel working purely topological techniques introduced and developed in the first chapters to several notions regarding discrete semi-flows and dynamical systems.

For researchers devoted to Numerical Analysis, it is important to know as many aspects as possible concerning basins of attraction associated with fixed or periodic points of a rational function different from the identity, since these functions appear naturally when the most common numerical iterative methods (such as Newton-Raphson, Tchebychev, etcetera) are used in order to find polynomial roots. The knowledge of the size and shape of these basins allows them to more easily find initial iteration points that converge quickly to one of the polynomial roots. For all of these reasons, given a semi-flow structure on the Riemann sphere induced by the iteration of a rational map, we are interested in visualizing basins of attraction of end points and periodic sets, as well as in calculating, up to a certain precision, the measure of such basins.

In this context, chapter 5 serves as a bridge between the topological and the computational part of this thesis, since the results proved so far help establish a theoretical framework in which the algorithms that have been designed and implemented for these purposes make sense.

The representation of basins of end points requires of the measurement of distances between points to check whether an orbit has converged or not; owing to this, we focus along section 5.1 on the development of the study of discrete semi-flows on metric spaces. In this study, we contribute a brand-new notion of end point, defined by the authors in [48, 44], which depends on the metrics of the subjacent space. Furthermore, we analyze the connection between the end points associated with the metrics and those which are related to the externology; in that sense, Theorem 5.1.2 shows that the basins of both types of end points can be related and coincide.

Besides, in order to establish a theory-based method to quantify the areas of the different basins of end points, we need a way to create a new procedure to introduce measures which allow us to estimate in a computational manner and compare the sizes of those basins. To that end, following the techniques devised in [3], we use in section 5.2 the properties of a regular CW-complex and its subdivisions to construct a subdivision algebra and a subdivision pre-measure and then, using the extension theorem developed by Carathéodory in [13], we extend this pre-measure to a cellular-extension measure defined in a cellular-extension σ-algebra. In particular, this method allows us to introduce measures on S^1 (angles) and on S^2 (solid angles) that make it possible to construct algorithms for estimating the size of basins on the surface on the 2-sphere, as we will see in chapter 7.

Relying on the theorems shown in the previous chapters and on the geometry and the complex structure of the Riemann sphere, chapter 6 aims to present a new program written in *Sage* and *Mathematica* which allows us to visualize the basins of attraction associated with the end points of a discrete semi-flow induced by a rational function different from the identity defined on the surface of the sphere S^2. Not only can this program draw these basins, just assigning a color to the points on the sphere depending on the fixed points to which the corresponding iteration sequences converge, but it is also able to use other different strategies of graphic representation based on the total number of iterations until convergence, up to a given tolerance.

Unlike earlier programs created by other authors for similar purposes, one interesting novelty brought by the developed program is that it is able to plot fractals not only in a determined rectangular area of the complex plane, but also on the whole surface of the Riemann sphere. Another advantage of our program is that it permits to visualize not only basins of attraction of fixed points, but also basins of end points associated with periodic points.

The chapter is divided into three sections: section 6.1 explains the theoretical underpinnings of why our algorithms are well designed, section 6.2 shows the source codes of the algorithms in the referred programming languages and section 6.3 includes a user manual of the program.

In chapter 7, we develop and implement an algorithm for computing the measure of the basins of attraction that can be plotted by means of the program described along chapter 6. These algorithms are based on the subdivisions of cubic decompositions of a sphere and they have been made by using the computational environments considered in this work, *Sage* and *Mathematica*, as well as two different subdivision methods: the one described in section 5.2, based on the iterated subdivision of the projection of the boundary of the 3-cube onto the surface of the sphere, and another one consisting of the projection of the iterated subdivisions of the cube –a comparison between this pair of processes is made. The theoretical aspects backing up rigorously the measure constructions on the surface of the sphere from these subdivision methods are considered along chapter 5, specifically in subsections 5.2.1 and 5.2.2, as well as in section 7.1.

As an application, we study the basins of attraction of the fixed points of the rational functions obtained when Newton's method is applied to a polynomial with two roots of multiplicities m and n. We focus our attention on the analysis of the influence of the multiplicities m and n on the measure of the two induced basins of attraction. As a consequence of the numerical results given in this chapter, we conclude that, if $m > n$, the probability that a point in the Riemann sphere belongs to the basin of the root with multiplicity m is bigger than the other case. In addition, if n is fixed and m tends to infinity, the probability of reaching the root with multiplicity n tends to zero.

Finally, we end this thesis with a conclusion paragraph that includes the main ideas described along this work, the techniques developed and possible further work, plus the bibliographical references.

Chapter 0

Preliminaries

This initial chapter is devoted to recalling some basic notions regarding the topics that we are considering throughout this work, introducing and fixing in the meantime some relevant notation. First of all, in sections 0.1 and 0.2, we shall remind some concepts about topology and categories. After that, in the subsequent sections, we will remember certain already known aspects about exterior spaces and externologies; in section 0.4, some interesting end sets and some limit spaces of exterior spaces will be revisited, so that we can study the connections among them (which will be done in chapter 1) and use them as a tool afterwards. Section 0.5 is reserved for discrete semi-flows. Finally, in the last sections of this chapter, we will bring to mind some basic definitions about measure theory, infinite sums and regular CW-complexes, as well as Carathéodory's extension theorem, in order to establish a theoretically based method to measure the area of the spherical quadrilaterals given by a determined iterated subdivision of the surface of the sphere; these notions and results will be widely used along section 5.2.

0.1 Topological spaces

Let X be a set. A *topology* (or *topological structure*) in X is a family $\mathbf{t}_X$ of subsets of X which is closed by union and finite intersection. A couple $(X, \mathbf{t}_X)$ consisting of a set X and a topology $\mathbf{t}_X$ in X is called a *topological space*.

When it is not necessary to specify $\mathbf{t}_X$ directly, we simply say, "X is a space" (to distinguish from "X is a set"). Elements of topological spaces are called *points*. The members of $\mathbf{t}_X$ are called the *open sets* of the topological space $(X, \mathbf{t}_X)$ (or of the topology $\mathbf{t}_X$). Given a space $(X, \mathbf{t}_X)$, by a *neighborhood* of a point $x \in X$ is meant a subset N containing x such that there exists an open set U (that is, member of $\mathbf{t}_X$) satisfying $x \in U \subset N$.

There are a lot of well-known examples of topologies, some of which will be referred to later. For instance, if $\mathbb{R}$ is the set of real numbers, a subset $G \subset \mathbb{R}$ is said to be "open" if, for each $x \in G$, $\exists r > 0$ such that the interval $B(x; r) = \{y \in \mathbb{R} \mid |y - x| < r\} \subset G$; the family $\mathbf{t}_{\mathbb{R}}$ of sets declared "open" by this criterion is actually a topology in the set $\mathbb{R}$, called the *usual topology* (or *Euclidean topology*) of $\mathbb{R}$. The topological space given by the pair $(\mathbb{R}, \mathbf{t}_{\mathbb{R}})$ is called the Euclidean 1-space.

This example can be generalized to any dimension: let $\mathbb{R}^n$ be the set of all ordered n-tuples of real numbers and call "ball of center $\mathbf{x}$ and radius r" the set $B(\mathbf{x}; r) = \{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{y} - \mathbf{x}\| < r\}$, where $\|\mathbf{x}\|$ represents the Euclidean norm of the vector $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$; the *usual topology* (or *Euclidean topology*) in $\mathbb{R}^n$ is determined by calling $G \subset \mathbb{R}^n$ "open" if, for each $\mathbf{x} \in G$, there is some $r > 0$ such that $B(\mathbf{x}; r) \subset G$. With this topology, $\mathbb{R}^n$ is called the Euclidean n-space. Another example is the *discrete topology*, in which every set is an open set; that is to say, $\mathbf{t}_X$ is equal to the power set 2^X (which is the set of all subsets of X, including the empty set and X itself).

Given a topological space $(X, \mathbf{t}_X)$ and a subset S of X, the *subspace topology* (or *relative topology*) on S is defined by the family of sets $\{S \cap U\}_{U \in \mathbf{t}_X}$. If S is equipped with the subspace topology, then it is a topological space in its own right, and is called a *subspace* of $(X, \mathbf{t}_X)$.

A particular topology could have perhaps too many open sets to cope. The task of specifying a topology is simplified by giving only enough open sets to generate all the open sets. Given a topological space $(X, \mathbf{t}_X)$, a family $\mathcal{B} \subset \mathbf{t}_X$ is called a *basis* for $\mathbf{t}_X$ if each open set (that is, a member of $\mathbf{t}_X$) is the union of members of $\mathcal{B}$. $\mathcal{B}$ is also called a "basis for the space X", and its members the "basic open sets of the topology $\mathbf{t}_X$". At the same time, a *subbasis* of $\mathbf{t}_X$ is usually defined as a subcollection ς of $\mathbf{t}_X$ satisfying that the collection of open sets consisting of all finite intersections of elements of ς forms a basis for $\mathbf{t}_X$. Let $\mathcal{V}_p$ be the family of neighborhoods of a point $p \in X$. A subset $\mathfrak{B}_p \subset \mathcal{V}_p$ is called a *neighborhood basis* of p if for every $V \in \mathcal{V}_p$ there is $B \in \mathfrak{B}_p$ with $B \subset V$. One says that a topological space is *first-countable* if it satisfies the first axiom of countability, that is to say, if each point has a countable neighborhood basis.

A topological space X may satisfy some nice properties. Regarding separability, X is said to be T_2 (or *Hausdorff*) if any two distinct points of it can be separated by disjoint neighborhoods. In relation to connectedness, X is a *connected* space if it cannot be represented as a union of two or more disjoint non-empty open subsets, and it is said to be *path-connected* if any two points of it can be joined by a path. In the case that each point of X has a neighborhood basis of (path-)connected sets, then it is a *locally (path-)connected* space. Finally, X is said to be *compact* if each of its open covers has a finite subcover, and X is *locally compact* if every point of X has a neighborhood basis of compact neighborhoods.

Given a topological space X with a topology $\mathbf{t}_X$ and a subset $A \subset X$, the closure of A in X will be denoted by $\overline{A}$ and the interior of A in X will be denoted by $\mathrm{Int}(A)$.

Most of what has been said in this section on general topology can be enriched by topology books such as [17], [84] or [23] if it becomes necessary.

0.2 Categories

0.2.1 Categories, functors and natural transformations

A *category* $\mathscr{C}$ consists of three things: a collection of *objects* (often denoted by capital letters $A, B, C, \dots$), denoted by $|\mathscr{C}|$; for every pair A, B of objects, a set $\hom_{\mathscr{C}}(A, B)$ (or $\hom(A, B)$, or $\mathscr{C}(A, B)$), whose elements $f, g, \dots$ are called *morphisms* (or *maps*, or *arrows*) from A to B; and a binary operation $\circ$ defined on compatible pairs of morphisms f, g called *composition*.

The morphisms must obey the following laws:

(1) Composition of morphisms yields a morphism: given the morphisms $f \in \hom_{\mathscr{C}}(A, B)$ and $g \in \hom_{\mathscr{C}}(B, C)$, then there is a morphism $g \circ f$ (also denoted by gf) from A to C.

(2) Composition of morphisms, where defined, is associative, so if $f \in \hom_{\mathscr{C}}(A, B)$, $g \in \hom_{\mathscr{C}}(B, C)$ and $h \in \hom_{\mathscr{C}}(C, D)$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

(3) For every object $B \in |\mathscr{C}|$, there is an *identity morphism* $1_B \in \hom_{\mathscr{C}}(B, B)$ (also denoted by Id_B) so that, given morphisms $f \in \hom_{\mathscr{C}}(A, B)$ and $g \in \hom_{\mathscr{C}}(B, C)$, $1_B \circ f = f$ and $g \circ 1_B = g$.

A morphism $f \in \hom_{\mathscr{C}}(A, B)$ will often be represented by the notation $f\colon A \to B$; A is called the *domain* of f and B is called the *codomain* of f.

For instance, **Set** denotes the category whose objects are sets and whose morphisms are maps with the usual composition. Similarly, topological spaces and continuous maps between them form a category denoted by **Top**, as do groups and homomorphisms, or vector spaces over a field and linear maps.

In an arbitrary category $\mathscr{C}$, a morphism $f\colon A \to B$ in $\mathscr{C}$ is called an *isomorphism* if there exists a morphism $g\colon B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$. This determines g uniquely, and g is called the *inverse* of f. If such a morphism f exists, one says that A is isomorphic to B, and one writes $A \cong B$.

Categories are related by using functors. A *functor* $F\colon \mathscr{A} \to \mathscr{B}$ carries any object $A \in |\mathscr{A}|$ to an object $F(A) \in |\mathscr{B}|$ and a morphism $f\colon A \to B$ is mapped into a morphism $F(f)\colon F(A) \to F(B)$. This correspondence has to preserve composition and identities.

For a category $\mathscr{A}$, there is an *identity functor* $\mathrm{Id}_{\mathscr{A}}\colon \mathscr{A} \to \mathscr{A}$, and for two functors $F\colon \mathscr{A} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{C}$, one can form a new functor $G \circ F\colon \mathscr{A} \to \mathscr{C}$ by *composition*.

Now, we will see some examples of categories and functors. For instance, every set I can be viewed as a category $\mathscr{I}$ whose objects are the elements of I and the only morphisms are identities ($\hom_{\mathscr{I}}(i, j)$ is a singleton when $i = j$ and is empty otherwise). A category whose only morphisms are the identities is called a *discrete category*.

Also, given a category $\mathscr{C}$, there is a category $\mathscr{C}^{\mathrm{op}}$ called *opposite* (or *dual*) with the same collection of objects and such that, $\forall A, B \in |\mathscr{C}|$, $\hom_{\mathscr{C}^{\mathrm{op}}}(A, B) = \hom_{\mathscr{C}}(B, A)$.

Moreover, from a given category $\mathscr{C}$, one very often constructs new categories such as, fixed an object $I \in |\mathscr{C}|$, the category $\mathscr{C}/I$ of "arrows over I", whose objects are the arrows of $\mathscr{C}$ with codomain I and whose morphisms are given by the commutative triangles over I. Here, the composition law is induced by the composition of $\mathscr{C}$.

Similarly, again fixing an object $I \in |\mathscr{C}|$, we can also define the category $I/\mathscr{C}$ of "arrows under I" whose objects are this time the arrows of $\mathscr{C}$ with domain I and whose morphisms are given by the commutative triangles under I. The composition law is induced by that of $\mathscr{C}$.

Yet another example of these kind of constructions is the category $\mathrm{Ar}(\mathscr{C})$ of arrows of $\mathscr{C}$ that has for objects all the arrows of $\mathscr{C}$; in this category, a morphism from the object $f\colon A \to B$ to the object $g\colon C \to D$ is a pair of morphisms of $\mathscr{C}$, $h\colon A \to C$ and $k\colon B \to D$, with the property $k \circ f = g \circ h$ (a commutative square):

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Again, the composition law is induced pointwise by the composition in $\mathscr{C}$.

Example 0.2.1. *Given a category $\mathscr{C}$ and a fixed object $C \in |\mathscr{C}|$, we define a functor*

$$\mathscr{C}(C, \cdot)\colon \mathscr{C} \longrightarrow \mathbf{Set}$$

from $\mathscr{C}$ to the category of sets (also denoted by $\hom_{\mathscr{C}}(C, \cdot)$) by first putting

$$\mathscr{C}(C, \cdot)(A) = \mathscr{C}(C, A).$$

Now if $f\colon A \to B$ is a morphism of $\mathscr{C}$, the corresponding mapping

$$\mathscr{C}(C, \cdot)(f) \equiv \mathscr{C}(C, f)\colon \mathscr{C}(C, A) \to \mathscr{C}(C, B)$$

is defined by the formula

$$\mathscr{C}(C, f)(g) = f \circ g$$

for an arrow $g \in \mathscr{C}(C, A)$. Such a functor is called a representable functor *(the functor is represented by the object C).*

The following lemma will be really useful to prove that the definitions of several of the functors which will appear in section 2.3 make sense.

Lemma 0.2.1. *Let $F\colon \mathscr{C} \to \mathbf{Set}$ be a functor. For each object $X \in |\mathscr{C}|$, define a subobject $G(X) \subset F(X)$ and, given a morphism $f\colon X \to Y$ in $\mathscr{C}$, suppose that $F(f)(G(X)) \subset G(Y)$. Define $G(f)\colon G(X) \to G(Y)$ such that $G(f) = F(f)|_{G(X)}$. Then, G is a functor from $\mathscr{C}$ to* **Set**.

Given two functors from a category to another one, there is the notion of "natural transformation" between those two functors, which allows you to connect one functor with the other: let F and G be two functors from a category $\mathscr{A}$ to a category $\mathscr{B}$; a *natural transformation* α from F to G, written $\alpha\colon F \to G$, is a correspondence associating with each object A of $|\mathscr{A}|$ a morphism $\alpha_A\colon F(A) \to G(A)$ of $\mathscr{B}$, in such a way that, for every morphism $f\colon A \to A'$ in $\mathscr{A}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(A') \arrow[r, "\alpha_{A'}"'] & G(A') \end{tikzcd}$$

commutes, i.e., $\alpha_{A'} \circ F(f) = G(f) \circ \alpha_A$.

The morphism α_A is called the *component* of α at A. If every component of α is an isomorphism, α is said to be a *natural isomorphism*. If $\alpha\colon F \to G$ and $\beta\colon G \to H$ are two natural transformations between functors $\mathscr{A} \to \mathscr{B}$, one can define a composite natural transformation $\beta \circ \alpha$ by setting

$$(\beta \circ \alpha)_A = \beta_A \circ \alpha_A.$$

For fixed categories $\mathscr{A}$ and $\mathscr{B}$, this yields a new category $\mathscr{B}^{\mathscr{A}}$: the objects of $\mathscr{B}^{\mathscr{A}}$ are functors from $\mathscr{A}$ to $\mathscr{B}$, while the morphisms of $\mathscr{B}^{\mathscr{A}}$ are natural transformations between such functors. Categories so constructed are called *functor categories*.

The following results will be really useful in later chapters.

Proposition 0.2.1. *If $\mathscr{A}$ is a category and $g\colon A \to B$ is a morphism in $\mathscr{A}$, for an object X in $\mathscr{A}$ we can consider $R^g_X\colon \hom_{\mathscr{A}}(B, X) \to \hom_{\mathscr{A}}(A, X)$ given by $R^g_X(f) = fg$. Then, R^g_X, $X \in |\mathscr{A}|$, gives a natural transformation*

$$R^g\colon \hom_{\xi}(B, \cdot) \to \hom_{\xi}(A, \cdot).$$

Let $F, G\colon \mathscr{C} \to \mathbf{Set}$ be functors. If there is a natural transformation $\alpha\colon G \to F$ such that $\alpha_X\colon G(X) \to F(X)$ is an inclusion for every X in $|\mathscr{C}|$, then it is said that G is a *subfunctor* of F.

Lemma 0.2.2. *Suppose that F_1 and F_2 are functors from $\mathscr{C}$ to* **Set** *and that G_1 and G_2 are subfunctors of F_1 and F_2, respectively. Let $\theta\colon F_1 \to F_2$ be a natural transformation. If $\theta_X(G_1(X)) \subset G_2(X)$ and we define $\Theta_X = \theta_X|_{G_1(X)}$ for each object $X \in |\mathscr{C}|$, then*

$$\Theta\colon G_1 \to G_2$$

is a natural transformation.

Consider a functor $F\colon \mathscr{A} \to \mathscr{B}$ and, for every pair of objects $A, A' \in |\mathscr{A}|$, the mapping

$$\hom_{\mathscr{A}}(A, A') \longrightarrow \hom_{\mathscr{B}}(F(A), F(A')), \quad f \mapsto F(f).$$

If the above-mentioned mappings are injective (resp. surjective) for all A, A', then one says that F is *faithful* (resp. *full*). It is obvious that, if these mappings are bijective for A, A', then F is both full and faithful; in that case, if in addition any object of $\mathscr{B}$ is isomorphic to an object in the image of F, then F is an *equivalence of categories*.

We now recall the notion of adjoint functors. Consider two categories $\mathscr{A}$ and $\mathscr{X}$ and two functors between them in opposite directions, say

$$F\colon \mathscr{X} \to \mathscr{A}, \qquad G\colon \mathscr{A} \to \mathscr{X}.$$

One says that F is *left adjoint* to G (and that G is *right adjoint* to F) when, for any two objects $X \in |\mathscr{X}|$ and $A \in |\mathscr{A}|$, there is a natural bijection between morphisms

$$\frac{X \xrightarrow{f} G(A)}{F(X) \xrightarrow{h} A}, \tag{0.1}$$

in the sense that each morphism f, as displayed, uniquely determines a morphism h, and conversely. This bijection is to be natural in the following sense: given any morphisms $\alpha\colon A \to A'$ in $\mathscr{A}$ and $\xi\colon X' \to X$ in $\mathscr{X}$, and corresponding arrows f and h as in (0.1), the composites also correspond under the bijection (0.1):

$$\frac{X' \xrightarrow{\xi} X \xrightarrow{f} G(A) \xrightarrow{G(\alpha)} G(A')}{F(X') \xrightarrow{F(\xi)} F(X) \xrightarrow{h} A \xrightarrow{\alpha} A'}.$$

If we write this bijective correspondence as

$$\theta_{X,A}\colon \hom_{\mathscr{X}}(X, G(A)) \xrightarrow{\sim} \hom_{\mathscr{A}}(F(X), A),$$

then this naturality condition can be expressed by the equation

$$\theta_{X',A'}(G(\alpha) \circ f \circ \xi) = \alpha \circ \theta_{X,A}(f) \circ F(\xi).$$

In other words, if we recognize $\hom_{\mathscr{X}}(\cdot, G(\cdot))$ and $\hom_{\mathscr{A}}(F(\cdot), \cdot)$ as functors from $\mathscr{X}^{\mathrm{op}} \times \mathscr{A}$ to **Set**, the naturality of θ means that, for all morphisms $\alpha\colon A \to A'$ in $\mathscr{A}$ and $\xi\colon X' \to X$ in $\mathscr{X}$, the following diagram commutes:

$$\begin{tikzcd}
\hom_{\mathscr{X}}(X, G(A)) \arrow[r, "\theta_{X,A}"] \arrow[d, "\hom_{\mathscr{X}}(\xi, G(\alpha))"'] & \hom_{\mathscr{A}}(F(X), A) \arrow[d, "\hom_{\mathscr{A}}(F(\xi), \alpha)"] \\
\hom_{\mathscr{X}}(X', G(A')) \arrow[r, "\theta_{X',A'}"'] & \hom_{\mathscr{A}}(F(X'), A')
\end{tikzcd}$$

0.2.2 Limits

Given a functor $F\colon \mathscr{D} \to \mathscr{C}$, a *cone* on F consists of an object $C \in |\mathscr{C}|$ and, for every object $D \in |\mathscr{D}|$, a morphism $p_D\colon C \to F(D)$ in $\mathscr{C}$, in such a way that, for every morphism $d\colon D \to D'$ in $\mathscr{D}$, $p_{D'} = F(d) \circ p_D$.

Definition 0.2.1. *Given a functor $F\colon \mathscr{D} \to \mathscr{C}$, a* limit *of F is a cone $(L, \{p_D\}_{D \in \mathscr{D}})$ on F such that, for every cone $(M, \{q_D\}_{D \in \mathscr{D}})$ on F, there exists a unique morphism $m\colon M \to L$ such that, for every object $D \in |\mathscr{D}|$,*

$$q_D = p_D \circ m.$$

We shall remember now the dual notions of cocone and colimit. Given a functor $F\colon \mathscr{D} \to \mathscr{C}$, a *cocone* on F consists of an object $C \in |\mathscr{C}|$ and, for every object $D \in |\mathscr{D}|$, a morphism $s_D\colon F(D) \to C$ in $\mathscr{C}$, in such a way that, for every morphism $d\colon D' \to D$ in $\mathscr{D}$, $s_{D'} = s_D \circ F(d)$.

Definition 0.2.2. *Given a functor $F\colon \mathscr{D} \to \mathscr{C}$, a* colimit *of F is a cocone $(L, \{s_D\}_{D \in \mathscr{D}})$ on F such that, for every cocone $(M, \{t_D\}_{D \in \mathscr{D}})$ on F, there exists a unique morphism $m\colon L \to M$ such that, for every object $D \in |\mathscr{D}|$,*

$$t_D = m \circ s_D.$$

Remind that products, equalizers and pullbacks are special cases of the notion of limit. For instance, given a set I regarded as a discrete category $\mathscr{I}$, giving a functor $F\colon \mathscr{I} \to \mathscr{C}$ to a category $\mathscr{C}$ is just giving a family $F_i \in \mathscr{C}$, $i \in I$ of objects, and defining the limit of F is actually just defining the product $\prod_{i\in I} F_i$.

Similarly, consider the category $\mathscr{K}$ defined by

$$|\mathscr{K}| = \{A, B\},$$

$$\mathscr{K}(A,A) = \{1_A\}, \quad \mathscr{K}(B,B) = \{1_B\}, \quad \mathscr{K}(A,B) = \{\alpha, \beta\}, \quad \mathscr{K}(B,A) = \emptyset$$

and sketched in the following diagram:

$$\mathscr{K}: \qquad \begin{tikzcd} A \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & B \end{tikzcd}$$

Let $\mathscr{C}$ be a category. Giving a functor $F\colon \mathscr{K} \to \mathscr{C}$ is just giving two arrows $F(\alpha), F(\beta)\colon F(A) \rightrightarrows F(B)$ in $\mathscr{C}$ and defining the limit of F is actually just defining the equalizer of $F(\alpha), F(\beta)$.

Finally, consider the category $\mathscr{P}$ defined by

$$|\mathscr{P}| = \{A, B, C\},$$

$$\mathscr{P}(A,A) = \{1_A\}, \quad \mathscr{P}(B,B) = \{1_B\}, \quad \mathscr{P}(C,C) = \{1_C\},$$

$$\mathscr{P}(A,C) = \{\alpha\}, \quad \mathscr{P}(B,C) = \{\beta\}, \quad \mathscr{P}(C,A) = \mathscr{P}(C,B) = \mathscr{P}(A,B) = \mathscr{P}(B,A) = \emptyset$$

and sketched in the following diagram:

$$\mathscr{P}: \qquad \begin{tikzcd} & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"'] & C \end{tikzcd}$$

Giving a functor F from $\mathscr{P}$ to a category $\mathscr{C}$ is just giving a pair

$$F(\alpha)\colon F(A) \to F(C), \quad F(\beta)\colon F(B) \to F(C)$$

of arrows in $\mathscr{C}$ and defining the limit of F is actually just defining the pullback of $F(\alpha), F(\beta)$, respectively.

These previous examples can be dualized to present the notions of coproduct, coequalizer and pushout as special cases of the general notion of colimit.

What we are intending by introducing all these notions is to recall that a category has limits (resp. colimits) if and only if it has products and equalizers (resp. coproducts and coequalizers). Remind that, if every functor F from a small category (i.e., a category such that its class $|\mathscr{D}|$ of objects is a set) to a category $\mathscr{C}$ has a limit, then one says that the category $\mathscr{C}$ is *complete*.

Theorem 0.2.1. *A category $\mathscr{C}$ is complete precisely when each family of objects has a product and each pair of parallel arrows has an equalizer.*

0.2.3 Directed sets

In order to define the concept of directed set, we need the notion of a preordered set. We will also require the definition of a partially ordered set later on.

A *preordered set* $(\Lambda, \leq)$ is a set Λ together with a binary relation $\leq$ over Λ satisfying:

(1) $\lambda \leq \lambda$, $\forall \lambda \in \Lambda$ (reflexive condition).

(2) If $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$, then $\lambda_1 \leq \lambda_3$, $\forall \lambda_1, \lambda_2, \lambda_3 \in \Lambda$ (transitive condition).

If a preordered set fulfills in addition the antisymmetric condition,

(3) If $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_1$, then $\lambda_1 = \lambda_2$, $\forall \lambda_1, \lambda_2 \in \Lambda$,

then it is called a *partially ordered set.*

Let $a, b \in \Lambda$, where Λ is a partially ordered set. For $a \leq b$, the closed interval $[a, b]$ is the set of elements x satisfying $a \leq x \leq b$ (i.e., $a \leq x$ and $x \leq b$). It contains, at least, the elements a and b. Using the corresponding strict relation "$<$", the open interval (a, b) is the set of elements x satisfying $a < x < b$ (i.e., $a < x$ and $x < b$). An open interval may be empty, even if $a < b$. For example, with the usual order in $\mathbb{R}$, the open interval $(1, 2)$ on the integers is empty, since there are no integers i such that $1 < i < 2$. The half-open intervals $[a, b)$ and $(a, b]$ are defined similarly.

From now on, given $a, b \in \mathbb{N}$ and when no confusion with the closed interval $[a, b] \subset \mathbb{R}$ is possible, we shall denote the subset of natural numbers $\{n \in \mathbb{N} \mid a \leq n \leq b\} = [a, b] \cap \mathbb{N}$ by $[a, b]$. Similarly, we shall denote $\{n \in \mathbb{N} \mid a < n < b\} = (a, b) \cap \mathbb{N}$ by (a, b). An analogous notation will be used when considering the intersection of half-open intervals and the set of natural numbers.

Definition 0.2.3. *Let Λ be a preordered set with upper bounds for finite subsets, that is, for any elements i, j in Λ there is an element k such that $i \leq k$ and $j \leq k$. Such a set is called a* directed set*.*

Example 0.2.2. *Any partially ordered set $(\Lambda, \leq)$ gives rise to a category, with the elements of Λ as objects and with precisely one morphism from λ_1 to λ_2 if and only if $\lambda_1 \leq \lambda_2$. Thus, the composition operation for Λ is uniquely determined by the transitivity of the order relation $\leq$. We will also consider the category $\underleftarrow{\Lambda}$ whose objects are the elements of Λ and such that there is a unique morphism from λ_2 to λ_1 if $\lambda_1 \leq \lambda_2$.*

0.2.4 Inverse systems

Definition 0.2.4. *Given a directed set Λ and a category $\mathscr{C}$, consider the functor $F\colon \underleftarrow{\Lambda} \to \mathscr{C}$, which can be seen just as a collection of objects F_i indexed by Λ and one morphism $F_i^j : F_j \to F_i$ for each $i \leq j$ such that $F_i^i = \mathrm{Id}_{F_i}$ and $F_i^k = F_i^j \circ F_j^k$. This functor is called an* inverse system *over Λ in $\mathscr{C}$.*

The limit of this functor is called the inverse limit *of the inverse system, and it is written $\varprojlim F$ or $\lim_{i \in \Lambda} F_i$.*

This definition of inverse limit really means that it is an object A with morphisms $\pi_i\colon A \to F_i$ for each $i \in \Lambda$, such that $F_i^j \circ \pi_j = \pi_i$. The object A has to be universal in the sense that if A' is another object with morphisms $\pi_i'\colon A' \to F_i$ and $F_i^j \circ \pi_j' = \pi_i'$, then there is a unique morphism $f\colon A' \to A$ such that $\pi_i \circ f = \pi_i'$, for all $i \in \Lambda$.

Remark 0.2.1. *Using the notation above, in the categories that we are going to consider in this work (topological spaces and continuous maps, exterior spaces and exterior maps, sets and functions), we can also give an alternative definition of inverse limit as follows:*

$$\varprojlim F = \{\lambda \in \prod_{i\in\Lambda} F_i \mid \forall i,j \in \Lambda,\ [i \leq j] \Rightarrow [\pi_i(\lambda) = (F_i^j \circ \pi_j)(\lambda)]\}.$$

Definition 0.2.5. *Let Λ_1, Λ_2 be directed sets. A map $\varphi\colon \Lambda_1 \to \Lambda_2$ is said to be* cofinal *if verifies:*

(i) For all $i_1, i_2 \in \Lambda_1$ such that $i_1 \geq i_2$, $\varphi(i_1) \geq \varphi(i_2)$.

(ii) For all $i_2 \in \Lambda_2$, $\exists i_1 \in \Lambda_1$ such that $\varphi(i_1) \geq i_2$.

One says that $\kappa \subset \Lambda$ is a *cofinal set* if the inclusion function $\iota\colon \kappa \to \Lambda$ is cofinal. For instance, for $k \geq 1$, the ordered set $k\mathbb{N} = \{kj \mid j \in \mathbb{N}\}$ is cofinal in $\mathbb{N}$.

A cofinal map $\varphi\colon \Lambda_1 \to \Lambda_2$ induces a functor $\underleftarrow{\varphi}\colon \underleftarrow{\Lambda_1} \to \underleftarrow{\Lambda_2}$. Then, given an inverse system $X\colon \underleftarrow{\Lambda_2} \to \mathscr{C}$, one has an induced inverse system $X \circ \underleftarrow{\varphi}\colon \underleftarrow{\Lambda_1} \to \mathscr{C}$ satisfying

$$(X \circ \underleftarrow{\varphi})_{i_1} = X_{\varphi(i_1)}, \quad \forall i_1 \in \Lambda_1.$$

Proposition 0.2.2. *If $\varphi\colon \Lambda_1 \to \Lambda_2$ is a cofinal map and $X\colon \underleftarrow{\Lambda_2} \to \mathscr{C}$ is an inverse system, then the induced map*

$$\lim_{i_2\in\Lambda_2} X_{i_2} \longrightarrow \lim_{i_1\in\Lambda_1} X_{\varphi(i_1)}$$

is an isomorphism.

Most of the notions seen in this section might be complemented with the definitions appearing along the first chapters of [10]. For a broader look at inverse systems, see [64]. For those who wish to delve further into category theory, see [57] or [58]. Throughout the next preliminary sections, we shall remember the constructions of exterior spaces and discrete semi-flows done in [62, 38]. At the same time, the notations that we are going to use will be given.

0.3 Exterior spaces

Definition 0.3.1. *Let $(X, \mathbf{t}_X)$ be a topological space. An* externology *on $(X, \mathbf{t}_X)$ is a non-empty collection $\varepsilon \subset \mathbf{t}_X$ which is closed under finite intersections and such that, if $E \in \varepsilon$, $U \in \mathbf{t}_X$ and $E \subset U$, then $U \in \varepsilon$. If an open subset is a member of ε, it is said to be an* exterior open subset.

An exterior space *$(X, \varepsilon, \mathbf{t}_X)$ consists of a topological space $(X, \mathbf{t}_X)$ together with an externology ε.*

A map $f\colon (X, \varepsilon, \mathbf{t}_X) \to (X', \varepsilon', \mathbf{t}_{X'})$ is said to be an exterior map *if it is continuous and $f^{-1}(E') \in \varepsilon$, for all $E' \in \varepsilon'$.*

An exterior space $(X, \varepsilon, \mathbf{t}_X)$ will often be denoted as (X, ε) or X for short when no confusion is possible.

For a given topological space $(X, \mathbf{t}_X)$, we can consider the *trivial* externology, constituted by a unique exterior open subset $\varepsilon = \{X\}$, and the *total* externology, $\varepsilon = \mathbf{t}_X$. Note that an externology ε is a topology if and only if it contains the empty set, which happens if and only if $\varepsilon = \mathbf{t}_X$. Given an exterior space $(X, \varepsilon, \mathbf{t}_X)$, the *relative* externology in $A \subset X$ is given by $\{E \cap A\}_{E \in \varepsilon}$. Obviously, the inclusion $A \hookrightarrow X$ becomes exterior. For a topological space X, we can also consider the co-compact externology $\varepsilon^c(X) = \{E \in \mathbf{t}_X \mid X \setminus E \text{ is compact and closed}\}$. We denote $\mathbb{R}_+$ and $\mathbb{N}$ the exterior spaces determined by the usual topology and the co-compact externology in the sets of non-negative real numbers $\mathbb{R}_+$ and natural numbers, respectively.

Example 0.3.1. *Let A be a subset of a topological space X. Denote by*

$$\varepsilon(X, A) = \{U \in \mathbf{t}_X \mid A \subset U\}.$$

It is easy to check that $\varepsilon(X, A)$ is an externology.

Definition 0.3.2. *Let $(X, \varepsilon, \mathbf{t}_X)$ be an exterior space. An* exterior basis *for $(X, \varepsilon, \mathbf{t}_X)$ is a collection $\mathcal{E}$ of subsets of X satisfying that, for every exterior open subset E, there exists $B \in \mathcal{E}$ such that $B \subset E$ and, for every $B' \in \mathcal{E}$, there exists an exterior open subset E' such that $E' \subset B'$.*

If an exterior space X has a countable exterior basis $\mathcal{E} = \{X_n\}_{n \in \mathbb{N}}$, then we say that X is first-countable at infinity*.*

Observe that, for these exterior spaces, we can suppose without loss of generality that there is a countable exterior basis satisfying that:

$$X = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_n \supset \cdots$$

Since the composition of exterior maps is exterior and the identity map on an exterior space is exterior, we have the category of exterior spaces and exterior maps. This category is denoted by $\mathbf{E}$.

We can consider the functor

$$(\cdot)\bar{\times}(\cdot)\colon \mathbf{E} \times \mathbf{Top} \to \mathbf{E}$$

given by the following construction: let $(X, \varepsilon^X, \mathbf{t}_X)$ be an exterior space, $(Y, \mathbf{t}_Y)$ a topological space and, for $y \in Y$, denote by $\{\mathbf{t}_Y\}_y$ the family of open neighborhoods of Y at y. We consider on $X \times Y$ the product topology $\mathbf{t}_{X \times Y}$ and the externology $\varepsilon^{X \bar{\times} Y}$ given by those $E \in \mathbf{t}_{X \times Y}$ such that, for each $y \in Y$, there exist $U_y \in \{\mathbf{t}_Y\}_y$ and $T^y \in \varepsilon^X$ such that $T^y \times U_y \subset E$. This exterior space will be denoted by $X \bar{\times} Y$ in order to avoid a possible confusion with the product externology.

Let $E \in \mathbf{t}_{X \times Y}$. Note that, if Y is a compact space, one has that $E \in \varepsilon^{X \bar{\times} Y}$ if and only if there exists $T \in \varepsilon^X$ such that $T \times Y \subset E$. The functor

$$(\cdot)\bar{\times}(\cdot)\colon \mathbf{E} \times \mathbf{Top} \to \mathbf{E}$$

allows us to give the next definition.

Definition 0.3.3. *Given $f, g\colon X \to Y$ in $\mathbf{E}$, it is said that f is* exterior homotopic *to g if there exists an exterior homotopy $H\colon X \bar{\times} I \to Y$ from f to g, and it will be denoted by $f \simeq_e g$.*

Denote by $\pi\mathbf{E}$ and $\pi\mathbf{Top}$ the exterior homotopy category and the usual homotopy category corresponding to $\mathbf{E}$ and $\mathbf{Top}$, respectively. Denote $\pi\mathbf{E}(X, Y) = [X, Y]$. In the next section, we shall deal with special limit constructions –for more on homotopy categories, see [75], for example.

0.4 Limit spaces and end sets of exterior spaces

Note that, if $X = (X, \varepsilon(X))$ is an exterior space, its externology $\varepsilon(X)$ can be seen as an inverse system of spaces by taking the set $\varepsilon(X)$ directed by reverse inclusion and the functor $\varepsilon(X)\colon \underleftarrow{\varepsilon(X)} \to \mathbf{Top}$ given by $\varepsilon(X)_E = E$, so that if $E \geq E'$, then $\varepsilon(X)^E_{E'}$ is the inclusion $\mathrm{in}^E_{E'}\colon E \to E'$. We are using the same symbol $\varepsilon(X)$ to denote the externology and its corresponding inverse system.

In addition, if we define

$$\bar{\varepsilon}(X) = \{\overline{E} \mid E \in \varepsilon(X)\},$$

then $\bar{\varepsilon}(X)$ can also be seen as an inverse system of spaces by taking this set directed by reverse inclusion and the functor $\bar{\varepsilon}(X)\colon \underleftarrow{\bar{\varepsilon}(X)} \to \mathbf{Top}$ given by $\bar{\varepsilon}(X)_{\overline{E}} = \overline{E}$, so that if $\overline{E} \geq \overline{E'}$, then $\bar{\varepsilon}(X)^{\overline{E}}_{\overline{E'}}$ is the inclusion $\mathrm{in}^{\overline{E}}_{\overline{E'}}\colon \overline{E} \to \overline{E'}$. As before, we shall use the same symbol $\bar{\varepsilon}(X)$ to denote the family of sets and its inverse system.

Definition 0.4.1. *Given an exterior space X with externology $\varepsilon(X)$, the topological space*

$$L(X) = \varprojlim \varepsilon(X)$$

will be called the limit space *of X and*

$$\bar{L}(X) = \varprojlim \bar{\varepsilon}(X)$$

will be called the bar-limit space *of X.*

Proposition 0.4.1. *For each exterior space $X = (X, \varepsilon(X))$, there are canonical homeomorphisms*

$$L(X) \cong \bigcap_{E \in \varepsilon(X)} E, \qquad \bar{L}(X) \cong \bigcap_{E \in \varepsilon(X)} \overline{E}.$$

Note that if $f\colon (X, \varepsilon(X)) \to (Y, \varepsilon(Y))$ is an exterior map, then $f(L(X)) \subset L(Y)$. Thus, we can define $L(f) = f|_{L(X)}$ so that $L(f)(x) = f(x)$. It is easy to check that L respects composition and identity morphisms; therefore,

$$L\colon \mathbf{E} \to \mathbf{Top}$$

is a functor. Observe that given a continuous map $g\colon A \to B$ and a subset $S \subset A$, one has that $g(\overline{S}) \subset \overline{g(S)}$. That implies $f(\bar{L}(X)) \subset \bar{L}(Y)$. Thus, we have that

$$\bar{L}\colon \mathbf{E} \to \mathbf{Top}$$

is also a functor.

By using the forgetful functor $U\colon \mathbf{Top} \to \mathbf{Set}$, one can consider the composite

$$U \circ L\colon \mathbf{E} \to \mathbf{Set},$$

which assigns to each topological space the underlying set and to each continuous map the underlying map, "forgetting" its continuity. Similarly, one has the functor

$$U \circ \bar{L}\colon \mathbf{E} \to \mathbf{Set}.$$

When no confusion is possible, the functor $U \circ L$ will be denoted by L, and $U \circ \bar{L}$ by $\bar{L}$.

Definition 0.4.2. *Let X be an exterior space with externology $\varepsilon(X)$. The* π_0^{BG}-end set *of X is given by*

$$\pi_0^{\mathrm{BG}}(X) = [\mathbb{N}, X] = \hom_{\pi\mathbf{E}}(\mathbb{N}, X).$$

The π_0^{S}-end set *of X is given by*

$$\pi_0^{\mathrm{S}}(X) = [\mathbb{R}_+, X] = \hom_{\pi\mathbf{E}}(\mathbb{R}_+, X).$$

The $\check{\pi}_0$-end set *of X is given by*

$$\check{\pi}_0(X) = \lim_{E \in \varepsilon(X)} \pi_0(E),$$

where $\pi_0(E)$ denotes the set of path components of E.

The $\check{\bar{\pi}}_0$-end set *of X is given by*

$$\check{\bar{\pi}}_0(X) = \lim_{E \in \varepsilon(X)} \pi_0(\overline{E}).$$

An element of any given end set will be called end point *of that end set.*

As can be seen in [22, 62, 38], the constructions π_0^{BG}, π_0^{S}, $\check{\pi}_0$ and $\check{\bar{\pi}}_0$ given in definition above are functors. Using the canonical functor $\gamma\colon \mathbf{E} \to \pi\mathbf{E}$, which is the identity on objects and is yielded by the obvious quotient map on morphism sets, we have the induced functors $\pi_0^{\mathrm{BG}} \circ \gamma\colon \mathbf{E} \to \mathbf{Set}$ and $\pi_0^{\mathrm{S}} \circ \gamma\colon \mathbf{E} \to \mathbf{Set}$ associated with the objects $\mathbb{N}$ and $\mathbb{R}_+$ in $\mathbf{E}$. When no confusion is possible, we usually will denote the composites $\pi_0^{\mathrm{BG}} \circ \gamma$ and $\pi_0^{\mathrm{S}} \circ \gamma$ by π_0^{BG} and π_0^{S}, respectively.

Next, we shall present how the functor $\check{\pi}_0\colon \mathbf{E} \to \mathbf{Set}$ works, explicitly. Let $f \in \hom_{\mathbf{E}}(X, Y)$ and note that $\check{\pi}_0$ is a function which maps each exterior space $(X, \varepsilon(X))$ to the inverse limit $\check{\pi}_0(X) = \lim_{E^X \in \varepsilon(X)} \pi_0(E^X)$. To see how $\check{\pi}_0(f)$ is defined, take $a = (C_{E^X})_{E^X \in \varepsilon(X)} \in \check{\pi}_0(X)$, where C_{E^X} is a path component of the exterior open subset E^X. Given $E^Y \in \varepsilon(Y)$, we have that $f^{-1}(E^Y) \in \varepsilon(X)$. Then, there exists a unique path component $C_{f^{-1}(E^Y)}$ in $f^{-1}(E^Y)$ representing $a \in \check{\pi}_0(X)$. Since $f(C_{f^{-1}(E^Y)}) \subset E^Y$ is path-connected, there is a unique path component C_{E^Y} such that $f(C_{f^{-1}(E^Y)}) \subset C_{E^Y}$; take

$$b = (C_{E^Y})_{E^Y \in \varepsilon(Y)} \in \check{\pi}_0(Y).$$

One has that $\check{\pi}_0(f)(a) = b$.

One can describe analogously the functor $\check{\bar{\pi}}_0\colon \mathbf{E} \to \mathbf{Set}$.

Example 0.4.1. *Given $E_0 \in \varepsilon^c(\mathbb{R}_+)$, there is $n_0 \in \mathbb{N}$ such that $\mathbb{R}_+ \setminus E_0 \subset [0, n_0]$; hence, $(n_0, +\infty) \subset E_0$. This implies that $\{(n, +\infty)\}_{n\in\mathbb{N}}$ is a basis for $\varepsilon^c(\mathbb{R}_+)$. Therefore, $((n, +\infty))_{n\in\mathbb{N}}$ is cofinal in $\varepsilon^c(\mathbb{R}_+)$. By Proposition 0.2.2, it follows that*

$$\lim_{E\in\varepsilon^c(\mathbb{R}_+)} \pi_0(E) \cong \lim_{n\in\mathbb{N}} \pi_0((n, +\infty)).$$

The inverse limit of an inverse system of a one-point set is also a one-point set, and the unique point of the limit will be denoted by $+\infty_{\mathbb{R}_+}$.

0.5 Discrete semi-flows

Next we recall some basic notions and properties about discrete semi-flows. These notions can be given for a set or for a topological space.

Definition 0.5.1. *A* discrete semi-flow *on a set (topological space) X is a map (continuous map) $\varphi\colon \mathbb{N} \times X \to X$ such that:*

(i) $\varphi(0, x) = x$, $\forall x \in X$;

(ii) $\varphi(n, \varphi(m, x)) = \varphi(n + m, x)$, $\forall x \in X$, $\forall n, m \in \mathbb{N}$.

A discrete semi-flow on X will be denoted by (X, φ) and, when no confusion is possible, we will use X and $n \cdot x = \varphi(n, x)$ for short.

Given discrete semi-flows (X, φ) and (Y, ψ), a discrete semi-flow morphism *$h\colon (X, \varphi) \to (Y, \psi)$ is a (continuous) map $h\colon X \to Y$ such that $h(n \cdot x) = n \cdot h(x)$, for every $(n, x) \in \mathbb{N} \times X$.*

We shall denote by $\mathbf{F}(\mathbb{N})$ the category of discrete semi-flows.

Given a discrete semi-flow $\varphi\colon \mathbb{N}\times X \to X$ and $n_0 \in \mathbb{N}$, we have the induced map $\varphi^{n_0}\colon X \to X$ given by $\varphi^{n_0}(x) = \varphi(n_0, x)$. The *trajectory* (or *orbit*) of a point $x_0 \in X$ is defined via the map $\varphi_{x_0}\colon \mathbb{N} \to X$ given by $\varphi_{x_0}(n) = \varphi(n, x_0)$.

It is interesting to note that a discrete semi-flow (X, φ) induces a (continuous) map $\varphi^1\colon X \to X$ and conversely a (continuous) map $h\colon X \to X$ induces a discrete semi-flow $\varphi\colon \mathbb{N}\times X \to X$, $\varphi(n, x) = h^n(x)$, where h^n denotes the function composition $h \circ \underbrace{\cdots}_{n \text{ times}} \circ h$ and $h^0 = \mathrm{Id}_X$.

For a discrete semi-flow (X, φ), a subset $A \subset X$ is said to be *right-invariant* if $\varphi^1(A) \subset A$ and it is said to be *left-invariant* if $(\varphi^1)^{-1}(A) \subset A$. A subset which is left-invariant and right-invariant is said to be *completely invariant.*

Given two points $x, y \in X$, we have the following equivalence relation: $x \sim y$ if there exist $k, l \in \mathbb{N}$ such that $\varphi^k(x) = \varphi^l(y)$. If $[x]$ is the equivalence class of x, note that $[x]$ is a completely invariant subset. In fact, if S is a subset of X, there is the notion of *minimal completely invariant subset* that contains S, which is $[S]$; in particular, given a point $a \in X$, the minimal completely invariant subset that contains the one-point set $\{a\}$ is $[\{a\}]$, which we write $[a]$ by abusing of notation. Denote by $X/\sim$ the quotient set, which has a trivial induced action.

We will use the following result.

Lemma 0.5.1. *Given a discrete semi-flow morphism h, the inverse image of a completely invariant subset under h is also a completely invariant subset.*

Definition 0.5.2. *Let X be a discrete semi-flow and let x be a point of X.*

(i) x is a fixed *point if, for every $n \in \mathbb{N}$, $n \cdot x = x$.*

(ii) x is a periodic or cyclic *point if there exists $n \in \mathbb{N}$, $n \neq 0$ such that $n \cdot x = x$.*

*(iii) For $m \in \mathbb{N}$, x is a m-*periodic *point if $m \cdot x = x$.*

*(iv) For $m \in \mathbb{N}$, x is a m-*cyclic *point if $m \cdot x = x$ and, if $0 < k < m$, then $k \cdot x \neq x$.*

(v) For $m \in \mathbb{N}$, x is a pre-m-periodic *point if there exists $l \geq 0$ such that $m \cdot (l \cdot x) = (l \cdot x)$.*

The right-invariant subsets of fixed, periodic, m-periodic, m-cyclic and pre-m-periodic points of X are denoted by $\mathrm{Fix}(X)$, $P(X)$, $P_m(X)$, $C_m(X)$ and $\mathrm{pre}P_m(X)$, respectively. From the definition, it is clear that $C_m(X) \subset P_m(X) \subset \mathrm{pre}P_m(X)$.

From now on, let us denote $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Given a positive integer $m \in \mathbb{N}^*$, let $\mathrm{Div}(m) = \{n \in \mathbb{N} \colon n \text{ divides } m\}$. It is easy to check that

$$P_m(X) = \bigsqcup_{n \in \mathrm{Div}(m)} C_n(X).$$

We denote a net of a topological space X by (t_δ), where we suppose that δ describes a directed preordered set. The following notions are given for topological spaces with a given semi-flow structure.

Definition 0.5.3. *For a discrete semi-flow (X, φ), the $\omega^{\mathbf{r}}$-*limit set of a point $x \in X$ *(or right-limit set, or positive limit set) is given as follows:*

$$\omega^{\mathbf{r}}(x) = \{y \in X \mid \textit{there is a net } (t_\delta) \to +\infty,\ t_\delta \in \mathbb{N},\ \textit{such that } t_\delta \cdot x \to y\}.$$

Note that the subset $\omega^{\mathbf{r}}(x)$ admits the alternative definition

$$\omega^{\mathbf{r}}(x) = \bigcap_{t \geq 0} \overline{[t, +\infty) \cdot x},$$

where $[t, +\infty)$ denotes the subset $\{n \in \mathbb{N} \mid n \geq t\}$.

The set

$$\omega^{\mathbf{r}}(X) = \bigcup_{x \in X} \omega^{\mathbf{r}}(x)$$

*is called the $\omega^{\mathbf{r}}$-*limit set *of X.*

It is easy to check that $\omega^{\mathbf{r}}(x)$ and $\omega^{\mathbf{r}}(X)$ are right-invariant subsets of X.

Definition 0.5.4. *Let X be a discrete semi-flow and $S \subset X$. The* region of pseudo-weak attraction *of S will be defined as:*

$$\mathrm{PWA}(S) = \{x \in X \mid \omega^{\mathbf{r}}(x) \subset S\}.$$

On its part, the region of weak attraction *of S is defined as:*

$$\mathrm{WA}(S) = \{x \in X \mid \omega^{\mathbf{r}}(x) \cap S \neq \emptyset\}.$$

Finally, the region of attraction *is known as:*

$$\mathrm{A}(S) = \{x \in X \mid \omega^{\mathbf{r}}(x) \neq \emptyset,\, \omega^{\mathbf{r}}(x) \subset S\}.$$

From the definition above, note that

$$\mathrm{A}(S) = \mathrm{PWA}(S) \cap \mathrm{WA}(S).$$

0.6 On measures and Carathéodory's extension theorem

The Carathéodory extension method [13, 14] is a technique that consists in defining an algebra of subsets with a pre-measure and the posterior extension of the algebra to a σ-algebra and the pre-measure to a measure. For a topological space X, it is interesting to consider measures defined on σ-algebras containing the Borel σ-algebra generated by the topology $\mathbf{t}_X$. In these cases, we can assign a measure to an open subset or to a closed subset; moreover, a countable intersection of open subsets or a countable union of closed subsets can also be measured. For more results about Borel sets, measure theory, construction and extension of measures, we refer to [45, 46].

Considering the foregoing, Carathéodory's extension theorem is essential for the construction of countably additive probability measures. As we are interested in creating a measure for a given iterated subdivision on a CW-complex defined on the surface of the 2-sphere, this theorem will be very useful for this purpose.

We will also recall some interesting properties of infinite sums along subsection 0.6.2, since we will be coping with measures satisfying the countable additivity property.

0.6.1 Carathéodory's measure-extension theorem

Let X be a set. In the power set 2^X, we can consider the usual operations: finite union, countable union and arbitrary union; we have similar operations for the intersection of subsets and one can take the complement of a subset.

An *algebra* $\mathcal{A}$ of subsets of X is a collection $\mathcal{A} \subset 2^X$ such that X is in $\mathcal{A}$; if A is in $\mathcal{A}$, then so is $X \setminus A$ (which is the complement of A), and if A_1, A_2 are elements of $\mathcal{A}$, then $A_1 \cup A_2$ is in $\mathcal{A}$. When $\mathcal{A}$ is also closed by countable unions (i.e., if $(A_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathcal{A}$, then the union $\bigcup_{n\in\mathbb{N}} A_n$ is in $\mathcal{A}$), it is said that $\mathcal{A}$ is a *σ-algebra*.

If $\mathcal{S}$ is any collection of subsets of X, we can always find a σ-algebra containing $\mathcal{S}$, namely the power set 2^X of X. Taking the intersection of all σ-algebras containing $\mathcal{S}$, we obtain the smallest such σ-algebra $\sigma(\mathcal{S})$, which is said to be the σ-algebra generated by $\mathcal{S}$ –similar arguments prove that, for $\mathcal{S}$, there is a (minimal) algebra $a(\mathcal{S}) \subset 2^X$ generated by $\mathcal{S}$. In particular, if X is a topological space and $\mathbf{t}_X$ is its topology, the σ-algebra $\sigma(\mathbf{t}_X)$ is called the *Borel σ-algebra* of X. Note that the open subsets and the closed subsets of $\mathbf{t}_X$ are members of the σ-algebra $\sigma(\mathbf{t}_X)$.

Given an algebra $\mathcal{A}$ of subsets of X, a *pre-measure* is a set map $\mu\colon \mathcal{A} \to [0,\infty]$ satisfying the following conditions:

(1) $\mu(\emptyset) = 0$.

(2) The *finite additivity* property of μ on the algebra $\mathcal{A}$: if $A_1, \dots, A_n \in \mathcal{A}$ are disjoint, then its disjoint union $\bigsqcup_{i=1}^n A_i$ verifies
$$\mu\left(\bigsqcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i).$$

(3) The *countable additivity* property of μ on the algebra $\mathcal{A}$: if $A_1, A_2, \dots \in \mathcal{A}$ are disjoint and $\bigsqcup_{i=1}^\infty A_i \in \mathcal{A}$, then
$$\mu\left(\bigsqcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i).$$

Notice that, if $\mathcal{A}$ is a σ-algebra of subsets of X, then a pre-measure μ is always countable additive and, in this case, it is said that μ is a *measure*. If $\mu(X) < \infty$, μ is said to be a *finite pre-measure* (*finite measure*, when $\mathcal{A}$ is a σ-algebra). If there exists a sequence $(A_n)_{n\in\mathbb{N}}$, with $A_n \in \mathcal{A}$, such that $X = \bigcup_{n\in\mathbb{N}} A_n$ and $\mu(A_n) < \infty$, μ is said to be a *σ-finite pre-measure* (*σ-finite measure* when $\mathcal{A}$ is a σ-algebra).

A *measurable space* is a pair $(X, \mathcal{E})$ consisting of a non-empty set X together with a σ-algebra $\mathcal{E}$ of subsets of X. If $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ are measurable spaces, then a mapping $f\colon X \to Y$ is said to be *measurable* if $f^{-1}(\mathcal{F}) \subset \mathcal{E}$. For example, if $(X, \mathbf{t}_X)$ and $(Y, \mathbf{t}_Y)$ are topological spaces, one has that a continuous map $f\colon (X, \mathbf{t}_X) \to (Y, \mathbf{t}_Y)$ is a measurable map
$$f\colon (X, \sigma(\mathbf{t}_X)) \to (Y, \sigma(\mathbf{t}_Y)).$$
A *measure space* is a triple $(X, \mathcal{E}, \mu)$, where $(X, \mathcal{E})$ is a measurable space and μ is a measure on $\mathcal{E}$. If the measure is finite, we say that the measure space is *finite*.

If $\mathcal{A}$ is an algebra and $\mathcal{E}$ is a σ-algebra of subsets of X such that $\mathcal{A} \subset \mathcal{E}$, $\mu\colon \mathcal{A} \to [0,\infty]$ is a pre-measure and $\mathrm{in}\colon \mathcal{A} \to \mathcal{E}$ is the canonical inclusion, the extension problem consists of finding a measure $\bar{\mu}\colon \mathcal{E} \to [0,\infty]$ such that the diagram
$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathrm{in}"] \arrow[d, "\mu"'] & \mathcal{E} \arrow[dl, dashed, "\bar{\mu}"] \\ {[0,\infty]} & \end{tikzcd}$$
commutes. If such a map exists, it is said that $\bar{\mu}$ is an *extension* of μ.

Carathéodory's extension theorem [13, 80, 14] gives, under some conditions, a positive answer to this question.

Theorem 0.6.1. *Let $\mathcal{A}$ be an algebra of subsets of a set X and suppose that $\mu\colon \mathcal{A} \to [0,\infty]$ is a pre-measure. Consider the map $\mu^{outer}\colon 2^X \to [0,\infty]$ given by*

$$\mu^{outer}(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) \mid A_1,\dots,A_n,\dots \in \mathcal{A} \text{ cover } E\right\}$$

and take the family

$$\mathcal{E} = \{B \subset X \mid \text{ for every } E \subset X,\ \mu^{outer}(B\cap E) + \mu^{outer}(B\setminus E) = \mu^{outer}(B)\}.$$

Then, $\mathcal{E}$ is a σ-algebra, $\mathcal{A} \subset \mathcal{E}$ and $\bar{\mu} = \mu^{outer}|_{\mathcal{E}}$ is a measure map. Moreover, if μ is σ-finite, then $\bar{\mu}$ is the unique extension measure of μ to $\mathcal{E}$.

If $\mathcal{E}$ is the σ-algebra given in Theorem 0.6.1, we say that $\mathcal{E}$ is the *Carathéodory extension σ-algebra* of $(\mathcal{A},\mu)$ and $\bar{\mu}$ is the *Carathéodory extension measure* of μ.

0.6.2 Some basic properties of infinite sums

The space $[0,\infty]$ is provided with the canonical sum of $[0,\infty)$, which can be extended by using the formulas $\infty + r = r + \infty = \infty$, for $r \in [0,\infty)$, and $\infty + \infty = \infty$. Given any set I, we can consider the directed set $\mathrm{fin}(I)$ of the finite subsets of I, where, if $F, F' \in \mathrm{fin}(I)$, $F \geq F'$ if $F' \subset F$. It is interesting to note that, for a map $r\colon I \to [0,\infty]$, $i \mapsto r_i$, if the net $\left(\sum_{i\in F} r_i\right)_{F\in\mathrm{fin}(I)}$ has a limit in $[0,\infty]$ (considering the Alexandroff compactification topology of the usual topology of $[0,\infty)$), this limit is unique and it will be denoted by $\sum_{i\in I} r_i$.

Lemma 0.6.1. *For every set I and for every map $r\colon I \to [0,\infty]$, $i \mapsto r_i$, there exists the sum $\sum_{i\in I} r_i$ in $[0,\infty]$.*

Proof. It suffices to check that the map $\mathrm{fin}(I) \to [0,\infty]$, $F \mapsto \sum_{i\in F} r_i$, is monotone. □

Proposition 0.6.1. *Let I be a set and suppose that $I = \bigsqcup_{j\in J} I_j$. Then, for every map $r\colon I \to [0,\infty]$, $i \mapsto r_i$,*

$$\sum_{i\in I} r_i = \sum_{j\in J}\left(\sum_{i\in I_j} r_i\right).$$

This sum can be interpreted as an integral with respect to the counting measure.

0.7 Regular CW-complexes and subdivisions

The notion of CW-complex was introduced by J. H. C. Whitehead in [83]. These spaces admit a cellular decomposition as a disjoint union of cells of several dimensions. When the dimension of the cells is less than or equal to n and there is a cell of dimension n, one has an n-dimensional CW-complex.

Some interesting classes of CW-complexes are the regular CW-complexes, which have injective characteristic maps and the "boundary" of each cell has a canonical structure of CW-complex. In the same way that for a simplicial complex, one can consider the canonical barycentric subdivision: for a regular CW-complex, one can also subdivide in many different ways all the cells to obtain a new regular CW-structure, which is a subdivision of the initial CW-structure. When one iterates this process, one obtains a consecutive sequence of subdivisions, having more new "small" cells. In this work, a regular iterated subdivision of a regular CW-complex X is given by a sequence of regular CW-structures $\Gamma_*^0, \Gamma_*^1, \dots$ such that Γ_*^0 is the initial cellular decomposition of X and, for every $r \in \mathbb{N}$, Γ_*^{r+1} is a subdivision of Γ_*^r.

Let $D^n = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| \leq 1\}$ be the unit n-disk, where $|\cdot|$ is the Euclidean norm of the n-dimensional Euclidean space $\mathbb{R}^n$ and $\mathring{D}^n = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| < 1\}$.

Given a topological space X and a continuous map $\gamma\colon D^n \to X$, we denote $\mathring{\gamma} = \gamma(\mathring{D}^n)$, $\bar{\gamma} = \gamma(D^n)$ and $\dot{\gamma} = \bar{\gamma} \setminus \mathring{\gamma}$ (note that, for $n = 0$, $\mathring{\gamma} = \bar{\gamma}$). If the map γ verifies that $\gamma|_{\mathring{D}^n}$ is injective, it is said that γ is an *n-characteristic map*. An *n-cell* is a space homeomorphic to $\mathring{D}^n$ and a *cell* is an n-cell for some n.

A *CW-structure* on a Hausdorff space X consists of a graduate family $\Gamma_*(X) = \bigsqcup_{n\in\mathbb{N}} \Gamma_n(X)$ such that:

(1) If $\gamma \in \Gamma_n(X)$, then $\gamma\colon D^n \to X$ is an n-characteristic map.

(2) $X = \bigsqcup_{\gamma\in\Gamma_*(X)} \mathring{\gamma}$.

(3) X has the finite closure property: for each $\gamma \in \Gamma_*(X)$, the family $\{\beta \in \Gamma_*(X) \mid \bar{\gamma} \cap \mathring{\beta} \neq \emptyset\}$ is finite.

(4) X has the weak topology: $F \subset X$ is closed if, for every $\gamma \in \Gamma_*(X)$, $F \cap \bar{\gamma}$ is closed in $\bar{\gamma}$ ($\bar{\gamma}$ is provided with the quotient topology).

A *CW-complex* is a pair $(X, \Gamma_*(X))$, where X is a Hausdorff space and $\Gamma_*(X)$ is a CW-structure on X.

Given a CW-complex $X = (X, \Gamma_*(X))$, for $\Gamma'_* \subset \Gamma_*(X)$ the subspace $Y = \bigsqcup_{\gamma'\in\Gamma'_*} \mathring{\gamma}'$ is said to be a *subcomplex* of X if it satisfies that, if $\gamma' \in \Gamma'_*$, then $\bar{\gamma}' \subset Y$. A subcomplex Y is a closed subset of X and the family of cells Γ'_* is a CW-structure of Y when the relative topology is taken in Y. For each $n \in \mathbb{N}$, the subcomplex induced by

$$\Gamma_*(X)_n = \{\gamma \in \Gamma_*(X) \mid \gamma \text{ is a } q\text{-characteristic map with } q \leq n\}$$

is called the *n-skeleton* of X and it is denoted by X_n (X_{-1} is usually taken as the empty set). It is said that a CW-complex X is *n-dimensional* if $X = X_n$ and $X \neq X_{n-1}$. Given a subset $A \subset X$, the *star* $\mathrm{st}(A, \Gamma_*(X))$ of A is the minimal subcomplex of $\Gamma_*(X)$ that contains the set $\{\gamma \in \Gamma_*(X) \mid A \cap \bar{\gamma} \neq \emptyset\}$.

We shall use the following regularity notion: suppose that $\Gamma_*(X)$ is a CW-structure on a Hausdorff space X. $\Gamma_*(X)$ is said to be *regular* if, for every $\gamma \in \Gamma_*(X)$, γ is an injective map (this implies that $\bar{\gamma}$ is a subcomplex of X, see [55]).

Suppose that $\Gamma_*(X)$ and $\Gamma'_*(X)$ are CW-structures on X. $\Gamma'_*(X)$ is said to be a *subdivision* of $\Gamma_*(X)$ if, for every $\alpha \in \Gamma_*(X)$, there is $\mathrm{Sd}(\alpha) \subset \Gamma'_*(X)$ such that $\bar{\alpha} = \bigsqcup_{\beta \in \mathrm{Sd}(\alpha)} \mathring{\beta}$. An *iterated subdivision* on a Hausdorff space X is a sequence

$$\Gamma^0_*(X), \Gamma^1_*(X), \Gamma^2_*(X), \ldots$$

of CW-structures on X such that, for every $r \in \mathbb{N}$, $\Gamma^{r+1}_*(X)$ is a subdivision of $\Gamma^r_*(X)$.

We can consider the bi-graduate family $\Gamma^*_*(X) = \bigsqcup_{r \in \mathbb{N}} \Gamma^r_*(X)$ and its associated subdivision operator $\mathrm{Sd}\colon 2^{\Gamma^r_*(X)} \to 2^{\Gamma^{r+1}_*(X)}$. If each $\Gamma^r_*(X)$ is regular, we say that $\Gamma^*_*(X)$ is a *regular iterated subdivision* on X. An *iterated subdivision on a CW-complex* X is an iterated subdivision $\Gamma^*_*(X)$ such that $\Gamma^0_*(X)$ is the initial CW-structure on X. It is interesting to note that $\Gamma^*_*(X)$ is countable if and only if $\Gamma^0_*(X)$ is countable. A *regular iterated subdivision* $\Gamma^*_*(X)$ of a CW-complex X whose topology $\mathbf{t}_X$ is induced by a metric $d\colon X \times X \to [0, \infty)$ can satisfy the following *vanishing-star property*: for every sequence of characteristic maps $\gamma_r \in \Gamma^r_*(X)$ such that, for each $r \in \mathbb{N}$, $\mathring{\gamma}_{r+1} \subset \mathring{\gamma}_r$, one has that

$$\lim_{r \to \infty} \mathrm{diam}(\mathrm{st}(\mathring{\gamma}_r, \Gamma^r_*(X))) = 0,$$

where $\mathrm{diam}(Y)$ denotes the diameter of the subcomplex Y (similarly, we shall denote the diameter of a cell σ by $\mathrm{diam}(\sigma)$). Note that, in this case, the family of interiors of stars

$$\{\mathrm{Int}(\mathrm{st}(v, \Gamma^r_*(X)))\}_{v \in \Gamma^*_0(X)}$$

is a base for the topology $\mathbf{t}_X$.

Chapter 1

Connections between limit spaces and end sets of exterior spaces

Once we have remembered some of the end sets and limit spaces of an exterior space in the section 0.4 of the preceding chapter, the objective of this chapter is to construct natural transformations that connect them. Towards the end of the chapter, we shall study the connections among three of these end sets, π_0^{BG}, π_0^{S} and $\check{\pi}_0$, as well as certain interesting properties of the map $R_X\colon \pi_0^{\mathrm{S}}(X) \to \pi_0^{\mathrm{BG}}(X)$ such as its canonical decomposition into the maps $\phi_X\colon \pi_0^{\mathrm{S}}(X) \to \check{\pi}_0(X)$ and $\theta_X\colon \check{\pi}_0(X) \to \pi_0^{\mathrm{BG}}(X)$ when X is a first-countable at infinity exterior space.

1.1 Natural transformations between limit and end spaces

Just by applying the notion of inverse limit and its universal property, we can induce immediately several natural transformations between some of the functors seen in section 0.4.

For instance, given an exterior space $X = (X, \varepsilon(X))$, we can consider, for all $E \in \varepsilon(X)$, the inclusion map $\imath_E\colon E \to \overline{E}$ given by $\imath(x) = x$, $\forall x \in E$. Taking inverse limits, we have an induced map $\Phi_X\colon L(X) \to \bar{L}(X)$. Now, the family $\{\Phi_X\}_{X\in|\mathbf{E}|}$ defines the natural transformation

$$\Phi\colon L \to \bar{L}.$$

Remind that, by Proposition 0.4.1, the functors L and $\bar{L}$ from the category $\mathbf{E}$ to the category of sets can also be given by $L(X) = \bigcap_{E\in\varepsilon(X)} E$ and $\bar{L}(X) = \bigcap_{E\in\varepsilon(X)} \overline{E}$, respectively.

Given again an exterior space $X = (X, \varepsilon(X))$, we can consider, for all $E \in \varepsilon(X)$, the map $\jmath_E\colon E \to \pi_0(E)$ given by $\jmath_E(x) = C_E^x$, where C_E^x is the unique path component of E containing the point x, $\forall x \in E$. Applying inverse limits, one has the induced map $\Psi_X\colon L(X) \to \check{\pi}_0(X)$, and the family $\{\Psi_X\}_{X\in|\mathbf{E}|}$ gives the natural transformation

$$\Psi\colon L \to \check{\pi}_0.$$

Similarly, the maps $\bar{\jmath}_E\colon \overline{E} \to \pi_0(\overline{E})$, $E \in \varepsilon(X)$, induce the map $\bar{\Psi}_X\colon \bar{L}(X) \to \check{\bar{\pi}}_0(X)$ and the family $\{\bar{\Psi}_X\}_{X\in|\mathbf{E}|}$ defines the natural transformation

$$\bar{\Psi}\colon \bar{L} \to \check{\bar{\pi}}_0.$$

Moreover, when one considers for all $E \in \varepsilon(X)$ the map (induced by the inclusion $E \hookrightarrow \overline{E}$) $\pi_0(\imath_E)\colon \pi_0(E) \to \pi_0(\overline{E})$ given by $\pi_0(\imath_E)(C_E) = C'_{\overline{E}}$, where $C'_{\overline{E}}$ is the unique path component of $\overline{E}$ containing C_E, one has the induced map $\Delta_X\colon \check{\pi}_0(X) \to \check{\bar{\pi}}_0(X)$ by taking inverse limits. Now, the family $\{\Delta_X\}_{X\in|\mathbf{E}|}$ gives the natural transformation

$$\Delta\colon \check{\pi}_0 \to \check{\bar{\pi}}_0.$$

What is more, since the diagram

$$\begin{tikzcd} E \arrow[r, "\imath_E"] \arrow[d, "\jmath_E"'] & \overline{E} \arrow[d, "\bar{\jmath}_E"] \\ \pi_0(E) \arrow[r, "\pi_0(\imath_E)"'] & \pi_0(\overline{E}) \end{tikzcd}$$

is commutative, for all $E \in \varepsilon(X)$, one obtains the following commutative diagram of natural transformations:

$$\begin{tikzcd} L \arrow[r, "\Phi"] \arrow[d, "\Psi"'] & \bar{L} \arrow[d, "\bar{\Psi}"] \\ \check{\pi}_0 \arrow[r, "\Delta"'] & \check{\bar{\pi}}_0 \end{tikzcd}$$

1.2 The natural transformation $R\colon \pi_0^{\mathrm{S}} \to \pi_0^{\mathrm{BG}}$

Remind that, by definition, one has that $\pi_0^{\mathrm{S}}(X) = \hom_{\pi\mathbf{E}}(\mathbb{R}_+, X)$ and $\pi_0^{\mathrm{BG}}(X) = \hom_{\pi\mathbf{E}}(\mathbb{N}, X)$. Let $f\colon (X, \varepsilon(X)) \to (Y, \varepsilon(Y))$ be an exterior map. Taking $\mathscr{A} = \pi\mathbf{E}$ and $g = \mathrm{in}\colon \mathbb{N} \to \mathbb{R}_+$, by Proposition 0.2.1 the diagram

$$\begin{tikzcd} \pi_0^{\mathrm{S}}(X) \arrow[r, "R_X^{\mathrm{in}}"] \arrow[d, "\pi_0^{\mathrm{S}}(f)"'] & \pi_0^{\mathrm{BG}}(X) \arrow[d, "\pi_0^{\mathrm{BG}}(f)"] \\ \pi_0^{\mathrm{S}}(Y) \arrow[r, "R_Y^{\mathrm{in}}"'] & \pi_0^{\mathrm{BG}}(Y) \end{tikzcd}$$

is commutative. Therefore, taking $R = \{R_X^{\mathrm{in}}\}_{X\in|\mathbf{E}|}$, we have the following.

Proposition 1.2.1. *The inclusion map* $\mathrm{in}\colon \mathbb{N} \to \mathbb{R}_+$ *induces a natural transformation*

$$R\colon \pi_0^{\mathrm{S}} \to \pi_0^{\mathrm{BG}}.$$

1.3 The natural transformation $\phi\colon \pi_0^{\mathrm{S}} \to \check{\pi}_0$

In order to give a natural transformation $\phi\colon \pi_0^{\mathrm{S}} \to \check{\pi}_0$, for each exterior space $X = (X, \varepsilon(X))$ define $\phi_X\colon \pi_0^{\mathrm{S}}(X) \to \check{\pi}_0(X)$ as follows: given an element $[\alpha] \in \pi_0^{\mathrm{S}}(X)$ represented by the exterior map $\alpha\colon \mathbb{R}_+ \to X$, $\phi_X([\alpha]) = (C_E)_{E\in\varepsilon(X)}$, where for each $E \in \varepsilon(X)$, C_E is the unique path component of E such that there is $r_E \in \mathbb{R}_+$ satisfying $\alpha([r_E, +\infty)) \subset C_E \subset E$.

Note that, given $E', E'' \in \varepsilon(X)$ such that $E'' \subset E'$, we can suppose that $r_{E'} \leq r_{E''}$ and hence $\alpha([r_{E''}, +\infty)) \subset \alpha([r_{E'}, +\infty))$. In order to see that this map is actually well-defined, we have to show that $C_{E''} \subset C_{E'}$ and that it does not depend on the chosen element of the exterior homotopy class $[\alpha]$.

To prove the first statement, observe that $\alpha([r_{E''}, +\infty)) \subset \alpha([r_{E'}, +\infty)) \subset C_{E'}$ and that $\alpha([r_{E''}, +\infty)) \subset C_{E''}$, so it follows that $C_{E'} \cap C_{E''} \supset \alpha([r_{E''}, +\infty)) \neq \emptyset$ and thus $C_{E'} \cup C_{E''}$ is path-connected; therefore, since $C_{E'}$ is a maximal arcwise connected subset, we can deduce that $C_{E''} \subset C_{E'}$. For the second statement, let $\beta\colon \mathbb{R}_+ \to X$ be an exterior map such that $[\alpha] = [\beta]$. Then, there exists an exterior homotopy $F\colon \mathbb{R}_+ \bar{\times} I \to X$ satisfying $F(r, 0) = \alpha(r)$ and $F(r, 1) = \beta(r)$, $\forall r \in \mathbb{R}_+$. Now, given $E \in \varepsilon(X)$, $\exists r_0 \in \mathbb{R}_+$ such that $F([r_0, +\infty) \times I) \subset E$. Thus,

$$F([r_0, +\infty) \times \{0\}) = \alpha([r_0, +\infty)) \subset C_E^\alpha \subset E,$$

$$F([r_0, +\infty) \times \{1\}) = \beta([r_0, +\infty)) \subset C_E^\beta \subset E.$$

Since $F([r_0, +\infty) \times I)$ is path-connected, it follows that $C_E^\alpha = C_E^\beta$ for every $E \in \varepsilon(X)$. Therefore, $\phi_X([\alpha]) = \phi_X([\beta])$.

Observe that, given $[\alpha] \in \pi_0^{\mathrm{S}}(X)$, we can give an alternative definition of $\phi_X([\alpha])$. Since $\check{\pi}_0$ is a functor, the exterior map $\alpha\colon \mathbb{R}_+ \to X$ induces a map $\check{\pi}_0(\alpha)\colon \check{\pi}_0(\mathbb{R}_+) \to \check{\pi}_0(X)$. Now, in order to define $\phi_X([\alpha])$, set:

$$\phi_X([\alpha]) = \check{\pi}_0(\alpha)(+\infty_{\mathbb{R}_+}),$$

where $+\infty_{\mathbb{R}_+}$ was defined in Example 0.4.1. Let us show that this gives the same result as the previous definition. Remind that $\phi_X([\alpha]) = (C_E)_{E\in\varepsilon(X)}$ so that, for each $E \in \varepsilon(X)$, $\exists r_E \in \mathbb{R}_+$ such that $\alpha([r_E, +\infty)) \subset C_E$. We have to prove that

$$\check{\pi}_0(\alpha)(+\infty_{\mathbb{R}_+}) = (C_E)_{E\in\varepsilon(X)}.$$

Note that, given $E \in \varepsilon(X)$, $\alpha^{-1}(E) \in \varepsilon^c(\mathbb{R}_+)$. Hence, there exists $n \in \mathbb{N}$ such that $(n, +\infty) \subset \alpha^{-1}(E)$. Then, $\alpha((n, +\infty)) \subset E$. Let C_E^α be the unique path component of E such that $\alpha((n, +\infty)) \subset C_E^\alpha$. We have that

$$\check{\pi}_0(\alpha)(+\infty_{\mathbb{R}_+}) = \check{\pi}_0(\alpha)((n, +\infty))_{n\in\mathbb{N}} = (C_E^\alpha)_{E\in\varepsilon(X)} \in \check{\pi}_0(X).$$

It is obvious that, if $\alpha((n, +\infty)) \subset C_E^\alpha$, then $\alpha([n+1, +\infty)) \subset C_E^\alpha$. Thus, there exists $r_E = n+1$ such that

$$\alpha([r_E, +\infty)) \subset C_E^\alpha \subset E, \quad \forall E \in \varepsilon(X).$$

Therefore, $(C_E)_{E\in\varepsilon(X)} = (C_E^\alpha)_{E\in\varepsilon(X)}$, as we wanted to show.

Proposition 1.3.1. *The family of maps* $\{\phi_X\}_{X\in|\mathbf{E}|}$ *defines a natural transformation*

$$\phi\colon \pi_0^{\mathrm{S}} \to \check{\pi}_0.$$

Proof. Let $f\colon X \to Y$ be a morphism in $\mathbf{E}$ and let $[\alpha]$ be an element of $\pi_0^{\mathrm{S}}(X)$ represented by the exterior map $\alpha\colon \mathbb{R}_+ \to X$. Then,

$$\begin{aligned}(\check{\pi}_0(f)\circ\phi_X)([\alpha]) &= \check{\pi}_0(f)(\check{\pi}_0(\alpha)(+\infty_{\mathbb{R}_+}))\\ &= (\check{\pi}_0(f)\circ\check{\pi}_0(\alpha))(+\infty_{\mathbb{R}_+})\\ &= \check{\pi}_0(f\circ\alpha)(+\infty_{\mathbb{R}_+}).\end{aligned}$$

Besides,

$$(\phi_Y\circ\pi_0^{\mathrm{S}}(f))([\alpha]) = \phi_Y([f\circ\alpha]) = \check{\pi}_0(f\circ\alpha)(+\infty_{\mathbb{R}_+}).$$

□

Proposition 1.3.2. *Let* $X = (X, \varepsilon(X))$ *be an exterior space which is first-countable at infinity. Then,*

$$\phi_X\colon \pi_0^{\mathrm{S}}(X) \to \check{\pi}_0(X)$$

is surjective.

Proof. Since X is first-countable at infinity, there exists a sequence $(E_i)_{i\in\mathbb{N}}$ with $E_i \in \varepsilon(X)$ and $E_i \supset E_{i+1}$, $\forall i \in \mathbb{N}$, which satisfies that, $\forall E \in \varepsilon(X)$, $\exists i_E \in \mathbb{N}$ such that $E \supset E_{i_E}$. Then, $\check{\pi}_0(X) = \lim_{E\in\varepsilon(X)} \pi_0(E) = \lim_{i\in\mathbb{N}} \pi_0(E_i)$. Let $b \in \check{\pi}_0(X)$, which can be represented in the form

$$b = (C_i)_{i\in\mathbb{N}} \in \lim_{i\in\mathbb{N}} \pi_0(E_i).$$

For each $i \in \mathbb{N}$, choose $p_i \in C_i$. Because $C_i \supset C_{i+1}$, $p_i, p_{i+1} \in C_i$, which is path-connected, so there is a path $f_i\colon [0,1] \to C_i$ such that $f_i(0) = p_i$ and $f_i(1) = p_{i+1}$.

Now, define a map $\alpha\colon [0,+\infty) \to X$ given by $\alpha(t) = f_i(t-i)$, whenever $i \le t < i+1$. Because for $E' \in \varepsilon(X)$ there exists $n_{E'} \in \mathbb{N}$ such that $E' \supset E_{n_{E'}}$, $\alpha([n_{E'},+\infty)) \subset C_{n_{E'}} \subset E_{n_{E'}} \subset E'$. This implies that α is exterior and we can take $[\alpha] \in \pi_0^{\mathrm{S}}(X)$. Furthermore, observe that $\alpha([i,+\infty)) \subset C_i \subset E_i$, for all $i \in \mathbb{N}$, so $\phi_X([\alpha]) = b$ and ϕ_X is surjective, as we wanted to show. □

1.4 The natural transformation $\theta\colon \check{\pi}_0 \to \pi_0^{\mathrm{BG}}$

Consider the category $\mathbf{E_{fc}}$ whose objects are exterior spaces which are first-countable at infinity and whose morphisms are exterior maps –that is, we consider $\mathbf{E_{fc}}$ as a full subcategory of $\mathbf{E}$. When there is no chance of confusion, the restriction functors $\pi_0^{\mathrm{BG}}|_{\mathbf{E_{fc}}}$, $\pi_0^{\mathrm{S}}|_{\mathbf{E_{fc}}}$ and $\check{\pi}_0|_{\mathbf{E_{fc}}}$ will be denoted by π_0^{BG}, π_0^{S} and $\check{\pi}_0$, respectively.

In order to construct a natural transformation from $\check{\pi}_0$ to π_0^{BG}, for each $X \in \mathbf{E_{fc}}$ note that there exists a countable basis $\{E_i\}_{i\in\mathbb{N}} \subset \varepsilon(X)$, $E_0 \supset E_1 \supset E_2 \ldots$, and define a map

$$\theta_X\colon \check{\pi}_0(X) \to \pi_0^{\mathrm{BG}}(X)$$

given as follows: let $a \in \check{\pi}_0(X)$, which can be represented by $a = (C_i)_{i\in\mathbb{N}}$ considering the basis above, where C_i is a path component of E_i verifying that $C_{i+1} \subset C_i$, and choose a point $p_i \in C_i$ for each $i \in \mathbb{N}$. Now, define $\alpha\colon \mathbb{N} \to X$ by $\alpha(i) = p_i$. We shall take $\theta_X(a) = [\alpha]$.

Let $X, Y \in \mathbf{E_{fc}}$. Suppose that $f\colon (X, \varepsilon(X)) \to (Y, \varepsilon(Y))$ is an exterior map and let $\{E_i^X\}_{i\in\mathbb{N}}$, $\{E_i^Y\}_{i\in\mathbb{N}}$ be bases for $\varepsilon(X)$ and $\varepsilon(Y)$, respectively, such that $E_i^X \supset E_{i+1}^X$ and $E_i^Y \supset E_{i+1}^Y$, for all $i \in \mathbb{N}$. Since f is exterior, given $i \in \mathbb{N}$ there exists n_i such that $f(E_{n_i}^X) \subset E_i^Y$. The election of the sequence $(n_i)_{i\in\mathbb{N}}$ can be chosen verifying $n_{i+1} > n_i$ so that $E_{n_i}^X \supset E_{n_{i+1}}^X$.

Note that, if $\{E_n^X\}_{n\in\mathbb{N}}$ is a basis for $\varepsilon(X)$, taking $E_i' = E_{n_i}^X$ for all $i \in \mathbb{N}$ one has that $\{E_i'\}_{i\in\mathbb{N}}$ is a basis of $\varepsilon(X)$ and $f(E_i') \subset E_i^Y$, $\forall i \in \mathbb{N}$. Thus, in the case that $f\colon X \to Y$ is exterior, there exist countable bases $\{E_n^X\}_{n\in\mathbb{N}}$ of $\varepsilon(X)$ and $\{E_n^Y\}_{n\in\mathbb{N}}$ of $\varepsilon(Y)$ such that $f(E_n^X) \subset E_n^Y$. This fact will allow us to simplify the calculations when studying if there is a natural transformation between the considered functors.

The following results (Lemma 1.4.1 and Proposition 1.4.1) state that the correspondence θ_X neither depends on the basis of the exterior space which has been chosen nor on the choice of points in the path components.

Lemma 1.4.1. *Let $X = (X, \varepsilon(X))$ be an exterior space. Suppose that X is first-countable at infinity and let $E_0 \supset E_1 \supset \dots$ be a countable basis of $\varepsilon(X)$. Assume that an end point $a \in \check{\pi}_0(X)$ is represented by the sequence $C_0 \supset C_1 \supset \dots$, where C_i is a path component of E_i. If $\alpha\colon \mathbb{N} \to X$ is an exterior map such that $\alpha(i) \in C_i$ and $n\colon \mathbb{N} \to \mathbb{N}$ is an increasing sequence (i.e., $n_i < n_{i+1}$), the map $\beta\colon \mathbb{N} \to X$ given by $\beta(i) = \alpha(n_i)$ is exterior and α is exterior homotopic to β.*

Proof. Let $\alpha\colon \mathbb{N} \to X$ be an exterior map. That means that, given $E \in \varepsilon(X)$, there exists $i \in \mathbb{N}$ such that $\alpha([i, +\infty)) \subset E$. Note that $i \leq n_i$, $\forall i \in \mathbb{N}$. Therefore, $[n_i, +\infty) \subset [i, +\infty)$ and $\alpha([n_i, +\infty)) \subset \alpha([i, +\infty)) \subset E$. Since the map $\beta\colon \mathbb{N} \to X$ is given by $\beta(i) = \alpha(n_i)$, it follows that $\beta([i, +\infty)) \subset E$ and β is exterior.

Suppose, moreover, that $\alpha(i) \in C_i$, for all $i \in \mathbb{N}$. Then, $\alpha(n_i) \in C_{n_i}$. Since $i \leq n_i$, $C_i \supset C_{n_i}$ and $\alpha(n_i) \in C_i$. Hence, there exists a path $f_i\colon I \to C_i$ such that $f_i(0) = \alpha(i)$ and $f_i(1) = \beta(i) = \alpha(n_i)$. Define $F\colon \mathbb{N}\bar{\times} I \to X$ given by $F(i, t) = f_i(t)$. We have that F is a homotopy from α to β. In order to prove that F is exterior, for $E' \in \varepsilon(X)$ there is $E_{i_{E'}} \in \{E_i\}_{i\in\mathbb{N}}$ such that $E' \supset E_{i_{E'}}$. Now, observe that $\alpha([i_{E'}, +\infty)) \subset C_{i_{E'}} \subset E_{i_{E'}}$ and then $\beta([i_{E'}, +\infty)) \subset C_{i_{E'}} \subset E_{i_{E'}}$, because $\beta([i_{E'}, +\infty)) \subset \alpha([n_{i_{E'}}, +\infty)) \subset \alpha([i_{E'}, +\infty))$. Therefore,

$$F([i, +\infty) \times I) \subset C_i \subset E_i \subset E', \quad \forall i \geq i_{E'},$$

and the proof is completed. □

Proposition 1.4.1. *Let $X = (X, \varepsilon(X))$ be an exterior space. Suppose that X is first-countable at infinity and let $\{E_i\}_{i\in\mathbb{N}}$ and $\{E_i'\}_{i\in\mathbb{N}}$, with $E_0 \supset E_1 \supset \dots$ and $E_0' \supset E_1' \supset \dots$, be countable bases of $\varepsilon(X)$. Assume that an end point $a \in \check{\pi}_0(X)$ is represented by the sequence $C_0 \supset C_1 \supset \dots$, where C_i is a path component of E_i, and by the sequence $C_0' \supset C_1' \supset \dots$, where C_i' is a path component of E_i'. For a given choice of points $\alpha_i \in C_i$ and $\alpha_i' \in C_i'$, the maps $\alpha, \alpha'\colon \mathbb{N} \to X$ given by $\alpha(i) = \alpha_i$ and $\alpha'(i) = \alpha_i'$ are exterior, and $\alpha \simeq_e \alpha'$.*

Proof. By definition of basis, one can find increasing sequences (n_i) and (n'_i) such that $E_i \cap E'_i \supset E'_{n'_i} \supset E_{n_i} \supset E'_{n'_{i+1}}$. By Lemma 1.4.1, $\alpha \simeq_e \beta$, where $\beta(i) = \alpha(n_i)$, and $\alpha' \simeq_e \beta'$, where $\beta'(i) = \alpha'(n'_i)$. Since $\beta(i), \beta'(i) \in C'_{n'_i}$ we can choose a path $f_i \colon I \to C'_{n'_i}$ such that $f_i(0) = \beta(i)$ and $f_i(1) = \beta'(i)$. Now, if we define $F \colon \mathbb{N}\bar{\times} I \to X$ by $F(i,t) = f_i(t)$, we have that F is an exterior map from β to β'.

Indeed, given $E \in \varepsilon(X)$, $\exists i_E \in \mathbb{N}$ such that $E \supset E'_{n'_{i_E}}$. We know that, for all $i \geq i_E$, $\beta(i), \beta'(i) \in C'_{n'_{i_E}} \subset E'_{n'_{i_E}}$, and hence we have that

$$F(i,t) = f_i(t) \subset C'_{n'_i} \subset E'_{n'_i} \subset E'_{n'_{i_E}}, \quad \forall i \geq i_E, \forall t \in I.$$

Therefore, $F([i_E, +\infty) \times I) \subset E'_{n'_{i_E}} \subset E$ so that $F(i,0) = \beta(i)$ and $F(i,1) = \beta'(i)$, $\forall i \in \mathbb{N}$. Then, one has $\alpha \simeq_e \beta \simeq_e \beta' \simeq_e \alpha'$. □

The previous results prove that $\theta_X \colon \check{\pi}_0(X) \to \pi_0^{\mathrm{BG}}(X)$ is well-defined. Let us suppose that X, Y are exterior spaces which are first-countable at infinity and let $f \colon X \to Y$ be an exterior map. If we take a countable basis for Y, $\{E_n^Y\}_{n\in\mathbb{N}}$, such that $E_n^Y \supset E_{n+1}^Y$, then we can suppose that there exists a basis for X, $\{E_n^X\}_{n\in\mathbb{N}}$, such that $f(E_n^X) \subset E_n^Y$, $\forall n \in \mathbb{N}$. We note that $a \in \check{\pi}_0(X)$ can be represented by $a = (C_n^X)_{n\in\mathbb{N}}$ and $\check{\pi}_0(f)(a) = (C_n^Y)_{n\in\mathbb{N}}$, where $C_n^Y \supset f(C_n^X)$.

We shall show that the family of maps $\theta_X \colon \check{\pi}_0(X) \to \pi_0^{\mathrm{BG}}(X)$ defines a natural transformation. For, we have to prove that $\theta_Y \circ \check{\pi}_0(f) = \pi_0^{\mathrm{BG}}(f) \circ \theta_X$. Let $a = (C_n^X)_{n\in\mathbb{N}} \in \check{\pi}_0(X)$. Then, $\check{\pi}_0(f)(a) = (C_n^Y)_{n\in\mathbb{N}}$ and $\theta_Y(\check{\pi}_0(f)(a)) = \theta_Y((C_n^Y)_{n\in\mathbb{N}}) = [\beta]$, which satisfies that $\beta(n) \in C_n^Y$, $\forall n \in \mathbb{N}$. Besides, $\pi_0^{\mathrm{BG}}(f)(\theta_X(a)) = \pi_0^{\mathrm{BG}}(f)([\alpha]) = [f \circ \alpha]$, which satisfies that $\alpha(n) \in C_n^X$, $\forall n \in \mathbb{N}$. Let us see that $[\beta] = [f \circ \alpha]$. Since $f(C_n^X) \subset C_n^Y$ and $\alpha(n) \in C_n^X$, we have that $f(\alpha(n)) \in f(C_n^X) \subset C_n^Y$. Hence, there is a path $g_n \colon I \to C_n^Y \subset E_n^Y$ such that $g_n(0) = (f \circ \alpha)(n)$, $g_n(1) = \beta(n)$. Therefore, there exists an exterior homotopy $G \colon \mathbb{N}\bar{\times} I \to X$ from $f \circ \alpha$ to β given by $G(n,t) = g_n(t)$, $\forall n \in \mathbb{N}$ and then $[f \circ \alpha] = [\beta]$. Thus, we have just proved the following.

Proposition 1.4.2. *The family of maps $\theta = \{\theta_X\}_{X \in |\mathbf{E_{fc}}|}$ is a natural transformation from $\check{\pi}_0|_{\mathbf{E_{fc}}}$ to $\pi_0^{\mathrm{BG}}|_{\mathbf{E_{fc}}}$.*

In addition, we also have the following interesting result.

Proposition 1.4.3. *Let $X = (X, \varepsilon(X))$ be an exterior space which is first-countable at infinity. Then,*

$$\theta_X \colon \check{\pi}_0(X) \to \pi_0^{\mathrm{BG}}(X)$$

is injective.

Proof. Suppose that $E_0 \supset E_1 \supset \dots$ is a countable basis for $\varepsilon(X)$. Take $a, b \in \check{\pi}_0(X)$, $a = (C_{E_n})_{E_n \in \varepsilon(X)}$, $b = (C'_{E_n})_{E_n \in \varepsilon(X)}$ and set $\theta_X(a) = [\alpha]$, $\alpha(n) \in C_{E_n}$ for all $n \in \mathbb{N}$ and $\theta_X(b) = [\alpha']$, $\alpha'(n) \in C'_{E_n}$ for all $n \in \mathbb{N}$. If $[\alpha] = [\alpha']$, then there is an exterior homotopy $F \colon \mathbb{N}\bar{\times} I \to X$ such that $F(n,0) = \alpha(n)$ and $F(n,1) = \alpha'(n)$, $\forall n \in \mathbb{N}$. Given $E_n \in \varepsilon(X)$, there is $m \geq n$ such that $F(\{k\} \times I) \subset E_n$ for every $k \geq m$. This implies that C_{E_n} contains $F(\{m\} \times I)$ and C'_{E_n} contains $F(\{m\} \times I)$. Thus, $C_{E_n} = C'_{E_n}$, $\forall n \in \mathbb{N}$. Therefore, $a = b$. □

1.5 Connections among π_0^{BG}, π_0^{S} and $\check{\pi}_0$

Definition 1.5.1. *Let* $\mathrm{sh}\colon \mathbb{N} \to \mathbb{N}$ *be the shift operator given by* $\mathrm{sh}(i) = i+1$, $\forall i \in \mathbb{N}$. *We shall define the shift of a given map* $f\colon \mathbb{N} \to X$ *as the composition* $\mathrm{Sh}(f) = f \circ \mathrm{sh}$.

Note that, if $\alpha\colon \mathbb{N} \to X$ is an exterior map, by definition we have that, given $E \in \varepsilon(X)$, there is $n_0 \in \mathbb{N}$ such that $\alpha([n_0, +\infty)) \subset E$. Thus, $\mathrm{Sh}(\alpha)$ is also an exterior map, since $\mathrm{Sh}(\alpha)(n_0) = \alpha \circ \mathrm{sh}(n_0) = \alpha(n_0+1)$ and then $\mathrm{Sh}(\alpha)([n_0, +\infty)) = \alpha([n_0+1, +\infty)) \subset \alpha([n_0, +\infty)) \subset E$.

Definition 1.5.2. *Let* $X = (X, \varepsilon(X))$ *be an exterior space. We define* $\mathrm{Sh}_X\colon \pi_0^{\mathrm{BG}}(X) \to \pi_0^{\mathrm{BG}}(X)$ *by*

$$\mathrm{Sh}_X([\alpha]) = [\mathrm{Sh}(\alpha)], \quad [\alpha] \in \pi_0^{\mathrm{BG}}(X).$$

Observe that the map $\mathrm{Sh}_X\colon \pi_0^{\mathrm{BG}}(X) \to \pi_0^{\mathrm{BG}}(X)$ is well-defined, since if $\alpha, \beta\colon \mathbb{N} \to X$ are exterior maps such that $\alpha \simeq_e \beta$, there is an exterior homotopy $F\colon \mathbb{N}\bar{\times} I \to X$ such that $F(n,0) = \alpha(n)$ and $F(n,1) = \beta(n)$, for all $n \in \mathbb{N}$. Taking $G(n,t) = F(n+1,t)$, we have that the map $G\colon \mathbb{N}\bar{\times} I \to X$ is an exterior homotopy such that $G(n,0) = \alpha(n+1) = \mathrm{Sh}(\alpha)(n)$ and $G(n,1) = \beta(n+1) = \mathrm{Sh}(\beta)(n)$, for all $n \in \mathbb{N}$, and therefore $\mathrm{Sh}(\alpha) \simeq_e \mathrm{Sh}(\beta)$.

Theorem 1.5.1. *Let* R *and* ϕ *be the natural transformations defined in sections 1.2 and 1.3, respectively. Let* $X = (X, \varepsilon(X))$ *be an exterior space and let* $\mathrm{Id}_X, \mathrm{Sh}_X\colon \pi_0^{\mathrm{BG}}(X) \to \pi_0^{\mathrm{BG}}(X)$ *be maps given by* $\mathrm{Id}_X([\alpha]) = [\alpha]$ *and* $\mathrm{Sh}_X([\alpha]) = [\mathrm{Sh}(\alpha)]$, $[\alpha] \in \pi_0^{\mathrm{BG}}(X)$. *Then:*

(i) $\mathrm{Id}_X \circ R_X = \mathrm{Sh}_X \circ R_X$. *Furthermore,* $R_X(\pi_0^{\mathrm{S}}(X)) = \mathrm{Eq}(\mathrm{Id}_X, \mathrm{Sh}_X)$.

(ii) Let θ *be the natural transformation defined in section 1.4. If* X *is first-countable at infinity, then in the diagram*

$$\begin{tikzcd}
\pi_0^{\mathrm{S}}(X) \arrow[rr, "R_X"] \arrow[dr, "\phi_X"'] & & \pi_0^{\mathrm{BG}}(X) \arrow[r, shift left, "\mathrm{Id}_X"] \arrow[r, shift right, "\mathrm{Sh}_X"'] & \pi_0^{\mathrm{BG}}(X) \\
& \check{\pi}_0(X) \arrow[ur, "\theta_X"'] & &
\end{tikzcd}$$

we have that $R_X = \theta_X \circ \phi_X$, ϕ_X *is surjective and* θ_X *is injective. As a consequence,* $\theta_X\colon \check{\pi}_0(X) \to \pi_0^{\mathrm{BG}}(X)$ *is the equalizer of* Id_X *and* Sh_X.

Proof. (i) Let $\alpha\colon \mathbb{R}_+ \to X$ be an exterior map; therefore, given an exterior open subset $E \in \varepsilon(X)$, $\exists r_E \geq 0 \mid \alpha([r_E, +\infty)) \subset E$. Observe that the natural transformation $R\colon \pi_0^{\mathrm{S}} \to \pi_0^{\mathrm{BG}}$ is defined as follows: $R_X([\alpha]) = [\alpha|_{\mathbb{N}}]$, where the restriction $\alpha|_{\mathbb{N}}\colon \mathbb{N} \to X$ is given by $\alpha|_{\mathbb{N}}(n) = \alpha(n)$, for all $n \in \mathbb{N}$. We have to prove that $[\alpha|_{\mathbb{N}}] = [\mathrm{Sh}(\alpha|_{\mathbb{N}})] = [\alpha|_{\mathbb{N}} \circ \mathrm{sh}]$. Note that, for every $n \in \mathbb{N}$, there exists a path $f_n\colon I \to X$ given by $f_n(k) = \alpha(n+k)$, $k \in [0,1]$. Taking $F\colon \mathbb{N}\bar{\times} I \to X$ so that $F(n,k) = f_n(k)$ for each $n \in \mathbb{N}$, one has that

$$F(n,0) = f_n(0) = \alpha(n) = \alpha|_{\mathbb{N}}(n),$$

$$F(n,1) = f_n(1) = \alpha(n+1) = \alpha|_{\mathbb{N}}(n+1),$$

and F is exterior. Indeed, fix an exterior open subset $E' \in \varepsilon(X)$. Since α is exterior, there exists $r' \geq 0$ satisfying $\alpha([r', +\infty)) \subset E'$. Note that we can find $n' \in \mathbb{N}$ such that $n' \geq r'$; hence, $\alpha([n', +\infty)) \subset \alpha([r', +\infty)) \subset E'$ and we have that

$$F(\{n\} \times I) = \alpha([n, n+1)) \subset E', \quad \forall n \geq n',$$

since $F(n \times I) = f_n(I) \subset E'$, $\forall n \geq n'$. Thus, F is an exterior homotopy and $[\alpha|_{\mathbb{N}}] = [\mathrm{Sh}(\alpha|_{\mathbb{N}})]$, as we wanted to show.

In fact, we have just shown that $R_X(\pi_0^{\mathrm{S}}(X)) \subset \mathrm{Eq}(\mathrm{Id}_X, \mathrm{Sh}_X)$. To show the other inclusion, let $[\alpha] \in \pi_0^{\mathrm{BG}}(X)$ such that $[\alpha] = \mathrm{Sh}_X([\alpha]) = [\alpha \circ \mathrm{sh}]$. Then, there is an exterior homotopy $F\colon \mathbb{N}\bar{\times}I \to X$ such that $F(n, 0) = \alpha(n)$, $F(n, 1) = \mathrm{Sh}(\alpha)(n) = (\alpha \circ \mathrm{sh})(n) = \alpha(n+1)$, for all $n \in \mathbb{N}$. Hence, there exists a continuous map $\beta\colon [0, +\infty) \to X$ given by $\beta(t) = F(n, t-n)$, whenever $n \leq t < n+1$, $n \in \mathbb{N}$. Furthermore, β is exterior. In order to show that, observe that given $E_0 \in \varepsilon(X)$, there exists $n_0 \in \mathbb{N} \subset \mathbb{R}_+$ such that $F(\{n\} \times I) \subset E_0$, $\forall n \geq n_0$. Then, $\beta([n_0, +\infty)) \subset E_0$. Thus, $[\beta] \in \pi_0^{\mathrm{S}}(X)$ and verifies that $\beta(n) = F(n, 0) = \alpha(n) = \beta|_{\mathbb{N}}(n)$, $\forall n \in \mathbb{N}$. So $[\alpha] \in R_X(\pi_0^{\mathrm{S}}(X))$ and we also have that $\mathrm{Eq}(\mathrm{Id}_X, \mathrm{Sh}_X) \subset R_X(\pi_0^{\mathrm{S}}(X))$. It follows that $R_X(\pi_0^{\mathrm{S}}(X)) = \mathrm{Eq}(\mathrm{Id}_X, \mathrm{Sh}_X)$.

(ii) Let $\{E_i\}_{i \in \mathbb{N}}$ be a countable basis for $\varepsilon(X)$ which satisfies that $E_0 = X$ and $E_i \supset E_{i+1}$ for every $i \in \mathbb{N}$, and let $\alpha\colon \mathbb{R}_+ \to X$ be an exterior map, $[\alpha] \in \pi_0^{\mathrm{S}}(X)$. Suppose that $\phi_X([\alpha]) = (C_i)_{i \in \mathbb{N}} \in \check{\pi}_0(X)$. For each $i \in \mathbb{N}$, choose a point $\alpha_i' \in C_i$ and define a sequence $\alpha'\colon \mathbb{N} \to X$ given by $\alpha'(i) = \alpha_i'$. Then, one has that $(\theta_X \circ \phi_X)([\alpha]) = [\alpha']$ and $R([\alpha]) = [\alpha|_{\mathbb{N}}]$. We have to show that $[\alpha'] = [\alpha|_{\mathbb{N}}]$.

Given $i \in \mathbb{N}$, let $n_i \in \mathbb{N}$, $n_i \geq i$ such that $\alpha([n_i, +\infty)) \subset E_i$. We can suppose that $n_{i+1} > n_i$ and $n_0 = 0$. Note that

$$\alpha([n_i, +\infty)) \subset C_i \subset E_i \supset E_{i+1} \supset C_{i+1} \supset \alpha([n_{i+1}, +\infty)).$$

For each $k \in \mathbb{N}$ such that $n_i \leq k < n_{i+1}$, let $f_k\colon I \to C_i \subset E_i$ be a path such that $f_k(0) = \alpha(k)$ and $f_k(1) = \alpha_i'$. Define $F\colon \mathbb{N}\bar{\times}I \to X$ by $F(j, t) = f_{n_j}(t)$, $\forall j \in \mathbb{N}$. Hence, we have that, for every $j \in \mathbb{N}$, $F(j, 0) = f_{n_j}(0) = \alpha(n_j) = \alpha|_{\mathbb{N}}(n_j)$ and $F(j, 1) = f_{n_j}(1) = \alpha'(j)$. Furthermore, F is exterior, since $F(j, t) = f_{n_j}(t) \subset C_j \subset E_j$ and $C_j \supset C_{j+1}$, $\forall j \in \mathbb{N}$. Now, by Lemma 1.4.1, if we denote $n\colon \mathbb{N} \to \mathbb{N}$, $i \mapsto n_i$, we have:

$$\alpha|_{\mathbb{N}} \simeq_e \alpha|_{n(\mathbb{N})} \simeq_e \alpha'.$$

Therefore, $[\alpha|_{\mathbb{N}}] = [\alpha']$ and $R_X = \theta_X \circ \phi_X$. The fact that ϕ_X and θ_X are respectively surjective and injective follows from Proposition 1.3.2 and Proposition 1.4.3, and it immediately allows us to state that θ_X is the equalizer of Id_X and Sh_X.

□

Chapter 2

Limit spaces and end sets of an exterior discrete semi-flow

In this chapter, we remember the notion of exterior discrete semi-flow, which is one of the original contributions that were introduced by the author in [62], and we study it in more depth. This concept arises from the combination of the notions of exterior space and discrete semi-flow. That permits to apply the constructions and properties of exterior spaces to the study of the dynamics of discrete semi-flows.

In particular, the notions of limit space and end set of an exterior space can be used with a view to produce limit spaces of exterior discrete semi-flows. For the construction of different kinds of end points of an exterior discrete semi-flow X, we analyze a **d**-exterior discrete semi-flow $D \subset X$ so that the orbit of each point in D reaches an end point of the exterior discrete semi-flow X. We call "basin of an end point" the set of points in D which reach that end point. This construction allows us to decompose D into a disjoint union of basins.

The new functors that we are going to obtain will be defined from those presented in the previous chapters by restriction, that is to say, as subfunctors of them. As in those chapters, we will also study the connections among these new functors. When presenting them along this chapter, we shall first give their respective definitions and then explain why they are well-defined.

2.1 Exterior discrete semi-flows

Definition 2.1.1. *Let X be an exterior space, $X_{\mathbf{t}}$ the subjacent topological space and $X_{\mathbf{d}}$ the set X provided with the discrete topology. An* **exterior discrete semi-flow** *is a discrete semi-flow $\varphi\colon \mathbb{N} \times X_{\mathbf{t}} \to X_{\mathbf{t}}$ such that:*

(i) For any $n \in \mathbb{N}$, $\varphi^n\colon X \to X$ is exterior (this is equivalent to the simpler condition $\varphi^1\colon X \to X$ is exterior).

An exterior discrete semi-flow X is said to be a **d-exterior discrete semi-flow** *if satisfies the additional condition:*

(ii) The canonical map $\varphi\colon \mathbb{N}\bar{\times}X_{\mathbf{d}} \to X$ *is exterior.*

Given two exterior discrete semi-flows $(X, \varphi, \varepsilon(X))$ *and* $(Y, \psi, \varepsilon(Y))$*, an* **exterior discrete semi-flow morphism** *from* $(X, \varphi, \varepsilon(X))$ *to* $(Y, \psi, \varepsilon(Y))$ *consists of a discrete semi-flow morphism* $f\colon (X, \varphi) \to (Y, \psi)$ *such that* f *is exterior.*

Denote by $\mathbf{EF}(\mathbb{N})$ the category of exterior discrete semi-flows and by $\mathbf{E^dF}(\mathbb{N})$ the full subcategory of $\mathbf{d}$-exterior discrete semi-flows.

We shall adopt the following notational convention: an exterior discrete semi-flow will be denoted by a triplet $(X, \varphi, \varepsilon(X))$. Nevertheless, when the action φ or the externology is clear in a determined context, we will short the notation and we will use $(X, \varepsilon(X))$ or (X, φ); moreover, in many cases the notation will be reduced to X.

Example 2.1.1. *For a given discrete semi-flow* $X = (X, \varphi)$*, we can consider the externology* $\varepsilon^{\mathbf{r}}(X)$ *given by all the open subsets* E *such that, for every* $x \in X$*, there is* $n_x \in \mathbb{N}$ *such that, for* $n \geq n_x$*,* $\varphi(n, x) \in E$*. It is easy to check that* $(X, \varphi, \varepsilon^{\mathbf{r}}(X))$ *is a* $\mathbf{d}$*-exterior discrete semi-flow.*

Example 2.1.2. *Denote* $\mathbb{Z}_- = \{z \in \mathbb{Z} \mid z < 0\} = \{-1, -2, -3, \dots\}$*. Given an discrete semi-flow* $X = (X, \varphi)$*, a sequence* $x\colon \mathbb{Z}_- \to X$ *is said to be a* **backward sequence** *if* $\varphi^1(x_{k+1}) = x_k$*,* $\forall k \in \mathbb{Z}_-$*. We can consider the externology* $\varepsilon^{\mathbf{l}}(X)$ *given by all the open subsets* E *such that, for every backward sequence* $x\colon \mathbb{Z}_- \to X$*, there is* $z_x \in \mathbb{Z}_-$ *such that, for* $z \leq z_x$*,* $x_z \in E$*. We can check that* φ^1 *is an exterior map: if* $x\colon \mathbb{Z}_- \to X$ *is a backward sequence, then* $\varphi^1 \circ x\colon \mathbb{Z}_- \to X$ *is also a backward sequence; hence, there is* $z_{\varphi^1 \circ x}$ *such that, for all* $z \leq z_{\varphi^1 \circ x}$*,* $(\varphi^1 \circ x)_z = \varphi^1(x_z) \in E$*, and this implies that* $x_z \in (\varphi^1)^{-1}(E)$*,* $\forall z \leq z_{\varphi^1 \circ x}$*. Thus,* $(\varphi^1)^{-1}(E) \in \varepsilon^{\mathbf{l}}(X)$*. Therefore,* $(X, \varphi, \varepsilon^{\mathbf{l}}(X))$ *is an exterior discrete semi-flow.*

We have defined above the limit space of an exterior space. In particular, since an exterior discrete semi-flow X is an exterior space, we can consider its limit space $L(X)$ and its bar-limit space $\bar{L}(X)$.

Proposition 2.1.1. *Let* $X = (X, \varphi, \varepsilon(X))$ *be an exterior discrete semi-flow. Then, the spaces* $L(X), \bar{L}(X)$ *are right-invariant.*

Proof. Denote $f = \varphi^1$. One has that

$$f^{-1}(L(X)) = f^{-1}\left(\bigcap_{E \in \varepsilon(X)} E\right) = \bigcap_{E \in \varepsilon(X)} f^{-1}(E) \supset \bigcap_{E \in \varepsilon(X)} E = L(X).$$

This implies that $f(L(X)) \subset L(X)$. For $\bar{L}(X)$, the proof is similar. □

For an exterior discrete semi-flow X one has that the exterior map $f = \varphi^1\colon X \to X$ induce the maps $\pi_0^{\mathrm{BG}}(f)$, $\check{\pi}_0(f)$, $\check{\check{\pi}}_0(f)$ that give canonical discrete semi-flow structures on the corresponding sets $\pi_0^{\mathrm{BG}}(X)$, $\check{\pi}_0(X)$, $\check{\check{\pi}}_0(X)$. In this paper these exterior homotopy invariants are always taken as a set with a discrete semi-flow structure.

2.2 The region of exterior attraction of an exterior discrete semi-flow

Suppose that $X = (X, \varphi, \varepsilon(X))$ is an exterior discrete semi-flow. Consider

$$D(X) = D(X, \varphi, \varepsilon(X)) = \{x \in X \mid \varphi_x \text{ is exterior}\}.$$

Proposition 2.2.1. *Given $X = (X, \varphi, \varepsilon(X)) \in |\mathbf{EF}(\mathbb{N})|$, the subspace $D(X)$ is completely invariant and with the relative externology is a* $\mathbf{d}$*-exterior discrete semi-flow.*

Proof. Denote $D = D(X)$. Observe that $\varphi_{\varphi^1(x)} = \varphi_x \circ \mathrm{sh}$ (where the shift sh, given by $\mathrm{sh}(n) = n + 1$, is an exterior map). This implies that, if $x \in D$, then $\varphi^1(x) \in D$.

Now, in order to prove that D is left-invariant, suppose that $\varphi^1(x) = y$ and $y \in D$. Given an exterior open subset E, there is $n_0 \in \mathbb{N}$ such that $\varphi^m(y) \in E$, for every $m \geq n_0$. Then, $\varphi^k(x) \in E$, for every $k \geq n_0 + 1$. This implies that $x \in D$.

Finally, since X is an exterior discrete semi-flow, it follows that $\varphi^n \colon D \to D$ is exterior, for every $n \in \mathbb{N}$. From the definition of D, one has that $\varphi \colon \mathbb{N} \bar{\times} D_{\mathbf{d}} \to D$ is exterior. Therefore, D with the relative externology is a $\mathbf{d}$-exterior discrete semi-flow. □

Definition 2.2.1. *Suppose that $X = (X, \varphi, \varepsilon(X))$ is an exterior discrete semi-flow. Then, the* $\mathbf{d}$*-exterior discrete semi-flow $D(X)$ is said to be the* **region of exterior attraction** *of the exterior discrete semi-flow X.*

Example 2.2.1. *Let $X = \mathbb{R}$, $\varepsilon(X) = \varepsilon^c(\mathbb{R})$ and $\varphi \colon \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ such that $\varphi^1 \colon \mathbb{R} \to \mathbb{R}$ is given by $\varphi^1(x) = t \cdot x$, $t > 0$, $t \in \mathbb{R}$. Then,*

$$D(X) = \begin{cases} \mathbb{R} \setminus \{0\}, & \text{if } t > 1, \\ \emptyset, & \text{if } t \leq 1. \end{cases}$$

Example 2.2.2. *If we consider the externology $\varepsilon(X) = \{E \in \varepsilon^c(\mathbb{R}) \mid 0 \in E\}$ in the previous example, in this case we have that:*

$$D(X) = \begin{cases} \mathbb{R}, & \text{if } t \neq 1, \\ \{0\}, & \text{if } t = 1. \end{cases}$$

The following proposition states that the subspace $D(X)$ of an exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$ can be also seen as a functor.

Proposition 2.2.2. *The construction $D \colon \mathbf{EF}(\mathbb{N}) \to \mathbf{EF}(\mathbb{N})$ given by $D(X) = \{x \in X \mid \varphi_x$ is exterior$\}$, for $X \in |\mathbf{EF}(\mathbb{N})|$, and $D(f) = f|_{D(X)} \colon D(X) \to D(Y)$, for a morphism $f \colon X \to Y$ in $\mathbf{EF}(\mathbb{N})$, is a functor.*

Proof. Let $X = (X, \varphi, \varepsilon(X))$ and $Y = (Y, \psi, \varepsilon(Y))$ be exterior discrete semi-flows. Let $f \colon X \to Y$ be an exterior discrete semi-flow morphism. We have to prove that $D(f) = f|_{D(X)}$ is an exterior discrete semi-flow morphism from $D(X)$ to $D(Y)$: indeed, if $x \in D(X)$, then the composite $f \circ \varphi_x \colon \mathbb{N} \to Y$ is exterior and we have that

$$(f \circ \varphi_x)(n) = f(\varphi_x(n)) = f(\varphi(n, x)) = \psi(n, f(x)) = \psi_{f(x)}(n),$$

for all $n \in \mathbb{N}$. Hence, $f \circ \varphi_x = \psi_{f(x)}$ and it follows that $f(x) \in D(Y)$. □

Note that, for an exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$, each trajectory of a point of $D(X)$ has an end point given as follows: if $x \in D(X)$, we have the exterior map $\varphi_x \colon \mathbb{N} \to X$, $\varphi_x(n) = \varphi(n, x)$, which determines an end point $[\varphi_x] \in \pi_0^{\mathrm{BG}}(X)$. Then, the following canonical map is obtained:

$$\omega \colon D(X) \to \pi_0^{\mathrm{BG}}(X),$$

where $\omega(x) = [\varphi_x]$.

Definition 2.2.2. *An end point $a \in \pi_0^{\mathrm{BG}}(X)$ is said to be* **ω-representable** *if there exists $x \in D(X)$ such that $\omega(x) = a$.*

Also, observe that the inclusion $D(X) \subset X$ of exterior spaces induces the transformation $L(D(X)) \to L(X)$.

Proposition 2.2.3. *Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow. Then,*

$$L(D(X)) = L(X),$$

so that the following diagram is commutative up to natural isomorphism:

$$\begin{tikzcd}
\mathbf{EF}(\mathbb{N}) \arrow[rr, "D"] \arrow[dr, "L"'] & & \mathbf{E^dF}(\mathbb{N}) \arrow[dl, "L"] \\
& \mathbf{F}(\mathbb{N}) &
\end{tikzcd}$$

Proof. It suffices to keep in mind that if $x \in L(X)$, then $\varphi^n(x) \in E$, for all $n \in \mathbb{N}$ and for all $E \in \varepsilon(X)$. $\square$

Remark 2.2.1. *For a given exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$ such that $X \in |\mathbf{E_{fc}}|$, the diagram*

$$\begin{tikzcd}
L(X) \arrow[r, "\Psi"] \arrow[d] & \check{\pi}_0(X) \arrow[d, "\theta"] \\
D(X) \arrow[r, "\omega"'] & \pi_0^{\mathrm{BG}}(X)
\end{tikzcd}$$

is not commutative in general. Nevertheless, the diagram commutes if we have the following additional condition: "for every $x \in L(X)$, the points $x, \varphi^1(x)$ are in the same path component of $L(X)$". For instance, if $L(X) \subset \mathrm{Fix}(X)$, this condition is satisfied.

Remark 2.2.2. *Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow. Then:*

(i) Similarly as before, one can define

$$\bar{D}(X) = \bar{D}(X, \varphi, \varepsilon(X)) = \{x \in X \mid \forall E \in \varepsilon(X),\ \exists n_E \in \mathbb{N} \text{ such that } \varphi^n(x) \in \overline{E},\ \forall n \geq n_E\}.$$

Note that $D(X) \subset \bar{D}(X)$.

(ii) It is interesting to remark that $\bar{D}(X)$ is contained in the region of pseudo-weak attraction of $\bar{L}(X)$.

In order to see (ii), let $x \in \bar{D}(X)$. Then, for each $E \in \varepsilon(X)$, $\exists n_E \in \mathbb{N}$ such that $[n_E, +\infty) \cdot x \subset \overline{E}$, so $\overline{[n_E, +\infty) \cdot x} \subset \overline{E}$. Hence, one has that $\omega^{\mathbf{r}}(x) = \bigcap_{t \in \mathbb{N}} \overline{[t, +\infty) \cdot x} \subset \overline{E}$, $\forall E \in \varepsilon(X)$, and this implies that $\omega^{\mathbf{r}}(x) \subset \bigcap_{E \in \varepsilon(X)} \overline{E} = \bar{L}(X)$. Therefore, $x \in \mathrm{PWA}(\bar{L}(X))$.

Theorem 2.2.1. *The inclusion functor* $\mathrm{in}\colon \mathbf{E^dF}(\mathbb{N}) \to \mathbf{EF}(\mathbb{N})$ *is left adjoint to the functor* $D\colon \mathbf{EF}(\mathbb{N}) \to \mathbf{E^dF}(\mathbb{N})$.

Proof. Let $X = (X, \varphi, \varepsilon(X)) \in \mathbf{EF}(\mathbb{N})$ and $X' = (X', \psi, \varepsilon(X')) \in \mathbf{E^dF}(\mathbb{N})$. Let $f\colon X' \to D(X)$ be a morphism in $\mathbf{E^dF}(\mathbb{N})$ and let $g\colon \mathrm{in}(X') \to X$ be a morphism in $\mathbf{EF}(\mathbb{N})$. Define

$$\theta_{X',X}\colon \hom_{\mathbf{E^dF}(\mathbb{N})}(X', D(X)) \longrightarrow \hom_{\mathbf{EF}(\mathbb{N})}(\mathrm{in}(X'), X),$$

given by $\theta_{X',X}(f) = \iota_X \circ f$, where $\iota_X\colon D(X) \to X$ is given by $\iota_X(x) = x$. Besides, define

$$\tilde{\theta}_{X',X}\colon \hom_{\mathbf{EF}(\mathbb{N})}(\mathrm{in}(X'), X) \longrightarrow \hom_{\mathbf{E^dF}(\mathbb{N})}(X', D(X)),$$

given by $\tilde{\theta}_{X',X}(g) = D(g) = g|_{D(\mathrm{in}(X'))} = g|_{X'}$.

We are seeing now that $\tilde{\theta}_{X',X}$ is actually the inverse of $\theta_{X',X}$. For, note that:

$$(\theta_{X',X} \circ \tilde{\theta}_{X',X})(g) = \theta_{X',X}(g|_{X'}) = \iota_X \circ g|_{X'} = g,$$

$$(\tilde{\theta}_{X',X} \circ \theta_{X',X})(f) = \tilde{\theta}_{X',X}(\iota_X \circ f) = (\iota_X \circ f)|_{X'} = f.$$

It remains to prove that the "naturality condition" is satisfied. For, let $Y \in \mathbf{EF}(\mathbb{N})$ and $Y' \in \mathbf{E^dF}(\mathbb{N})$. Suppose that $\alpha\colon X \to Y$ and $\xi\colon Y' \to X'$ are morphisms in $\mathbf{EF}(\mathbb{N})$ and $\mathbf{E^dF}(\mathbb{N})$, respectively. Given $y \in Y' = \mathrm{in}(Y')$, we have that:

$$(\alpha \circ \theta_{X',X}(f) \circ \mathrm{in}(\xi))(y) = (\alpha \circ \iota_X \circ f \circ \mathrm{in}(\xi))(y) = (\alpha \circ \iota_X \circ f)(\xi(y)) = \alpha(f(\xi(y))),$$

$$(\theta_{X',X}(D(\alpha) \circ f \circ \xi))(y) = (\iota_Y \circ D(\alpha) \circ f \circ \xi)(y) = \iota_Y(\alpha|_{D(X)}(f(\xi(y)))) = \alpha(f(\xi(y))).$$

□

Observe that we have a forgetful functor $U\colon \mathbf{EF}(\mathbb{N}) \to \mathbf{E}$ which associates to each morphism of exterior discrete semi-flows $f\colon (X, \varphi, \varepsilon(X)) \to (X', \varphi', \varepsilon(X'))$ the morphism of exterior spaces $U(f) = f$, $U(f)\colon (X, \varepsilon(X)) \to (X', \varepsilon(X'))$. Hence, we can consider the following composites:

$$\mathbf{EF}(\mathbb{N}) \xrightarrow{U} \mathbf{E} \xrightarrow{L, \bar{L}, \pi_0^{\mathrm{S}}, \pi_0^{\mathrm{BG}}, \check{\pi}_0, \check{\bar{\pi}}_0} \mathbf{Set}$$

As a consequence, we can obtain limit functors and end sets for the category of exterior discrete semi-flows by taking the composites $L \circ U$, $\bar{L} \circ U$, $\pi_0^{\mathrm{S}} \circ U$, $\pi_0^{\mathrm{BG}} \circ U$, $\check{\pi}_0 \circ U$ and $\check{\bar{\pi}}_0 \circ U$, which will be denoted respectively just by L, $\bar{L}$, π_0^{S}, π_0^{BG}, $\check{\pi}_0$ and $\check{\bar{\pi}}_0$ when no confusion is possible.

2.3 ω-end sets of an exterior discrete semi-flow

Given an exterior discrete semi-flow, we can define some other different functors from $\mathbf{EF}(\mathbb{N})$ to $\mathbf{Set}$ by using the ones we have just constructed. We will start defining the functor ${}^{\omega}\pi_0^{\mathrm{BG}}$.

Definition 2.3.1. *We define the functor ${}^{\omega}\pi_0^{\mathrm{BG}}$ from $\mathbf{EF}(\mathbb{N})$ to $\mathbf{Set}$ as one that maps a given exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$ to the set*

$$ {}^{\omega}\pi_0^{\mathrm{BG}}(X) = \{a \in \pi_0^{\mathrm{BG}}(X) \mid \exists x \in D(X) \text{ such that } [\varphi_x] = a\} $$

and assigns each exterior discrete semi-flow morphism

$$ f\colon (X, \varphi, \varepsilon(X)) \to (Y, \psi, \varepsilon(Y)) $$

to the map

$$ {}^{\omega}\pi_0^{\mathrm{BG}}(f) = \pi_0^{\mathrm{BG}}(f)|_{{}^{\omega}\pi_0^{\mathrm{BG}}(X)}. $$

Remark 2.3.1. *We note that ${}^{\omega}\pi_0^{\mathrm{BG}}(X)$ is the image of the canonical map $\omega\colon D(X) \to \pi_0^{\mathrm{BG}}(X)$; that is, ${}^{\omega}\pi_0^{\mathrm{BG}}(X)$ is the set of ω-representable end points of $\pi_0^{\mathrm{BG}}(X)$.*

Let us check that the functor ${}^{\omega}\pi_0^{\mathrm{BG}}$ is well-defined. Indeed, let $a \in {}^{\omega}\pi_0^{\mathrm{BG}}(X)$; hence, there exists $x \in D(X)$ such that $a = [\varphi_x]$. Moreover, we have that, for all $n \in \mathbb{N}$,

$$ (f \circ \varphi_x)(n) = (f \circ \varphi)(n, x) = \psi(n, f(x)) = \psi_{f(x)}(n) $$

since f is an exterior discrete semi-flow morphism between X and Y. Thus, $[f \circ \varphi_x] = [\psi_{f(x)}]$ and it follows that

$$ {}^{\omega}\pi_0^{\mathrm{BG}}(f)(a) = \pi_0^{\mathrm{BG}}(f)(a) = [f \circ \varphi_x] = [\psi_{f(x)}] \in {}^{\omega}\pi_0^{\mathrm{BG}}(Y), $$

which is true because $f(D(X)) \subset D(Y)$. By Lemma 0.2.1, ${}^{\omega}\pi_0^{\mathrm{BG}}$ preserves identity morphisms and composites of morphisms.

In addition, there is a natural transformation from ${}^{\omega}\pi_0^{\mathrm{BG}}$ to π_0^{BG} given by

$$ \mathrm{in}_X^{\mathrm{BG}}\colon {}^{\omega}\pi_0^{\mathrm{BG}}(X) \to \pi_0^{\mathrm{BG}}(X), $$

where X runs through $\mathbf{EF}(\mathbb{N})$.

Remember that, for a given exterior discrete semi-flow $(X, \varphi, \varepsilon(X))$, an exterior map $f = \varphi^1\colon X \to X$ induces a map $\pi_0^{\mathrm{BG}}(f)$ which gives a discrete semi-flow structure on the set $\pi_0^{\mathrm{BG}}(X)$.

Proposition 2.3.1. *Suppose that $X = (X, \varphi, \varepsilon(X))$ is an exterior discrete semi-flow. Then:*

(i) The map $\omega\colon D(X, \varphi, \varepsilon(X)) \to \pi_0^{\mathrm{BG}}(X)$ is a discrete semi-flow morphism (between discrete semi-flow sets);

(ii) ${}^{\omega}\pi_0^{\mathrm{BG}}(X)$ is right-invariant;

(iii) $\mathrm{Sh} \circ \pi_0^{\mathrm{BG}}(\varphi^1) = \pi_0^{\mathrm{BG}}(\varphi^1) \circ \mathrm{Sh}$*; furthermore,*

$$^{\omega}\pi_0^{\mathrm{BG}}(X) \subset \{a \in \pi_0^{\mathrm{BG}}(X) \mid \mathrm{Sh}(a) = \pi_0^{\mathrm{BG}}(\varphi^1)(a)\}.$$

Proof. (i) Given $x \in D(X, \varphi, \varepsilon(X))$, we have that

$$\begin{aligned}\varphi_{\varphi^1(x)}(n) &= \varphi(n, \varphi^1(x)) = \varphi(n, \varphi(1, x)) = \varphi(1 + n, x) = \varphi^1(\varphi(n, x)) \\ &= (\varphi^1 \circ \varphi_x)(n).\end{aligned}$$

Then,

$$\begin{aligned}(\omega \circ \varphi^1)(x) &= \omega(\varphi^1(x)) = [\varphi_{\varphi^1(x)}] = [\varphi^1 \circ \varphi_x] = \pi_0^{\mathrm{BG}}(\varphi^1)([\varphi_x]) = \pi_0^{\mathrm{BG}}(\varphi^1)(\omega(x)) \\ &= (\pi_0^{\mathrm{BG}}(\varphi^1) \circ \omega)(x).\end{aligned}$$

Therefore, $\omega \circ \varphi^1 = \pi_0^{\mathrm{BG}}(\varphi^1) \circ \omega$.

(ii) The image of a semi-flow morphism is right-invariant.

(iii) $(\mathrm{Sh} \circ \pi_0^{\mathrm{BG}}(\varphi^1))([\alpha]) = [(\varphi^1 \circ \alpha) \circ \mathrm{sh}] = [\varphi^1 \circ (\alpha \circ \mathrm{sh})] = (\pi_0^{\mathrm{BG}}(\varphi^1) \circ \mathrm{Sh})([\alpha])$. Note that, if $x \in D$,

$$\mathrm{Sh}(\omega(x)) = \mathrm{Sh}([\varphi_x]) = [\varphi^1 \circ \varphi_x] = \pi_0^{\mathrm{BG}}(\varphi^1)([\varphi_x]) = \pi_0^{\mathrm{BG}}(\varphi^1)(\omega(x)).$$

□

Definition 2.3.2. *Given an exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$, the subspace denoted by*

$$D_a = \omega^{-1}(a), \quad a \in {}^{\omega}\pi_0^{\mathrm{BG}}(X),$$

will be called the **basin of** a.

If S is a subset of $^{\omega}\pi_0^{\mathrm{BG}}(X)$,

$$D_S = \omega^{-1}(S)$$

will be called the **basin of** S.

The map ω permits us to divide an exterior discrete semi-flow.

Corollary 2.3.1. *Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow. Then, there are the following induced partitions of X:*

$$X = (X \setminus D) \sqcup \left(\bigsqcup_{a \in {}^{\omega}\pi_0^{\mathrm{BG}}(X)} D_a \right),$$

$$X = (X \setminus D) \sqcup \left(\bigsqcup_{[a] \in {}^{\omega}\pi_0^{\mathrm{BG}}(X)/\sim} D_{[a]} \right),$$

where $D = D(X, \varphi, \varepsilon(X))$ and each $D_{[a]}$ is a completely invariant sub-flow of X (remember the notation employed in section 0.5: $[a]$ denotes the minimal completely invariant subset that contains the point $a \in {}^{\omega}\pi_0^{\mathrm{BG}}(X)$).

Remark 2.3.2. *Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow. If X is path-connected, the union of the basins of each cyclic point $a_1, \dots, a_n$ of a n-cycle in ${}^{\omega}\pi_0^{\mathrm{BG}}(X)$ is just the basin of the n-cycle $[a_1] = [a_2] = \cdots = [a_n]$.*

The following example will allow us to distinguish between the functors π_0^{BG} and ${}^{\omega}\pi_0^{\mathrm{BG}}$.

Example 2.3.1. *Let S be a set and define in $\prod_{\mathbb{N}} S$ the following equivalence relation: given $x, y \in \prod_{\mathbb{N}} S$, $x \sim y$ if there is $i_0 \in \mathbb{N}$ such that $x_i = y_i$ for every $i \geq i_0$. Consider the quotient set*

$$I(S) = \prod_{\mathbb{N}} S/\sim,$$

which is called the reduced product *of S.*

With this notion in mind, let $X = \mathbb{R}$ and $\varepsilon(X) = \{E \in \varepsilon^c(\mathbb{R}) \mid 0 \in E\}$. One has that

$$\pi_0^{\mathrm{BG}}(X) = I(\{-\infty, 0, +\infty\}).$$

Let $\{\varphi(t)\}_{t\in\mathbb{R}_+}$ be a one-parameter family of discrete semi-flows $\varphi(t)\colon \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ such that $(\varphi(t))^1\colon \mathbb{R} \to \mathbb{R}$ is given by $(\varphi(t))^1(x) = t \cdot x$. Associated with $\{\varphi(t)\}_{t\in\mathbb{R}_+}$, there is a family of exterior discrete semi-flows $\{(X, \varphi(t), \varepsilon(X))\}_{t\in\mathbb{R}_+}$ and the corresponding family of sets $\{{}^{\omega}\pi_0^{\mathrm{BG}}(X)(t)\}_{t\in\mathbb{R}_+}$, one for each discrete semi-flow. We will see that the cardinality of the sets ${}^{\omega}\pi_0^{\mathrm{BG}}(X)(t)$ is much lower. Note that, when $t > 1$, $[(\varphi(t))_x] = [(\varphi(t))_{x'}] \in {}^{\omega}\pi_0^{\mathrm{BG}}(X)(t)$ if and only if $\operatorname{sgn}(x) = \operatorname{sgn}(x')$. We have that:

$${}^{\omega}\pi_0^{\mathrm{BG}}(X)(t) = \begin{cases} \{[(\varphi(t))_0], a_{-\infty}, a_{+\infty}\}, & \text{if } t > 1, \\ \{[(\varphi(t))_0]\}, & \text{if } t \leq 1, \end{cases}$$

where $a_{+\infty} = [(\varphi(t))_x] \in {}^{\omega}\pi_0^{\mathrm{BG}}(X)(t)$ if $x > 0$ and $a_{-\infty} = [(\varphi(t))_x] \in {}^{\omega}\pi_0^{\mathrm{BG}}(X)(t)$ if $x < 0$.

Remark 2.3.3. *Example 2.3.1 shows that the invariant ${}^{\omega}\pi_0^{\mathrm{BG}}$ can differentiate between distinct types of discrete semi-flows belonging to the family $\{\varphi(t)\}_{t\in\mathbb{R}_+}$ and it is able to find singular values of the parameter t: for instance, when $t = 1$ the number of ω-representable end points is no longer 1 and becomes to be 3. Therefore, the parameter space $[0, +\infty)$ is divided into three intervals: $[0, +\infty) = [0, 1) \cup \{1\} \cup (1, +\infty)$. The one-parameter system is stable when $t \in [0, 1)$ and when $t \in (1, +\infty)$, and a trifurcation value appears at $t = 1$.*

We now continue defining the functor ${}^{\omega}\pi_0^{\mathrm{S}}$.

Definition 2.3.3. *We define the functor ${}^{\omega}\pi_0^{\mathrm{S}}$ from $\mathbf{EF}(\mathbb{N})$ to $\mathbf{Set}$ as one that maps a given exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$ to the set*

$${}^{\omega}\pi_0^{\mathrm{S}}(X) = \{[\alpha] \in \pi_0^{\mathrm{S}}(X) \mid \exists y \in D(X) \text{ such that } [\alpha|_{\mathbb{N}}] = [\varphi_y]\}$$

and assigns every exterior discrete semi-flow morphism

$$f\colon (X, \varphi, \varepsilon(X)) \to (Y, \psi, \varepsilon(Y))$$

to the map ${}^{\omega}\pi_0^{\mathrm{S}}(f)$, which is given by the formula

$${}^{\omega}\pi_0^{\mathrm{S}}(f)([\alpha]) = \pi_0^{\mathrm{S}}(f)([\alpha]) = [f \circ \alpha].$$

In order to see that ${}^{\omega}\pi_0^{\mathrm{S}}$ is well-defined, consider $[\alpha] \in {}^{\omega}\pi_0^{\mathrm{S}}(X)$; hence, there exists $x \in D(X)$ such that $[\alpha|_{\mathbb{N}}] = [\varphi_x]$. Moreover, we have that $f \circ \varphi_x = \psi_{f(x)}$, since f is an exterior discrete semi-flow morphism between X and Y. We also have that $\pi_0^{\mathrm{S}}(f)([\alpha]) = [f \circ \alpha]$; since $f(D(X)) \subset D(Y)$ and

$$[(f \circ \alpha)|_{\mathbb{N}}] = [f(\alpha|_{\mathbb{N}})] = [f \circ \varphi_x] = [\psi_{f(x)}],$$

it follows that $f(x) \in D(Y)$ and $[f \circ \alpha] \in {}^{\omega}\pi_0^{\mathrm{S}}(Y)$. By Lemma 0.2.1, ${}^{\omega}\pi_0^{\mathrm{S}}$ preserves identity morphisms and composites of morphisms.

In addition, there is a natural transformation from ${}^{\omega}\pi_0^{\mathrm{S}}$ to π_0^{S} given by the inclusions

$$\mathrm{in}_X^{\mathrm{S}} \colon {}^{\omega}\pi_0^{\mathrm{S}}(X) \to \pi_0^{\mathrm{S}}(X),$$

where X runs through $|\mathbf{EF}(\mathbb{N})|$.

In the lines below, we will define the functor ${}^{\omega}\check{\pi}_0$.

Definition 2.3.4. *We define the functor ${}^{\omega}\check{\pi}_0$ from $\mathbf{EF}(\mathbb{N})$ to $\mathbf{Set}$ as one that maps a given exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$ to the set*

$$\begin{aligned} {}^{\omega}\check{\pi}_0(X) = \{a = (C_E)_{E \in \varepsilon(X)} \in \check{\pi}_0(X) \mid \exists x \in D(X) \text{ such that} \\ \forall E \in \varepsilon(X),\ \exists n_E \in \mathbb{N} \text{ with } \varphi_x(n) \in C_E \subset E,\ \forall n \geq n_E\} \end{aligned}$$

and assigns every exterior discrete semi-flow morphism

$$f \colon (X, \varphi, \varepsilon(X)) \to (Y, \psi, \varepsilon(Y))$$

to the map ${}^{\omega}\check{\pi}_0(f)$, which is given by the formula

$${}^{\omega}\check{\pi}_0(f) = \check{\pi}_0(f)|_{{}^{\omega}\check{\pi}_0(X)}.$$

Let us check that the functor ${}^{\omega}\check{\pi}_0$ is well-defined. Let $a \in {}^{\omega}\check{\pi}_0(X)$ and suppose that $\check{\pi}_0(f)(a) = (C_{E^Y})_{E^Y \in \varepsilon(Y)}$. Given $E^Y \in \varepsilon(Y)$, since f is exterior we have that $f^{-1}(E^Y) \in \varepsilon(X)$. Then, there is $x \in X$ and $n_{f^{-1}(E^Y)}$ such that

$$\varphi_x(n) \in C_{f^{-1}(E^Y)}, \quad \forall n \geq n_{f^{-1}(E^Y)},$$

where $C_{f^{-1}(E^Y)}$ is the path component of a in $f^{-1}(E^Y)$. Take $f(x) \in Y$ and $n_{E^Y} = n_{f^{-1}(E^Y)}$. We have that

$$\psi_{f(x)}(n) = \psi(n, f(x)) = f(\varphi(n, x)) = f(\varphi_x(n)) \in f(C_{f^{-1}(E^Y)}) \subset C_{E^Y},$$

for every $n \geq n_{E^Y}$. This implies that $\check{\pi}_0(f)(a) \in {}^{\omega}\check{\pi}_0(Y)$. Since $\check{\pi}_0(f)({}^{\omega}\check{\pi}_0(X)) \subset {}^{\omega}\check{\pi}_0(Y)$, by applying Lemma 0.2.1 it follows that ${}^{\omega}\check{\pi}_0$ preserves identity morphisms and composites of morphisms.

In addition, there is a natural transformation $\check{\mathrm{in}}$ from ${}^{\omega}\check{\pi}_0$ to $\check{\pi}_0$ given by the inclusions

$$\check{\mathrm{in}}_X \colon {}^{\omega}\check{\pi}_0(X) \to \check{\pi}_0(X),$$

where X runs through $|\mathbf{EF}(\mathbb{N})|$.

As we stated in Remark 2.3.3, the functor ${}^{\omega}\pi_0^{\mathrm{BG}}$ was able to distinguish between different members of a one-parameter family of discrete semi-flows and it could find critical values for the parameter. So can functors ${}^{\omega}\pi_0^{\mathrm{S}}$ and ${}^{\omega}\check{\pi}_0$, as we will check in the next example.

Example 2.3.2. *Let $X = \mathbb{R}$, $\varepsilon(X) = \{E \in \varepsilon^c(\mathbb{R}) \mid 0 \in E\}$ and let $\{\varphi(t)\}_{t\in\mathbb{R}_+}$ be a one-parameter family of discrete semi-flows $\varphi(t)\colon \mathbb{N}\times\mathbb{R} \to \mathbb{R}$ such that $(\varphi(t))^1\colon \mathbb{R} \to \mathbb{R}$ is given by $(\varphi(t))^1(x) = t \cdot x$. Associated with $\{\varphi(t)\}_{t\in\mathbb{R}_+}$, there are two family of sets, $\{{}^{\omega}\pi_0^{\mathrm{S}}(X)(t)\}_{t\in\mathbb{R}_+}$ and $\{{}^{\omega}\check{\pi}_0(X)(t)\}_{t\in\mathbb{R}_+}$, with one set for each discrete semi-flow.*

Let us suppose that $[\alpha], [\alpha'] \in {}^{\omega}\pi_0^{\mathrm{S}}(X)(t)$ so that they satisfy $[\alpha|_{\mathbb{N}}] = [(\varphi(t))_x]$ and $[\alpha'|_{\mathbb{N}}] = [(\varphi(t))_{x'}]$. Note that, in the case $t > 1$, $[\alpha] = [\alpha']$ if and only if $\operatorname{sgn}(x) = \operatorname{sgn}(x')$. Then, we have that:

$$ {}^{\omega}\pi_0^{\mathrm{S}}(X)(t) = \begin{cases} \{a_0, a_{-\infty}, a_{+\infty}\}, & \text{if } t > 1, \\ \{a_0\}, & \text{if } t \leq 1, \end{cases} $$

where $a_{+\infty} = [\alpha] \in {}^{\omega}\pi_0^{\mathrm{S}}(X)(t)$ satisfying $[\alpha|_{\mathbb{N}}] = [(\varphi(t))_x]$ if $x > 0$, $a_{-\infty} = [\beta] \in {}^{\omega}\pi_0^{\mathrm{S}}(X)(t)$ satisfying $[\beta|_{\mathbb{N}}] = [(\varphi(t))_x]$ if $x < 0$ and $a_0 = [\gamma] \in {}^{\omega}\pi_0^{\mathrm{S}}(X)(t)$ satisfying $[\gamma|_{\mathbb{N}}] = [(\varphi(t))_0]$.

Something similar happens with functor ${}^{\omega}\check{\pi}_0$ when considering the same family of discrete semi-flows:

$$ {}^{\omega}\check{\pi}_0(X)(t) = \begin{cases} \{b_0, b_{-\infty}, b_{+\infty}\}, & \text{if } t > 1, \\ \{b_0\}, & \text{if } t \leq 1, \end{cases} $$

where, if we take $\{(-\infty, -n) \cup \left(-\frac{1}{n}, \frac{1}{n}\right) \cup (n, +\infty)\}_{n\in\mathbb{N}^*}$ as a basis for the externology, the elements $b_{+\infty}$, $b_{-\infty}$ and b_0 can be represented by $b_{+\infty} = ((n, +\infty))_{n\in\mathbb{N}^*}$, $b_{-\infty} = ((-\infty, -n))_{n\in\mathbb{N}^*}$ and $b_0 = \left(\left(-\frac{1}{n}, \frac{1}{n}\right)\right)_{n\in\mathbb{N}^*}$.

In the previous examples, when considering the family of discrete semi-flows $\varphi(t)\colon \mathbb{N}\times\mathbb{R} \to \mathbb{R}$ given by $(\varphi(t))^1(x) = t\cdot x$, one has probably noticed that the cardinalities of the sets ${}^{\omega}\pi_0^{\mathrm{BG}}(X)(t)$, ${}^{\omega}\pi_0^{\mathrm{S}}(X)(t)$ and ${}^{\omega}\check{\pi}_0(X)(t)$ are always the same, regardless of parameter t. Nevertheless, this fact happens completely by chance, as there are many cases (such as the next example) in which it does not occur.

Example 2.3.3. *Let $X = \mathbb{R}$, $\varepsilon(X) = \{E \in \varepsilon^c(\mathbb{R}) \mid 0 \in E\}$ and let $\{\varphi(t)\}_{t\in\mathbb{R}_+}$ be a one-parameter family of discrete semi-flows $\varphi(t)\colon \mathbb{N}\times\mathbb{R} \to \mathbb{R}$ such that $(\varphi(t))^1\colon \mathbb{R} \to \mathbb{R}$ is given by $(\varphi(t))^1(x) = -t \cdot x$. In this case,*

$$ {}^{\omega}\pi_0^{\mathrm{BG}}(X)(t) = \begin{cases} \{[\varphi_0], a_{\infty}^{+}, a_{\infty}^{-}\}, & \text{if } t > 1, \\ \{[\varphi_0]\}, & \text{if } t \leq 1, \end{cases} $$

where $a_{\infty}^{+} = [\varphi_x] \in {}^{\omega}\pi_0^{\mathrm{BG}}(X)(t)$ if $x > 0$, $a_{\infty}^{-} = [\varphi_x] \in {}^{\omega}\pi_0^{\mathrm{BG}}(X)(t)$ if $x < 0$ and $[\varphi_0] \in {}^{\omega}\pi_0^{\mathrm{BG}}(X)(t)$ is the class of equivalence of $\varphi_0\colon \mathbb{N} \to X$, which is given by $\varphi_0(n) = \varphi(t)(n, 0) = 0$.

However, same is not true when considering functors ${}^{\omega}\pi_0^{\mathrm{S}}$ and ${}^{\omega}\check{\pi}_0$, for every $t \in \mathbb{R}_+$, because ${}^{\omega}\pi_0^{\mathrm{S}}(X)(t) = \{a_0\}$ and ${}^{\omega}\check{\pi}_0(X)(t) = \{b_0\}$, where a_0 and b_0 are the end points which were defined in Example 2.3.2.

Remark 2.3.4. *Observe that, given an exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$, if $D(X) = \emptyset$, then ${}^{\omega}\pi_0^{\mathrm{S}}(X) = {}^{\omega}\check{\pi}_0(X) = \emptyset$; however, the converse is not true. For instance, consider $X = \mathbb{R} \setminus \{0\}$, $\varepsilon(X) = \{E \cap (\mathbb{R} \setminus \{0\}) \mid E \in \varepsilon^c(\mathbb{R})\}$ (which is the relative externology in $\mathbb{R} \setminus \{0\} \subset \mathbb{R}$) and $\varphi\colon \mathbb{N} \times \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ such that $\varphi^1\colon \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ is given by $\varphi^1(x) = -2x$. In this case, ${}^{\omega}\pi_0^{\mathrm{S}}(X) = {}^{\omega}\check{\pi}_0(X) = \emptyset$, but $D(X) = X \neq \emptyset$.*

As in chapter 1, we will show that there is a natural transformation between each pair of functors which have just been defined. For example, a natural transformation ${}^{\omega}R\colon {}^{\omega}\pi_0^{\mathrm{S}} \to {}^{\omega}\pi_0^{\mathrm{BG}}$ can be given by the family of maps

$$ {}^{\omega}R_X\colon {}^{\omega}\pi_0^{\mathrm{S}}(X) \to {}^{\omega}\pi_0^{\mathrm{BG}}(X), $$

where, for $X = (X, \varphi, \varepsilon(X))$ an exterior discrete semi-flow and $[\alpha] \in {}^{\omega}\pi_0^{\mathrm{S}}(X)$, ${}^{\omega}R_X([\alpha]) = [\alpha|_{\mathbb{N}}]$. Note that ${}^{\omega}R_X$ is well-defined, since, if $[\alpha|_{\mathbb{N}}] = [\varphi_x]$ with $x \in D(X)$, one has that ${}^{\omega}R_X([\alpha]) = [\alpha|_{\mathbb{N}}] = [\varphi_x]$, which is in ${}^{\omega}\pi_0^{\mathrm{BG}}(X)$ and, by Lemma 0.2.2, ${}^{\omega}R$ is actually a natural transformation.

In order to give a natural transformation ${}^{\omega}\phi\colon {}^{\omega}\pi_0^{\mathrm{S}} \to {}^{\omega}\check{\pi}_0$, for each exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$ define

$$ {}^{\omega}\phi_X\colon {}^{\omega}\pi_0^{\mathrm{S}}(X) \to {}^{\omega}\check{\pi}_0(X) $$

by ${}^{\omega}\phi_X = \phi_X|_{{}^{\omega}\pi_0^{\mathrm{S}}(X)}$.

To see that ${}^{\omega}\phi_X$ is well-defined, we have to check that $\phi_X({}^{\omega}\pi_0^{\mathrm{S}}(X)) \subset {}^{\omega}\check{\pi}_0(X)$. Take $a \in {}^{\omega}\pi_0^{\mathrm{S}}(X)$ such that $a = [\alpha]$, where $\alpha\colon \mathbb{R}_+ \to X$ is exterior. Let

$$ \phi_X(a) = (C_E)_{E \in \varepsilon(X)} $$

so that, for a given $E \in \varepsilon(X)$, there exists $n_E \in \mathbb{N}$ such that $\alpha([n_E, +\infty)) \subset C_E \subset E$. Because $a \in {}^{\omega}\pi_0^{\mathrm{S}}(X)$, then there is $x \in D(X)$ such that $[\alpha|_{\mathbb{N}}] = [\varphi_x]$. Hence, there exists an exterior homotopy $F\colon \mathbb{N}\bar{\times}I \to X$ such that $F(n, 0) = \alpha(n)$ and $F(n, 1) = \varphi_x(n)$, $\forall n \in \mathbb{N}$. Since F is exterior, there is $n'_E \in \mathbb{N}$ such that

$$ F(\{n\} \times I) \subset E, \forall n \geq n'_E. $$

Take $n''_E = \max\{n_E, n'_E\}$. If $n \geq n''_E$, we have that $\alpha([n, +\infty)) \subset C_E$ and $F(\{n\} \times I) \subset E$, satisfying $\alpha([n, +\infty)) \cap F(\{n\} \times I) \neq \emptyset$. Thus, $F(\{n\} \times I) \subset C_E$ and it follows that

$$ \varphi_x(n) \in C_E, \quad \forall n \geq n''_E. $$

Therefore, ${}^{\omega}\phi_X([\alpha]) \in {}^{\omega}\check{\pi}_0(X)$. By Lemma 0.2.2, the family of maps ${}^{\omega}\phi_X\colon {}^{\omega}\pi_0^{\mathrm{S}}(X) \to {}^{\omega}\check{\pi}_0(X)$ defines a natural transformation.

Proposition 2.3.2. *Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow whose subjacent exterior space X is first-countable at infinity. Then,*

$$ {}^{\omega}\phi_X\colon {}^{\omega}\pi_0^{\mathrm{S}}(X) \to {}^{\omega}\check{\pi}_0(X) $$

is surjective.

Proof. Since X is first-countable at infinity, there exists a sequence $(E_i)_{i\in\mathbb{N}}$ with $E_i \in \varepsilon(X)$ and $E_i \supset E_{i+1}$, $\forall i \in \mathbb{N}$, which satisfies that, $\forall E \in \varepsilon(X)$, $\exists i_E \in \mathbb{N}$ such that $E \supset E_{i_E}$. Let $b \in {}^{\omega}\check{\pi}_0(X)$, which can be represented in the form $b = (C_i)_{i\in\mathbb{N}} \in \lim_{i\in\mathbb{N}} \pi_0(E_i)$. Since b is ω-representable, there exists $x \in D(X)$ such that, for all $i \in \mathbb{N}$, $\exists n_i \in \mathbb{N}$ satisfying $\varphi_x(n) \in C_i \subset E_i$, $\forall n \geq n_i$. Note that we can suppose that $n_0 < n_1 < \dots$ Particularly, taking $i = 0$, one has that $\exists n_0 \in \mathbb{N}$ such that $\varphi_x(n) \in C_0 \subset E_0$, $\forall n \geq n_0$. Let $x' = n_0 \cdot x \in D(X)$. Then, for all $i \in \mathbb{N}$, $\exists n'_i = (n_i - n_0) \in \mathbb{N}$ such that $\varphi_{x'}(n) \in C_i \subset E_i$, $\forall n \geq n'_i$, and also $\varphi_{x'}(n) \in C_0 \subset E_0$, $\forall n \in \mathbb{N}$. Similarly as before, we have $0 = n'_0 < n'_1 < \dots$, too.

Let $j_i = \max\{j \in \mathbb{N} \mid \varphi_{x'}(i) \in C_j\}$, so that $C_{j_i} \supset C_{j_{i+1}}$ with $\varphi_{x'}(i) \in C_{j_i}$, $\forall i \in \mathbb{N}$. Note that $\lim_i j_i = +\infty$ and that, given $i \in \mathbb{N}$, we have $\varphi_{x'}(i), \varphi_{x'}(i+1) \in C_{j_i}$. Hence, there is a path $f_i \colon [0,1] \to C_{j_i}$ such that $f_i(0) = \varphi_{x'}(i)$ and $f_i(1) = \varphi_{x'}(i+1)$, $\forall i \in \mathbb{N}$. Now, define a map $\alpha \colon [0,+\infty) \to X$ given by $\alpha(t) = f_i(t-i)$ whenever $i \leq t < i+1$. Note that, by construction, $\alpha([i,+\infty)) \subset C_{j_i}$. This implies that α is exterior and $\alpha(i) = i \cdot x' = \varphi_{x'}(i)$. Then, $[\alpha] \in {}^{\omega}\pi_0^{\mathrm{S}}(X)$ and ${}^{\omega}\phi_X([\alpha]) = (C_i)_{i\in\mathbb{N}} = b$, since $\alpha([n_i,+\infty)) \subset C_i$. Therefore, ${}^{\omega}\phi_X$ is surjective, as we wanted to show. □

Next, in order to construct a natural transformation from ${}^{\omega}\check{\pi}_0$ to ${}^{\omega}\pi_0^{\mathrm{BG}}$, consider the category $\mathbf{EF_{fc}}(\mathbb{N})$ whose objects are exterior discrete semi-flows $(X, \varphi, \varepsilon(X))$ whose subjacent exterior spaces X are first-countable at infinity and whose morphisms are exterior discrete semi-flow morphisms. Let ${}^{\omega}\check{\pi}_0$ and ${}^{\omega}\pi_0^{\mathrm{BG}}$ be functors from $\mathbf{EF_{fc}}(\mathbb{N})$ to the category of sets, which are defined in the same way as we did in that section.

For each $X = (X, \varphi, \varepsilon(X)) \in \mathbf{EF_{fc}}(\mathbb{N})$, note that there exists a countable basis $\{E_i\}_{i\in\mathbb{N}} \subset \varepsilon(X)$ such that $E_0 \supset E_1 \supset E_2 \dots$ Recall that $\theta_X \colon \check{\pi}_0(X) \to \pi_0^{\mathrm{BG}}(X)$ is given as follows: let $a \in \check{\pi}_0(X)$, which can be represented by $a = (C_i)_{i\in\mathbb{N}}$, where C_i is a path component of E_i verifying that $C_{i+1} \subset C_i$, $\forall i \in \mathbb{N}$. Take $x_i \in C_i$ and define the exterior map $\alpha \colon \mathbb{N} \to X$, $\alpha(i) = x_i$; then, $\theta_X(a) = [\alpha]$. We also proved that this definition neither depends on the choice of the basis $\{E_i\}_{i\in\mathbb{N}}$ nor on the choice of the elements x_i.

If $a \in {}^{\omega}\check{\pi}_0(X)$, there is $x \in D(X)$ and there is an increasing sequence $n_0 < n_1 < \dots$ such that $\varphi_x(n) \in C_i \subset E_i$, $\forall n \geq n_i$. Consider a new countable basis $\{E'_j\}_{j\in\mathbb{N}}$ of the externology given by $E'_j = E_0$ if $j < n_0$ and $E'_j = E_i$ if $n_i \leq j < n_{i+1}$, $j \in \mathbb{N}$. In this new basis, we have that a is represented by $a = (C'_j)_{j\in\mathbb{N}}$, where $C'_j = C_0$ if $j < n_0$ and $C'_j = C_i$ if $n_i \leq j < n_{i+1}$, $j \in \mathbb{N}$. Take $x' = n_0 \cdot x \in D(X)$. Observe that, if $j < n_0$, then $\varphi_{x'}(j) = j \cdot x' = (j+n_0) \cdot x = \varphi_x(j+n_0) \in C_0 = C'_j$, and if $n_i \leq j < n_{i+1}$, then $\varphi_{x'}(j) = j \cdot x' = (j+n_0) \cdot x = \varphi_x(j+n_0) \in C_i = C'_j$. This implies that, with the new choice of basis $\{E'_j\}_{j\in\mathbb{N}}$ and the new choice of elements $\varphi_{x'}(j) \in C'_j$, one has that $\theta_X(a) = [\varphi_{x'}]$, $x' \in D(X)$. Therefore, $\theta_X(a) \in {}^{\omega}\pi_0^{\mathrm{BG}}(X)$.

Since we have just shown that $\theta_X({}^{\omega}\check{\pi}_0(X)) \subset {}^{\omega}\pi_0^{\mathrm{BG}}(X)$, we can consider the restriction

$$ {}^{\omega}\theta_X = \theta_X|_{{}^{\omega}\check{\pi}_0(X)}. $$

By Lemma 0.2.2, the family of maps ${}^{\omega}\theta_X \colon {}^{\omega}\check{\pi}_0(X) \to {}^{\omega}\pi_0^{\mathrm{BG}}(X)$, $X \in |\mathbf{EF_{fc}}(\mathbb{N})|$, defines a natural transformation from ${}^{\omega}\check{\pi}_0$ to ${}^{\omega}\pi_0^{\mathrm{BG}}$.

Now, we can apply that the restriction of an injective map is also injective to obtain the following result.

Proposition 2.3.3. *Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow whose subjacent exterior space X is first-countable at infinity. Then,*

$$^{\omega}\theta_X \colon {}^{\omega}\check{\pi}_0(X) \to {}^{\omega}\pi_0^{\mathrm{BG}}(X)$$

is injective.

Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow. Then, observe that the map $^{\omega}\mathrm{Sh}_X \colon {}^{\omega}\pi_0^{\mathrm{BG}}(X) \to {}^{\omega}\pi_0^{\mathrm{BG}}(X)$ given by $^{\omega}\mathrm{Sh}_X = \mathrm{Sh}_X|_{^{\omega}\pi_0^{\mathrm{BG}}(X)}$ is well-defined. For, let $x \in D(X)$ such that $a = [\varphi_x] \in {}^{\omega}\pi_0^{\mathrm{BG}}(X)$; then, there exists $x' = 1 \cdot x \in D(X)$ such that

$$[\varphi_{x'}] = [\varphi_{1\cdot x}] = [\varphi_x \circ \mathrm{sh}] = \mathrm{Sh}_X([\varphi_x]) = \mathrm{Sh}_X(a)$$

and then $^{\omega}\mathrm{Sh}_X(a) \in {}^{\omega}\pi_0^{\mathrm{BG}}(X)$.

Theorem 2.3.1. *Let $^{\omega}R$ and $^{\omega}\phi$ be the natural transformations that we have just defined in this section and let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow. Then:*

(i) $^{\omega}\mathrm{Id}_X \circ {}^{\omega}R_X = {}^{\omega}\mathrm{Sh}_X \circ {}^{\omega}R_X$. *Furthermore,* $^{\omega}R_X({}^{\omega}\pi_0^{\mathrm{S}}(X)) = \mathrm{Eq}({}^{\omega}\mathrm{Id}_X, {}^{\omega}\mathrm{Sh}_X)$.

(ii) Let $^{\omega}\theta$ be the natural transformation defined on $\mathbf{EF_{fc}}(\mathbb{N})$. If the subjacent exterior space X is first-countable at infinity, then in the diagram

$$\begin{tikzcd}
{}^{\omega}\pi_0^{\mathrm{S}}(X) \arrow[rr, "{}^{\omega}R_X"] \arrow[dr, two heads, "{}^{\omega}\phi_X"'] & & {}^{\omega}\pi_0^{\mathrm{BG}}(X) \arrow[r, shift left, "{}^{\omega}\mathrm{Id}_X"] \arrow[r, shift right, "{}^{\omega}\mathrm{Sh}_X"'] & {}^{\omega}\pi_0^{\mathrm{BG}}(X) \\
& {}^{\omega}\check{\pi}_0(X) \arrow[ur, "{}^{\omega}\theta_X"'] & &
\end{tikzcd}$$

we have that $^{\omega}R_X = {}^{\omega}\theta_X \circ {}^{\omega}\phi_X$, $^{\omega}\phi_X$ is surjective and $^{\omega}\theta_X$ is injective. As a consequence, $^{\omega}\theta_X \colon {}^{\omega}\check{\pi}_0(X) \to {}^{\omega}\pi_0^{\mathrm{BG}}(X)$ is the equalizer of $^{\omega}\mathrm{Id}_X$ and $^{\omega}\mathrm{Sh}_X$.

Proof. (i) By Theorem 1.5.1(i),

$$^{\omega}\mathrm{Sh}_X \circ {}^{\omega}R_X = (\mathrm{Sh}_X \circ R_X)|_{^{\omega}\pi_0^{\mathrm{S}}(X)} = (\mathrm{Id}_X \circ R_X)|_{^{\omega}\pi_0^{\mathrm{S}}(X)} = (\mathrm{Id}_X)|_{^{\omega}\pi_0^{\mathrm{S}}(X)} \circ {}^{\omega}R_X = {}^{\omega}\mathrm{Id}_X \circ {}^{\omega}R_X.$$

We have just shown that $^{\omega}R_X({}^{\omega}\pi_0^{\mathrm{S}}(X)) \subset \mathrm{Eq}({}^{\omega}\mathrm{Id}_X, {}^{\omega}\mathrm{Sh}_X)$. To show the other inclusion, let $[\alpha] \in {}^{\omega}\pi_0^{\mathrm{BG}}(X)$ such that $[\alpha] = {}^{\omega}\mathrm{Sh}_X([\alpha]) = \mathrm{Sh}_X|_{^{\omega}\pi_0^{\mathrm{BG}}(X)}([\alpha]) = [\alpha \circ \mathrm{sh}]$ and $[\alpha] = [\varphi_x]$, for some suitable $x \in D(X)$. Observe that there is an exterior homotopy $F \colon \mathbb{N} \bar{\times} I \to X$ such that $F(n, 0) = \alpha(n)$, $F(n, 1) = \mathrm{Sh}_X(\alpha)(n) = (\alpha \circ \mathrm{sh})(n) = \alpha(n+1)$, $\forall n \in \mathbb{N}$. Hence, there exists a continuous map $\beta \colon [0, +\infty) \to X$ given by $\beta(t) = F(n, t-n)$, whenever $n \leq t < n+1$, $n \in \mathbb{N}$.

Furthermore, β is exterior. In order to show that, observe that, given $E_0 \in \varepsilon(X)$, there exists $n_0 \in \mathbb{N} \subset \mathbb{R}_+$ such that

$$F(\{n\} \times I) \subset E_0, \quad \forall n \geq n_0.$$

Then, $\beta([n_0, +\infty)) \subset E_0$. What is more, $\beta(n) = F(n, 0) = \alpha(n) = \beta|_{\mathbb{N}}(n)$, $\forall n \in \mathbb{N}$, which also implies that $[\beta|_{\mathbb{N}}] = [\alpha] = [\varphi_x]$. Thus, $[\beta] \in {}^{\omega}\pi_0^{\mathrm{S}}(X)$ and $[\alpha] \in {}^{\omega}R_X({}^{\omega}\pi_0^{\mathrm{S}}(X))$. It follows that $\mathrm{Eq}({}^{\omega}\mathrm{Id}_X, {}^{\omega}\mathrm{Sh}_X) \subset {}^{\omega}R_X({}^{\omega}\pi_0^{\mathrm{S}}(X))$. Hence, we have that

$$ {}^{\omega}R_X({}^{\omega}\pi_0^{\mathrm{S}}(X)) = \mathrm{Eq}({}^{\omega}\mathrm{Id}_X, {}^{\omega}\mathrm{Sh}_X). $$

(ii) Let $X \in |\mathbf{EF_{fc}}(\mathbb{N})|$. By Theorem 1.5.1, ${}^{\omega}R_X = R_X|_{{}^{\omega}\pi_0^{\mathrm{S}}(X)} = (\theta_X \circ \phi_X)|_{{}^{\omega}\pi_0^{\mathrm{S}}(X)} = {}^{\omega}\theta_X \circ {}^{\omega}\phi_X$ and ϕ_X is surjective. Moreover, by Proposition 2.3.3, ${}^{\omega}\theta_X$ is injective.

In order to show that ${}^{\omega}\phi_X$ is surjective, take $b \in {}^{\omega}\check{\pi}_0(X)$. Notice that ${}^{\omega}\mathrm{Sh}_X({}^{\omega}\theta_X(b)) = \mathrm{Sh}_X(\theta_X(b)) = \mathrm{Id}_X(\theta_X(b)) = {}^{\omega}\mathrm{Id}_X({}^{\omega}\theta_X(b))$. This implies ${}^{\omega}\theta_X(b) \in \mathrm{Eq}({}^{\omega}\mathrm{Id}_X, {}^{\omega}\mathrm{Sh}_X)$. Since we proved that ${}^{\omega}R_X({}^{\omega}\pi_0^{\mathrm{S}}(X)) = \mathrm{Eq}({}^{\omega}\mathrm{Id}_X, {}^{\omega}\mathrm{Sh}_X)$, it follows that there exists $a \in {}^{\omega}\pi_0^{\mathrm{S}}(X)$ such that ${}^{\omega}R_X(a) = {}^{\omega}\theta_X(b)$. As ${}^{\omega}R_X(a) = R_X(a) = \theta_X(\phi_X(a)) = {}^{\omega}\theta_X({}^{\omega}\phi_X(a))$, we have that ${}^{\omega}\theta_X(b) = {}^{\omega}\theta_X({}^{\omega}\phi_X(a))$. Taking into account that ${}^{\omega}\theta_X$ is injective, it follows that ${}^{\omega}\phi_X(a) = b$. Therefore, ${}^{\omega}\phi_X$ is surjective.

$\square$

Chapter 3

Intrinsic topology and Ω-end sets of an exterior discrete semi-flow

A path component of a topological space X is an equivalence class of X under the equivalence relation which makes a pair of points $x, y \in X$ equivalent if there is a path from x to y. If an exterior discrete semi-flow $(X, \varphi, \varepsilon(X))$ is given and there exists a path $h\colon I \to X$ from x to y such that, for a big enough n, $\varphi^n(h(I))$ is eventually contained in every exterior open subset, then we say that x and y belong to the same intrinsic path component.

We use this new notion in the current chapter to create more functors from the category of exterior discrete semi-flows to the category of sets. We also study the connections among several of them.

3.1 Intrinsic paths and intrinsic topology

Definition 3.1.1. *Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow. We say that a path $h\colon [a, b] \to X$ is* **intrinsic** *if, for all $E \in \varepsilon(X)$, there exists $n_E \in \mathbb{N}$ such that $\varphi^n(h([a, b])) \subset E$, $\forall n \geq n_E$.*

Lemma 3.1.1. *Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow. Then:*

(i) If $\alpha\colon I \to X$ is an intrinsic path, then the inverse path $\bar{\alpha}\colon I \to X$ given by $\bar{\alpha}(t) = \alpha(1-t)$, $t \in I$, is intrinsic.

(ii) If $\alpha_i\colon I \to X$ are intrinsic paths such that $\alpha_i(1) = \alpha_{i+1}(0)$, for each $i = 0, \ldots, k-1$, the path $\alpha\colon I \to X$ given by

$$\alpha(t) = \begin{cases} \alpha_i(kt - i), & \text{if } \frac{i}{k} \leq t < \frac{i+1}{k}, \\ \alpha_{k-1}(1), & \text{if } t = 1 \end{cases}$$

is intrinsic.

Proof. The proof for (i) is highly trivial; let us show (ii). Since the path α_i is intrinsic for each $i = 0, \ldots, k-1$, given $E \in \varepsilon(X)$ there exist $n_i^E \in \mathbb{N}$ such that $\varphi^n(\alpha_i(t)) \subset E$, $\forall n \geq n_i^E$, $t \in [0, 1]$.

Take $n_E = \max_{i\in\{0,\dots,k-1\}}\{n_i^E\}$; then, $\varphi^n(\alpha_i(t)) \subset E$, $\forall n \geq n_E$ and $\forall i \in \{0,\dots,k-1\}$. Thus, $\varphi^n(\alpha(t)) \subset E$, $\forall n \geq n_E$, $t \in [0,1]$. Therefore, α is an intrinsic path. □

Definition 3.1.2. *Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow and let S be a subset of $D(X)$. We define*

$$\pi_0^{\text{int}}(S) = S/\sim_{\text{int}},$$

where, given $s_0, s_1 \in S$, $s_0 \sim_{\text{int}} s_1$ if there is an intrinsic path $h\colon I \to X$ such that $h(I) \subset S$, $h(0) = s_0$ and $h(1) = s_1$. These equivalence classes are called the **intrinsic path components** *of S.*

Given a discrete semi-flow (X, φ) and $S \subset X$, denote

$$\text{inv}^{\mathbf{r}}(S) = \{x \in S \mid \varphi^n(x) \in S,\ \forall n \in \mathbb{N}\}.$$

Note that, if $\text{inv}^{\mathbf{r}}(S) = S$, then S is a right-invariant subset. At the same time, let $(X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow. Consider the following family of subsets of X:

$$\varsigma = \mathbf{t}_X \cup \{\text{inv}^{\mathbf{r}}(E) \mid E \in \varepsilon(X)\}.$$

Remind that any finite intersection of open subsets is an open subset. Moreover, it is easy to check that $\text{inv}^{\mathbf{r}}(S) \cap \text{inv}^{\mathbf{r}}(T) = \text{inv}^{\mathbf{r}}(S \cap T)$. Bearing this in mind, the following basis is obtained by taking finite intersections of the sets which form the subbasis ς:

$$\mathcal{B} = \{U \cap \text{inv}^{\mathbf{r}}(E) \mid U \in \mathbf{t}_X,\ E \in \varepsilon(X)\}.$$

We will denote the topology generated by the basis $\mathcal{B}$ (or by the subbasis ς) by $\mathbf{t}_X^{\text{int}}$, and it will be called the **intrinsic topology** of the exterior discrete semi-flow $(X, \varphi, \varepsilon(X))$. Note that $\varepsilon(X) \subset \mathbf{t}_X^{\text{int}}$.

If we denote $X = (X, \varphi, \varepsilon(X))$, we will denote by X^{int} the exterior discrete semi-flow $((X, \mathbf{t}_X^{\text{int}}), \varphi, \varepsilon(X))$.

Proposition 3.1.1. *Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow and let $\mathbf{t}_X^{\text{int}}$ be its intrinsic topology.*

(i) Let

$$\varepsilon^{\mathbf{r}}(X) = \{U \in \mathbf{t}_X \mid \forall x \in X,\ \exists n_x \in \mathbb{N} \text{ such that } \varphi^n(x) \in U,\ \forall n \geq n_x\}.$$

If $\alpha\colon I \to X^{\text{int}}$ is a continuous map and $\varepsilon(X) \subset \varepsilon^{\mathbf{r}}(X)$, then $\alpha\colon I \to X$ is an intrinsic path.

(ii) $\alpha\colon I \to X$ is an intrinsic path if and only if $\alpha\colon I \to X^{\text{int}}$ is a continuous map and $\alpha(I) \subset D(X)$.

Proof.

(i) Let $E \in \varepsilon(X)$. Suppose that $t_0 \in I$. Since $E \in \varepsilon^{\mathbf{r}}(X)$, there exists $n_{t_0} \in \mathbb{N}$ such that, for all $n \geq n_{t_0}$, $\varphi^n(\alpha(t_0)) \in E$. Note that $\varphi^n(\alpha(t_0)) \in E \Leftrightarrow \alpha(t_0) \in (\varphi^n)^{-1}(E)$. Then, $\alpha(t_0) \in (\varphi^{m+n_{t_0}})^{-1}(E)$, $\forall m \in \mathbb{N}$, which implies that

$$\varphi^m(\alpha(t_0)) \in (\varphi^{n_{t_0}})^{-1}(E), \quad \forall m \in \mathbb{N}.$$

Thus,

$$\alpha(t_0) \in \mathrm{inv}^{\mathbf{r}}((\varphi^{n_{t_0}})^{-1}(E)).$$

Since $\mathrm{inv}^{\mathbf{r}}((\varphi^{n_{t_0}})^{-1}(E))$ is a (subbasic) open subset of X^{int} and $\alpha\colon I \to X^{\mathrm{int}}$ is a continuous map, there is an open neighborhood of t_0, V_{t_0}, such that

$$\alpha(V_{t_0}) \subset \mathrm{inv}^{\mathbf{r}}((\varphi^{n_{t_0}})^{-1}(E)).$$

What is more, we can find suitable open neighborhoods $V_{t_1}, \ldots, V_{t_k}$ of the respective points $t_1, \ldots, t_k \in I$ satisfying $I = \bigcup_{i=1}^k V_{t_i}$, because I is compact. Then,

$$\alpha(I) = \alpha\left(\bigcup_{i=1}^k V_{t_i}\right) = \bigcup_{i=1}^k \alpha(V_{t_i}) \subset \bigcup_{i=1}^k \mathrm{inv}^{\mathbf{r}}((\varphi^{n_{t_i}})^{-1}(E)),$$

where $\{n_{t_i}\}_{i\in\{1,\ldots,k\}}$ are natural numbers that satisfy $\alpha(t_i) \in \mathrm{inv}^{\mathbf{r}}((\varphi^{n_{t_i}})^{-1}(E))$, $i = 1, \ldots, k$. Taking $n_0 = \max\{n_{t_1}, \ldots, n_{t_k}\}$, we have that

$$\alpha(I) \subset \mathrm{inv}^{\mathbf{r}}((\varphi^{n_0})^{-1}(E)).$$

Therefore,

$$\varphi^n(\alpha(I)) \subset (\varphi^{n_0})^{-1}(E),\ \forall n \in \mathbb{N} \Longrightarrow \varphi^{n+n_0}(\alpha(I)) \subset E,\ \forall n \in \mathbb{N} \Longrightarrow \varphi^n(\alpha(I)) \subset E,\ \forall n \geq n_0.$$

This implies that $\alpha\colon I \to X$ is an intrinsic path.

(ii) We will show that the preimage of any subbasic open subset $S \in \varsigma$ under α is open, as long as α is an intrinsic path. If $U \in \mathbf{t}_X$, then $\alpha^{-1}(U)$ is open, because $\alpha\colon I \to X$ is continuous. Besides, given an exterior open subset E, we will prove that

$$\alpha^{-1}(\mathrm{inv}^{\mathbf{r}}(E)) = \{t \in I \mid \alpha(t) \in \mathrm{inv}^{\mathbf{r}}(E)\}$$

is an open subset of I: for, let $t_0 \in \alpha^{-1}(\mathrm{inv}^{\mathbf{r}}(E))$. Then, $\alpha(t_0) \in \mathrm{inv}^{\mathbf{r}}(E) \subset E$. Since α is an intrinsic path, there exists n_0 such that $\varphi^n(\alpha(I)) \subset E$, $\forall n \geq n_0$. In addition, $\alpha(t_0) \in \mathrm{inv}^{\mathbf{r}}(E)$, which implies that $\varphi^k(\alpha(t_0)) \in E$, $k = 0, 1, \ldots, n_0 - 1$. Now, α and φ^k, where $k \in \{0, 1, \ldots, n_0 - 1\}$, are continuous maps, so there are open neighborhoods of t_0, $V_{t_0}^k \subset I$, such that $\varphi^k(\alpha(V_{t_0}^k)) \subset E$. Hence, $V_{t_0} = \bigcap_{k=0}^{n_0-1} V_{t_0}^k$ is another open neighborhood of t_0 that satisfies $\varphi^k(\alpha(V_{t_0})) \subset E$, $k \in \{0, 1, \ldots, n_0 - 1\}$. Thus, $\varphi^k(\alpha(V_{t_0})) \subset E$, $\forall k \in \mathbb{N}$. As a matter of fact,

$$V_{t_0} \subset \alpha^{-1}(\mathrm{inv}^{\mathbf{r}}(E)),$$

and it follows that $\alpha^{-1}(\mathrm{inv}^{\mathbf{r}}(E))$ is open. Therefore, $\alpha\colon I \to X^{\mathrm{int}}$ is a continuous map.

To prove that $\alpha(I) \subset D(X)$, choose any $t_0 \in I$; since α is intrinsic, given $E \in \varepsilon(X)$ there exists $n_E \in \mathbb{N}$ such that

$$\varphi_{\alpha(t_0)}(n) = \varphi^n(\alpha(t_0)) \in E, \quad \forall n \geq n_E.$$

It follows that $\varphi_{\alpha(t_0)}$ is exterior, and hence $\alpha(t_0) \in D(X)$.

To show the converse, denote $D = D(X)$ and let us consider the relative externology $\varepsilon(D) = \{E \cap D\}_{E \in \varepsilon(X)}$. Since D is completely invariant, one has that the intrinsic topology induced by the relative externology is also the relative intrinsic topology, as the respective bases which have been chosen for both topologies are the same:

$$(D \cap U) \cap \mathrm{inv}^{\mathbf{r}}(E \cap D) = D \cap U \cap \mathrm{inv}^{\mathbf{r}}(E) \cap \mathrm{inv}^{\mathbf{r}}(D) = D \cap U \cap \mathrm{inv}^{\mathbf{r}}(E) \cap D = D \cap (U \cap \mathrm{inv}^{\mathbf{r}}(E)).$$

If $\alpha\colon I \to X^{\mathrm{int}}$ is continuous and $\alpha(I) \subset D$, then $\alpha\colon I \to D^{\mathrm{int}}$ is continuous. Notice that $\varepsilon(D) \subset \varepsilon^{\mathbf{r}}(D)$; indeed, if $x \in D(X)$, then owing to the fact that D is completely invariant, one has that $\varphi^n(x) \in D(X)$, for all $n \in \mathbb{N}$ and, given $E \in \varepsilon(X)$, there is $n_x \in \mathbb{N}$ such that $\varphi^n(x) \in E$, $\forall n \geq n_x$, so $\varphi^n(x) \in E \cap D$, $\forall n \geq n_x$. Therefore, one has by (i) that $\alpha\colon I \to D$ is an intrinsic path. Since $\varepsilon(D)$ is the relative externology, we have that $\alpha\colon I \to X$ is also intrinsic. □

3.2 Ω-end sets of an exterior discrete semi-flow

Given an exterior discrete semi-flow X, we modify the definitions of ${}^{\omega}\pi_0^{\mathrm{BG}}(X)$ and ${}^{\omega}\pi_0^{\mathrm{S}}(X)$ to obtain new sets ${}^{\Omega}\pi_0^{\mathrm{BG}}(X)$ and ${}^{\Omega}\pi_0^{\mathrm{S}}(X)$, respectively. Along this section, to avoid any confusion, we will denote by $[\,\cdot\,]_\omega$ the equivalence classes taken as elements of ${}^{\omega}\pi_0^{\mathrm{BG}}(X)$ or ${}^{\omega}\pi_0^{\mathrm{S}}(X)$, as appropriate.

Definition 3.2.1. *We define the functor ${}^{\Omega}\pi_0^{\mathrm{BG}}$ from* $\mathbf{EF}(\mathbb{N})$ *to* **Set** *as one that maps a given exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$ to the set*

$${}^{\Omega}\pi_0^{\mathrm{BG}}(X) = \{\varphi_x \mid x \in D(X)\}/\sim_\Omega,$$

where $\varphi_x \sim_\Omega \varphi_{x'}$ if there exists an exterior homotopy $F\colon \mathbb{N}\bar{\times}I \to X$ such that, for all $n \in \mathbb{N}$, $F(n,0) = \varphi_x(n)$, $F(n,1) = \varphi_{x'}(n)$ and $F(n,t) = \varphi^n(F(0,t))$, $t \in I$; and assigns each exterior discrete semi-flow morphism $f\colon (X, \varphi, \varepsilon(X)) \to (Y, \psi, \varepsilon(Y))$ to the map ${}^{\Omega}\pi_0^{\mathrm{BG}}(f)\colon {}^{\Omega}\pi_0^{\mathrm{BG}}(X) \to {}^{\Omega}\pi_0^{\mathrm{BG}}(Y)$ given by

$${}^{\Omega}\pi_0^{\mathrm{BG}}(f)([\varphi_x]) = [f \circ \varphi_x] = [\psi_{f(x)}].$$

We denote by $[\varphi_x]_\Omega$ the elements (equivalence classes) of ${}^{\Omega}\pi_0^{\mathrm{BG}}(X)$ (or sometimes just by $[\varphi_x]$ if no confusion is possible).

Observe that, if $F\colon \mathbb{N}\bar{\times}I \to X$ is an exterior homotopy such that $F(n,0) = \varphi_x(n)$, $F(n,1) = \varphi_{x'}(n)$ and $F(n,t) = \varphi^n(F(0,t))$, then the map ${}^{\Omega}\pi_0^{\mathrm{BG}}(f)$ is well-defined, as the exterior homotopy $G = f \circ F\colon \mathbb{N}\bar{\times}I \to Y$ satisfies that, for all $n \in \mathbb{N}$, $G(n,0) = (f \circ F)(n,0) = f(F(n,0)) = f(\varphi_x(n)) = \psi_{f(x)}(n)$, $G(n,1) = (f \circ F)(n,1) = f(F(n,1)) = f(\varphi_{x'}(n)) = \psi_{f(x')}(n)$ and

$$\begin{aligned} G(n,t) = (f \circ F)(n,t) = f(F(n,t)) = f(\varphi^n(F(0,t))) = f(\varphi(n, F(0,t))) = f(\varphi_{F(0,t)}(n)) = \\ = \psi_{f(F(0,t))}(n) = \psi_{(f\circ F)(0,t)}(n) = \psi_{G(0,t)}(n) = \psi(n, G(0,t)) = \psi^n(G(0,t)), \end{aligned}$$

where $t \in I$.

Remark 3.2.1. *Given an exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$, note that one can also regard the set ${}^{\omega}\pi_0^{\mathrm{BG}}(X)$ as a quotient:*

$$ {}^{\omega}\pi_0^{\mathrm{BG}}(X) = \{\varphi_x \mid x \in D(X)\}/\sim_{\omega}, $$

where $\varphi_x \sim_{\omega} \varphi_{x'}$ if there exists an exterior homotopy $F\colon \mathbb{N}\bar{\times}I \to X$ such that, for all $n \in \mathbb{N}$, $F(n,0) = \varphi_x(n)$ and $F(n,1) = \varphi_{x'}(n)$.

Proposition 3.2.1. *There is a natural transformation $\rho^{\mathrm{BG}}\colon {}^{\Omega}\pi_0^{\mathrm{BG}} \to {}^{\omega}\pi_0^{\mathrm{BG}}$ such that, for each exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$, the map $\rho_X^{\mathrm{BG}}\colon {}^{\Omega}\pi_0^{\mathrm{BG}}(X) \to {}^{\omega}\pi_0^{\mathrm{BG}}(X)$ is given by $\rho_X^{\mathrm{BG}}([\varphi_x]_{\Omega}) = [\varphi_x]_{\omega}$ and it is surjective.*

Proof. Observe that ρ^{BG} is well-defined, since, given $x, x' \in D(X)$,

$$ \varphi_x \sim_{\Omega} \varphi_{x'} \Longrightarrow \varphi_x \sim_{\omega} \varphi_{x'}. $$

To prove that ρ^{BG} is a natural transformation, given an exterior discrete semi-flow morphism $f\colon (X, \varphi, \varepsilon(X)) \to (Y, \psi, \varepsilon(Y))$ we must see that the following diagram is commutative:

$$ \begin{tikzcd} {}^{\Omega}\pi_0^{\mathrm{BG}}(X) \arrow[r, "\rho_X^{\mathrm{BG}}"] \arrow[d, "{}^{\Omega}\pi_0^{\mathrm{BG}}(f)"'] & {}^{\omega}\pi_0^{\mathrm{BG}}(X) \arrow[d, "{}^{\omega}\pi_0^{\mathrm{BG}}(f)"] \\ {}^{\Omega}\pi_0^{\mathrm{BG}}(Y) \arrow[r, "\rho_Y^{\mathrm{BG}}"'] & {}^{\omega}\pi_0^{\mathrm{BG}}(Y) \end{tikzcd} $$

For, let $[\varphi_x]_{\Omega} \in {}^{\Omega}\pi_0^{\mathrm{BG}}(X)$. We have that

$$ ({}^{\omega}\pi_0^{\mathrm{BG}}(f) \circ \rho_X^{\mathrm{BG}})([\varphi_x]_{\Omega}) = {}^{\omega}\pi_0^{\mathrm{BG}}(f)(\rho_X^{\mathrm{BG}}([\varphi_x]_{\Omega})) = {}^{\omega}\pi_0^{\mathrm{BG}}(f)([\varphi_x]_{\omega}) = [f \circ \varphi_x]_{\omega} = [\psi_{f(x)}]_{\omega}; $$

besides,

$$ (\rho_Y^{\mathrm{BG}} \circ {}^{\Omega}\pi_0^{\mathrm{BG}}(f))([\varphi_x]_{\Omega}) = \rho_Y^{\mathrm{BG}}({}^{\Omega}\pi_0^{\mathrm{BG}}(f)([\varphi_x]_{\Omega})) = \rho_Y^{\mathrm{BG}}([f \circ \varphi_x]_{\Omega}) = \rho_Y^{\mathrm{BG}}([\psi_{f(x)}]_{\Omega}) = [\psi_{f(x)}]_{\omega}. $$

Finally, observe that, if we take the quotient maps q_{ω} and q_{Ω}, the following diagram is commutative:

$$ \begin{tikzcd} \{\varphi_x \mid x \in D(X)\} \arrow[d, "q_{\Omega}"'] \arrow[dr, "q_{\omega}"] & \\ {}^{\Omega}\pi_0^{\mathrm{BG}}(X) \arrow[r, "\rho_X^{\mathrm{BG}}"'] & {}^{\omega}\pi_0^{\mathrm{BG}}(X) \end{tikzcd} $$

Then, the surjectivity of ρ_X^{BG} follows from the surjectivity of the quotient map q_{ω}. □

Proposition 3.2.2. *Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow and consider $D = D(X, \varphi, \varepsilon(X))$ provided with the restriction $\varphi|_{\mathbb{N}\times D}$ and the relative externology. Then:*

(i) The map ${}^{\Omega}\pi_0^{\mathrm{BG}}(D) \to {}^{\Omega}\pi_0^{\mathrm{BG}}(X)$ induced by the canonical inclusion $D \hookrightarrow X$ is bijective.

(ii) $\pi_0^{\mathrm{int}}(D) \cong {}^{\Omega}\pi_0^{\mathrm{BG}}(D) \cong {}^{\omega}\pi_0^{\mathrm{BG}}(D^{\mathrm{int}}) \cong {}^{\Omega}\pi_0^{\mathrm{BG}}(X)$.

Proof. (i) Take $[\varphi_x] \in {}^{\Omega}\pi_0^{\mathrm{BG}}(X)$, $x \in D(X)$. Then, $[\varphi_x] \in {}^{\Omega}\pi_0^{\mathrm{BG}}(D)$, because $\varphi_x(n) = \varphi^n(x) \in D(X)$, for all $n \in \mathbb{N}$, as $D(X)$ is a completely invariant subset. Thus, the map is surjective.

Moreover, if $x, x' \in D(X)$ and $[\varphi_x] = [\varphi_{x'}] \in {}^{\Omega}\pi_0^{\mathrm{BG}}(X)$, then there exists an exterior homotopy $F\colon \mathbb{N}\bar{\times}I \to X$ such that $F(n,0) = \varphi_x(n)$, $F(n,1) = \varphi_{x'}(n)$ and $F(n,t) = \varphi^n(F(0,t))$, $\forall n \in \mathbb{N}$, $t \in I$. Since F is exterior, so is $\varphi_{F(0,t)}$; hence, $F(0,t) \in D(X)$, $\forall t \in I$. Now, since D is completely invariant, $F(n,t) = \varphi^n(F(0,t)) \in D(X)$, $\forall n \in \mathbb{N}$, $t \in I$. Therefore, $F(\mathbb{N}\bar{\times}I) \subset D(X)$ and $[\varphi_x] = [\varphi_{x'}] \in {}^{\Omega}\pi_0^{\mathrm{BG}}(D)$, so the map is also injective.

(ii) ${}^{\Omega}\pi_0^{\mathrm{BG}}(D) \cong {}^{\Omega}\pi_0^{\mathrm{BG}}(X)$ has already been proved. We shall show firstly that ${}^{\Omega}\pi_0^{\mathrm{BG}}(D) \cong {}^{\omega}\pi_0^{\mathrm{BG}}(D^{\mathrm{int}})$, and secondly that $\pi_0^{\mathrm{int}}(D) \cong {}^{\Omega}\pi_0^{\mathrm{BG}}(D)$.

To prove the first statement, let $[\varphi_x] = [\varphi_{x'}] \in {}^{\Omega}\pi_0^{\mathrm{BG}}(D)$, with $x, x' \in D(X)$. Then, there exists an exterior homotopy $F\colon \mathbb{N}\bar{\times}I \to D$ such that $F(n,0) = \varphi_x(n)$, $F(n,1) = \varphi_{x'}(n)$ and $F(n,t) = \varphi^n(F(0,t))$, $\forall n \in \mathbb{N}$, $t \in I$. Notice that, for all $n \in \mathbb{N}$, the path $F|_{\{n\}\times I}$ is intrinsic: indeed, F is exterior and, hence, for each $E \in \varepsilon(X)$ there exists $n_E \in \mathbb{N}$, $n_E \geq n$, such that, $\forall m \geq n_E$, $F(\{m\} \times I) \subset E$, so $\exists n'_E = n_E - n \in \mathbb{N}$ such that, $\forall m' \geq n'_E$,

$$F(\{n + m'\} \times I) = \varphi^{m'}(F(\{n\} \times I)) \subset E.$$

Thus, by Proposition 3.1.1(ii), $F|_{\{n\}\times I}$ is continuous in D^{int}, for all $n \in \mathbb{N}$, and then $[\varphi_x] = [\varphi_{x'}] \in {}^{\omega}\pi_0^{\mathrm{BG}}(D^{\mathrm{int}})$. Conversely, if $[\varphi_x] = [\varphi_{x'}] \in {}^{\omega}\pi_0^{\mathrm{BG}}(D^{\mathrm{int}})$, then there is an exterior homotopy $G\colon \mathbb{N}\bar{\times}I \to D^{\mathrm{int}}$ such that $G(n,0) = \varphi_x(n)$ and $G(n,1) = \varphi_{x'}(n)$.

Since, for each $n \in \mathbb{N}$, $G|_{\{n\}\times I}$ is continuous in D^{int} and $G(n,t) \subset D(X)$, $\forall t \in I$, we have by Proposition 3.1.1(ii) that $G|_{\{n\}\times I}$ is intrinsic in D. In particular, $\alpha\colon I \to D$ given by $\alpha(t) = G(0,t)$ is intrinsic. Now, let us construct a new homotopy $F'\colon \mathbb{N}\bar{\times}I \to D$ given by

$$F'(n,t) = \varphi^n(\alpha(t)).$$

Note that, for all $n \in \mathbb{N}$ and $t \in I$, the homotopy F' satisfies

$$F'(n,0) = \varphi^n(\alpha(0)) = \varphi^n(G(0,0)) = \varphi^n(\varphi_x(0)) = \varphi^n(\varphi^0(x)) = \varphi^n(x) = \varphi_x(n),$$

$$F'(n,1) = \varphi^n(\alpha(1)) = \varphi^n(G(0,1)) = \varphi^n(\varphi_{x'}(0)) = \varphi^n(\varphi^0(x')) = \varphi^n(x') = \varphi_{x'}(n),$$

$$F'(n,t) = \varphi^n(\alpha(t)) = \varphi^n(\varphi^0(\alpha(t))) = \varphi^n(F'(0,t))$$

and, since α is intrinsic, for each $E \in \varepsilon(X)$ there exists $n_E \in \mathbb{N}$ such that, for all $n \geq n_E$,

$$\varphi^n(\alpha(I)) = F'(\{n\} \times I) \subset E,$$

so F' is exterior. That means that $[\varphi_x] = [\varphi_{x'}] \in {}^{\Omega}\pi_0^{\mathrm{BG}}(D)$.

Finally, let us show the second statement. Note that D is bijective to $\{\varphi_x \mid x \in D\}$. Now, given $x, y \in D$, if $x \sim_{\mathrm{int}} y$, then there is an intrinsic path $h\colon I \to D$ such that $h(0) = x$ and $h(1) = y$. From this path, we can create a homotopy $F\colon \mathbb{N}\bar{\times}I \to D$ given by

$$F(n,t) = \varphi^n(h(t)).$$

For all $n \in \mathbb{N}$ and $t \in I$, the homotopy F satisfies

$$F(n,0) = \varphi^n(h(0)) = \varphi^n(x) = \varphi_x(n),$$

$$F(n,1) = \varphi^n(h(1)) = \varphi^n(y) = \varphi_y(n),$$

$$F(n,t) = \varphi^n(h(t)) = \varphi^n(\varphi^0(h(t))) = \varphi^n(F(0,t))$$

and is exterior, as h is intrinsic. Therefore, $\varphi_x \sim_\Omega \varphi_y$. Besides, if $[\varphi_x] = [\varphi_y] \in {}^\Omega\pi_0^{\mathrm{BG}}(D)$, then there is an exterior homotopy $G\colon \mathbb{N}\bar{\times}I \to D$ such that $G(n,0) = \varphi_x(n)$, $G(n,1) = \varphi_y(n)$ and $G(n,t) = \varphi^n(G(0,t))$, $\forall n \in \mathbb{N}$, $t \in I$. Define $\alpha\colon I \to D$ so that $\alpha(t) = G(0,t)$; hence,

$$\alpha(0) = G(0,0) = \varphi_x(0) = \varphi^0(x) = x,$$

$$\alpha(1) = G(0,1) = \varphi_y(0) = \varphi^0(y) = y$$

and α is intrinsic, since, for each $E \in \varepsilon(X)$, there exists $n_E \in \mathbb{N}$ such that, for all $n \geq n_E$,

$$G(\{n\} \times I) = \varphi^n(G(\{0\} \times I)) = \varphi^n(\alpha(I)) \subset E,$$

as G is exterior. Thus, $x \sim_{\mathrm{int}} y$.

□

Definition 3.2.2. *We define the functor ${}^\Omega\pi_0^{\mathrm{S}}$ from* $\mathbf{EF}(\mathbb{N})$ *to* **Set** *as one that maps a given exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$ to the set*

$$\begin{aligned} {}^\Omega\pi_0^{\mathrm{S}}(X) = \{\alpha\colon \mathbb{R}_+ \to X \text{ exterior} \mid \forall n \in \mathbb{N},\ \varphi^n(\alpha(0)) = \alpha(n) \\ \text{and each path } \alpha|_{[n,n+1]} \text{ is intrinsic}\}/\sim_\Omega, \end{aligned}$$

where $\alpha \sim_\Omega \beta$ if there exists an exterior homotopy $F\colon \mathbb{R}_+\bar{\times}I \to X$ such that, for all $r \in \mathbb{R}_+$, $F(r,0) = \alpha(r)$, $F(r,1) = \beta(r)$, and $F(n,t) = \varphi^n(F(0,t))$, $\forall n \in \mathbb{N}$, $t \in I$; and assigns each exterior discrete semi-flow morphism $f\colon (X,\varphi,\varepsilon(X)) \to (Y,\psi,\varepsilon(Y))$ to the map ${}^\Omega\pi_0^{\mathrm{S}}(f)\colon {}^\Omega\pi_0^{\mathrm{S}}(X) \to {}^\Omega\pi_0^{\mathrm{S}}(Y)$ given by

$${}^\Omega\pi_0^{\mathrm{S}}(f)([\alpha]) = [f \circ \alpha].$$

We denote by $[\alpha]_\Omega$ the elements (equivalence classes) of ${}^\Omega\pi_0^{\mathrm{S}}(X)$ (or sometimes just by $[\alpha]$ if no confusion is possible).

Note that, if $[\alpha]_\Omega \in {}^\Omega\pi_0^{\mathrm{S}}(X)$, then $[f \circ \alpha]_\Omega \in {}^\Omega\pi_0^{\mathrm{S}}(Y)$ indeed, because $f \circ \alpha$ is exterior (composite of exterior maps),

$$\begin{aligned} (f \circ \alpha)(n) &= f(\alpha(n)) = f(\varphi^n(\alpha(0))) = f(\varphi(n, \alpha(0))) = f(\varphi_{\alpha(0)}(n)) = \psi_{f(\alpha(0))}(n) = \psi_{(f\circ\alpha)(0)}(n) \\ &= \psi(n, (f \circ \alpha)(0)) = \psi^n((f \circ \alpha)(0)) \end{aligned}$$

and, given any $m \in \mathbb{N}$, the path $f(\alpha(m+t))$, $t \in [0,1]$, is intrinsic. To see this, observe that, given $E^Y \in \varepsilon(Y)$, there exist $E^X = f^{-1}(E^Y) \in \varepsilon(X)$ and $n_{E^Y} \in \mathbb{N}$ such that, for all $n \geq n_{E^Y}$, $\varphi^n(\alpha(m+t)) \subset E^X$, $t \in [0,1]$; therefore,

$$f(\varphi^n(\alpha(m+t))) = \psi^n(f(\alpha(m+t))) \subset f(E^X) \subset E^Y, \quad \forall n \geq n_{E^Y},\ t \in [0,1].$$

Furthermore, if $F\colon \mathbb{R}_+\bar{\times}I \to X$ is an exterior homotopy such that $F(r,0)=\alpha(r)$, $F(r,1)=\beta(r)$ and $F(n,t)=\varphi^n(F(0,t))$, $\forall n\in\mathbb{N}$ and $t\in I$, then the map ${}^{\Omega}\pi_0^{\mathrm{S}}(f)$ is well-defined, since the exterior homotopy $G=f\circ F\colon \mathbb{R}_+\bar{\times}I\to Y$ satisfies that, for all $r\in\mathbb{R}_+$, there exist exterior maps $\gamma=f\circ\alpha$ and $\delta=f\circ\beta$ such that

$$G(r,0)=(f\circ F)(r,0)=f(F(r,0))=f(\alpha(r))=(f\circ\alpha)(r)=\gamma(r),$$

$$G(r,1)=(f\circ F)(r,1)=f(F(r,1))=f(\beta(r))=(f\circ\beta)(r)=\delta(r)$$

and, for all $n\in\mathbb{N}$ and $t\in I$,

$$\begin{aligned}G(n,t)&=(f\circ F)(n,t)=f(F(n,t))=f(\varphi^n(F(0,t)))=f(\varphi(n,F(0,t)))=f(\varphi_{F(0,t)}(n))=\\&=\psi_{f(F(0,t))}(n)=\psi_{(f\circ F)(0,t)}(n)=\psi_{G(0,t)}(n)=\psi(n,G(0,t))=\psi^n(G(0,t)).\end{aligned}$$

Lemma 3.2.1. *We can give an alternative definition of the set ${}^{\omega}\pi_0^{\mathrm{S}}(X)$ as follows:*

$${}^{\omega}\pi_0^{\mathrm{S}}(X)=\{\alpha\colon\mathbb{R}_+\to X \text{ exterior} \mid \varphi^n(\alpha(0))=\alpha(n),\forall n\in\mathbb{N}\}/\sim_\omega,$$

where $\alpha\sim_\omega\beta$ if there exists an exterior homotopy $F\colon\mathbb{R}_+\bar{\times}I\to X$ such that, for all $r\in\mathbb{R}_+$, $F(r,0)=\alpha(r)$ and $F(r,1)=\beta(r)$.

Proof. Denote

$$S_1(X)=\{[\alpha]\in\pi_0^{\mathrm{S}}(X)\mid\exists x\in D(X)\text{ such that }[\alpha|_{\mathbb{N}}]=[\varphi_x]\},$$

$$S_2(X)=\{\alpha\colon\mathbb{R}_+\to X\text{ exterior}\mid\varphi^n(\alpha(0))=\alpha(n),\forall n\in\mathbb{N}\}/\sim_\omega.$$

Let $a\in S_1(X)$. Then, there exists an exterior map $\alpha\colon\mathbb{R}_+\to X$ such that $a=[\alpha]$ and $\exists x\in D(X)$ such that $[\alpha|_{\mathbb{N}}]=[\varphi_x]$. Therefore, there is an exterior homotopy $F\colon\mathbb{N}\bar{\times}I\to X$ such that $F(n,0)=\alpha(n)$ and $F(n,1)=\varphi_x(n)$, $\forall n\in\mathbb{N}$. Note that, for all $n\in\mathbb{N}$, there exists a path $f_n\colon I\to X$ given by

$$f_n(t)=\begin{cases}F(n,1-3t), & \text{if } 0\le t<\frac{1}{3};\\ \alpha(n+3t-1), & \text{if } \frac{1}{3}\le t<\frac{2}{3};\\ F(n+1,3t-2), & \text{if } \frac{2}{3}\le t\le 1.\end{cases}$$

Given a non-negative real number x, note that there is a unique natural number n_x such that $n_x\le x<n_x+1$. Now, define $\beta\colon\mathbb{R}_+\to X$ so that $\beta(r)=f_{n_r}(r-n_r)$, whenever $n_r\le r<n_r+1$, $n_r\in\mathbb{N}$. β is continuous and, by construction, it is also exterior, since α and F are exterior. We have that $\beta\sim_\omega\alpha$, because there exists an exterior homotopy $G\colon\mathbb{R}_+\bar{\times}I\to X$ given, for every $r\in\mathbb{R}_+$, by

$$G(r,t)=\beta\left(r+\frac{1-2(r-n_r)}{3}\,t\right).$$

Note that $G(r,0)=\beta(r)$ and

$$\begin{aligned}G(r,1)&=\beta\left(r+\frac{1-2(r-n_r)}{3}\right)=\beta\left(n_r+\frac{r-n_r+1}{3}\right)=f_{n_r}\left(\frac{r-n_r+1}{3}\right)\\&=\alpha(n_r+r-n_r+1-1)=\alpha(r).\end{aligned}$$

Moreover,

$$\beta(n) = f_n(0) = F(n,1) = \varphi_x(n) = \varphi(n,x) = \varphi^n(x), \quad n \in \mathbb{N},$$

and since $\beta(0) = \varphi_x(0) = x$, it follows that $\beta(n) = \varphi^n(\beta(0))$. Thus, $a = [\alpha]_\omega = [\beta]_\omega \in S_2(X)$.

To show the other inclusion, let $[\alpha]_\omega \in S_2(X)$. Then, $\alpha\colon \mathbb{R}_+ \to X$ is an exterior map and $\varphi^n(\alpha(0)) = \alpha(n)$, $\forall n \in \mathbb{N}$. Particularly, $\alpha|_{\mathbb{N}}\colon \mathbb{N} \to X$ is exterior. Hence, $\alpha(0) \in D(X)$ and it follows trivially that $[\varphi_{\alpha(0)}] = [\alpha|_{\mathbb{N}}]$, since

$$\alpha(n) = \varphi^n(\alpha(0)) = \varphi(n, \alpha(0)) = \varphi_{\alpha(0)}(n).$$

Moreover, note that $[\alpha]_\omega \in \pi_0^{\mathrm{S}}(X)$. Thus, $[\alpha]_\omega \in S_1(X)$. Then, $S_1(X) = S_2(X)$. □

Proposition 3.2.3. *There is a natural transformation $\rho^{\mathrm{S}}\colon {}^{\Omega}\pi_0^{\mathrm{S}} \to {}^{\omega}\pi_0^{\mathrm{S}}$ such that, for each exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$, $\rho_X^{\mathrm{S}}\colon {}^{\Omega}\pi_0^{\mathrm{S}}(X) \to {}^{\omega}\pi_0^{\mathrm{S}}(X)$ is given by $\rho_X^{\mathrm{S}}([\alpha]_\Omega) = [\alpha]_\omega$.*

Proof. Note that ρ^{S} is well-defined, since, given exterior maps $\alpha, \beta\colon \mathbb{R}_+ \to X$ which satisfy $\varphi^n(\alpha(0)) = \alpha(n)$ and $\varphi^n(\beta(0)) = \beta(n)$, $\forall n \in \mathbb{N}$, one has that:

$$\alpha \sim_\Omega \beta \Longrightarrow \alpha \sim_\omega \beta.$$

In order to prove that ρ^{S} is a natural transformation, given an exterior discrete semi-flow morphism $f\colon (X, \varphi, \varepsilon(X)) \to (Y, \psi, \varepsilon(Y))$ we must see that the following diagram is commutative:

$$\begin{tikzcd}
{}^{\Omega}\pi_0^{\mathrm{S}}(X) \arrow[r, "\rho_X^{\mathrm{S}}"] \arrow[d, "{}^{\Omega}\pi_0^{\mathrm{S}}(f)"'] & {}^{\omega}\pi_0^{\mathrm{S}}(X) \arrow[d, "{}^{\omega}\pi_0^{\mathrm{S}}(f)"] \\
{}^{\Omega}\pi_0^{\mathrm{S}}(Y) \arrow[r, "\rho_Y^{\mathrm{S}}"'] & {}^{\omega}\pi_0^{\mathrm{S}}(Y)
\end{tikzcd}$$

For, let $[\alpha]_\Omega \in {}^{\Omega}\pi_0^{\mathrm{S}}(X)$. We have that

$$({}^{\omega}\pi_0^{\mathrm{S}}(f) \circ \rho_X^{\mathrm{S}})([\alpha]_\Omega) = {}^{\omega}\pi_0^{\mathrm{S}}(f)(\rho_X^{\mathrm{S}}([\alpha]_\Omega)) = {}^{\omega}\pi_0^{\mathrm{S}}(f)([\alpha]_\omega) = [f \circ \alpha]_\omega;$$

besides,

$$(\rho_Y^{\mathrm{S}} \circ {}^{\Omega}\pi_0^{\mathrm{S}}(f))([\alpha]_\Omega) = \rho_Y^{\mathrm{S}}({}^{\Omega}\pi_0^{\mathrm{S}}(f)([\alpha]_\Omega)) = \rho_Y^{\mathrm{S}}([f \circ \alpha]_\Omega) = [f \circ \alpha]_\omega.$$

□

We can also regard the set $\check{\pi}_0(X)$ in the way shown below to define a new functor $\check{\pi}_0^{\mathrm{int}}(X)$. Let $X = (X, \varepsilon(X))$ be an exterior space and let S be a subset of X. Observe that the set $\pi_0(S)$ can be seen in the following way:

$$\pi_0(S) = S/\sim,$$

where, given $s_0, s_1 \in S$, $s_0 \sim s_1$ if there is a path $h\colon I \to S$ such that $h(0) = s_0$ and $h(1) = s_1$. Now, as before,

$$\check{\pi}_0(X) = \lim_{E \in \varepsilon(X)} \pi_0(E).$$

Mimicking what we have just done, we can give the concept of the functor $\check{\pi}_0^{\text{int}}(X)$. Nonetheless, in order to do so, given an exterior discrete semi-flow $(X, \varphi, \varepsilon(X))$ we need firstly to extend the definition of the quotient set $\pi_0^{\text{int}}(S)$ for any subset $S \subset X$ (not necessarily contained in $D = D(X)$) by taking intrinsic paths in $S \cap D$ and creating one additional equivalence class $[\{x\}]_{\text{int}}$ for each point $x \in S \setminus D$ to define an actual equivalence relation.

Definition 3.2.3. *Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow, let $D = D(X)$ and let S be a subset of X. We define*

$$\pi_0^{\text{int}}(S) = (S \cap D)/\sim_{\text{int}} \cup \{\{x\} \mid x \in S \setminus D\},$$

where, given $s_0, s_1 \in S$, $s_0 \sim_{\text{int}} s_1$ if there is an intrinsic path $h\colon I \to X$ such that $h(I) \subset S$, $h(0) = s_0$ and $h(1) = s_1$.

Observe that, if $S \subset D$, then the Definition 3.2.3 above constructs the same invariant as the given previously in Definition 3.1.2.

Lemma 3.2.2. *Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow and $D = D(X)$. Then,*

$$\lim_{E \in \varepsilon(X)} \pi_0^{\text{int}}(E) \cong \check{\pi}_0(D^{\text{int}}).$$

Proof. Firstly, notice that $L(X) \subset D(X)$ since, by Proposition 2.2.3, $L(X) = L(D(X))$. Now,

$$\begin{aligned}
\lim_{E \in \varepsilon(X)} \pi_0^{\text{int}}(E) &= \lim_{E \in \varepsilon(X)} ((E \cap D)/\sim_{\text{int}} \cup \{\{x\} \mid x \in E \setminus D\}) \\
&\cong \lim_{E \in \varepsilon(X)} \left(((E \cap D)/\sim_{\text{int}}) \cup \left(\left(\textstyle\bigcap_{E \in \varepsilon(X)} E\right) \setminus D\right)\right) \\
&= \check{\pi}_0(D^{\text{int}}) \cup (L(X) \setminus D) \\
&= \check{\pi}_0(D^{\text{int}}),
\end{aligned}$$

as we wanted to show. □

Definition 3.2.4. *We define the functor $\check{\pi}_0^{\text{int}}$ from $\mathbf{EF}(\mathbb{N})$ to $\mathbf{Set}$ as one that maps a given exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$ to the set*

$$\check{\pi}_0^{\text{int}}(X) = \lim_{E \in \varepsilon(X)} \pi_0^{\text{int}}(E) \cong \check{\pi}_0(D^{\text{int}})$$

and assigns each exterior discrete semi-flow morphism $f\colon X = (X, \varphi, \varepsilon(X)) \to Y = (Y, \psi, \varepsilon(Y))$ to the map $\check{\pi}_0^{\text{int}}(f)\colon \check{\pi}_0^{\text{int}}(X) \to \check{\pi}_0^{\text{int}}(Y)$ constructed as follows. Let $a = (C_{E^X}^{\text{int}})_{E^X \in \varepsilon(X)} \in \check{\pi}_0^{\text{int}}(X)$. Remind that, given $E^Y \in \varepsilon(Y)$, we have that $f^{-1}(E^Y) \in \varepsilon(X)$. Take $C_{E^Y}^{\text{int}}$ as the intrinsic path component that contains $f(C_{f^{-1}(E^Y)}^{\text{int}})$. Then,

$$\check{\pi}_0^{\text{int}}(f)(a) = (C_{E^Y}^{\text{int}})_{E^Y \in \varepsilon(Y)}.$$

Given $s, s_0 \in S \subset X$ such that $[s]_{\text{int}} = [s_0]_{\text{int}}$, one has that s and s_0 belong to the same path component of S, that is, $s \sim s_0$. Therefore, there exists a surjective map $\rho_S \colon \pi_0^{\text{int}}(S) \to \pi_0(S)$ given by $\rho_S([s]_{\text{int}}) = [s]$. Then, for each exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$, the surjective maps $\pi_0^{\text{int}}(E) \to \pi_0(E)$, $E \in \varepsilon(X)$, induce the map $\rho_X^{\text{int}} \colon \check{\pi}_0^{\text{int}}(X) \to \check{\pi}_0(X)$ given by $\rho_X^{\text{int}}((C_E^{\text{int}})_{E\in\varepsilon(X)}) = (C_E)_{E\in\varepsilon(X)}$, so that C_E is the unique path component of E such that $C_E^{\text{int}} \subset C_E$, $\forall E \in \varepsilon(X)$.

Proposition 3.2.4. *There is a natural transformation*

$$\rho^{\text{int}} \colon \check{\pi}_0^{\text{int}} \to \check{\pi}_0$$

given by the family of maps $\rho^{\text{int}} = \{\rho_X^{\text{int}} \colon \check{\pi}_0^{\text{int}}(X) \to \check{\pi}_0(X)\}_{X\in|\mathbf{EF}(\mathbb{N})|}$.

Proof. Let $f \colon X = (X, \varphi, \varepsilon(X)) \to Y = (Y, \psi, \varepsilon(Y))$ be a morphism in $\mathbf{EF}(\mathbb{N})$. We have to show that the following diagram is commutative:

$$\begin{tikzcd}
\check{\pi}_0^{\text{int}}(X) \arrow[r, "\rho_X^{\text{int}}"] \arrow[d, "\check{\pi}_0^{\text{int}}(f)"'] & \check{\pi}_0(X) \arrow[d, "\check{\pi}_0(f)"] \\
\check{\pi}_0^{\text{int}}(Y) \arrow[r, "\rho_Y^{\text{int}}"'] & \check{\pi}_0(Y)
\end{tikzcd}$$

Let a be an element of $\check{\pi}_0^{\text{int}}(X)$. Taking into account Lemma 3.2.2, a can be represented as $a = (C_{E^X}^{\text{int}})_{E^X\in\varepsilon(X)}$, where $C_{E^X}^{\text{int}}$ is an intrinsic path component of $D \cap E^X$. Given $E^Y \in \varepsilon(Y)$, we have that $f^{-1}(E^Y) \in \varepsilon(X)$ and we can take $C_{f^{-1}(E^Y)}^{\text{int}}$ as the unique intrinsic path component of a in $D \cap f^{-1}(E^Y)$, $C_{f^{-1}(E^Y)}$ as the unique path component of $\rho_X^{\text{int}}(a)$ in $f^{-1}(E^Y)$ and $C_{E^Y}^{\text{int}}$ as the unique path component of $\check{\pi}_0^{\text{int}}(f)(a)$ in E^Y. Notice that $C_{E^Y}^{\text{int}} \supset f(C_{f^{-1}(E^Y)}^{\text{int}}) \subset f(C_{f^{-1}(E^Y)})$. Take C_{E^Y} as the unique path component of E^Y containing $f(C_{f^{-1}(E^Y)}^{\text{int}})$. Since $f(C_{f^{-1}(E^Y)})$ and $C_{E^Y}^{\text{int}}$ are path-connected, one has that $C_{E^Y}^{\text{int}} \subset C_{E^Y} \supset f(C_{f^{-1}(E^Y)})$. This implies that $\rho_Y^{\text{int}}(\check{\pi}_0^{\text{int}}(f)(a)) = \check{\pi}_0(f)(\rho_X^{\text{int}}(a))$, for every $a \in \check{\pi}_0^{\text{int}}(X)$. Therefore, $\rho^{\text{int}} = \{\rho_X^{\text{int}}\}_{X\in|\mathbf{EF}(\mathbb{N})|}$ is a natural transformation. □

Finally, we will give the definition of a new functor ${}^{\Omega}\check{\pi}_0(X)$, based on the definition of the functor $\check{\pi}_0^{\text{int}}(X)$.

Definition 3.2.5. *We define the functor* ${}^{\Omega}\check{\pi}_0$ *from* $\mathbf{EF}(\mathbb{N})$ *to* **Set** *as one that maps a given exterior discrete semi-flow* $X = (X, \varphi, \varepsilon(X))$ *to the set*

$$\begin{aligned}
{}^{\Omega}\check{\pi}_0(X) = \{a = (C_E^{\text{int}})_{E\in\varepsilon(X)} \in \check{\pi}_0^{\text{int}}(X) \mid \exists x \in D(X) \text{ such that} \\
\forall E \in \varepsilon(X), \exists n_E \in \mathbb{N} \text{ with } \varphi_x(n) \in C_E^{\text{int}} \subset E, \forall n \geq n_E\}
\end{aligned}$$

and assigns every exterior discrete semi-flow morphism $f \colon X = (X, \varphi, \varepsilon(X)) \to Y = (Y, \psi, \varepsilon(Y))$ *to the map* ${}^{\Omega}\check{\pi}_0(f) \colon {}^{\Omega}\check{\pi}_0(X) \to {}^{\Omega}\check{\pi}_0(Y)$, *which is given by the formula*

$${}^{\Omega}\check{\pi}_0(f) = \check{\pi}_0^{\text{int}}(f)|_{{}^{\Omega}\check{\pi}_0(X)}.$$

To check that ${}^{\Omega}\check{\pi}_0(f)$ is well-defined, let us see that $\check{\pi}_0^{\mathrm{int}}(f)(a) \in {}^{\Omega}\check{\pi}_0(Y)$ if $a \in {}^{\Omega}\check{\pi}_0(X)$. Let $E^Y \in \varepsilon(Y)$; hence, $f^{-1}(E^Y) \in \varepsilon(X)$. Since $a \in {}^{\Omega}\check{\pi}_0(X)$, $a = (C^{\mathrm{int}}_{E^X})_{E^X \in \varepsilon(Y)}$, and then there exist $x \in D(X)$ and $n_{f^{-1}(E^Y)} \in \mathbb{N}$ such that $\varphi_x(n) \in C^{\mathrm{int}}_{f^{-1}(E^Y)} \subset f^{-1}(E^Y)$, for all $n \geq n_{f^{-1}(E^Y)}$. Thus,

$$(f \circ \varphi_x)(n) = \psi_{f(x)}(n) \in f(C^{\mathrm{int}}_{f^{-1}(E^Y)}) \subset C^{\mathrm{int}}_{E^Y} \subset E^Y, \quad \forall n \geq n_{f^{-1}(E^Y)},$$

for the intrinsic path component $C^{\mathrm{int}}_{E^Y}$ of the exterior open subset E^Y that contains $f(C^{\mathrm{int}}_{f^{-1}(E^Y)})$. Remind that $f(x) \in f(D(X)) \subset D(Y)$. Therefore, $\check{\pi}_0^{\mathrm{int}}(f)(a) \in {}^{\Omega}\check{\pi}_0(Y)$.

As a consequence, the following diagram is commutative:

$$\begin{tikzcd}
{}^{\Omega}\check{\pi}_0(X) \arrow[r, hook] \arrow[d, "{}^{\Omega}\check{\pi}_0(f)"'] & \check{\pi}_0^{\mathrm{int}}(X) \arrow[d, "\check{\pi}_0^{\mathrm{int}}(f)"] \\
{}^{\Omega}\check{\pi}_0(Y) \arrow[r, hook] & \check{\pi}_0^{\mathrm{int}}(Y)
\end{tikzcd}$$

By Lemma 0.2.1, this implies that ${}^{\Omega}\check{\pi}_0$ is a functor. In fact, ${}^{\Omega}\check{\pi}_0$ is a subfunctor of $\check{\pi}_0^{\mathrm{int}}$; that is, there is a monomorphic natural transformation ${}^{\Omega}\check{\pi}_0 \longrightarrow \check{\pi}_0^{\mathrm{int}}$.

Moreover, if $a = (C^{\mathrm{int}}_E)_{E \in \varepsilon(X)} \in {}^{\Omega}\check{\pi}_0(X) \subset \check{\pi}_0^{\mathrm{int}}(X)$, then $\rho_X^{\mathrm{int}}(a)$ belongs to ${}^{\omega}\check{\pi}_0(X)$. For, let $E \in \varepsilon(X)$; hence, there exist $x \in D(X)$ and $n_E \in \mathbb{N}$ such that $\varphi_x(n) \in C^{\mathrm{int}}_E \subset E$, for all $n \geq n_E$ and for each intrinsic path component C^{int}_E. Suppose that $\rho_X^{\mathrm{int}}(a) = (C_E)_{E \in \varepsilon(X)}$. Then, since $C^{\mathrm{int}}_E \subset C_E$ for each $E \in \varepsilon(X)$, we have that $\varphi_n(x) \in C_E \subset E$, $\forall n \geq n_E$, as we wanted to show.

Given $X \in |\mathbf{EF}(\mathbb{N})|$, define the map ${}^{\Omega}\rho_X \colon {}^{\Omega}\check{\pi}_0(X) \to {}^{\omega}\check{\pi}_0(X)$ given by ${}^{\Omega}\rho_X = \rho_X^{\mathrm{int}}|_{{}^{\Omega}\check{\pi}_0(X)}$. As a result of this definition, the following diagram is commutative:

$$\begin{tikzcd}
{}^{\Omega}\check{\pi}_0(X) \arrow[r, hook] \arrow[d, "{}^{\Omega}\rho_X"'] & \check{\pi}_0^{\mathrm{int}}(X) \arrow[d, "\rho_X^{\mathrm{int}}"] \\
{}^{\omega}\check{\pi}_0(X) \arrow[r, hook, "\check{\mathrm{in}}_X"'] & \check{\pi}_0(X)
\end{tikzcd}$$

Since ρ^{int} is a natural transformation, ${}^{\Omega}\check{\pi}_0$ is a subfunctor of $\check{\pi}_0^{\mathrm{int}}$ and ${}^{\omega}\check{\pi}_0$ is a subfunctor of $\check{\pi}_0$, by Lemma 0.2.2 we have the following result.

Proposition 3.2.5. *There is a natural transformation*

$${}^{\Omega}\rho \colon {}^{\Omega}\check{\pi}_0 \to {}^{\omega}\check{\pi}_0$$

given by the family of maps $\{{}^{\Omega}\rho_X \colon {}^{\Omega}\check{\pi}_0(X) \to {}^{\omega}\check{\pi}_0(X)\}_{X \in |\mathbf{EF}(\mathbb{N})|}$.

Remark 3.2.2. *By definition and from the fact that* $\check{\pi}_0^{\mathrm{int}}(X) = \check{\pi}_0(D^{\mathrm{int}}) = \check{\pi}_0^{\mathrm{int}}(D)$, *it easily follows that*

$${}^{\Omega}\check{\pi}_0(X) = {}^{\Omega}\check{\pi}_0(D) = {}^{\omega}\check{\pi}_0(D^{\mathrm{int}}).$$

The following example is quite suitable in order to highlight some differences among most of the end functors that we have studied so far: ${}^{\omega}\pi_0^{\mathrm{BG}}$, ${}^{\Omega}\pi_0^{\mathrm{BG}}$, ${}^{\omega}\pi_0^{\mathrm{S}}$, ${}^{\Omega}\pi_0^{\mathrm{S}}$, ${}^{\omega}\check{\pi}_0$, $\check{\pi}_0^{\mathrm{int}}$ and ${}^{\Omega}\check{\pi}_0$.

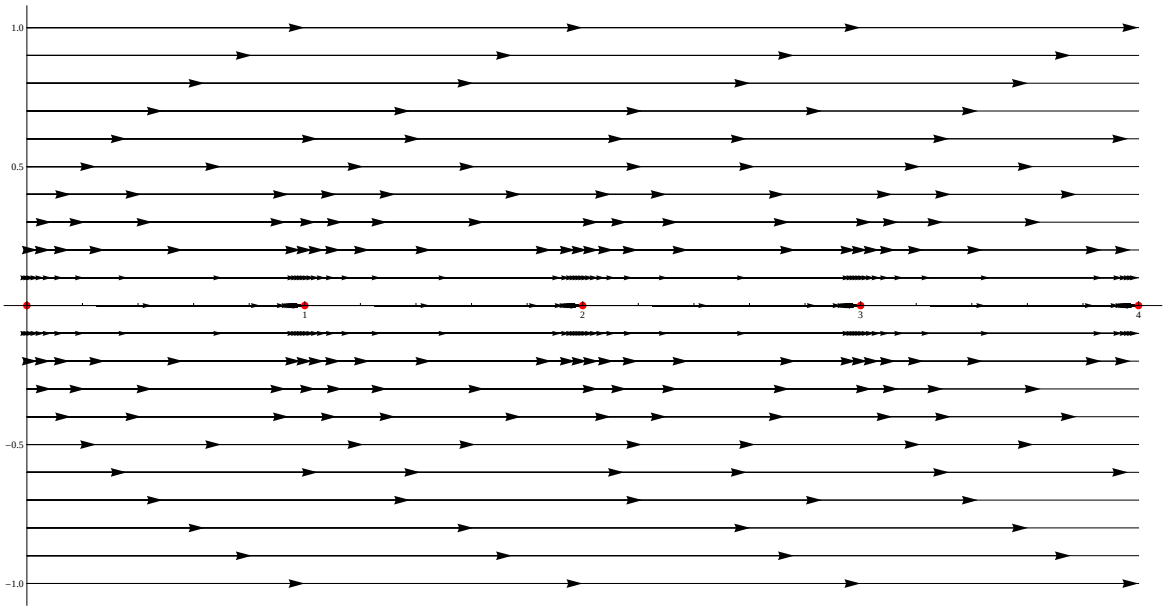


Figure 3.1: Representation of the action of the exterior discrete semi-flow $(X, \varphi, \varepsilon(X))$ on the points $\left(0, \pm\frac{k}{10}\right)$, $k = 1, 2, \ldots, 10$, and on the points $\left(l + \frac{1}{4}, 0\right)$, $l = 0, 1, 2, 3$.

Example 3.2.1. *Let* $X = \mathbb{R}_+ \times [-1, 1]$, $\varepsilon(X) = \{E \in \varepsilon^c(X) \mid \mathbb{N} \times \{0\} \subset E\}$ *and* $\varphi \colon \mathbb{N} \times X \to X$ *such that* $\varphi^1 \colon X \to X$ *is given by*

$$\varphi^1((x, y)) = \left(x + \frac{1 - \cos(2\pi x)}{5} + y^2, y\right).$$

The action of the exterior discrete semi-flow $(X, \varphi, \varepsilon(X))$ *on some points of the* x*-axis and* y*-axis is represented in Figure 3.1.*

Note that $\mathrm{Fix}(X) = \{(n, 0) \mid n \in \mathbb{N}\}$. *This is because*

$$\begin{aligned}
\varphi^1((x, 0)) = (x, 0) &\iff x + \frac{1 - \cos(2\pi x)}{5} = x \\
&\iff \cos(2\pi x) = 1 \\
&\iff 2\pi x = 2k\pi,\ k \in \mathbb{N} \\
&\iff x = k,\ k \in \mathbb{N}
\end{aligned}$$

and, if $y \neq 0$, *the equation*

$$\varphi^1((x, y)) = (x, y) \iff x + \frac{1 - \cos(2\pi x)}{5} + y^2 = x \iff \cos(2\pi x) = 1 + 5y^2$$

has no solutions, since $1 + 5y^2 > 1$, $\forall y \in [-1, 1] \setminus \{0\}$.

Also, if $(x,0) \in (n, n+1] \times \{0\}$*, then the trajectory* $(\varphi_{(x,0)}(k))_{k\in\mathbb{N}}$ *converges to the fixed point* $(n+1, 0)$*, for every* $n \in \mathbb{N}$*. In order to see that, it suffices to prove that, given* $x \in (0,1]$*, the sequence* $a_n \colon \mathbb{N} \to \mathbb{R}_+$ *such that* $a_0 = x$ *and*

$$a_{n+1} = a_n + \frac{1-\cos(2\pi a_n)}{5}$$

is monotonic and bounded (that sequence will converge to 1*, because it is the unique fixed point in* $(0,1]$*). In that case, what we want to show would follow from the fact that, for each* $n \in \mathbb{N}$*,*

$$\begin{aligned}\varphi^1((x+n,0)) &= \left(x+n+\frac{1-\cos(2\pi(x+n))}{5}, 0\right) = \left(x+n+\frac{1-\cos(2\pi x+2\pi n)}{5}, 0\right)\\ &= \left(x+\frac{1-\cos(2\pi x)}{5}+n, 0\right) = \varphi^1((x,0)) + (n,0).\end{aligned}$$

Notice that (a_n) *is monotonically increasing, because* $1-\cos(2\pi a_n) \geq 0$*,* $\forall n \in \mathbb{N}$*, and then*

$$a_{n+1} = a_n + \frac{1-\cos(2\pi a_n)}{5} \geq a_n, \ \forall n \in \mathbb{N}.$$

Besides, we will show that $a_n \in (0,1]$*,* $\forall n \in \mathbb{N}$*. For, define a map* $f \colon (0,1] \to \mathbb{R}_+$ *given by*

$$f(x) = x + \frac{1-\cos(2\pi x)}{5}.$$

One has that

$$f'(x) = 1 + \frac{2\pi}{5}\sin(2\pi x), \quad f''(x) = \frac{4\pi^2}{5}\cos(2\pi x);$$

$$f'(x) = 0 \iff \begin{cases} x = x_1 = 1 - \frac{\arcsin\left(\frac{5}{2\pi}\right)}{2\pi} \\ x = x_2 = \frac{1}{2} + \frac{\arcsin\left(\frac{5}{2\pi}\right)}{2\pi} \end{cases}$$

Since $f''(x_1) = \frac{2\pi}{5}\sqrt{4\pi^2-25} > 0$ *and* $f''(x_2) = -\frac{2\pi}{5}\sqrt{4\pi^2-25} < 0$*, we have exactly one local maximum at* x_2 *in* $(0,1]$*. The value of the map at* x_2 *is*

$$f(x_2) = \frac{7}{10} + \frac{\arcsin\left(\frac{5}{2\pi}\right)}{2\pi} + \frac{\sqrt{4\pi^2-25}}{10\pi} \approx 0.967587 < 1.$$

Moreover, $f(1) = 1$ *and* $\lim_{x\to 0^+} f(x) = 0$*. Therefore,* $0 < f(x) \leq 1$*,* $\forall x \in (0,1]$*, and* (a_n) *is bounded. A plot of the map* f *is shown in Figure 3.2.*

Suppose that we consider a continuous path $h \colon I \to X$ *with components* $h(t) = (x(t), y(t))$*. We have the following characterization of intrinsic paths in* X*:*

- *If* $h(I) \subset \mathbb{R}_+ \times \{0\}$*, we have that* h *is not intrinsic if and only if there exist* $r_0, r_1 \in I$ *such that* $x(r_0) \in \mathbb{N}$ *and* $x(r_1) > x(r_0)$*.*
- *If* $h(I) \not\subset \mathbb{R}_+ \times \{0\}$*,* h *is not intrinsic if and only if there exists* $t_0 \in I$ *such that* $y(t_0) = 0$*.*

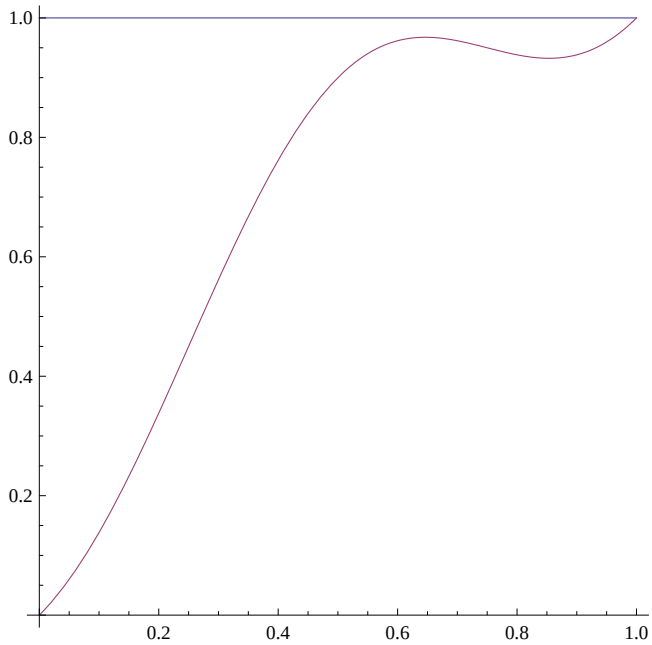


Figure 3.2: Graphic representation of the function $f\colon (0,1] \to (0,1]$ given by $f(x) = x + \dfrac{1-\cos(2\pi x)}{5}$.

In this example, $D(X) = X$. Now, we will calculate the ω-end sets and Ω-end sets of the exterior discrete semi-flow X:

$$^{\omega}\pi_0^{\mathrm{BG}}(X) = \{a_\infty\} \cup \{[\varphi_{(n,0)}]_\omega \mid n \in \mathbb{N}\},$$

where $a_\infty = [\varphi_{(x,y)}]_\omega \in {}^{\omega}\pi_0^{\mathrm{BG}}(X)$ if $y \neq 0$;

$$^{\Omega}\pi_0^{\mathrm{BG}}(X) = \{a_\infty^+, a_\infty^-\} \cup \{[\varphi_{(n,0)}]_\Omega \mid n \in \mathbb{N}\},$$

where $a_\infty^+ = [\varphi_{(x,y)}]_\Omega \in {}^{\Omega}\pi_0^{\mathrm{BG}}(X)$ if $y > 0$ and $a_\infty^- = [\varphi_{(x,y)}]_\Omega \in {}^{\Omega}\pi_0^{\mathrm{BG}}(X)$ if $y < 0$;

$$^{\omega}\pi_0^{\mathrm{S}}(X) = \{b_\infty\} \cup \{b_n \mid n \in \mathbb{N}\},$$

where $b_\infty = [\alpha]_\omega \in {}^{\omega}\pi_0^{\mathrm{S}}(X)$ satisfying $[\alpha|_{\mathbb{N}}]_\omega = [\varphi_{(x,y)}]_\omega$ if $y \neq 0$ and $b_n = [\beta]_\omega \in {}^{\omega}\pi_0^{\mathrm{S}}(X)$ satisfying $[\beta|_{\mathbb{N}}]_\omega = [\varphi_{(n,0)}]_\omega$;

$$^{\Omega}\pi_0^{\mathrm{S}}(X) = \{b_\infty^+, b_\infty^-\} \cup \{b_n \mid n \in \mathbb{N}\},$$

where $b_\infty^+ = [\alpha]_\Omega \in {}^{\Omega}\pi_0^{\mathrm{S}}(X)$ satisfying $[\alpha|_{\mathbb{N}}]_\Omega = [\varphi_{(x,y)}]_\Omega$ if $y > 0$ (in such a way that, if $\alpha(t) = (x(t), y(t))$, then $y(t) > 0$, $\forall t \in \mathbb{R}_+$), $b_\infty^- = [\beta]_\Omega \in {}^{\Omega}\pi_0^{\mathrm{S}}(X)$ satisfying $[\beta|_{\mathbb{N}}]_\Omega = [\varphi_{(x,y)}]_\Omega$ if $y < 0$ (in such a way that, if $\beta(t) = (x(t), y(t))$, then $y(t) < 0$, $\forall t \in \mathbb{R}_+$) and $b_n = [\gamma]_\Omega \in {}^{\Omega}\pi_0^{\mathrm{S}}(X)$ satisfying $[\gamma|_{\mathbb{N}}]_\Omega = [\varphi_{(n,0)}]_\Omega$ (in such a way that $\gamma(t) = (x(t), 0)$, $\forall t \in \mathbb{R}_+$);

$$^{\omega}\check{\pi}_0(X) = \{c_\infty, c_0\} \cup \{c_n \mid n \in \mathbb{N}^*\},$$

where $c_\infty = ((n,+\infty)\times[-1,1])_{n\in\mathbb{N}}$, $c_0 = (C_E)_{E\in\varepsilon(X)} \in {}^{\omega}\check{\pi}_0(X)$ *such that* $(0,0)\in C_E \subset E$, $\forall E\in\varepsilon(X)$ *and*

$$c_n = \left(\left(n-\frac{1}{k+1}, n+\frac{1}{k+1}\right)\times\left(-\frac{1}{k+1},\frac{1}{k+1}\right)\right)_{k\in\mathbb{N}};$$

finally,

$$\check{\pi}_0^{\mathrm{int}}(X) = \{c_\infty^+, c_\infty^-, c_0, c_0^+, c_0^-, c_0^{\mathrm{r}}\}\cup\{c_n^+, c_n^-, c_n^{\mathrm{l}}, c_n^{\mathrm{r}} \mid n\in\mathbb{N}^*\},$$
$${}^{\Omega}\check{\pi}_0(X) = \{c_\infty^+, c_\infty^-, c_0\}\cup\{c_n^{\mathrm{l}} \mid n\in\mathbb{N}^*\},$$

with $c_\infty^+ = ((n,+\infty)\times(0,1])_{n\in\mathbb{N}}$, $c_\infty^- = ((n,+\infty)\times[-1,0))_{n\in\mathbb{N}}$, $c_0 = (\{(0,0)\})$, $c_0^+ = \left(\left[0,\frac{1}{k+1}\right)\times\left(0,\frac{1}{k+1}\right)\right)_{k\in\mathbb{N}}$, $c_0^- = \left(\left[0,\frac{1}{k+1}\right)\times\left(-\frac{1}{k+1},0\right)\right)_{k\in\mathbb{N}}$, $c_0^{\mathrm{r}} = \left(\left(0,\frac{1}{k+1}\right)\times\{0\}\right)_{k\in\mathbb{N}}$, $c_n^+ = \left(\left(n-\frac{1}{k+1},n+\frac{1}{k+1}\right)\times\left(0,\frac{1}{k+1}\right)\right)_{k\in\mathbb{N}}$, $c_n^- = \left(\left(n-\frac{1}{k+1},n+\frac{1}{k+1}\right)\times\left(-\frac{1}{k+1},0\right)\right)_{k\in\mathbb{N}}$, $c_n^{\mathrm{l}} = \left(\left(n-\frac{1}{k+1},n\right]\times\{0\}\right)_{k\in\mathbb{N}}$ *and* $c_n^{\mathrm{r}} = \left(\left(n,n+\frac{1}{k+1}\right)\times\{0\}\right)_{k\in\mathbb{N}}$.

Taking advantage of the study of the exterior discrete semi-flow given in Example 3.2.1, the next example will be used to put into practice what was claimed in Proposition 3.1.1.

Example 3.2.2. *Let* $X=\mathbb{R}_+$,

$$\varepsilon(\mathbb{R}_+) = \{E\in\varepsilon^c(\mathbb{R}_+) \mid \mathbb{N}\subset E\}$$

and $\varphi\colon \mathbb{N}\times\mathbb{R}_+\to\mathbb{R}_+$ *such that* $\varphi^1\colon\mathbb{R}_+\to\mathbb{R}_+$ *is given by*

$$\varphi^1(x) = x + \frac{1-\cos(2\pi x)}{5}.$$

A local basis for the intrinsic topology $\mathbf{t}_X^{\mathrm{int}}$ *of the exterior discrete semi-flow* X^{int} *at any point* $x\in\mathbb{R}_+$ *is given by:*

$$\mathcal{B}(x) = \begin{cases}\{(x-\epsilon,x+\epsilon)\cap[0,+\infty)\}_{\epsilon\in\mathbb{R}_+}, & \text{if } x\notin\mathbb{N},\\ \{(x-\epsilon,x]\cap[0,+\infty)\}_{\epsilon\in\mathbb{R}_+}, & \text{if } x\in\mathbb{N}.\end{cases}$$

At the same time, a path $\alpha\colon I\to\mathbb{R}_+$ *is intrinsic if either* $\alpha(I)\subseteq(n,n+1]$, *for some* $n\in\mathbb{N}$, *or* $\alpha(t)=0$, *for all* $t\in I$. *In this example, since* $D(X)=X$, *we have that* $\alpha\colon I\to X$ *is an intrinsic path if and only if* $\alpha\colon I\to X^{\mathrm{int}}$ *is a continuous map.*

Remark 3.2.3. *Given an exterior discrete semi-flow* X, *the map* $\rho_X^{\mathrm{int}}\colon\check{\pi}_0^{\mathrm{int}}(X)\to\check{\pi}_0(X)$ *does not have to be either injective or surjective.*

- ρ_X^{int} *is not necessarily injective: in Example 3.2.1, we have that*

$$\rho_X^{\mathrm{int}}(c_\infty^+) = \rho_X^{\mathrm{int}}(c_\infty^-) = c_\infty \in {}^{\omega}\check{\pi}_0(X)\subset\check{\pi}_0(X).$$

  As a result, ρ_X^{int} *is not injective.*

- ρ_X^{int} *is not necessarily surjective: in Example 3.2.2, we have that*

$$\check{\pi}_0(\mathbb{R}_+) = \{c_\infty\} \cup \{c_n \mid n \in \mathbb{N}\},$$

where $c_\infty = ((n, +\infty))_{n\in\mathbb{N}}$ *and* $c_n = (C_E)_{E\in\varepsilon(\mathbb{R}_+)} \in \check{\pi}_0(\mathbb{R}_+)$ *such that* $n \in C_E \subset E$, $\forall E \in \varepsilon(X)$;

$$\check{\pi}_0^{\mathrm{int}}(\mathbb{R}_+) = \{c_0, c_0^{\mathrm{r}}\} \cup \{c_n^{\mathrm{l}}, c_n^{\mathrm{r}} \mid n \in \mathbb{N}^*\},$$

where $c_0 = (\{0\})$, $c_0^{\mathrm{r}} = \left(\left(0, \frac{1}{k+1}\right)\right)_{k\in\mathbb{N}}$, $c_n^{\mathrm{l}} = \left(\left(n - \frac{1}{k+1}, n\right]\right)_{k\in\mathbb{N}}$ *and* $c_n^{\mathrm{r}} = \left(\left(n, n + \frac{1}{k+1}\right)\right)_{k\in\mathbb{N}}$. *Observe that*

$$(\rho_X^{\mathrm{int}})^{-1}(c_\infty) = \emptyset.$$

Subsequently, ρ_X^{int} *is not surjective.*

Now, we shall prove that there exists a natural transformation between each pair of Ω-end functors.

Proposition 3.2.6. *There is a natural transformation*

$${}^{\Omega}R\colon {}^{\Omega}\pi_0^{\mathrm{S}} \to {}^{\Omega}\pi_0^{\mathrm{BG}}$$

given by the family of maps $\{{}^{\Omega}R_X\colon {}^{\Omega}\pi_0^{\mathrm{S}}(X) \to {}^{\Omega}\pi_0^{\mathrm{BG}}(X)\}_{X\in|\mathbf{EF}(\mathbb{N})|}$ *such that, for every exterior discrete semi-flow* $X = (X, \varphi, \varepsilon(X))$, *the following formula is satisfied:*

$${}^{\Omega}R_X([\alpha]) = [\alpha|_{\mathbb{N}}] = [\varphi_{\alpha(0)}], \quad [\alpha] \in {}^{\Omega}\pi_0^{\mathrm{S}}(X).$$

Proof. Note that ${}^{\Omega}R$ is actually well-defined, because $\alpha(0) \in D(X)$ (since $\alpha|_{\mathbb{N}}$ is exterior) and $\alpha \sim_\Omega \beta \Rightarrow \varphi_{\alpha(0)} \sim_\Omega \varphi_{\beta(0)}$ being $\alpha, \beta\colon \mathbb{R}_+ \to X$ exterior maps. Indeed, ${}^{\Omega}R$ is also a natural transformation. For, given an exterior discrete semi-flow morphism $f\colon (X, \varphi, \varepsilon(X)) \to (Y, \psi, \varepsilon(Y))$, one has that, for all $[\alpha] \in {}^{\Omega}\pi_0^{\mathrm{S}}(X)$,

$$\begin{aligned}({}^{\Omega}R_Y \circ {}^{\Omega}\pi_0^{\mathrm{S}}(f))([\alpha]) &= {}^{\Omega}R_Y({}^{\Omega}\pi_0^{\mathrm{S}}(f)([\alpha])) = {}^{\Omega}R_Y([f\circ\alpha]) = [(f\circ\alpha)|_{\mathbb{N}}] = [f\circ(\alpha|_{\mathbb{N}})]\\ &= [f\circ\varphi_{\alpha(0)}] = [\psi_{f(\alpha(0))}],\end{aligned}$$

$$\begin{aligned}({}^{\Omega}\pi_0^{\mathrm{BG}}(f) \circ {}^{\Omega}R_X)([\alpha]) &= {}^{\Omega}\pi_0^{\mathrm{BG}}(f)({}^{\Omega}R_X([\alpha])) = {}^{\Omega}\pi_0^{\mathrm{BG}}(f)([\alpha|_{\mathbb{N}}]) = {}^{\Omega}\pi_0^{\mathrm{BG}}(f)([\varphi_{\alpha(0)}])\\ &= [f\circ\varphi_{\alpha(0)}] = [\psi_{f(\alpha(0))}];\end{aligned}$$

hence, ${}^{\Omega}R_Y \circ {}^{\Omega}\pi_0^{\mathrm{S}}(f) = {}^{\Omega}\pi_0^{\mathrm{BG}}(f) \circ {}^{\Omega}R_X$. □

In order to give a natural transformation ${}^{\Omega}\phi\colon {}^{\Omega}\pi_0^{\mathrm{S}} \to {}^{\Omega}\check{\pi}_0$, for each exterior discrete semi-flow $X = (X, \varphi, \varepsilon(X))$ define ${}^{\Omega}\phi_X\colon {}^{\Omega}\pi_0^{\mathrm{S}}(X) \to {}^{\Omega}\check{\pi}_0(X)$ as follows: given an element $[\alpha] \in {}^{\Omega}\pi_0^{\mathrm{S}}(X)$, set ${}^{\Omega}\phi_X([\alpha]) = a = ({}^{\alpha}C_E^{\mathrm{int}})_{E\in\varepsilon(X)}$, where for each $E \in \varepsilon(X)$, ${}^{\alpha}C_E^{\mathrm{int}}$ is the unique intrinsic path component of E such that there is $r_E \in \mathbb{R}_+$ satisfying $\alpha([r_E, +\infty)) \subset {}^{\alpha}C_E^{\mathrm{int}} \subset E$.

Let us see that the maps ${}^{\Omega}\phi_X$ are well-defined. If $[\alpha] \in {}^{\Omega}\pi_0^{\mathrm{S}}(X)$, then $\varphi^n(\alpha(0)) = \alpha(n)$, for all $n \in \mathbb{N}$, and α is exterior, so $\alpha(0) \in D(X)$. If we suppose that $\alpha([r_E, +\infty)) \subset {}^{\alpha}C_E^{\mathrm{int}} \subset E$, then there exists $n_E \in \mathbb{N}$ satisfying $n_E \geq r_E$ and $\varphi_{\alpha(0)}(n) = \varphi^n(\alpha(0)) = \alpha|_{\mathbb{N}}(n) \in {}^{\alpha}C_E^{\mathrm{int}} \subset E$, $\forall n \geq n_E$. Therefore, $a = {}^{\Omega}\phi_X([\alpha]) \in {}^{\Omega}\check{\pi}_0(X)$.

Moreover, if $[\beta] \in {}^{\Omega}\pi_0^{\mathrm{S}}(X)$ is such that $[\beta] = [\alpha]$, then ${}^{\Omega}\phi_X([\alpha]) = {}^{\Omega}\phi_X([\beta])$. To prove this, suppose that $F\colon \mathbb{R}_+ \bar{\times} I \to X$ is an exterior homotopy such that, for all $u \in \mathbb{R}_+$, $F(u,0) = \alpha(u)$, $F(u,1) = \beta(u)$ and $F(n,t) = \varphi^n(F(0,t))$, $\forall n \in \mathbb{N}$, $t \in I$; also, let ${}^{\Omega}\phi_X([\alpha]) = ({}^{\alpha}C_E^{\mathrm{int}})_{E\in\varepsilon(X)}$ and ${}^{\Omega}\phi_X([\beta]) = ({}^{\beta}C_E^{\mathrm{int}})_{E\in\varepsilon(X)}$, so that there exist $r_E, s_E \in \mathbb{R}_+$ such that $\alpha([r_E, +\infty)) \subset {}^{\alpha}C_E^{\mathrm{int}}$ and $\beta([s_E, +\infty)) \subset {}^{\beta}C_E^{\mathrm{int}}$, $\forall E \in \varepsilon(X)$. Take any $E_0 \in \varepsilon(X)$ and take a natural number m such that $r_{E_0} \leq m$, $s_{E_0} \leq m$ and $\varphi^m(F(0,t)) \subset E_0$, $\forall t \in I$. It suffices to observe that, given $r, s \in \mathbb{R}_+$ with $r_{E_0} \leq r$ and $s_{E_0} \leq s$, there is an intrinsic path $h\colon I \to X$ with $h(I) \subset E_0$ given by

$$h(t) = \begin{cases} \alpha(3(m-r)t + r), & \text{if } 0 \leq t < \frac{1}{3}, \\ F(m, 3t-1), & \text{if } \frac{1}{3} \leq t < \frac{2}{3}, \\ \beta(3(s-m)t + 3m - 2s), & \text{if } \frac{2}{3} \leq t \leq 1 \end{cases}$$

satisfying $h(0) = \alpha(r)$, $h(1) = \beta(s)$. Since $\alpha|_{[n,n+1]}$ and $\beta|_{[n,n+1]}$ are intrinsic paths, $\forall n \in \mathbb{N}$, note that $\alpha|_{[a,b]}$ and $\beta|_{[a,b]}$ are intrinsic paths, for any $[a,b] \subset \mathbb{R}_+$. Then, by Lemma 3.1.1, the path h is intrinsic. This implies that ${}^{\alpha}C_{E_0}^{\mathrm{int}} = {}^{\beta}C_{E_0}^{\mathrm{int}}$. Therefore, ${}^{\alpha}C_E^{\mathrm{int}} = {}^{\beta}C_E^{\mathrm{int}}$, for all $E \in \varepsilon(X)$, and then ${}^{\Omega}\phi_X$ is well-defined.

Let $X = (X, \varphi, \varepsilon(X))$ and $Y = (Y, \psi, \varepsilon(Y))$ be exterior discrete semi-flows, and let $f\colon X \to Y$ be an exterior discrete semi-flow morphism. It remains to prove that the following diagram is commutative:

$$\begin{tikzcd} {}^{\Omega}\pi_0^{\mathrm{S}}(X) \arrow[r, "{}^{\Omega}\phi_X"] \arrow[d, "{}^{\Omega}\pi_0^{\mathrm{S}}(f)"'] & {}^{\Omega}\check{\pi}_0(X) \arrow[d, "{}^{\Omega}\check{\pi}_0(f)"] \\ {}^{\Omega}\pi_0^{\mathrm{S}}(Y) \arrow[r, "{}^{\Omega}\phi_Y"'] & {}^{\Omega}\check{\pi}_0(Y) \end{tikzcd}$$

Take $[\alpha] \in {}^{\Omega}\pi_0^{\mathrm{S}}(X)$. Suppose that ${}^{\Omega}\phi_X([\alpha]) = (C_{E^X}^{\mathrm{int}})_{E^X\in\varepsilon(X)}$ and that ${}^{\Omega}\check{\pi}_0(f)((C_{E^X}^{\mathrm{int}})_{E^X\in\varepsilon(X)}) = (C_{E^Y}^{\mathrm{int}})_{E^Y\in\varepsilon(Y)}$, where $f(C_{f^{-1}(E^Y)}^{\mathrm{int}}) \subset C_{E^Y}^{\mathrm{int}}$. Given $E^Y \in \varepsilon(Y)$, then $f^{-1}(E^Y) \in \varepsilon(X)$. Hence, there is $n_{f^{-1}(E^Y)} \in \mathbb{N}$ such that $\alpha([n_{f^{-1}(E^Y)}, +\infty)) \subset C_{f^{-1}(E^Y)}^{\mathrm{int}} \subset f^{-1}(E^Y)$. This implies that $f(\alpha([n_{f^{-1}(E^Y)}, +\infty))) \subset f(C_{f^{-1}(E^Y)}^{\mathrm{int}}) \subset C_{E^Y}^{\mathrm{int}} \subset E^Y$. Then,

$$({}^{\Omega}\check{\pi}_0(f) \circ {}^{\Omega}\phi_X)([\alpha]) = (C_{E^Y}^{\mathrm{int}})_{E^Y\in\varepsilon(Y)} = {}^{\Omega}\phi_Y([f \circ \alpha]) = ({}^{\Omega}\phi_Y \circ {}^{\Omega}\pi_0^{\mathrm{S}}(f))([\alpha]).$$

Thus, we have that the diagram is commutative, as well as the following result.

Proposition 3.2.7. *There is a natural transformation*

$${}^{\Omega}\phi\colon {}^{\Omega}\pi_0^{\mathrm{S}} \to {}^{\Omega}\check{\pi}_0$$

given by the family of maps $\{{}^{\Omega}\phi_X\colon {}^{\Omega}\pi_0^{\mathrm{S}}(X) \to {}^{\Omega}\check{\pi}_0(X)\}_{X\in|\mathbf{EF}(\mathbb{N})|}$.

In the case that the subjacent spaces of the exterior discrete semi-flows considered are first-countable at infinity, we also have the following proposition.

Proposition 3.2.8. *Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow whose subjacent exterior space X is first-countable at infinity. Then,*

$$^{\Omega}\phi_X \colon {}^{\Omega}\pi_0^{\mathrm{S}}(X) \to {}^{\Omega}\check{\pi}_0(X)$$

is surjective.

Proof. Since X is first-countable at infinity, there exists a sequence $(E_i)_{i\in\mathbb{N}}$ with $E_i \in \varepsilon(X)$ and $E_i \supset E_{i+1}$, $\forall i \in \mathbb{N}$, which satisfies that, $\forall E \in \varepsilon(X)$, $\exists i_E \in \mathbb{N}$ such that $E \supset E_{i_E}$. Let $b \in {}^{\Omega}\check{\pi}_0(X)$, which can be represented in the form $b = (C_i^{\mathrm{int}})_{i\in\mathbb{N}} \in \lim_{i\in\mathbb{N}} \pi_0^{\mathrm{int}}(E_i)$, where C_i^{int} is an intrinsic path component of E_i and $C_i^{\mathrm{int}} \supset C_{i+1}^{\mathrm{int}}$. Since $b \in {}^{\Omega}\check{\pi}_0(X)$, there exists $x \in D(X)$ and there exists an increasing sequence $n_0 < n_1 < n_2 < \dots$ such that, if $n_i \leq j < n_{i+1}$, then $\varphi_x(j) \in C_i^{\mathrm{int}}$. Taking $x' = \varphi_x(n_0)$ and $n_i' = n_i - n_0$, we have that $n_0' = 0$ and, for $n_i' \leq k < n_{i+1}'$, $\varphi_{x'}(k) \in C_i^{\mathrm{int}}$. For each $k \in \mathbb{N}$, if $n_i' \leq k < n_{i+1}'$, then $\varphi_{x'}(k), \varphi_{x'}(k+1) \in C_i^{\mathrm{int}}$; hence, there is an intrinsic path $f_k \colon [0,1] \to C_i^{\mathrm{int}}$ such that $f_k(0) = \varphi_{x'}(k)$ and $f_k(1) = \varphi_{x'}(k+1)$, $\forall k \in \mathbb{N}$. Now, define a map $\alpha \colon [0, +\infty) \to X$ given by $\alpha(t) = f_k(t-k)$, whenever $k \leq t < k+1$. Note that, by construction, $[\alpha] \in {}^{\Omega}\pi_0^{\mathrm{S}}(X)$: this is because

$$\varphi^n(\alpha(0)) = \varphi^n(f_0(0)) = \varphi^n(\varphi_{x'}(0)) = \varphi_{x'}(n) = f_n(0) = \alpha(n),$$

each path $\alpha|_{[k,k+1]} = f_k$ is intrinsic and $\alpha([k, k+1]) \subset C_i^{\mathrm{int}}$, if $n_i' \leq k$, $i \in \mathbb{N}$. Since ${}^{\Omega}\phi_X([\alpha]) = b$, then ${}^{\Omega}\phi_X$ is surjective, as we wanted to show. □

Next, to construct a natural transformation from ${}^{\Omega}\check{\pi}_0$ to ${}^{\Omega}\pi_0^{\mathrm{BG}}$, consider the category $\mathbf{EF_{fc}}(\mathbb{N})$ and let ${}^{\Omega}\check{\pi}_0$ and ${}^{\Omega}\pi_0^{\mathrm{BG}}$ be functors from $\mathbf{EF_{fc}}(\mathbb{N})$ to $\mathbf{Set}$.

For each $X = (X, \varphi, \varepsilon(X)) \in \mathbf{EF_{fc}}(\mathbb{N})$, remind that there exists a countable basis $\{E_i\}_{i\in\mathbb{N}} \subset \varepsilon(X)$, $E_0 \supset E_1 \supset E_2 \supset \dots$, and define a map

$$^{\Omega}\theta_X \colon {}^{\Omega}\check{\pi}_0(X) \to {}^{\Omega}\pi_0^{\mathrm{BG}}(X)$$

given in the following way: let $a \in {}^{\Omega}\check{\pi}_0(X)$, which can be represented, considering the basis above, by $a = (C_i^{\mathrm{int}})_{i\in\mathbb{N}}$, so that C_i^{int} is an intrinsic path component of E_i verifying that $C_{i+1}^{\mathrm{int}} \subset C_i^{\mathrm{int}}$ and, moreover, there exists $x \in D(X)$ and there exists an increasing sequence $n_0 < n_1 < n_2 < \dots$ such that, if $n_i \leq j < n_{i+1}$, then $\varphi_x(j) \in C_i^{\mathrm{int}}$. Taking $x' = \varphi_x(n_0)$ and $n_i' = n_i - n_0$, we have that $n_0' = 0$ and, for $n_i' \leq k < n_{i+1}'$, $\varphi_{x'}(k) \in C_i^{\mathrm{int}}$; in particular, $\varphi_x(n_0) \in C_0^{\mathrm{int}} \subset E_0$, and $x' = \varphi_x(n_0) \in D(X)$. We set ${}^{\Omega}\theta_X(a) = [\varphi_{x'}]$.

It is easy to check that

$$\begin{tikzcd}
{}^{\Omega}\check{\pi}_0(X) \arrow[r, "{}^{\Omega}\theta_X"] \arrow[d, "\cong"'] & {}^{\Omega}\pi_0^{\mathrm{BG}}(X) \arrow[d, "\cong"] \\
{}^{\omega}\check{\pi}_0(D^{\mathrm{int}}(X)) \arrow[r, "{}^{\omega}\theta_{D^{\mathrm{int}}}"'] & {}^{\omega}\pi_0^{\mathrm{BG}}(D^{\mathrm{int}}(X))
\end{tikzcd}$$

is a commutative diagram. This implies that ${}^{\Omega}\theta_X$ is well-defined. Since we have natural isomorphisms ${}^{\Omega}\check{\pi}_0(X) \cong {}^{\omega}\check{\pi}_0(D^{\mathrm{int}}(X))$, ${}^{\Omega}\pi_0^{\mathrm{BG}}(X) \cong {}^{\omega}\pi_0^{\mathrm{BG}}(D^{\mathrm{int}}(X))$ and ${}^{\omega}\theta_X$, we have that ${}^{\Omega}\theta_X$ is natural with respect to morphisms. Therefore, we have the following result.

Proposition 3.2.9. *There is a natural transformation*

$$ {}^{\Omega}\theta\colon {}^{\Omega}\check{\pi}_0 \to {}^{\Omega}\pi_0^{\mathrm{BG}} $$

given by the family of maps $\{{}^{\Omega}\theta_X\colon {}^{\Omega}\check{\pi}_0(X) \to {}^{\Omega}\pi_0^{\mathrm{BG}}(X)\}_{X\in\mathbf{EF_{fc}}(\mathbb{N})}$.

Moreover, one has the following proposition.

Proposition 3.2.10. *Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow whose subjacent exterior space X is first-countable at infinity. Then,*

$$ {}^{\Omega}\theta_X\colon {}^{\Omega}\check{\pi}_0(X) \to {}^{\Omega}\pi_0^{\mathrm{BG}}(X) $$

is injective.

Let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow. Then, observe that the map ${}^{\Omega}\mathrm{Sh}_X\colon {}^{\Omega}\pi_0^{\mathrm{BG}}(X) \to {}^{\Omega}\pi_0^{\mathrm{BG}}(X)$ given by ${}^{\Omega}\mathrm{Sh}_X([\varphi_x]) = [\varphi_x \circ \mathrm{sh}]$ is well-defined. Note that there exists $x' = 1 \cdot x \in D(X)$ such that

$$ [\varphi_{x'}] = [\varphi_{1\cdot x}] = [\varphi_x \circ \mathrm{sh}] = {}^{\Omega}\mathrm{Sh}_X([\varphi_x]), $$

and then ${}^{\Omega}\mathrm{Sh}_X([\varphi_x]) \in {}^{\Omega}\pi_0^{\mathrm{BG}}(X)$.

Moreover, if we suppose that $y \in D(X)$ so that $[\varphi_x] = [\varphi_y] \in {}^{\Omega}\pi_0^{\mathrm{BG}}(X)$, then there is an exterior homotopy $F\colon \mathbb{N}\bar{\times}I \to X$ such that $F(n,0) = \varphi_x(n)$, $F(n,1) = \varphi_y(n)$ and $F(n,t) = \varphi^n(F(0,t))$, for all $n \in \mathbb{N}$ and $t \in I$. Consider the exterior homotopy $G\colon \mathbb{N}\bar{\times}I \to X$ given by $G(n,t) = F(n+1,t)$, which satisfies

$$ \begin{aligned} G(n,0) &= F(n+1,0) = \varphi_x(n+1) = \varphi_x(\mathrm{sh}(n)) = (\varphi_x \circ \mathrm{sh})(n), \\ G(n,1) &= F(n+1,1) = \varphi_y(n+1) = \varphi_y(\mathrm{sh}(n)) = (\varphi_y \circ \mathrm{sh})(n), \\ G(n,t) &= F(n+1,t) = \varphi^{n+1}(F(0,t)) = \varphi^n(F(1,t)) = \varphi^n(G(0,t)), \end{aligned} $$

for all $n \in \mathbb{N}$, $t \in I$. This implies that $[\varphi_x \circ \mathrm{sh}] = [\varphi_y \circ \mathrm{sh}]$.

Theorem 3.2.1. *Let ${}^{\Omega}R$ and ${}^{\Omega}\phi$ be the natural transformations defined in this section and let $X = (X, \varphi, \varepsilon(X))$ be an exterior discrete semi-flow. Then:*

(i) ${}^{\Omega}\mathrm{Id}_X \circ {}^{\Omega}R_X = {}^{\Omega}\mathrm{Sh}_X \circ {}^{\Omega}R_X$. *Furthermore,* ${}^{\Omega}R_X({}^{\Omega}\pi_0^{\mathrm{S}}(X)) = \mathrm{Eq}({}^{\Omega}\mathrm{Id}_X, {}^{\Omega}\mathrm{Sh}_X)$.

(ii) Let ${}^{\Omega}\theta$ be the natural transformation defined in this section. If the subjacent exterior space X is first-countable at infinity, then in the diagram

$$ \begin{tikzcd} {}^{\Omega}\pi_0^{\mathrm{S}}(X) \arrow[rr, "{}^{\Omega}R_X"] \arrow[dr, "{}^{\Omega}\phi_X"'] & & {}^{\Omega}\pi_0^{\mathrm{BG}}(X) \arrow[r, shift left, "{}^{\Omega}\mathrm{Id}_X"] \arrow[r, shift right, "{}^{\Omega}\mathrm{Sh}_X"'] & {}^{\Omega}\pi_0^{\mathrm{BG}}(X) \\ & {}^{\Omega}\check{\pi}_0(X) \arrow[ur, "{}^{\Omega}\theta_X"'] & & \end{tikzcd} $$

we have that ${}^{\Omega}R_X = {}^{\Omega}\theta_X \circ {}^{\Omega}\phi_X$, where ${}^{\Omega}\phi_X$ is surjective and ${}^{\Omega}\theta_X$ is injective. As a consequence, ${}^{\Omega}\theta_X\colon {}^{\Omega}\check{\pi}_0(X) \to {}^{\Omega}\pi_0^{\mathrm{BG}}(X)$ is the equalizer of ${}^{\Omega}\mathrm{Id}_X$ and ${}^{\Omega}\mathrm{Sh}_X$.

Proof. (i) Let $[\alpha] \in {}^{\Omega}\pi_0^{\mathrm{S}}(X)$. Then, $\alpha\colon \mathbb{R}_+ \to X$ is an exterior map satisfying $\alpha(n) = \varphi^n(\alpha(0))$ and so that each path $\alpha|_{[n,n+1]}$ is intrinsic, $\forall n \in \mathbb{N}$. We have that

$$({}^{\Omega}\mathrm{Id}_X \circ {}^{\Omega}R_X)([\alpha]) = {}^{\Omega}\mathrm{Id}_X({}^{\Omega}R_X([\alpha])) = {}^{\Omega}R_X([\alpha]) = [\alpha|_{\mathbb{N}}].$$

Besides,

$$({}^{\Omega}\mathrm{Sh}_X \circ {}^{\Omega}R_X)([\alpha]) = {}^{\Omega}\mathrm{Sh}_X({}^{\Omega}R_X([\alpha])) = {}^{\Omega}\mathrm{Sh}_X([\alpha|_{\mathbb{N}}]).$$

We must prove that $[\alpha|_{\mathbb{N}}] = {}^{\Omega}\mathrm{Sh}_X([\alpha|_{\mathbb{N}}]) = [\alpha|_{\mathbb{N}} \circ \mathrm{sh}]$. For, define the homotopy $F\colon \mathbb{N}\bar{\times}I \to X$ given by $F(n,t) = \varphi^n(\alpha|_{[0,1]}(t))$ and such that

$$F(n,0) = \varphi^n(\alpha|_{[0,1]}(0)) = \varphi^n(\alpha(0)) = \alpha(n),$$

$$F(n,1) = \varphi^n(\alpha|_{[0,1]}(1)) = \varphi^n(\alpha(1)) = \varphi^{n+1}(\alpha(0)) = \alpha(n+1) = \alpha(\mathrm{sh}(n)) = (\alpha \circ \mathrm{sh})(n),$$

$$F(n,t) = \varphi^n(\alpha|_{[0,1]}(t)) = \varphi^n(\varphi^0(\alpha|_{[0,1]}(t))) = \varphi^n(F(0,t)).$$

Furthermore, F is exterior, since $F(0,t) = \alpha|_{[0,1]}(t)$ is intrinsic. Therefore, ${}^{\Omega}\mathrm{Id}_X \circ {}^{\Omega}R_X = {}^{\Omega}\mathrm{Sh}_X \circ {}^{\Omega}R_X$.

We have just shown that ${}^{\Omega}R_X({}^{\Omega}\pi_0^{\mathrm{S}}(X)) \subset \mathrm{Eq}({}^{\Omega}\mathrm{Id}_X, {}^{\Omega}\mathrm{Sh}_X)$. To show the other inclusion, let $[\varphi_x] \in {}^{\Omega}\pi_0^{\mathrm{BG}}(X)$ such that $[\varphi_x] = {}^{\Omega}\mathrm{Sh}_X([\varphi_x]) = [\varphi_x \circ \mathrm{sh}]$ for some suitable $x \in D(X)$. Observe that there is an exterior homotopy $F\colon \mathbb{N}\bar{\times}I \to X$ such that $F(n,0) = \varphi_x(n)$, $F(n,1) = \mathrm{Sh}_X(\varphi_x)(n) = (\varphi_x \circ \mathrm{sh})(n) = \varphi_x(\mathrm{sh}(n)) = \varphi_x(n+1)$ and $F(n,t) = \varphi^n(F(0,t))$, $\forall n \in \mathbb{N}$, $\forall t \in I$. Hence, there exists a continuous map $\beta\colon [0,+\infty) \to X$ given by $\beta(r) = F(n, r-n)$, whenever $n \le r < n+1$, $n \in \mathbb{N}$. We will see that $[\beta] \in {}^{\Omega}\pi_0^{\mathrm{S}}(X)$. For, observe that, since F is exterior, β is also exterior,

$$\beta(n) = F(n,0) = \varphi^n(F(0,0)) = \varphi^n(\beta(0))$$

and $\beta|_{[n,n+1]}$ is intrinsic, for all $n \in \mathbb{N}$. In order to prove this last statement, given $E \in \varepsilon(X)$, there exists $k_E \in \mathbb{N}$ such that $\varphi^k(F(0,t)) = F(k,t) \in E$, $\forall t \in I$ and $\forall k \ge k_E$. Now, if $k \ge k_E - n$,

$$\varphi^k(\beta|_{[n,n+1]}(t)) = \varphi^k(F(n, t-n)) = \varphi^{n+k}(F(0, t-n)) \in E;$$

then, $\beta|_{[n,n+1]}$ is intrinsic, $\forall n \in \mathbb{N}$. What is more, one has trivially that ${}^{\Omega}R_X([\beta]) = [\beta|_{\mathbb{N}}] = [\varphi_x]$, so $[\varphi_x] \in {}^{\Omega}R_X({}^{\Omega}\pi_0^{\mathrm{S}}(X))$. It follows that $\mathrm{Eq}({}^{\Omega}\mathrm{Id}_X, {}^{\Omega}\mathrm{Sh}_X) \subset {}^{\Omega}R_X({}^{\Omega}\pi_0^{\mathrm{S}}(X))$. Hence,

$${}^{\Omega}R_X({}^{\Omega}\pi_0^{\mathrm{S}}(X)) = \mathrm{Eq}({}^{\Omega}\mathrm{Id}_X, {}^{\Omega}\mathrm{Sh}_X).$$

(ii) Let $\{E_i\}_{i\in\mathbb{N}} \subset \varepsilon(X)$ be a basis for $\varepsilon(X)$ which satisfies that $E_i \supset E_{i+1}$ for every $i \in \mathbb{N}$, and let $[\alpha] \in {}^{\Omega}\pi_0^{\mathrm{S}}(X)$ such that $\alpha(n) = \varphi^n(\alpha(0))$ and $h_n\colon I \to X$ given by $h_n(t) = \alpha(n+t)$ is an intrinsic path, $\forall n \in \mathbb{N}$. Suppose that ${}^{\Omega}\phi_X([\alpha]) = a = ({}^{\alpha}C_i^{\mathrm{int}})_{i\in\mathbb{N}} \in {}^{\Omega}\check{\pi}_0(X)$. We can modify the basis to a new one (denoting it again by $\{E_i\}_{i\in\mathbb{N}}$) in order to find $x' \in D(X)$ such that ${}^{\Omega}\theta_X(a) = [\varphi_{x'}]$ and satisfying $\varphi_{x'}(n) \in {}^{\alpha}C_n^{\mathrm{int}} \subset E_n$, $\forall n \in \mathbb{N}$. Then, one has that $({}^{\Omega}\theta_X \circ {}^{\Omega}\phi_X)([\alpha]) = [\varphi_{x'}]$ and ${}^{\Omega}R([\alpha]) = [\alpha|_{\mathbb{N}}]$. We have to show that $[\alpha|_{\mathbb{N}}] = [\varphi_{x'}]$.

By the definition of ${}^{\Omega}\phi_X([\alpha])$, there exists $r_n \in \mathbb{R}_+$ such that $\alpha([r_n, +\infty)) \subset {}^{\alpha}C_n^{\text{int}} \subset E_n$, $\forall n \in \mathbb{N}$; particularly, $\exists r_0 \in \mathbb{R}_+ \mid \alpha([r_0, +\infty)) \subset {}^{\alpha}C_0^{\text{int}} \subset E_0$. Let $n' = \min\{n \in \mathbb{N} \mid n \geq r_0\}$; then, $\alpha(n') \in {}^{\alpha}C_0^{\text{int}}$. Since ${}^{\alpha}C_0^{\text{int}}$ is an intrinsic path component of E_0, there is an intrinsic path $\gamma\colon I \to {}^{\alpha}C_0^{\text{int}}$ such that $\gamma(0) = \alpha(n')$ and $\gamma(1) = x'$. Define a path $g\colon I \to X$ given by

$$g(t) = \begin{cases} h_i((n'+1)t - i), & \text{if } \frac{i}{n'+1} \leq t < \frac{i+1}{n'+1}; \\ \gamma((n'+1)t - n'), & \text{if } \frac{n'}{n'+1} \leq t \leq 1, \end{cases}$$

where $i \in \{0, \ldots, n'-1\}$. This path satisfies $g(0) = h_0(0) = \alpha(0)$, $g(1) = \gamma(1) = x'$ and, by Lemma 3.1.1, it is intrinsic, since so are both γ and h_i, where $i = 0, \ldots, n'-1$.

Now, there exists a homotopy $F\colon \mathbb{N}\bar{\times}I \to X$ given by $F(n,t) = \varphi^n(g(t))$ which is exterior (because g is intrinsic) and such that

$$F(n,0) = \varphi^n(g(0)) = \varphi^n(\alpha(0)) = \alpha(n), \quad F(n,1) = \varphi^n(g(1)) = \varphi^n(x') = \varphi_{x'}(n),$$

$$\varphi^n(F(0,t)) = \varphi^n(\varphi^0(g(t))) = \varphi^n(g(t)) = F(n,t),$$

for all $n \in \mathbb{N}$ and $t \in I$.

Therefore, $[\alpha|_{\mathbb{N}}] = [\varphi_{x'}]$ and ${}^{\Omega}R_X = {}^{\Omega}\theta_X \circ {}^{\Omega}\phi_X$. The fact that ${}^{\Omega}\phi_X$ and ${}^{\Omega}\theta_X$ are respectively surjective and injective follows from Proposition 3.2.8 and Proposition 3.2.10, and it immediately allows us to state that $\mathrm{Eq}({}^{\Omega}\mathrm{Id}_X, {}^{\Omega}\mathrm{Sh}_X) \cong {}^{\Omega}\check{\pi}_0(X)$.

□

Chapter 4

Basins of ω-representable end points induced by periodic points

Along this chapter, different ways of associating some exterior discrete semi-flows to a given discrete semi-flow are analyzed. Specifically, we consider those which are related to certain noteworthy sub-flows, such as those of fixed points or m-periodic points, and some results are proved concerning their structure and connections.

Given a subset $A \subset X$, suppose that we are working with the exterior space $(X, \varepsilon(X, A), \mathbf{t}_X)$, with $\varepsilon(X, A) = \{U \in \mathbf{t}_X \mid A \subset U\}$, which was defined in Example 0.3.1. In this chapter, we will consider exterior discrete semi-flows of the form $(X, \varphi, \varepsilon(X, A))$. Denote

$$D(X, A) = D(X, \varphi, \varepsilon(X, A)) = \{x \in X \mid \varphi_x \text{ is } \varepsilon(X, A)\text{-exterior}\}$$

the region of exterior attraction of the subset A.

In section 4.1, for a given discrete semi-flow (X, φ) and two right-invariant subsets $A \subset B \subset X$, we compare the induced exterior discrete semi-flows $(X, \varphi, \varepsilon(X, A))$ and $(X, \varphi, \varepsilon(X, B))$. Note that $\varepsilon(X, B) \subset \varepsilon(X, A)$ and that we have a canonical morphism $\mathrm{Id}_B^A \colon (X, \varphi, \varepsilon(X, A)) \to (X, \varphi, \varepsilon(X, B))$, which induces the maps ${}^{\omega}\pi_0^{\mathrm{BG}}(\mathrm{Id}_B^A) \colon {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, A)) \to {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, B))$ and $D(\mathrm{Id}_B^A) \colon D(X, A) \to D(X, B)$. The main result of this section gives sufficient conditions to ensure that ${}^{\omega}\pi_0^{\mathrm{BG}}(\mathrm{Id}_B^A)$ is injective and the basin in $D(X, B)$ of an end point of the form ${}^{\omega}\pi_0^{\mathrm{BG}}(\mathrm{Id}_B^A)(a)$, $a \in {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, A))$, coincides with the basin of a in $D(X, A)$.

Section 4.2 is devoted to the study of regions of exterior attraction of discrete semi-flows provided with externologies induced by the open neighborhoods of a given finite union of asymptotically stable cycles. Roughly speaking, given a discrete semi-flow (X, φ), an m-cycle is said to be asymptotically stable if the region of exterior attraction of this m-cycle is a neighborhood of it and, if we take a point close to any point of the m-cycle, then the image of this point under φ^1 is also close to some point in the m-cycle. In Corollary 4.2.2, we will prove that, under suitable conditions, we have a canonical isomorphism ${}_aP_m(X) \cong {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, {}_aP_m(X)))$; that is, all the ω-representable end points can be given by asymptotically stable m-periodic points.

Another central theorem of this chapter appears in section 4.2: if (X, f) is a discrete semi-flow induced by a continuous map $f\colon X \to X$, X is first-countable, locally path-connected and T_2 and the union of asymptotically stable m-cycles turns out to be a finite set, then we can decompose the region of exterior attraction of these asymptotically stable m-cycles into a disjoint union of regions of exterior attraction of the asymptotically stable fixed points of the composite f^m.

In section 4.3, we analyze the particular case in which $X = S^2$ and the action $\varphi^1 = f$ is induced by a continuous map $f\colon S^2 \to S^2$ such that, for all $m \in \mathbb{N}$, $P_m(S^2)$ is a finite set. This makes the set ${}_aP_m(S^2)$ finite; in fact, if x_0 is an attracting m-cyclic point, one has that $x_0 \in {}_aP_m(S^2)$.

It is interesting to note that, as a consequence of the results appearing in section 4.2, the study of the region of exterior attraction of $(S^2, f, \varepsilon(S^2, {}_aC_m(S^2, f)))$ can be reduced to the study of the region of exterior attraction of $(S^2, f^m, \varepsilon(S^2, {}_a\mathrm{Fix}(S^2, f^m)))$, and the basins of an asymptotically stable m-cycle (considering f) are the union of the basins of the asymptotically stable fixed points (with respect to f^m) of that m-cycle.

4.1 Comparison between externologies

In the present chapter, the set of connected components of a topological space X will be denoted by $C(X)$. If $a \in U$ and U is an open subset of X, denote by $C(a, U)$ the connected component of a in U. Note that, if X is locally connected, then $C(a, U)$ is also an open subset of X.

In addition, suppose that $(Z, \varepsilon(Z))$, $(X, \varepsilon_1(X))$ and $(X, \varepsilon_2(X))$ are exterior spaces. Along this chapter, given a continuous map $f\colon Z \to X$, if $f\colon (Z, \varepsilon(Z)) \to (X, \varepsilon_k(X))$ is an exterior map, then we will say that f is an $\varepsilon_k(X)$-exterior map, $k = 1, 2$.

Given two exterior spaces $(X, \varepsilon(X)), (Y, \varepsilon(Y))$, we use exponential notation $(Y, \varepsilon(Y))^{(X, \varepsilon(X))}$ or a shorter notation such as $(Y, \varepsilon(Y))^X$ or Y^X for denoting the space of exterior maps from $(X, \varepsilon(X))$ to $(Y, \varepsilon(Y))$.

Lemma 4.1.1. *Suppose that $A \subset B$ are subsets of a locally connected topological space X. Consider the following commutative pullback diagrams, where $d_0(F)(n) = F(n, 0)$, $\forall n \in \mathbb{N}$, and the maps $\imath_B^A\colon (X, \varepsilon(X, A))^{\mathbb{N}\bar{\times} I} \to (X, \varepsilon(X, B))^{\mathbb{N}\bar{\times} I}$ and $\jmath_B^A\colon (X, \varepsilon(X, A))^{\mathbb{N}} \to (X, \varepsilon(X, B))^{\mathbb{N}}$ are induced by the map $\mathrm{Id}_B^A\colon (X, \varepsilon(X, A)) \to (X, \varepsilon(X, B))$:*

$$\begin{tikzcd}
P(X,A,A) \arrow[rr] \arrow[dr] \arrow[dd] & & (X,\varepsilon(X,A))^{\mathbb{N}\bar{\times} I} \arrow[dr, "\imath_B^A"] \arrow[dd, "d_0"] & \\
& P(X,A,B) \arrow[rr] \arrow[dd] & & (X,\varepsilon(X,B))^{\mathbb{N}\bar{\times} I} \arrow[dd, "d_0"] \\
(X,\varepsilon(X,A))^{\mathbb{N}} \arrow[rr, "\jmath_A^A=\mathrm{Id}"] \arrow[dr, "\jmath_A^A=\mathrm{Id}"'] & & (X,\varepsilon(X,A))^{\mathbb{N}} \arrow[dr, "\jmath_B^A"] & \\
& (X,\varepsilon(X,A))^{\mathbb{N}} \arrow[rr, "\jmath_B^A"] & & (X,\varepsilon(X,B))^{\mathbb{N}}
\end{tikzcd}$$

If there is a sequence of open subsets $U_0 \supset U_1 \supset U_2 \supset \dots$ such that, for every integer $i \geq 0$, $B \subset U_i$ satisfying, for $a, b \in A$ and $a \neq b$, $C(a, U_i) \cap C(b, U_i) = \emptyset$ and $\{C_i(A)\}_{i\in\mathbb{N}}$ is a basis for the externology $\varepsilon(X, A)$, where $C_i(A) = \bigsqcup_{a\in A} C(a, U_i)$, then the canonical map

$$P(X, A, A) \to P(X, A, B)$$

is a bijection.

Proof. Let $\alpha\colon \mathbb{N} \to X$ be $\varepsilon(X, A)$-exterior and let $\beta\colon \mathbb{N} \to X$ be $\varepsilon(X, B)$-exterior. Then, one has that $\alpha\colon \mathbb{N} \to X$ is also $\varepsilon(X, B)$-exterior. Suppose that we have an $\varepsilon(X, B)$-exterior homotopy $F\colon \mathbb{N}\bar{\times}I \to X$ from α to β. Given $E \in \varepsilon(X, A)$, since $\{C_i(A)\}_{i\in\mathbb{N}}$ is a basis for the externology $\varepsilon(X, A)$, there is n_0 such that $C_{n_0}(A) \subset E$ and there is n_1 such that $\alpha(n) \in C_{n_0}(A) \subset E$, for every $n \geq n_1$. Since $U_{n_0} \in \varepsilon(X, B)$, there is $n_2 \in \mathbb{N}$ such that, for every $n \geq n_2$, $F(\{n\} \times I) \subset U_{n_0}$. Hence, for every $n \geq n_3 = \max\{n_1, n_2\}$, $\alpha(n) \in C_{n_0}(A)$ and $F(\{n\} \times I) \subset U_{n_0}$. Since $\{n\} \times I$ is connected, one has that $F(\{n\} \times I) \subset C_{n_0}(A)$, for $n \geq n_3$. Then, for every $n \geq n_3$, $F(\{n\} \times I) \subset E$. Therefore, $\beta\colon \mathbb{N} \to X$ and $F\colon \mathbb{N}\bar{\times}I \to X$ are $\varepsilon(X, A)$-exterior maps. □

As a consequence of Lemma 4.1.1, we have the following result.

Corollary 4.1.1. *Suppose that $A \subset B$ are subsets of a locally connected topological space X. If there is a sequence of open subsets $U_0 \supset U_1 \supset U_2 \supset \dots$ such that, for every integer $i \geq 0$, $B \subset U_i$ satisfying, for $a, b \in A$ and $a \neq b$, $C(a, U_i) \cap C(b, U_i) = \emptyset$ and $\{C_i(A)\}_{i\in\mathbb{N}}$ is a basis for the externology $\varepsilon(X, A)$, where $C_i(A) = \bigsqcup_{a\in A} C(a, U_i)$, then the canonical map $\pi_0^{\mathrm{BG}}(X, \varepsilon(X, A)) \to \pi_0^{\mathrm{BG}}(X, \varepsilon(X, B))$ is injective.*

Proof. Suppose that $[\alpha], [\alpha'] \in \pi_0^{\mathrm{BG}}(X, \varepsilon(X, A))$ and that there is an $\varepsilon(X, B)$-exterior homotopy $F\colon \mathbb{N}\bar{\times}I \to X$ from α to α'. Then, the pair $(\alpha, F) \in P(X, A, B)$. Applying Lemma 4.1.1, we have that $(\alpha, F) \in P(X, A, A)$. This implies that F is $\varepsilon(X, A)$-exterior. Therefore, it follows that $[\alpha] = [\alpha']$. □

Theorem 4.1.1. *Let $A \subset B$ be right-invariant subsets of a locally connected discrete semi-flow (X, φ). Suppose that there is a sequence of open subsets $U_0 \supset U_1 \supset U_2 \supset \dots$ such that, for every integer $i \geq 0$, $B \subset U_i$ satisfying, for $a, b \in A$ and $a \neq b$, $C(a, U_i) \cap C(b, U_i) = \emptyset$ and $\{C_i(A)\}_{i\in\mathbb{N}}$ is a basis for the externology $\varepsilon(X, A)$, where $C_i(A) = \bigsqcup_{a\in A} C(a, U_i)$. Consider the induced exterior semi-flows $(X, \varepsilon(X, A)) = (X, \varphi, \varepsilon(X, A))$ and $(X, \varepsilon(X, B)) = (X, \varphi, \varepsilon(X, B))$. Then:*

(i) The canonical map ${}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, A)) \to {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, B))$ is injective.

(ii) The following is a commutative pullback diagram:

$$\begin{tikzcd}
D(X, A) \arrow[r, "\omega_A"] \arrow[d] & {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, A)) \arrow[d] \\
D(X, B) \arrow[r, "\omega_B"'] & {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, B))
\end{tikzcd}$$

Equivalently, if $a \in {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, A))$, then $\omega_A^{-1}(a) = \omega_B^{-1}(a)$.

Proof. (i) Let us consider the following commutative diagram:

$$\begin{array}{ccc} {}^{\omega}\pi_0^{\mathrm{BG}}(X,\varepsilon(X,A)) & \longrightarrow & {}^{\omega}\pi_0^{\mathrm{BG}}(X,\varepsilon(X,B)) \\ \downarrow & & \downarrow \\ \pi_0^{\mathrm{BG}}(X,\varepsilon(X,A)) & \longrightarrow & \pi_0^{\mathrm{BG}}(X,\varepsilon(X,B)) \end{array}$$

By definition, the canonical map ${}^{\omega}\pi_0^{\mathrm{BG}}(X,\varepsilon(X,A)) \to \pi_0^{\mathrm{BG}}(X,\varepsilon(X,A))$ is injective and, by Corollary 4.1.1, the map $\pi_0^{\mathrm{BG}}(X,\varepsilon(X,A)) \to \pi_0^{\mathrm{BG}}(X,\varepsilon(X,B))$ is injective. This implies that

$${}^{\omega}\pi_0^{\mathrm{BG}}(X,\varepsilon(X,A)) \to {}^{\omega}\pi_0^{\mathrm{BG}}(X,\varepsilon(X,B))$$

is also injective.

(ii) Take $a = [\varphi_x] \in {}^{\omega}\pi_0^{\mathrm{BG}}(X,\varepsilon(X,A))$, with $x \in D(X,A)$. It is obvious that $\omega_A^{-1}(a) \subset \omega_B^{-1}(a)$. Now, take $y \in D(X,B)$ and suppose that $y \in \omega_B^{-1}(a)$. Hence, there is an $\varepsilon(X,B)$-exterior homotopy $F\colon \mathbb{N}\bar{\times}I \to X$ from φ_x to φ_y. Notice that the pair (φ_x, F) is in the pullback $P(X,A,B)$; therefore, by Lemma 4.1.1, we have that (φ_x, F) is in $P(X,A,A)$, and then F and φ_y are $\varepsilon(X,A)$-exterior maps. This implies that $y \in \omega_A^{-1}(a)$. Thus, one has that

$$\omega_A^{-1}(a) = \omega_B^{-1}(a).$$

□

4.2 Basins of end points related to periodic points

Although their proofs are quite trivial, the following pair of lemmas will be useful along this section.

Lemma 4.2.1. *Let $A = \{x_1, \ldots, x_n\}$ be a finite set in a T_2 topological space X. Then, there exist open neighborhoods $U_1, U_2, \ldots, U_n$ of the respective points $x_1, x_2, \ldots, x_n$ such that $U_i \cap U_j = \emptyset$, for any $i \neq j$, $i, j \in \{1, \ldots, n\}$.*

Lemma 4.2.2. *Let $X = (X, \varphi)$ be a discrete semi-flow. Suppose that $x \in P_m(X)$ and let $a \in \mathbb{N}$. Then,*

$$\varphi_x(a \cdot m) = x.$$

Proposition 4.2.1. *Let $X = (X, \varphi)$ be a discrete semi-flow. If A is a finite right-invariant subset and X is T_2, then the canonical map*

$$\begin{array}{ccc} A & \longrightarrow & {}^{\omega}\pi_0^{\mathrm{BG}}(X,\varepsilon(X,A)) \\ x & \longmapsto & [\varphi_x] \end{array}$$

is injective.

Proof. Let $A = \{x_1, \dots, x_N\}$. Since A is a finite right-invariant subset, there is $m \in \mathbb{N}$ such that $A \subset P_m(X)$. By Lemma 4.2.1, there exist open neighborhoods $U_1, U_2, \dots, U_N$ of the respective m-periodic points $x_1, x_2, \dots, x_N$ such that $U_i \cap U_j = \emptyset$, for any $i \neq j$, $i, j \in \{1, \dots, N\}$.

Now, given $i, j \in \{1, \dots, N\}$, let us suppose that there is an exterior homotopy $F\colon \mathbb{N}\bar{\times} I \to X$ such that $F(n, 0) = \varphi_{x_i}(n)$ and $F(n, 1) = \varphi_{x_j}(n)$, $\forall n \in \mathbb{N}$. Since F is exterior, for each $E \in \varepsilon(X, A)$ there exists $n_E \in \mathbb{N}$ such that, for all $n \geq n_E$ and $t \in I$, $F(n, t) \in E$. Note that we can take E as a disjoint union of open neighborhoods,

$$E = \bigsqcup_{k=1}^{N} V_k,$$

where $V_k \subset U_k$, $\forall k \in \{1, \dots, N\}$. Hence, $F(\{n\} \times [0, 1]) \in \bigsqcup_{k=1}^{N} V_k$, $\forall n \geq n_E$. Since the continuous images of connected sets are connected, one has that $F(\{n\} \times [0, 1])$ is connected and then there is $l_n \in \{1, \dots, N\}$ such that $F(n, t) \in U_{l_n}$, $\forall t \in I$, $\forall n \geq n_E$. This means that both $\varphi_{x_i}(n)$ and $\varphi_{x_j}(n)$ belongs to the same open neighborhood U_{l_n}, and since there can only be one m-periodic point in it, we have that $\varphi_{x_i}(n) = \varphi_{x_j}(n)$, $\forall n \geq n_E$. Notice that one can find $a \in \mathbb{N}$ such that $a \cdot m \geq n_E$. Then, as $x_i, x_j \in P_m(X)$, by Lemma 4.2.2 we have that:

$$x_i = \varphi_{x_i}(a \cdot m) = \varphi_{x_j}(a \cdot m) = x_j.$$

Thus, the map is injective. □

Corollary 4.2.1. *Let $X = (X, \varphi)$ be a discrete semi-flow. If $P_m(X)$ is finite and X is T_2, then the canonical map*

$$P_m(X) \to {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, P_m(X)))$$

is injective.

In order to find a bijective map similar to the above-mentioned canonical map, we can deal with asymptotically stable cycles. In addition, we need the discrete semi-flow X also to be first-countable, locally path-connected and path-connected.

Definition 4.2.1. *Let (X, φ) be a discrete semi-flow. An m-cycle $\{x_0, \dots, x_{m-1}\}$ is said to be* **attractor** *if $D(X, \{x_0, \dots, x_{m-1}\})$ is a neighborhood of $\{x_0, \dots, x_{m-1}\}$. An m-cycle $\{x_0, \dots, x_{m-1}\}$ is said to be* **stable** *if, for every neighborhood U of $\{x_0, \dots, x_{m-1}\}$, there is an open subset V such that $\varphi^1(V) \subset V$ and $\{x_0, \dots, x_{m-1}\} \subset V \subset U$. If an m-cycle is attractor and stable, then it is said to be* **asymptotically stable**. *A periodic point x is said to be asymptotically stable if so is the l-cycle $\{x, 1 \cdot x, \dots, l \cdot x = x\}$ that it generates. Given a right-invariant subset $S \subset X$, the union of asymptotically stable cycles contained in S will be denoted by ${}_aS$.*

Theorem 4.2.1. *Let X be a discrete semi-flow. If $A \subset X$ is a finite union of asymptotically stable cycles and X is T_2, first-countable, locally path-connected and path-connected, then the canonical map*

$$\begin{array}{ccc} A & \longrightarrow & {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, A)) \\ x & \longmapsto & [\varphi_x] \end{array}$$

is a bijection.

Proof. Applying Proposition 4.2.1, we have that the canonical map

$$\begin{array}{ccc} A & \longrightarrow & {}^{\omega}\pi_0^{\mathrm{BG}}(X,\varepsilon(X,A)) \\ x & \longmapsto & [\varphi_x] \end{array}$$

is injective.

Let $A=\{x_1,\dots,x_N\}$ and denote $D(X)=D(X,A)$ for short. Since the cycle generated by any $x_k\in A$ is attractor, for each $k\in\{1,\dots,N\}$ we can find an open neighborhood U_k such that $x_k\in U_k\subset D(X)$; what is more, since X is a T_2 space, we can take these open neighborhoods satisfying $U_i\cap U_{i'}=\emptyset$, $\forall i,i'\in\{1,\dots,N\}$, $i\neq i'$, and we can suppose that each U_k is path-connected, for all $k\in\{1,\dots,N\}$, as X is locally path-connected. Notice that, since X is first-countable, we can also take a countable neighborhood basis $\{V_{x_k}^j\}_{j\in\mathbb{N}}$ at x_k so that $V_{x_k}^{j+1}\subset V_{x_k}^j$ and $V_{x_k}^0=U_k$, being $V_{x_k}^j$ path-connected, $\forall k\in\{1,\dots,N\}$, $\forall j\in\mathbb{N}$. Moreover, taking into account that every cycle contained in A is asymptotically stable, we can find also a new countable neighborhood basis $\{W_{x_k}^j\}_{j\in\mathbb{N}}$ at x_k so that $W_{x_k}^{j+1}\subset W_{x_k}^j$ and $V_{x_k}^{j+1}\subset W_{x_k}^j\subset V_{x_k}^j$, $\varphi^1\left(\bigsqcup_{k=1}^N W_{x_k}^j\right)\subset\bigsqcup_{k=1}^N W_{x_k}^j$, $\forall j\in\mathbb{N}$.

To prove surjectivity, take $y\in D(X)$ so that $[\varphi_y]\in{}^{\omega}\pi_0^{\mathrm{BG}}(X,\varepsilon(X,A))$. Since the sequence $\varphi_y\colon\mathbb{N}\to X$ is $\varepsilon(X,A)$-exterior, we can assume that there are natural numbers $K_0<K_1<K_2<\dots$ such that $\varphi_y(n)\in\bigsqcup_{k=1}^N V_{x_k}^j$, $\forall n\geq K_j$; in particular, $\varphi_y(n)\in\bigsqcup_{k=1}^N V_{x_k}^1$, $\forall n\geq K_1$. Let $l_y\in\{1,\dots,N\}$ so that $\varphi_y(K_1)\in V_{x_{l_y}}^1$. Because of the fact that $V_{x_{l_y}}^1$ is path-connected and $x_{l_y}\in V_{x_{l_y}}^1$, there exists a path $h\colon I\to V_{x_{l_y}}^1$ such that $h(0)=\varphi_y(K_1)$ and $h(1)=x_{l_y}$. Note that $V_{x_{l_y}}^1\subset W_{x_{l_y}}^0$ and $\varphi^1\left(\bigsqcup_{k=1}^N W_{x_k}^0\right)\subset\bigsqcup_{k=1}^N W_{x_k}^0$. This implies that $\varphi^n(h(I))\subset\bigsqcup_{k=1}^N W_{x_k}^0\subset\bigsqcup_{k=1}^N V_{x_k}^0$, $\forall n\in\mathbb{N}$.

Since $\varphi^n(h(I))$ is connected, there exists a unique $l_n'\in\{1,\dots,N\}$ so that $\varphi^n(h(I))\subset V_{x_{l_n'}}^0$. What is more, for every $n\in\mathbb{N}$, there is $j_n\in\mathbb{N}$ such that $K_{j_n}\leq n+K_1<K_{j_n+1}$. Then,

$$\varphi^n(x_{l_y})=\varphi^n(h(1))=x_{l_n'}\in V_{\varphi^n(x_{l_y})}^{j_n};\quad \varphi^n(\varphi_y(K_1))=\varphi^n(h(0))\in V_{\varphi^n(x_{l_y})}^{j_n}.$$

Therefore, for all $r\in\mathbb{N}$ satisfying $K_{j_n}\leq r<K_{j_n+1}$, there is a path $\alpha_r\colon I\to V_{\varphi^{r-K_1}(x_{l_y})}^{j_n}$ such that $\alpha_r(0)=\varphi_y(r)$ and $\alpha_r(1)=\varphi^{r-K_1}(x_{l_y})$.

We can also consider the m-cycle generated by x_{l_y}, which is $\{x_{l_y},\varphi^1(x_{l_y}),\dots,\varphi^{m-1}(x_{l_y})\}$ explicitly. Since $\varphi^{K_1}\colon\{x_{l_y},\varphi^1(x_{l_y}),\dots,\varphi^{m-1}(x_{l_y})\}\to\{x_{l_y},\varphi^1(x_{l_y}),\dots,\varphi^{m-1}(x_{l_y})\}$ is a bijection, there is a unique $x\in\{x_{l_y},\varphi^1(x_{l_y}),\dots,\varphi^{m-1}(x_{l_y})\}\subset A$ such that $\varphi^{K_1}(x)=x_{l_y}$. At this stage, we are able to prove that $[\varphi_x]=[\varphi_y]$ and this would imply that the canonical map is surjective.

Notice that X is path-connected, so there are paths $\alpha_0,\dots,\alpha_{K_1-1}$ such that $\alpha_k(0)=\varphi_y(k)$ and $\alpha_k(1)=\varphi_x(k)$, $\forall k\in\{0,\dots,K_1-1\}$. Thus, we can define an exterior homotopy $F\colon\mathbb{N}\bar{\times}I\to X$ given by $F(r,t)=\alpha_r(t)$. This exterior homotopy verifies that $F(r,0)=\varphi_y(r)$ and $F(r,1)=\varphi_x(r)$, for every $r\in\mathbb{N}$. Therefore, the canonical map is surjective and the theorem is proved. □

Remark 4.2.1. *We can introduce the following notion: an m-cycle $\{x,\varphi^1(x),\dots,\varphi^{m-1}(x)\}$ is* **weakly stable** *if there are disjoint open neighborhoods $V_x,V_{\varphi^1(x)},\dots,V_{\varphi^{m-1}(x)}$ of the respective points $x,\varphi^1(x),\dots,\varphi^{m-1}(x)$ such that*

$$V=V_x\cup V_{\varphi^1(x)}\cup\dots\cup V_{\varphi^{m-1}(x)}\subset D(X,\{x,\varphi^1(x),\dots,\varphi^{m-1}(x)\}),\quad \varphi^1(V)\subset V.$$

If the conditions of Theorem 4.2.1 involved a finite union of weakly stable cycles instead of a finite union of asymptotically stable cycles, then the result would also hold.

Corollary 4.2.2. *Let $X = (X, \varphi)$ be a discrete semi-flow. Denote by ${}_aP_m(X)$ the union of k-cycles $\{x_0, \dots, x_k\}$ such that k divides m and $\{x_0, \dots, x_k\}$ is asymptotically stable. If ${}_aP_m(X)$ is finite and X is T_2, first-countable, locally path-connected and path-connected, then*

$${}_aP_m(X) \cong {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, {}_aP_m(X))).$$

Remark 4.2.2. *If we remove the path-connected condition from the theorem above, the structure of ${}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, {}_aP_m(X)))$ could be more complicated (being related with the completely invariant subset of pre-m-periodic points of X).*

Proposition 4.2.2. *Let $X = (X, \varphi)$ be a first-countable and locally path-connected discrete semi-flow and suppose that A, B are disjoint right-invariant finite subsets of X. If there are right-invariant open subsets U, V of X such that $A \subset U \subset D(X, A \sqcup B)$, $B \subset V \subset D(X, A \sqcup B)$ and $U \cap V = \emptyset$, then*

$$D(X, A \sqcup B) = D(X, A) \sqcup D(X, B).$$

Proof. It is clear that $D(X, A \sqcup B) \supset D(X, A) \sqcup D(X, B)$. To prove the other inclusion, suppose that $A = \{a_1, \dots, a_N\}$ and $B = \{b_1, \dots, b_{N'}\}$, and let $y \in D(X, A \sqcup B)$. Remember that X is locally path-connected, so for each $a_k \in A$ we can find an open path-connected neighborhood U_k such that $a_k \in U_k \subset U \subset D(X, A \sqcup B)$; similarly, for each $b_{k'} \in B$ there exists an open path-connected neighborhood $V_{k'}$ satisfying $b_{k'} \in V_{k'} \subset V \subset D(X, A \sqcup B)$.

Since X is first-countable, we can also take neighborhood bases $\{U_k^j\}_{j \in \mathbb{N}}$ at $a_k \in A$ and $\{V_{k'}^j\}_{j \in \mathbb{N}}$ at $b_{k'} \in B$ so that, $\forall j \in \mathbb{N}$, $U_k^j \subset U_k^{j-1}$, $U_k^0 = U_k$, $\forall k \in \{1, \dots, N\}$, and $V_{k'}^j \subset V_{k'}^{j-1}$, $V_{k'}^0 = V_{k'}$, $\forall k' \in \{1, \dots, N'\}$. Notice that $U_k \cap V_{k'} = \emptyset$, $\forall k \in \{1, \dots, N\}$ and $\forall k' \in \{1, \dots, N'\}$.

Suppose that $y \in D(X, A \sqcup B)$. Since the sequence $\varphi_y \colon \mathbb{N} \to X$ is $\varepsilon(X, A \sqcup B)$-exterior, we can assume that there are natural numbers $K_0 < K_1 < K_2 < \dots$ such that $\varphi_y(n) \in \bigcup_{k=1}^{N} U_k^j \sqcup \bigcup_{k'=1}^{N'} V_{k'}^j$, $\forall n \geq K_j$, $j \in \mathbb{N}$. Let us suppose that $\varphi_y(K_0) \in \bigcup_{k=1}^{N} U_k^0 \subset U$ (if $\varphi_y(K_0)$ was in V, the proof would be analogous). As a consequence, $\exists l_y \in \{1, \dots, N\}$ such that $\varphi_y(K_0) \in U_{l_y}^0$. The neighborhood $U_{l_y}^0 = U_{l_y}$ is path-connected and $a_{l_y} \in U_{l_y}^0$, so there exists a path $h \colon I \to U_{l_y}^0$ such that $h(0) = \varphi_y(K_0)$ and $h(1) = a_{l_y}$. In addition, this path is contained in U and U is right-invariant.

Since $\varphi^n(h(I))$ is connected, $\forall n \in \mathbb{N}$, it follows that

$$\varphi^n(h(I)) \subset U, \quad \forall n \geq 0.$$

This last condition implies that, for any $r \in N$ such that $K_j \leq r < K_{j+1}$, we have that $\varphi_y(r) \in \bigcup_{k=1}^{N} U_k^j$. Thus, $y \in D(X, A)$. Had we supposed that $\varphi_y(K_0) \in V$, we would have obtained that $y \in D(X, B)$. Either way, $y \in D(X, A) \sqcup D(X, B)$. □

One can easily split the set $P_m(X)$ into equivalence classes by the equivalence relation $\sim$, which associates two m-periodic points $x, x' \in P_m(X)$ if and only if there exists a positive integer k such that $\varphi^k(x) = x'$. In this way, each equivalence class would be of the form $[x] = \{x = x_0, x_1, \dots, x_{l-1}\} \in P_m(X)/\sim$, where $l \in \mathrm{Div}(m)$, $\varphi^1(x_{l-1}) = x$ and $\varphi^1(x_i) = x_{i+1}$, $\forall i \in \{0, \dots, l-2\}$.

Proposition 4.2.3. *Let (X, f) be a discrete semi-flow induced by a continuous map $\varphi^1 = f$ and let us suppose that X is first-countable, locally path-connected and T_2, $x \in {}_aP_m(X)$ and $[x] = \{x = x_0, \ldots, x_{l-1}\}$, where $l \in \mathrm{Div}(m)$. Then,*

$$D((X, f), [x]) = \bigsqcup_{y \in [x]} D((X, f^l), \{y\}).$$

Proof. Under the given conditions, since $[x]$ is an asymptotically stable l-cycle, we can find two bases of open neighborhoods $\{W_k^j\}_{j\in\mathbb{N}}$ and $\{V_k^j\}_{j\in\mathbb{N}}$ at $x_k \in [x]$, with $k \in \{0, \ldots, l-1\}$, such that, for all $j \in \mathbb{N}$, V_k^j is path-connected, $V_k^{j+1} \subset W_k^j \subset V_k^j$ and $V_k^0 \cap V_{k'}^0 = \emptyset$, as long as $k \neq k'$, and $\varphi^1\left(\bigcup_{k=0}^{l-1} W_k^j\right) \subset \bigcup_{k=0}^{l-1} W_k^j$.

As a consequence of Theorem 4.2.1, we have that the canonical map

$$\begin{array}{ccc} [x] & \longrightarrow & {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, [x])) \\ x' & \longmapsto & [\varphi_{x'}] \end{array}$$

is a bijection. Now, given $z \in D((X, f), [x])$, one has that $[\varphi_z] \in {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, [x]))$. Thus, there is $y \in [x]$ such that $[\varphi_z] = [\varphi_y]$. There also exists an $\varepsilon(X, [x])$-exterior homotopy $F\colon \mathbb{N}\bar{\times}I \to X$ such that there are positive integers $K_0 < K_1 < \ldots$ verifying that $F(\{r\} \times I) \subset W^j_{\varphi^r(y)}$, whenever $K_j \leq r < K_{j+1}$. Since $\varphi_y(r + l) = \varphi_y(r)$, we have that $W^j_{\varphi_y(r+l)} = W^j_{\varphi_y(r)}$. Therefore, $G(r, t) = F(lr, t)$ is an $\varepsilon(X, \{y\})$-exterior homotopy from $(\varphi^l)_z$ to $(\varphi^l)_y$. This implies that $z \in D((X, \varphi^l), \{y\})$. □

Theorem 4.2.2. *Let (X, f) be a discrete semi-flow induced by a continuous map $\varphi^1 = f$ and suppose that X is first-countable, locally path-connected and T_2.*

(i) If ${}_a\mathrm{Fix}(X)$ is finite, then $D(X, {}_a\mathrm{Fix}(X)) = \bigsqcup_{x \in {}_a\mathrm{Fix}(X)} D(X, \{x\})$.

(ii) If ${}_aP_m(X)$ is finite, then $D(X, {}_aP_m(X)) = \bigsqcup_{n \in \mathrm{Div}(m)} D(X, {}_aC_n(X))$.

(iii) If ${}_aC_n(X)$ is finite, then $D(X, {}_aC_n(X)) = \bigsqcup_{[x] \in {}_aC_n(X)/\sim} D(X, [x])$.

(iv) If ${}_aP_m(X)$ is finite, then $D(X, {}_aP_m(X)) = \bigsqcup_{[x] \in {}_aP_m(X)/\sim} D(X, [x])$.

(v) If ${}_aP_m(X, f) = {}_a\mathrm{Fix}(X, f^m)$ is finite, then

$$D((X, f), {}_aP_m(X, f)) = \bigsqcup_{x \in {}_a\mathrm{Fix}(X, f^m)} D((X, f^m), \{x\}).$$

Proof. (i), (ii), (iii) and (iv) are consequences of Proposition 4.2.2. By (iv) and Proposition 4.2.3, we have (v):

$$\begin{aligned} D((X, f), {}_aP_m(X, f)) &= \bigsqcup_{[x'] \in {}_aP_m(X,f)/\sim} D((X, f), [x']) = \bigsqcup_{[x'] \in {}_aP_m(X,f)/\sim} \left(\bigsqcup_{x \in [x']} D((X, f^m), \{x\}) \right) \\ &= \bigsqcup_{x \in {}_aP_m(X,f)} D((X, f^m), \{x\}) = \bigsqcup_{x \in {}_a\mathrm{Fix}(X, f^m)} D((X, f^m), \{x\}). \end{aligned}$$

□

Corollary 4.2.3. *Let X be a discrete semi-flow and suppose that X is first-countable, locally path-connected and T_2. If m, n are coprime, $m \neq n$, such that ${}_aP_m(X)$ and ${}_aP_n(X)$ are finite sets, then $D(X, {}_aP_m(X)) \cap D(X, {}_aP_n(X)) = D(X, {}_a\mathrm{Fix}(X))$.*

4.3 Basins of end points of discrete semi-flows on S^2

In this section, we remark that the topological space S^2, the 2-sphere, is T_2, path-connected and locally path-connected. Moreover, the conditions which are often required in the previous section are satisfied, provided that one considers on S^2 the structure of a discrete semi-flow (S^2, φ) induced by a continuous map $\varphi^1 = f$ such that, for all $m \in \mathbb{N}$, $P_m(S^2, f)$ is a finite set.

Lemma 4.3.1. *Let S^2 be the 2-sphere and let $A \subset B$ be subsets of S^2. If A and B are finite, then there is a sequence of open subsets $U_0 \supset U_1 \supset U_2 \supset \dots$ such that, for every integer $i \geq 0$, $B \subset U_i$ satisfying, for $a, b \in A$ and $a \neq b$, $C(a, U_i) \cap C(b, U_i) = \emptyset$, and $\{C_i(A)\}_{i\in\mathbb{N}}$ is a basis for the filter $\varepsilon(S^2, A)$, where $C_i(A) = \bigsqcup_{a\in A} C(a, U_i)$.*

Proof. Since B is finite and S^2 is T_2 and locally path-connected, we can take a basis of open neighborhoods $\{U_b^j\}_{j\in\mathbb{N}}$ at $b \in B$ such that U_b^j is path-connected, $U_b^{j+1} \subset U_b^j$ and $U_b^0 \cap U_{b'}^0 = \emptyset$, being $b \neq b'$. Now, take $U_i = \bigcup_{b\in B} U_b^i$ and let $a \in A$ such that $a \neq b$. We have that $C(a, U_i) = U_a^i$ is an open subset so that $C(a, U_i) \cap C(b, U_i) = U_a^i \cap U_b^i = \emptyset$, and since $C_i(A) = \bigcup_{a\in A} U_a^i$, then $\{C_i(A)\}_{i\in\mathbb{N}}$ is clearly a basis for $\varepsilon(S^2, A)$. □

Lemma 4.3.2. *Let S^2 be the 2-sphere and let $A \subset P$ be subsets of S^2. If A is finite and P is countable, then there is a sequence of open subsets $U_0 \supset U_1 \supset U_2 \supset \dots$ such that, for every integer $i \geq 0$, $P \subset U_i$ satisfying, for $a, b \in A$ and $a \neq b$, $C(a, U_i) \cap C(b, U_i) = \emptyset$, and $\{C_i(A)\}_{i\in\mathbb{N}}$ is a basis for the filter $\varepsilon(S^2, A)$, where $C_i(A) = \bigsqcup_{a\in A} C(a, U_i)$.*

Proof. It is not hard to find a triangulation T^1 of S^2 such that $P \cap T_1^1 = \emptyset$, where T_1^1 is the union of 1-cells of T^1 (the 1-skeleton). Now, we can find subdivisions T^2 of T^1 and, in general for every $i \in \mathbb{N}$, subdivisions T^{i+1} of T^i, such that $P \cap T_1^i = \emptyset$, where T_1^i is the union of 1-cells of T^i, and $\mathrm{diam}(\sigma) < \frac{1}{i}$, being σ a cell of T^i.

Let U_i be the union of the interior of the 2-cells of $T^i$and suppose that the triangulation T^0 has been chosen in such a way that $C(a, U_0) \cap C(b, U_0) = \emptyset$, for $a, b \in A$ and $a \neq b$. Then, the sequence (U_i) satisfies the conditions of the thesis of this lemma. □

Theorem 4.3.1. *Suppose that $X = S^2$ is provided with the structure of a discrete semi-flow induced by a continuous map $f\colon S^2 \to S^2$ such that, for every $m \in \mathbb{N}$, $P_m(S^2, f)$ is finite. Then:*

(i) If n divides m, the inclusion $P_n(X) \subset P_m(X)$ induces an injective map

$$^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, P_n(X))) \to {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, P_m(X)))$$

and one has the following commutative pullback diagram:

$$\begin{tikzcd}
D(X, \varepsilon(X, P_n(X))) \arrow[r, "\omega_n"] \arrow[d] & {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, P_n(X))) \arrow[d] \\
D(X, \varepsilon(X, P_m(X))) \arrow[r, "\omega_m"'] & {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, P_m(X)))
\end{tikzcd}$$

In particular, for $a \in {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, P_n(X)))$*, one has that*

$$\omega_n^{-1}(a) = \omega_m^{-1}(a).$$

(ii) For each $m \geq 1$*, the canonical map*

$${}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, P_m(X))) \to {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, P(X)))$$

is injective, and we have the following commutative pullback diagram:

$$\begin{tikzcd}
D(X, \varepsilon(X, P_m(X))) \arrow[r, "\omega_m"] \arrow[d] & {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, P_m(X))) \arrow[d] \\
D(X, \varepsilon(X, P(X))) \arrow[r, "\omega"'] & {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, P(X)))
\end{tikzcd}$$

In this case, if $a \in {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, P_m(X)))$*, one has that* $\omega_m^{-1}(a) = \omega^{-1}(a)$*.*

(iii) Denote by ${}_aP_m(X)$ *the union of* k*-cycles* $\{x_0, \ldots, x_k\}$ *such that* k *divides* m *and each* k*-cycle is asymptotically stable. Then,*

$${}_aP_m(X) \cong {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, {}_aP_m(X))).$$

Proof. (i) and (ii) are a consequence of Lemma 4.3.2 and Theorem 4.1.1; (iii) follows from Corollary 4.2.2. □

Chapter 5

Metrics and Borel measures on exterior discrete semi-flows

This chapter serves as a link between the most theoretical results of this work and its computational part, which will be developed along chapters 6 and 7, as we intend to adapt or take advantage of some of the concepts and theorems seen above to obtain either a method to compute some relevant aspects regarding exterior discrete semi-flows or a framework in which these calculations make sense. In particular, we are interested in representing basins of end points (to make it possible, it will be convenient, from a computational point of view, to measure distances between points to check whether an orbit has converged or not) and finding a way to construct a measure which allows us to estimate and compare the sizes of those basins.

Section 5.1 is focused on the development of the study of discrete semi-flows on metric spaces. In it, we contribute a brand-new notion of end point, which will depend on the metrics of the subjacent space and will bring about new decompositions of the basins associated with these end points. The main results of this section analyze the connection between the set of end points of a metric exterior discrete semi-flow and the set of ω-representable end points when the externology is given by the family of neighborhoods of a right-invariant finite subset. In this case, the set of end points induced by the externology can be considered as a subset of the set of end points induced by the metrics. Moreover, for an end point induced by the externology, the basins induced by the metrics match up with the basins induced by the externology.

In section 5.2, for a regular n-dimensional CW-complex with a sequence of consecutive subdivisions of its cellular structure, we prove that, giving only the measure of all the n-cells of the iterated subdivision verifying the subdivision invariance property (that is to say, the measure of an n-cell agrees with the sum of the measures of the n-cells of its subdivision), then there is a cellular-extension σ-algebra and an n-cellular-extension measure which extends the given measures of the cells. Moreover, for a metrizable regular CW-complex, if the iterated subdivision satisfies the vanishing-star property and has a countable number of vertexes, then the Borel σ-algebra generated by the topology of the CW-complex is contained in this cellular-extension σ-algebra. As an application, we see that the Lebesgue measure on $[-1,1]^n$, $\mathbb{R}$ and $\mathbb{R}^n$, as well as the measure of angles on the 1-sphere and solid angles on the 2-sphere, can be considered as particular cases of this cellular procedure of measure construction.

5.1 Discrete semi-flows on metric spaces

Along this section, we shall consider that (X, d) is a metric space with metric d. With this assumption, we intend to define some similar concepts to those seen above in order to adapt them to the particular cases in which we have to deal with topologies induced by metrics. Given a metric space $X = (X, d)$ and a discrete semi-flow $\varphi\colon \mathbb{N} \times X \to X$, it is said that the triple (X, d, φ) is a *metric discrete semi-flow.*

5.1.1 End points of a metric discrete semi-flow

Definition 5.1.1. *Given a metric discrete semi-flow $X = (X, d, \varphi)$, the* **space of end points** *of X is defined as the quotient set*

$$\Pi(X, d) = \frac{\{\varphi_x \mid x \in X\}}{\sim_d},$$

where, given $x, y \in X$, $\varphi_x \sim_d \varphi_y$ if and only if

$$(d(\varphi_x(n), \varphi_y(n))) \xrightarrow[n\to\infty]{} 0.$$

We will denote by $[\varphi_x]_d$ the equivalence class of φ_x induced by this relation.

An element $a = [\varphi_x]_d \in \Pi(X, d)$ is called an **end point** *of the metric discrete semi-flow.*

Given $x \in X$, we will sometimes denote the induced map $\varphi_x\colon \mathbb{N} \to X$ by $(x, \varphi_x(1), \varphi_x(2), \dots)$. For instance, if $y \in \mathrm{Fix}(X)$, we can interpret that y is an end point:

$$y \equiv [(y)]_d = [(y, y, \dots)]_d \in \Pi(X, d).$$

We can define the natural map

$$\omega_d\colon X \longrightarrow \Pi(X, d)$$

given by $\omega_d(x) = [\varphi_x]_d = [(\varphi_x(0) = x, \varphi_x(1), \varphi_x(2), \dots)]_d$.

The map ω_d allows us to decompose any metric discrete semi-flow.

Definition 5.1.2. *Let (X, d) be a metric discrete semi-flow. The subspace*

$$X_a = \omega_d^{-1}(a), \quad a \in \Pi(X, d)$$

will be called the **basin of the end point** *a.*

There exists an induced partition of X,

$$X = \bigsqcup_{a \in \Pi(X,d)} X_a,$$

*which will be called the ω_d-***decomposition** *of the metric discrete semi-flow (X, d).*

Definition 5.1.3. *Two metrics d and d' in a subset X are said to be* **strongly equivalent**, *which will be denoted by $d \approx d'$, if for each real number $\epsilon > 0$ there are real numbers $r \equiv r(\epsilon) > 0$ and $s \equiv s(\epsilon) > 0$ such that, for all $x \in X$, the following two conditions are satisfied:*

(i) $B_{d'}(x; r) \subset B_d(x; \epsilon)$;

(ii) $B_d(x; s) \subset B_{d'}(x; \epsilon)$,

where $B_\rho(x; \delta)$ denotes the ball of center x and radius δ in the metric $\rho \in \{d, d'\}$.

Remark 5.1.1. *Suppose that d and d' are strongly equivalent metrics in X and $x, y \in X$. One has that, if (X, φ) is a discrete semi-flow, then*

$$\varphi_x \sim_d \varphi_y \text{ in } (X, d) \Longleftrightarrow \varphi_x \sim_{d'} \varphi_y \text{ in } (X, d').$$

5.1.2 End points of exterior discrete semi-flows on metric spaces

Given a metric space (X, d) and an externology $\varepsilon(X) \subset \mathbf{t}_d$, where $\mathbf{t}_d$ is the topology induced by the metric d in X, the triple $(X, d, \varepsilon(X))$ will be called **exterior metric space**. In the same way, given an exterior metric space $(X, d, \varepsilon(X))$ and an exterior discrete semi-flow $(X, \varphi, \varepsilon(X))$, the 4-tuple $(X, d, \varphi, \varepsilon(X))$ will be called **metric exterior discrete semi-flow**. This definition will be key in chapter 6, and its notation will be shorten by $(X, d, \varepsilon(X))$ when the action φ of the discrete semi-flow is clear in the considered context.

Definition 5.1.4. *An exterior metric space $(X, d, \varepsilon(X))$ is said to be a d-***small exterior metric space** *if there exists a sequence of exterior open subsets $E_1 \supset E_2 \supset \dots$ (with $E_k \in \varepsilon(X)$, $\forall k \in \mathbb{N}^*$) such that, for all $k \in \mathbb{N}^*$, $\operatorname{diam}(U) < \frac{1}{k}$, for every path component U of E_k.*

*A metric exterior discrete semi-flow $(X, d, \varphi, \varepsilon(X))$ such that the exterior metric space $(X, d, \varepsilon(X))$ is d-small will be called d-***small metric exterior discrete semi-flow**.

Theorem 5.1.1. *If $X = (X, d, \varphi, \varepsilon(X))$ is a d-small metric exterior discrete semi-flow, then there exists a canonical map $h\colon {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X)) \longrightarrow \Pi(X, d)$ given by $h([\varphi_x]) = [\varphi_x]_d$.*

Proof. Let us suppose that $x, y \in D(X, \varphi, \varepsilon(X))$ and that $\varphi_x \simeq_e \varphi_y$. We have to show that $d(\varphi_x(n), \varphi_y(n)) \to 0$.

Let $\delta > 0$. Then, $\exists k \in \mathbb{N}$ such that $\frac{1}{k} < \delta$. Consider the exterior open subset E_k. Since φ_x, φ_y are exterior, $\exists n'_k \in \mathbb{N}$ such that, $\forall n \geq n'_k$, $\varphi_x(n), \varphi_y(n) \in E_k$. Moreover, there is an exterior homotopy $F\colon \mathbb{N} \bar{\times} I \to X$ from φ_x to φ_y, so $\exists n''_k \in \mathbb{N}$ such that $F(\{n\} \times I) \subset E_k$, $\forall n \geq n''_k$. Let $n_k = \max\{n'_k, n''_k\}$. Then, $\forall n \geq n_k$, $\varphi_x(n)$ and $\varphi_y(n)$ are in the same path component U of E_k. Since $\operatorname{diam}(U) < \frac{1}{k}$ as X is d-small, one has that

$$d(\varphi_x(n), \varphi_y(n)) < \frac{1}{k} < \delta, \quad \forall n \geq n_k.$$

□

Theorem 5.1.2. *Let $X = (X, d, \varphi)$ be a locally path-connected and locally compact metric discrete semi-flow, and let $A \subset X$ be a finite and right-invariant subset. Consider the externology $\varepsilon(X, A)$. Then, $(X, d, \varphi, \varepsilon(X, A))$ is a d-small metric exterior discrete semi-flow and the canonical map*

$$h\colon {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, A)) \longrightarrow \Pi(X, d)$$

is injective.

What is more, given $a = [\varphi_x] \in {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, A))$, with $x \in D(X, A) = D(X, \varepsilon(X, A))$, one has that

$$\omega_d^{-1}(h(a)) = \omega^{-1}(a).$$

Proof. Suppose that $A = \{x_1, \dots, x_l\}$. Note that, by hypothesis, the externology $\varepsilon(X, A)$ admits a countable basis $\{E_i\}_{i\in\mathbb{N}} \subset \varepsilon(X, A)$, $E_1 \supset E_2 \supset E_3 \supset \dots$, so that:

(1) $E_k = V_k^1 \cup \dots \cup V_k^l$, being V_k^i a path-connected open neighborhood of $x_i \in A$ such that $\overline{V_k^i}$ is compact.

(2) If $i \neq j$, $\overline{V_k^i} \cap \overline{V_k^j} = \emptyset$.

(3) $\operatorname{diam}(V_k^i) < \frac{1}{k}$.

From these properties, it follows that $(X, d, \varepsilon(X))$ is d-small. Thus, by Theorem 5.1.1, there exists the canonical map

$$h\colon {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, A)) \longrightarrow \Pi(X, d).$$

Let us prove that h is injective. Let $x, y \in D(X, A)$. One has that

$$h([\varphi_x]) = [\varphi_x]_d, \quad h([\varphi_y]) = [\varphi_y]_d.$$

Suppose that $d(\varphi_x(n), \varphi_y(n)) \to 0$. Since $\varphi_x, \varphi_y\colon \mathbb{N} \to X$ are exterior maps, for each $k \in \mathbb{N}^*$ there is n_k such that $n_k < n_{k+1}$ and $\forall n \geq n_k$, $\varphi_x(n), \varphi_y(n) \in E_k$.

Besides, let $m = \min\{d(V_1^i, V_1^j) \mid i \neq j\}$. Notice that $m > 0$. Let $k \in \mathbb{N}$ such that $m > \frac{1}{k} > 0$. Since $d(\varphi_x(n), \varphi_y(n)) \to 0$, $\exists N_k'$ such that $d(\varphi_x(n), \varphi_y(n)) < \frac{1}{k} < m$, for all $n \geq N_k'$. Take $N_k = \max\{n_k, N_k'\}$. Then, we have that, $\forall n \geq N_k$,

$$\varphi_x(n), \varphi_y(n) \in E_k \quad \text{and} \quad d(\varphi_x(n), \varphi_y(n)) < \frac{1}{k} < m.$$

This implies that $\exists V_k^{i(n)}$ such that $\varphi_x(n), \varphi_y(n) \in V_k^{i(n)}$. Now, since $V_k^{i(n)}$ is path-connected, there is a path $\alpha_n\colon I \to V_k^{i(n)}$ such that $\alpha_n(0) = \varphi_x(n)$ and $\alpha_n(1) = \varphi_y(n)$. Define the homotopy

$$F\colon \mathbb{N}\bar{\times} I \longrightarrow X$$

given by $F(n, t) = \alpha_n(t)$. Note that $F(\{n\} \times I) \subset V_k^{i(n)} \subset E_k$, $\forall n \geq N_k$. Therefore, F is an exterior homotopy from φ_x to φ_y and then $\varphi_x \simeq_e \varphi_y$. This is equivalent to the fact that $[\varphi_x] = [\varphi_y]$. Hence, h is injective.

Finally, let $a \in {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, A))$ and suppose that $a = [\varphi_x]$, being $\varphi_x \colon \mathbb{N} \to X$ an exterior map. Let $y \in \omega_d^{-1}(h(a))$; that is, such that

$$d(\varphi_x(n), \varphi_y(n)) \to 0.$$

Keeping in mind that $\varphi_x \colon \mathbb{N} \to X$ is exterior and that $d(\varphi_x(n), \varphi_y(n)) \to 0$, it follows that $\varphi_y \colon \mathbb{N} \to X$ is exterior. Thus, we have that

$$h([\varphi_x]) = [\varphi_x]_d = [\varphi_y]_d = h([\varphi_y]).$$

We had shown that h is injective, so $[\varphi_x] = [\varphi_y] \in {}^{\omega}\pi_0^{\mathrm{BG}}(X, \varepsilon(X, A))$ and $\omega(x) = [\varphi_x] = [\varphi_y] = \omega(y)$. Then, $[\varphi_x] = a = \omega(y)$, which implies that $y \in \omega^{-1}(a)$. □

5.2 Borel measures on exterior discrete semi-flows

In this section, we use the properties of an n-dimensional regular CW-complex and its subdivisions to construct a subdivision algebra and a subdivision pre-measure and then, using the extension technique developed by Carathéodory (remember subsection 0.6.1), we can extend this pre-measure to an n-cellular-extension measure defined in a cellular-extension σ-algebra.

The author had already devised in [48] some methods of subdivision on the surface of the sphere S^2 to calculate areas and associated probabilities of basins of end points. These procedures were generalized in [3] for regular n-dimensional CW-complexes. In the present thesis, we preferred to include directly the contents and methods developed in [3] instead of the particular case examined in the author's work [48].

5.2.1 Measures on regular CW-complexes

Let 2^{Γ} be the family of all the subsets of a set Γ. Given a map $\mu \colon \Gamma \to [0, \infty)$, by Lemma 0.6.1 there exists an induced map $\mu \colon 2^{\Gamma} \to [0, \infty]$ given by $\mu(\Gamma') = \sum_{\gamma \in \Gamma'} \mu(\gamma)$, where Γ' is a subset of Γ.

In this section, we suppose that X is an n-dimensional regular CW-complex and $\Gamma_*^*(X)$ is a regular iterated subdivision on X –see the appropriate definitions in section 0.7.

Definition 5.2.1. *An* n**-cellular measure** *on* $\Gamma_*^*(X)$ *consists of a family* $\mu_*^* = \{\mu_*^r\}_{r \in \mathbb{N}}$ *of maps* $\mu_*^r \colon \Gamma_*^r(X) \to [0, \infty)$ *verifying the following properties:*

(i) If $\beta \in \Gamma_q^r(X)$ *and* $q < n$*, then* $\mu_q^r(\beta) = 0$.

(ii) **(Subdivision invariance property)***: for every* $\gamma \in \Gamma_*^r(X)$,

$$\mu_*^r(\gamma) = \sum_{\beta \in \mathrm{Sd}(\gamma)} \mu_*^{r+1}(\beta) = \mu_*^{r+1}(\mathrm{Sd}(\gamma)).$$

Note that, since $\Gamma_*^*(X) \to \{\mathring{\gamma} \mid \gamma \in \Gamma_*^*(X)\}$, $\gamma \mapsto \mathring{\gamma}$ is a bijection, we have a canonical induced map $\{\mathring{\gamma} \mid \gamma \in \Gamma_*^*(X)\} \to [0, \infty]$, $\mathring{\gamma} \mapsto \mu_*^*(\gamma)$ that will also be denoted by μ_*^*.

Associated with a regular iterated subdivision $\Gamma_*^*(X)$ on X, we can consider the following family

$$\mathcal{A}_{\Gamma_*^*(X)}$$

of subsets of X: given $B \subset X$, $B \in \mathcal{A}_{\Gamma_*^*(X)}$ if there is $r \in \mathbb{N}$ and $\mathcal{B} \subset \Gamma_*^r(X)$ such that $B = \bigsqcup_{\gamma \in \mathcal{B}} \mathring{\gamma}$.

Proposition 5.2.1. *$\mathcal{A}_{\Gamma_*^*(X)}$ is an algebra of subsets of X.*

Proof. Since $X = \bigsqcup_{\gamma \in \Gamma_*^0(X)} \mathring{\gamma}$, it follows that $X \in \mathcal{A}_{\Gamma_*^*(X)}$.

If $B \in \mathcal{A}_{\Gamma_*^*(X)}$, there is $r \in \mathbb{N}$ and $\mathcal{B} \subset \Gamma_*^r(X)$ such that $B = \bigsqcup_{\gamma \in \mathcal{B}} \mathring{\gamma}$. Then, one has that $X \setminus B = \bigsqcup_{\gamma \in \Gamma_*^r(X) \setminus \mathcal{B}} \mathring{\gamma}$ and $X \setminus B$ is in $\mathcal{A}_{\Gamma_*^*(X)}$.

If $B_1, B_2 \in \mathcal{A}_{\Gamma_*^*(X)}$, there are $r_1, r_2 \in \mathbb{N}$ and $\mathcal{B}_1, \mathcal{B}_2$ subsets of $\Gamma_*^{\max\{r_1, r_2\}}(X)$ such that $B_1 = \bigsqcup_{\gamma \in \mathcal{B}_1} \mathring{\gamma}$ and $B_2 = \bigsqcup_{\gamma \in \mathcal{B}_2} \mathring{\gamma}$. Then, one has that $B_1 \cup B_2 = \bigsqcup_{\gamma \in (\mathcal{B}_1 \cup \mathcal{B}_2)} \mathring{\gamma}$ and $B_1 \cup B_2$ is in $\mathcal{A}_{\Gamma_*^*(X)}$. □

Definition 5.2.2. *Given an n-cellular measure $\mu_*^* = \{\mu_*^r\}_{r \in \mathbb{N}}$, we define $\mu \colon \mathcal{A}_{\Gamma_*^*(X)} \to [0, \infty]$ by the formula*

$$\mu(B) = \sum_{\gamma \in \mathcal{B}} \mu_*^r(\gamma),$$

where $\mathcal{B}$ is the unique subset $\mathcal{B} \subset \Gamma_*^r(X)$ such that $B = \bigsqcup_{\gamma \in \mathcal{B}} \mathring{\gamma}$.

It is important to remark that μ is well-defined. This follows by applying Proposition 0.6.1 and taking into account that μ_*^* has the subdivision invariance property.

Proposition 5.2.2. *The map $\mu \colon \mathcal{A}_{\Gamma_*^*(X)} \to [0, \infty]$ is a pre-measure map.*

Proof. It is easy to check that $\mu(\emptyset) = 0$. If $B_1, B_2, \dots \in \mathcal{A}_{\Gamma_*^*(X)}$ are mutually disjoint subsets and $\bigsqcup_{k \in \mathbb{N}} B_k \in \mathcal{A}_{\Gamma_*^*(X)}$, then there are $r \in \mathbb{N}$ and $\mathcal{B} \subset \Gamma_*^r(X)$ such that $\bigsqcup_{k \in \mathbb{N}} B_k = \bigsqcup_{\gamma \in \mathcal{B}} \mathring{\gamma}$. Thus, we have that $\mathcal{B} = \bigsqcup_{k \in \mathbb{N}} \mathcal{B}_k$, $B_k = \bigsqcup_{\gamma \in \mathcal{B}_k} \mathring{\gamma}$.

Applying Proposition 0.6.1, we have that

$$\mu\left(\bigsqcup_{k \in \mathbb{N}} B_k\right) = \sum_{\gamma \in \mathcal{B}} \mu_*^r(\gamma) = \sum_{k \in \mathbb{N}} \left(\sum_{\gamma \in \mathcal{B}_k} \mu_*^r(\gamma)\right) = \sum_{k \in \mathbb{N}} \mu(B_k).$$

Then, μ is countable additive. This implies that μ is a pre-measure. □

Definition 5.2.3. *$\mathcal{A}_{\Gamma_*^*(X)}$ is said to be the* **subdivision algebra** *of $\Gamma_*^*(X)$ and $\mu \colon \mathcal{A}_{\Gamma_*^*(X)} \to [0, \infty]$ is said to be the* **subdivision pre-measure** *of $\Gamma_*^*(X)$ induced by μ_*^*.*

Remark 5.2.1. *If we consider the canonical inclusion* $\mathrm{in}\colon \{\mathring{\gamma} \mid \gamma \in \Gamma_*^*(X)\} \to \mathcal{A}_{\Gamma_*^*(X)}$*, then in the diagram*

$$\begin{tikzcd} \{\mathring{\gamma} \mid \gamma \in \Gamma_*^*(X)\} \arrow[r, "\mathrm{in}"] \arrow[d, "\mu_*^*"'] & \mathcal{A}_{\Gamma_*^*(X)} \arrow[dl, "\mu"] \\ {[0,\infty]} & \end{tikzcd}$$

the pre-measure μ *is an extension. It is interesting to remark that, if* $\Gamma_*^*(X)$ *is countable, then* μ *is the unique extension pre-measure (as a consequence of the countable additivity property). Moreover,* $X = \bigsqcup_{\gamma \in \Gamma_*^0(X)} \mathring{\gamma}$*, with* $\mu(\mathring{\gamma}) < \infty$*. Therefore, in this case* μ *is a* σ*-finite pre-measure.*

The following result, Theorem 5.2.1, establishes that, in order to construct a measure on an n-dimensional regular CW-complex, it suffices to assign measures to all the n-cells of a regular iterated subdivision. This assignment has to be compatible with the subdivision operation: the sum of the measures assigned to the n-cells of the subdivision of any given n-cell has to be equal to the measure assigned to that n-cell. The construction of this n-cellular measure can be done in two steps: firstly, we construct an intermediate subdivision algebra by taking arbitrary unions of cells for each r-subdivision Γ_*^r and a subdivision pre-measure; after that, in the second step, we apply Carathéodory's extension theorem to obtain an n-cellular-extension measure defined on the cellular-extension σ-algebra.

Theorem 5.2.1. *Let* $\mu_*^* = \{\mu_*^r\}_{r\in\mathbb{N}}$ *be an* n*-cellular measure on* $\Gamma_*^*(X)$ *and* $\mu\colon \mathcal{A}_{\Gamma_*^*(X)} \to [0,\infty]$ *the induced subdivision pre-measure on* $\mathcal{A}_{\Gamma_*^*(X)}$*. Then, if* $\mathcal{E}_{\mu_*^*}(X)$ *is the Carathéodory extension* σ*-algebra induced by* $(\mathcal{A}_{\Gamma_*^*(X)}, \mu)$*, there is an extension measure* $\bar{\mu}\colon \mathcal{E}_{\mu_*^*}(X) \to [0,\infty]$ *(that is,* $\bar{\mu}(\mathring{\gamma}) = \mu_*^*(\gamma)$*, for every* $\gamma \in \Gamma_*^*(X)$*). Moreover, if* $\Gamma_*^*(X)$ *is countable, then* $\bar{\mu}$ *is the unique extension measure of* μ_*^* *and* $\bar{\mu}$ *is a* σ*-finite measure.*

Proof. By Proposition 5.2.1 and Proposition 5.2.2, we have that $\mu\colon \mathcal{A}_{\Gamma_*^*(X)} \to [0,\infty]$ is a pre-measure defined on an algebra of subsets of X. Applying Carathéodory's Extension Theorem 0.6.1, one obtains a σ-algebra $\mathcal{E} = \mathcal{E}_{\mu_*^*}(X)$ and a measure $\bar{\mu}\colon \mathcal{E} = \mathcal{E}_{\mu_*^*}(X) \to [0,\infty]$. The last part of the theorem follows from the observations given in Remark 5.2.1. □

Definition 5.2.4. *Given* μ_*^* *an* n*-cellular measure on* $\Gamma_*^*(X)$*,* $\mathcal{E}_{\mu_*^*}(X)$ *is said to be the* **cellular-extension** σ**-algebra** *of* μ_*^* *and* $\bar{\mu}\colon \mathcal{E}_{\mu_*^*}(X) \to [0,\infty]$ *is said to be the* n**-cellular-extension measure** *induced by* μ_*^**.*

Theorem 5.2.2 asserts that, if an n-dimensional metrizable CW-complex with a regular iterated subdivision having a countable number of vertexes satisfies the vanishing-star property, then the Borel σ-algebra is contained in the cellular-extension σ-algebra given above in Theorem 5.2.1.

Theorem 5.2.2. *Suppose that the* n*-dimensional CW-complex* X *is a metrizable space (with a metric* d*) satisfying one of the following two conditions:*

(i) The set of vertexes $\Gamma_0^*(X) = \bigcup_{r\in\mathbb{N}} \Gamma_0^r(X)$ *is countable and, for any sequence* $\gamma_r \in \Gamma_*^r(X)$*, with* $r \in \mathbb{N}$ *and* $\mathring{\gamma}_{r+1} \subset \mathring{\gamma}_r$*, one has that*

$$\lim_{r\to\infty} \mathrm{diam}(\mathrm{st}(\mathring{\gamma}_r, \Gamma_*^r(X))) = 0.$$

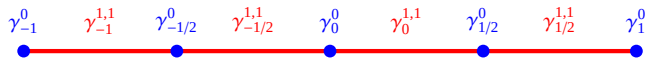


Figure 5.1: Iterated subdivision $\Gamma^1_*([-1,1])$.

(ii) The set $\Gamma^*_*(X)$ is countable and, for any sequence $\gamma_r \in \Gamma^r_*(X)$, with $r \in \mathbb{N}$ and $\mathring{\gamma}_{r+1} \subset \mathring{\gamma}_r$, one has that

$$\lim_{r\to\infty} \operatorname{diam}(\mathring{\gamma}_r) = 0.$$

Then, for every n-cellular measure μ^*_* on $\Gamma^*_*(X)$, the Borel σ-algebra $\sigma(\mathbf{t}_X)$ generated by the topology $\mathbf{t}_X$ of X is contained in the cellular-extension σ-algebra $\mathcal{E}_{\mu^*_*}(X)$ of μ^*_*.

Proof. Suppose that $U \in \mathbf{t}_X$ and $x \in U$. For each $r \in \mathbb{N}$, there is a unique $\gamma_r \in \Gamma^r_*(X)$ such that $x \in \mathring{\gamma}_r$. Notice that $\mathring{\gamma}_{r+1} \subset \mathring{\gamma}_r$. Then, under condition *(i)*, $\lim_{r\to\infty} \operatorname{diam}(\operatorname{st}(\mathring{\gamma}_r, \Gamma^r_*(X))) = 0$. This implies that there is r_0 such that x is in the interior of $\operatorname{st}(\mathring{\gamma}_{r_0}, \Gamma^{r_0}_*(X)) \subset U$. Note that, if v is a vertex of γ_{r_0}, one also has that $x \in \operatorname{st}(v, \Gamma^{r_0}_*(X)) \subset \operatorname{st}(\mathring{\gamma}_{r_0}, \Gamma^{r_0}_*(X)) \subset U$. Then, U is the union of a countable family of subsets of the form $\operatorname{st}(v, \Gamma^r_*(X))$. Taking into account that $\operatorname{st}(v, \Gamma^{r_0}_*(X)) \in \mathcal{A}_{\Gamma^*_*(X)} \subset \mathcal{E}_{\mu^*_*}(X)$, it follows that $U \in \mathcal{E}_{\mu^*_*}(X)$. For *(ii)*, the proof is similar. $\square$

5.2.2 Examples of cellular-extension measures

As canonical examples of the method of measure construction that we have just seen in subsection 5.2.1, we can obtain the Lebesgue measure on $[-1,1]^n$ using the canonical dyadic cubic subdivision of $[-1,1]^n$, the Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^n$ extending the same procedure to these sets and the standard measure of angles and solid angles using a dyadic iterated subdivision on the unit 1-sphere S^1 and a dyadic cubic iterated subdivision on the unit 2-sphere S^2.

Lebesgue measure on $[-1,1]$ and $[-1,1]^n$

Take $X = [-1,1]$ and consider the following iterated subdivision $\Gamma^*_* = \Gamma^*_*([-1,1])$ and the following 1-cellular measure μ^*_* (see Figure 5.1):

$$\Gamma^r_0 = \{\gamma^0_{\frac{z}{2^r}} \mid z \in \mathbb{Z},\ -2^r \le z \le 2^r\}, \quad \gamma^0_w(0) = w;$$

$$\Gamma^r_1 = \{\gamma^{r,1}_{\frac{z}{2^r}} \mid z \in \mathbb{Z},\ -2^r \le z < 2^r\}, \quad \gamma^{r,1}_w \colon [0,1] \to [-1,1],\ \gamma^{r,1}_w(t) = w + \frac{t}{2^r}.$$

The subdivision operator is given by

$$\operatorname{Sd}(\{\gamma^0_{\frac{z}{2^r}}\}) = \{\gamma^0_{\frac{z}{2^r}}\}; \quad \operatorname{Sd}(\{\gamma^{r,1}_{\frac{z}{2^r}}\}) = \{\gamma^{r+1,1}_{\frac{z}{2^r}}, \gamma^{r+1,1}_{\frac{2z+1}{2^{r+1}}}\}.$$

Define $\mu^*_* = \{\mu^r_*\}_{r\in\mathbb{N}}$, $\mu^r_1 \colon \Gamma^r_1 \to [0,\infty]$, $\mu^r_1(\gamma^{r,1}_{\frac{z}{2^r}}) = \frac{1}{2^r}$. Notice that μ^*_* has the subdivision invariance property:

$$\mu^{r+1}_1(\operatorname{Sd}(\{\gamma^{r,1}_{\frac{z}{2^r}}\})) = \mu^{r+1}_1(\{\gamma^{r+1,1}_{\frac{z}{2^r}}, \gamma^{r+1,1}_{\frac{2z+1}{2^{r+1}}}\}) = \frac{1}{2^{r+1}} + \frac{1}{2^{r+1}} = \frac{1}{2^r} = \mu^r_1(\{\gamma^{r,1}_{\frac{z}{2^r}}\}).$$

As a consequence of Theorem 5.2.1 and Theorem 5.2.2, we have that the Borel σ-algebra $\sigma(\mathbf{t}_{[-1,1]})$ is contained in the cellular-extension σ-algebra $\mathcal{E}_{\mu_*^*}([-1,1])$ of μ_*^* and we also have that the Lebesgue σ-algebra $\mathcal{L}([-1,1]) = \mathcal{E}_{\mu_*^*}([-1,1])$ and the 1-cellular-extension measure $\bar{\mu}$ agrees with the Lebesgue measure.

Now, taking $X = [-1,1]^n$, $\Gamma_*^r([-1,1]^n) = \prod_{k=1}^n \Gamma_*^r$ and $\mu_1^r\left(\prod_{k=1}^n \gamma^{r,1}_{\frac{z_k}{2^r}}\right) = \prod_{k=1}^n \frac{1}{2^r} = \frac{1}{2^{nr}}$, we also have that the cellular-extension σ-algebra $\mathcal{E}_{\mu_*^*}([-1,1]^n)$ is the Lebesgue σ-algebra $\mathcal{L}([-1,1]^n)$ of $[-1,1]^n$ and the n-cellular-extension measure is the Lebesgue measure.

Measures on $\mathbb{R}$ and $\mathbb{R}^n$

We can extend the procedure above to $\mathbb{R}$ by taking the following iterated subdivision $\Gamma_*^* = \Gamma_*^*(\mathbb{R})$ and 1-cellular measure μ_*^*:

$$\Gamma_0^r = \{\gamma^0_{\frac{z}{2^r}} \mid z \in \mathbb{Z}\}, \quad \gamma^0_w(0) = w;$$

$$\Gamma_1^r = \{\gamma^{r,1}_{\frac{z}{2^r}} \mid z \in \mathbb{Z}\}, \quad \gamma^{r,1}_w \colon [0,1] \to \mathbb{R},\ \gamma^{r,1}_w(t) = w + \frac{t}{2^r}.$$

The subdivision operator is given again by

$$\mathrm{Sd}(\{\gamma^0_{\frac{z}{2^r}}\}) = \{\gamma^0_{\frac{z}{2^r}}\}; \quad \mathrm{Sd}(\{\gamma^{r,1}_{\frac{z}{2^r}}\}) = \{\gamma^{r+1,1}_{\frac{z}{2^r}}, \gamma^{r+1,1}_{\frac{2z+1}{2^{r+1}}}\}.$$

Define $\mu_*^* = \{\mu_*^r\}_{r\in\mathbb{N}}$, $\mu_1^r \colon \Gamma_1^r \to [0,\infty]$, $\mu_1^r(\gamma^{r,1}_{\frac{z}{2^r}}) = \frac{1}{2^r}$. As before, μ_*^* has the subdivision invariance property, the Borel σ-algebra $\sigma(\mathbf{t}_{\mathbb{R}})$ is contained in the cellular-extension σ-algebra $\mathcal{E}_{\mu_*^*}(\mathbb{R})$ of μ_*^*, the Lebesgue σ-algebra $\mathcal{L}(\mathbb{R}) = \mathcal{E}_{\mu_*^*}(\mathbb{R})$ and the 1-cellular-extension measure $\bar{\mu}$ agrees with the Lebesgue measure.

Now, taking $X = \mathbb{R}^n$, $\Gamma_*^r(\mathbb{R}^n) = \prod_{k=1}^n \Gamma_*^r$ and $\mu_1^r\left(\prod_{k=1}^n \gamma^{r,1}_{\frac{z_k}{2^r}}\right) = \prod_{k=1}^n \frac{1}{2^r} = \frac{1}{2^{nr}}$, we also have that the cellular-extension σ-algebra $\mathcal{E}_{\mu_*^*}(\mathbb{R}^n)$ is the Lebesgue σ-algebra $\mathcal{L}(\mathbb{R}^n)$ of $\mathbb{R}^n$ and the n-cellular-extension measure is the Lebesgue measure.

Measures on S^1 and S^2

Take $X = S^1$ the unit 1-sphere and consider the following iterated subdivision $\Gamma_*^* = \Gamma_*^*(S^1)$ and the following 1-cellular measure μ_*^* (see Figure 5.2):

$$\Gamma_0^r = \{\gamma^0_{\frac{z}{2^r}} \mid z \in \mathbb{Z},\ 0 \le z < 2^{r+1}\}, \quad \gamma^0_w(0) = e^{i\pi w};$$

$$\Gamma_1^r = \{\gamma^{r,1}_{\frac{z}{2^r}} \mid z \in \mathbb{Z},\ 0 \le z < 2^{r+1}\}, \quad \gamma^{r,1}_w \colon [0,1] \to S^1,\ \gamma^{r,1}_w(t) = e^{i\pi\left(w + \frac{t}{2^r}\right)}.$$

The subdivision operator is given by

$$\mathrm{Sd}(\{\gamma^0_{\frac{z}{2^r}}\}) = \{\gamma^0_{\frac{z}{2^r}}\}; \quad \mathrm{Sd}(\{\gamma^{r,1}_{\frac{z}{2^r}}\}) = \{\gamma^{r+1,1}_{\frac{z}{2^r}}, \gamma^{r+1,1}_{\frac{2z+1}{2^{r+1}}}\}.$$

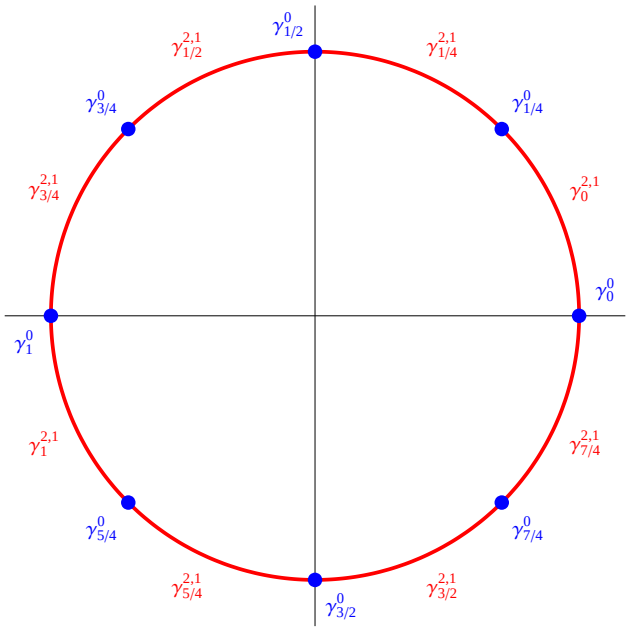


Figure 5.2: Iterated subdivision $\Gamma^2_*(S_1)$ on the 1-sphere.

Define $\mu^*_* = \{\mu^r_*\}_{r\in\mathbb{N}}$, $\mu^r_1\colon \Gamma^r_1 \to [0,\infty]$, $\mu^r_1(\gamma^{r,1}_{\frac{z}{2^r}}) = \frac{\pi}{2^r}$. Notice that μ^*_* has the subdivision invariance property:

$$\mu^{r+1}_1(\mathrm{Sd}(\{\gamma^{r,1}_{\frac{z}{2^r}}\})) = \mu^{r+1}_1(\{\gamma^{r+1,1}_{\frac{z}{2^r}}, \gamma^{r+1,1}_{\frac{2z+1}{2^{r+1}}}\}) = \frac{\pi}{2^{r+1}} + \frac{\pi}{2^{r+1}} = \frac{\pi}{2^r} = \mu^r_1(\{\gamma^{r,1}_{\frac{z}{2^r}}\}).$$

Once again, as a consequence of Theorem 5.2.1 and Theorem 5.2.2, we have that the Borel σ-algebra $\sigma(\mathbf{t}_{S^1})$ is contained in the cellular-extension σ-algebra $\mathcal{E}_{\mu^*_*}(S^1)$ of μ^*_* and the 1-cellular-extension measure $\bar{\mu}\colon \mathcal{E}_{\mu^*_*}(S^1) \to [0,\infty]$ of μ^*_* is the usual measure of angles.

Now, we are going to see how using an iterated subdivision on the unit 2-sphere S^2 and the measure of angles (on S^1) we can give a measure of solid angles, that is, a measure on S^2.

Recall that a spherical triangle ABC is formed by connecting three points on the surface of a sphere with great arcs, so that these three points do not lie on a great circle of the sphere –see Figure 5.3. The angle $\angle A$ at the vertex A is measured as the angle between the tangents to the incident sides in the vertex tangent plane. Note that a pair of unitary tangent vectors at a vertex determines a canonical arc in the unit 1-sphere S^1 contained in the tangent plane to the 2-sphere at this vertex, and the procedure above can be used to find the measure (angle) of this arc. If we add the condition that each angle of the triangle is smaller than π, we can avoid a possible ambiguity between the triangle and its complement on the 2-sphere.

Definition 5.2.5. *Let ABC be a triangle and $ABCD$ a quadrilateral on a 2-sphere of radius R with all their angles at vertexes smaller than π. Then, the non-negative real number $(\angle A+\angle B+\angle C-\pi)$ is called the* **excess of the spherical triangle**. *Similarly, the* **excess of a spherical quadrilateral** *$ABCD$ is the non-negative real number $(\angle A+\angle B+\angle C+\angle D-2\pi)$.*

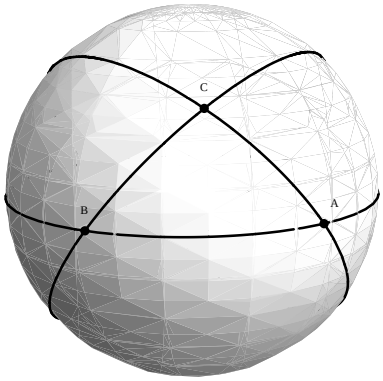


Figure 5.3: An example of a spherical triangle.

Remark 5.2.2. *There is a result, known as* Girard's theorem*, that asserts that the area of a spherical triangle (or a quadrilateral) is equal to the excess multiplied by R^2 and, in the case that $R = 1$, the area is equal to the excess. We recall that a similar formula is used in the hyperbolic plane to give the area of a triangle, but in this case one has to take the defect, given by $\pi - (\angle A + \angle B + \angle C)$.*

It is interesting to note that one has a canonical cubic structure Γ^0_* on the 2-sphere (with 8 vertexes, 12 edges and 6 faces). This can be obtained by taking the unit 2-sphere inside of the 3-cube $[-1, 1]^3$. Now, the canonical regular CW-structure on the boundary of $[-1, 1]^3$ can be projected onto the unit 2-sphere by dividing the vectors of the boundary by its norm –see Figure 5.4.

We can use the following basic properties of the spherical geometry:

(1) Given two non-antipodal points $A, B \in S^2$, with $A \neq B$, there is a unique spherical arc AB determined by A and B.

(2) Given a spherical arc AB, there is a unique middle point $(\widehat{AB}) = \frac{A+B}{|A+B|}$.

(3) Given a spherical quadrilateral $ABCD$, there is a unique middle point which can be obtained by the formula $(\widehat{ABCD}) = \frac{A+B+C+D}{|A+B+C+D|}$.

(4) Any spherical quadrilateral $ABCD$ admits a canonical subdivision given by the following four spherical quadrilaterals:

$$A(\widehat{AB})(\widehat{ABCD})(\widehat{DA}), \qquad (\widehat{AB})B(\widehat{BC})(\widehat{ABCD}),$$

$$(\widehat{ABCD})(\widehat{BC})C(\widehat{CD}), \qquad (\widehat{CD})D(\widehat{DA})(\widehat{ABCD}).$$

In this way, we can subdivide each quadrilateral of Γ^0_* to obtain the subdivision $\mathrm{Sd}(\Gamma^0_*) = \Gamma^1_*$. Applying consecutively the operator Sd for $r = 0, 1, 2, \ldots$, one has the r-th subdivision Γ^r_*, which for $r = 0, 1, 2$ can be seen in Figure 5.4.

Figure 5.4: Iterated subdivision $\Gamma_*^2(S_2)$ on the 2-sphere.

A sequence of consecutive subdivisions Γ_*^r of the 2-sphere permits us to consider the 2-sphere as a measure space and, dividing the measure function by the area of the 2-sphere, one can obtain a canonical structure of probability space. In order to assign an area to a spherical quadrilateral $ABCD \in \Gamma_*^r$ of a given subdivision, it suffices to associate to each quadrilateral of Γ_*^* its excess:

$$\mu_2^r(ABCD) = \angle A + \angle B + \angle C + \angle D - 2\pi.$$

Denote by $\mathrm{Sd}(ABCD)$ the family of the new spherical quadrilaterals of a subdivision of a spherical quadrilateral $ABCD$ and by $\mu_2^{r+1}(\mathrm{Sd}(ABCD))$ the sum of the areas of the quadrilaterals of the subdivision of $ABCD$, using the assignation above. Taking into account that the sum of the measures of the angles at some vertexes is π or 2π, it is easy to check the subdivision invariance:

$$\mu_2^{r+1}(\mathrm{Sd}(ABCD)) = \mu_2^r(ABCD).$$

Since μ_2^* has the subdivision invariance property, one has that μ_*^* is a 2-cellular measure for this iterated subdivision $\Gamma_*^* = \Gamma_*^*(S^2)$ on the 2-sphere. Then, applying Theorem 5.2.1 and Theorem 5.2.2, one has that the Borel σ-algebra $\sigma(\mathbf{t}_{S^2})$ is contained in the cellular-extension σ-algebra $\mathcal{E}_{\mu_*^*}(S^2)$ and we can consider the induced 2-cellular-extension measure $\bar{\mu}$.

5.2.3 Measure exterior discrete semi-flows

Just as we defined the notion of metric exterior discrete semi-flow in section 5.1, we can now define a new hybrid concept involving exterior discrete semi-flows and measures created from a Borel space induced by a σ-algebra. This new definition will be used throughout chapter 7.

Definition 5.2.6. *A* **measure exterior discrete semi-flow** $(X, \varphi, \varepsilon(X), \mathbf{t}_X, \mu)$ *is an exterior discrete semi-flow* $(X, \varphi, \varepsilon(X))$ *provided with a measure*

$$\mu\colon \sigma(\mathbf{t}_X) \to [0, +\infty],$$

where $(X, \sigma(\mathbf{t}_X))$ *is the Borel space induced by the* σ*-algebra* $\sigma(\mathbf{t}_X)$ *generated by the topology of* X*. Given two measure exterior discrete semi-flows*

$$X = (X, \varphi_X, \varepsilon(X), \mathbf{t}_X, \mu_X), \quad Y = (Y, \varphi_Y, \varepsilon(Y), \mathbf{t}_Y, \mu_Y),$$

a measurable morphism f *from* X *to* Y *is just an exterior discrete semi-flow morphism*

$$f\colon (X, \varphi_X, \varepsilon(X)) \to (Y, \varphi_Y, \varepsilon(Y)).$$

In some cases, given a measure exterior discrete semi-flow $(X, \varphi, \varepsilon(X), \mathbf{t}_X, \mu)$, we will shorten the notation and use $(X, \varepsilon(X), \mathbf{t}_X, \mu)$ when the action φ of the discrete semi-flow is clear.

We note that the maps $\varphi_X^n\colon X \to X$ are measurable because they are continuous maps and every continuous map is measurable. If $f\colon X \to Y$ is an exterior discrete semi-flow morphism, then f is also a measurable map.

Chapter 6

A computational study of the iteration of rational maps on the Riemann sphere

In the previous chapters, for a given exterior discrete semi-flow $(X, \varphi, \varepsilon(X))$, we have analyzed many properties about the set of ω-representable end points ${}^{\omega}\pi_0^{\mathrm{BG}}(X, \varphi, \varepsilon(X))$. In this chapter, we focus on the case $X = S^2 \cong \mathbb{C} \cup \{\infty\} \cong \mathbf{P}^1(\mathbb{C})$. This topological space has nice properties: it is a path-connected, locally path-connected, compact, locally compact, Hausdorff space. Moreover, it has got the structure of a 1-dimensional complex manifold, which is called the Riemann sphere.

In addition, in the present chapter, we study the semi-flow structure induced by a rational map –see subsections 6.1.2 and 6.1.5. A rational map defined on the Riemann sphere has the following important property: if $f \neq \mathrm{Id}_{S^2}$, then the set of m-periodic points is finite, for every integer $m \geq 1$. This implies that, for a rational map $f \neq \mathrm{Id}_{S^2}$, if we take the action φ with $\varphi^1 = f$, then the sets of m-cyclic points ${}_aC_m(S^2, f)$, $C_m(S^2, f)$ and m-periodic points ${}_aP_m(S^2, f)$, $P_m(S^2, f)$ are finite, and the set of periodic points $P(S^2, f)$ is countable.

If n divides m, one has the following inclusions:

$$\begin{array}{ccccc} {}_aC_n(S^2, f) & \subset & {}_aP_m(S^2, f) & \subset & {}_aP(S^2, f) \\ \cap & & \cap & & \cap \\ C_n(S^2, f) & \subset & P_m(S^2, f) & \subset & P(S^2, f). \end{array}$$

Using a simpler notation by denoting ${}_aC_n(S^2, f) = {}_aC_n$, ${}_aP_m(S^2, f) = {}_aP_m$, ${}_aP(S^2, f) = {}_aP$, $C_n(S^2, f) = C_n$, $P_m(S^2, f) = P_m$ and $P(S^2, f) = P$, we have the diagram of exterior discrete semi-flows

$$\begin{array}{ccccc} (S^2, \varphi, \varepsilon(S^2, {}_aC_n)) & \longrightarrow & (S^2, \varphi, \varepsilon(S^2, {}_aP_m)) & \longrightarrow & (S^2, \varphi, \varepsilon(S^2, {}_aP)) \\ \downarrow & & \downarrow & & \downarrow \\ (S^2, \varphi, \varepsilon(S^2, C_n)) & \longrightarrow & (S^2, \varphi, \varepsilon(S^2, P_m)) & \longrightarrow & (S^2, \varphi, \varepsilon(S^2, P)), \end{array} \tag{6.1}$$

the corresponding diagram of sets of ω-representable end points

$$\begin{tikzcd}
{}^{\omega}\pi_0^{\mathrm{BG}}(S^2,\varphi,\varepsilon(S^2,{}_aC_n)) \arrow[r] \arrow[d] & {}^{\omega}\pi_0^{\mathrm{BG}}(S^2,\varphi,\varepsilon(S^2,{}_aP_m)) \arrow[r] \arrow[d] & {}^{\omega}\pi_0^{\mathrm{BG}}(S^2,\varphi,\varepsilon(S^2,{}_aP)) \arrow[d] \\
{}^{\omega}\pi_0^{\mathrm{BG}}(S^2,\varphi,\varepsilon(S^2,C_n)) \arrow[r] & {}^{\omega}\pi_0^{\mathrm{BG}}(S^2,\varphi,\varepsilon(S^2,P_m)) \arrow[r] & {}^{\omega}\pi_0^{\mathrm{BG}}(S^2,\varphi,\varepsilon(S^2,P))
\end{tikzcd} \tag{6.2}$$

and the related diagram of regions of exterior attraction

$$\begin{tikzcd}
D(S^2,{}_aC_n) \arrow[r] \arrow[d] & D(S^2,{}_aP_m) \arrow[r] \arrow[d] & D(S^2,{}_aP) \arrow[d] \\
D(S^2,C_n) \arrow[r] & D(S^2,P_m) \arrow[r] & D(S^2,P).
\end{tikzcd} \tag{6.3}$$

As a consequence of the results achieved in chapter 4, one has that:

(1) The maps in diagram (6.2) are injective.

(2) The regions of exterior attraction in diagram (6.3) can be divided into basins of end points in a compatible way; that is to say, all the basins of end points belonging to some set appearing in diagram (6.2) can be computed in the corresponding region of exterior attraction. This means that the basins have no new points in larger regions of exterior attraction.

(3) For a union of asymptotically stable cycles, one has the following canonical isomorphisms:

$$ {}_aC_n \cong {}^{\omega}\pi_0^{\mathrm{BG}}(S^2,\varphi,\varepsilon(S^2,{}_aC_n)),$$

$$ {}_aP_m \cong {}^{\omega}\pi_0^{\mathrm{BG}}(S^2,\varphi,\varepsilon(S^2,{}_aP_m)).$$

(4) The calculation of the basins of an asymptotically stable n-cycle of $f=\varphi^1\neq \mathrm{Id}_{S^2}$ can be reduced to the calculation of an asymptotically stable fixed point of f^n.

On the Riemann sphere, we can also consider the structure of a metric space –see subsection 6.1.3. In this way, we have the metric exterior discrete semi-flows $(S^2,d,\varphi,\varepsilon(S^2,A))$ at our disposal, where A is one of the sets in diagram (6.1).

From the results in section 5.1, given any of the above-mentioned metric exterior discrete semi-flows $(S^2,d,\varphi,\varepsilon(S^2,A))$, we get that

$$h\colon {}^{\omega}\pi_0^{\mathrm{BG}}(S^2,\varphi,\varepsilon(S^2,A)) \to \Pi(S^2,d)$$

is injective. Moreover, the basin of an ω-representable end point $a\in {}^{\omega}\pi_0^{\mathrm{BG}}(S^2,\varphi,\varepsilon(S^2,A))$ can be computed as the basin of the end point $h(a)\in\Pi(S^2,d)$ in S^2. Therefore, for the case of a rational map $f\neq \mathrm{Id}_{S^2}$ defined on the Riemann sphere, the computation of ω-representable end points and their basins can be reduced to the study of end points and basins induced by the chordal (or Riemannian) metric on S^2.

Another view to look at is the graphic representation of the fractals that one can obtain from the iteration of a rational function different from the identity on the surface of the sphere S^2. In Numerical Analysis, some authors such as J. L. Varona in [82], O. Lewis in [54], M. McClure in [67] or W. T. Shaw in [77] developed graphic algorithms to study the basin of attraction of a root when a numerical method is employed. In this context, we set as a goal to improve some aspects of the existing programs for the visualization of basins of attraction and Julia sets associated with a rational function on the Riemann sphere by using its geometry and complex structure.

We present here a collection of algorithms based on the canonical bijection of the complex projective line and $\mathbb{C} \cup \{\infty\}$, which give us the following advantages:

a) the use of homogeneous coordinates which permits us to work at the point at infinity $\infty \in \mathbb{C} \cup \{\infty\}$,

b) the representation of a rational function by a pair of homogeneous polynomials of two variables and with the same degree that allows us to compute the numerical value of the function at any pole point and at the point at infinity,

c) the calculus of multiplicators which enables the usual classification of fixed points: super-attracting, attracting, indifferent or repelling,

d) the use of normalized homogeneous coordinates that avoids overflow and underflow errors in our algorithms.

Other of our subroutines are based on the stereographic bijection from the unit 2-sphere to $\mathbb{C} \cup \{\infty\}$, and it permits us:

e) to compute the distance from an ordinary point to the point at infinity on the 2-sphere by using the chordal metric,

f) to plot 3D-spherical global versions of the basins of attraction,

g) to draw global basins of attraction using two discs that correspond to the south and north hemispheres of the 2-sphere.

Finally, for a given positive integer n, our algorithms allow us:

h) to plot the basins of attraction of end points associated with n-periodic points.

The chapter is divided into three parts. In section 6.1, a mathematical theoretical basis for our program is given. Section 6.2 describes the tasks and source codes of our algorithms. In the end, section 6.3 includes a brief user manual that explains how to use the developed software properly.

6.1 Theoretical justification and mathematical framework of the algorithms

In order to create a theoretical basis to hold and justify the correct construction of our algorithms for the representation of basins of end points corresponding to rational maps different from the identity, we shall use the mathematical techniques described below in this section, apart from the notions of metric discrete semi-flow and end points previously explained in the preceding chapters. This study will be developed within the theoretical framework of complex dynamics on the Riemann sphere.

6.1.1 Smooth and complex structures on $\mathbb{C} \cup \{\infty\}$

Let $S^2 = \{(r_1, r_2, r_3) \in \mathbb{R}^3 \mid r_1^2 + r_2^2 + r_3^2 = 1\}$ be the unit 2-sphere and let $N = (0, 0, 1)$ be its north pole. Consider the stereographic atlas $\{\hat{x}, \hat{y}\}$ for S^2, where $\hat{x}\colon S^2 \setminus \{N\} \to \mathbb{R}^2$ and $\hat{y}\colon S^2 \setminus \{-N\} \to \mathbb{R}^2$ are both charts given by

$$\hat{x}(r_1, r_2, r_3) = \left(\frac{r_1}{1 - r_3}, \frac{r_2}{1 - r_3}\right),$$

$$\hat{y}(r_1, r_2, r_3) = \left(\frac{r_1}{1 + r_3}, \frac{r_2}{1 + r_3}\right).$$

The stereographic atlas gives a 2-dimensional smooth structure to S^2.

We can consider in a natural way a bijection $\tilde{\Theta}\colon S^2 \to \mathbb{C} \cup \{\infty\}$ given as follows:

$$\tilde{\Theta}(r_1, r_2, r_3) = \begin{cases} \frac{r_1}{1-r_3} + i\frac{r_2}{1-r_3}, & \text{if } r_3 < 1, \\ \infty, & \text{if } r_3 = 1. \end{cases}$$

In this way, we can also regard $\mathbb{C} \cup \{\infty\}$ as a 2-dimensional smooth manifold by using the bijection $\tilde{\Theta}$.

Take the following equivalence relation on $\mathbb{C}^2 \setminus \{(0, 0)\}$: $(z, t) \sim (z', t')$ iff there exists a $\lambda \in \mathbb{C} \setminus \{0\}$ such that $(z, t) = (\lambda z', \lambda t')$. The equivalence class of (z, t) is denoted by $[z, t]$ and the quotient set is denoted by $\mathbf{P}^1(\mathbb{C})$ and it is called the *complex projective line*.

Let x and y be functions from $\mathbf{P}^1(\mathbb{C})$ to $\mathbb{C}$ with domains $\operatorname{Dom} x = \{[z, t] \in \mathbf{P}^1(\mathbb{C}) \mid t \neq 0\}$ and $\operatorname{Dom} y = \{[z, t] \in \mathbf{P}^1(\mathbb{C}) \mid z \neq 0\}$ given by $x([z, t]) = z/t$ and $y([z, t]) = t/z$. Then, the atlas $\{x, y\}$ provides $\mathbf{P}^1(\mathbb{C})$ with a 1-dimensional complex structure. Given a point $[z, t] \in \mathbf{P}^1(\mathbb{C})$, the coordinates (z, t) are called the *homogeneous coordinates* of the point and t/z (or z/t where appropriate) is the *absolute coordinate* of that point. In our study, we often use *normalized homogeneous coordinates* for any point in $\mathbf{P}^1(\mathbb{C})$, which are given as follows:

$$[z, t] = \begin{cases} [z/t, 1] & \text{if } |t| \geq |z|, \\ [1, t/z] & \text{if } |t| < |z|, \end{cases}$$

where $|t|$ and $|z|$ represent the absolute value (or modulus) of the complex numbers t and z, respectively.

We also have the induced bijection $\Theta\colon \mathbf{P}^1(\mathbb{C}) \to \mathbb{C} \cup \{\infty\}$ given by

$$\Theta([z,t]) = \begin{cases} z/t, & \text{if } t \neq 0, \\ \infty, & \text{if } t = 0. \end{cases}$$

All the bijections above induce a new bijection $\Theta^{-1} \circ \tilde{\Theta}\colon S^2 \to \mathbf{P}^1(\mathbb{C})$, which can be defined as follows:

$$(\Theta^{-1} \circ \tilde{\Theta})(r_1, r_2, r_3) = [r_1 + ir_2, 1 - r_3].$$

The inverse map of this bijection $\tilde{\Theta}^{-1} \circ \Theta\colon \mathbf{P}^1(\mathbb{C}) \to S^2$ is given by the following formula:

$$(\tilde{\Theta}^{-1} \circ \Theta)([z,t]) = \left(\frac{\bar{z}t + z\bar{t}}{\bar{t}t + z\bar{z}}, \frac{i(\bar{z}t - z\bar{t})}{\bar{t}t + z\bar{z}}, \frac{-\bar{t}t + z\bar{z}}{\bar{t}t + z\bar{z}} \right). \tag{6.4}$$

Remark 6.1.1. *A surface with a* 1*-dimensional complex structure is said to be a* Riemann surface *and a Riemann surface of genus* 0 *is said to be a* Riemann sphere*. Using the bijections defined above, we have that* S^2 *and* $\mathbb{C} \cup \{\infty\}$ *are Riemann spheres.*

Remark 6.1.2. *We notice that the homogeneous coordinates presented in this subsection allow us to represent the point at infinity, and the use of normalized coordinates will avoid overflow and underflow errors in our computer programs.*

6.1.2 Complex rational maps

Consider a rational function $h\colon \mathbb{C} \to \mathbb{C}$ of the form $h(u) = a\,\dfrac{F(u)}{G(u)}$, where $u, a \in \mathbb{C}$, $a \neq 0$, $F(u) = (u - z_1)\cdots(u - z_p)$ and $G(u) = (u - l_1)\cdots(u - l_q)$. Suppose that $\{z_1, \ldots, z_p\} \cap \{l_1, \ldots, l_q\} = \emptyset$. Then, the function h induces an extension map $h^+\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, where $h^+(l_i) = \infty$ and $h^+(\infty)$ is given as follows:

$$h^+(\infty) = \begin{cases} \infty, & \text{if } q < p, \\ 0, & \text{if } q > p, \\ a, & \text{if } q = p. \end{cases}$$

Observe that the bijection

$$\Theta\colon \mathbf{P}^1(\mathbb{C}) \to \mathbb{C} \cup \{\infty\}$$

induces the map $h^1\colon \mathbf{P}^1(\mathbb{C}) \to \mathbf{P}^1(\mathbb{C})$ defined by $h^1 = \Theta^{-1} \circ h^+ \circ \Theta$, which is expressed in homogeneous coordinates as follows:

$$h^1([z,t]) = \begin{cases} [a(z - tz_1)\cdots(z - tz_p), t^{p-q}(z - tl_1)\cdots(z - tl_q)], & \text{if } p \geq q, \\ [at^{q-p}(z - tz_1)\cdots(z - tz_p), (z - tl_1)\cdots(z - tl_q)], & \text{if } p \leq q. \end{cases}$$

In this context, it is important to notice that, in the case that we take $u = z/t$ and $n = \max\{p, q\}$ in the expression

$$\frac{a(u^p + a_1 u^{p-1} + \cdots + a_p)}{(u^q + b_1 u^{q-1} + \cdots + b_q)},$$

we have

$$\frac{a(z^p t^{n-p} + a_1 z^{p-1} t^{n-p+1} + \cdots + a_p t^n)}{z^q t^{n-q} + b_1 z^{q-1} t^{n-q+1} + \cdots + b_q t^n}.$$

Now, take the following homogeneous polynomials of degree n in the variables z, t:

$$F_1(z,t) = a(z^p t^{n-p} + a_1 z^{p-1} t^{n-p+1} + \cdots + a_p t^n),$$

$$G_1(z,t) = z^q t^{n-q} + b_1 z^{q-1} t^{n-q+1} + \cdots + b_q t^n;$$

by using these polynomials, one has that

$$h^1([z,t]) = [F_1(z,t), G_1(z,t)].$$

Conversely, if $A(z,t)$ and $B(z,t)$ are homogeneous polynomials of degree n, then $A(\lambda z, \lambda t) = \lambda^n A(z,t)$ and $B(\lambda z, \lambda t) = \lambda^n B(z,t)$. This implies that the pair of homogeneous polynomials $A(z,t)$ and $B(z,t)$ induces the map $f\colon \mathbf{P}^1(\mathbb{C}) \to \mathbf{P}^1(\mathbb{C})$ defined as follows:

$$f([z,t]) = [A(z,t), B(z,t)].$$

The associated rational map is given by $F(z) = A(z,1)$ and $G(z) = B(z,1)$.

The next lemma discuss how to find all the fixed points of any rational map different from the identity which is represented by a pair of coprime homogeneous polynomials, even if it is composed with itself a certain number of times.

Lemma 6.1.1. *Let $f \neq \mathrm{Id}$ be a rational map represented by a pair of coprime homogeneous polynomials $A(z,t)$, $B(z,t)$ of degree n. Then:*

(i) The set $\{[z_1,t_1], \ldots, [z_{n+1}, t_{n+1}]\}$ of roots of $A(z,t)t - B(z,t)z$ is the set of fixed points of f,

(ii) f^r is a rational map of degree n^r which has $n^r + 1$ fixed points (taking into account its multiplicity).

Remark 6.1.3. *The representation of a rational function with a pair of homogeneous polynomials of two variables with the same degree combined with normalized homogeneous coordinates permits us to work with poles and the point at infinity, as well as to avoid overflows and underflows in our algorithms.*

6.1.3 Metrics on $S^2 \cong \mathbb{C} \cup \{\infty\} \cong \mathbf{P}^1(\mathbb{C})$

We have two natural metrics on S^2: since S^2 is a subspace of $\mathbb{R}^3$, the usual Euclidean metric of $\mathbb{R}^3$ induces a Euclidean metric d^E on S^2; besides, we have as well that S^2 inheres a Riemannian metric d^R from the canonical Riemannian structure of $S^2 \subset \mathbb{R}^3$. The connection between Riemannian metric d^R and Euclidean metric d^E on S^2 is given by the expression:

$$d^E((r_1,r_2,r_3),(r_1',r_2',r_3')) = 2\sin\left(\frac{d^R((r_1,r_2,r_3),(r_1',r_2',r_3'))}{2}\right).$$

Using the bijection $\tilde{\Theta}^{-1} \circ \Theta \colon \mathbf{P}^1(\mathbb{C}) \to S^2$ shown in (6.4), we can translate the metric structures from S^2 to $\mathbf{P}^1(\mathbb{C})$ with the following formulas:

$$d_1^E([z,t],[z',t']) = d^E((\tilde{\Theta}^{-1} \circ \Theta)([z,t]), (\tilde{\Theta}^{-1} \circ \Theta)([z',t'])),$$

$$d_1^R([z,t],[z',t']) = d^R((\tilde{\Theta}^{-1} \circ \Theta)([z,t]), (\tilde{\Theta}^{-1} \circ \Theta)([z',t'])).$$

An explicit formula for the *chordal metric* d_1^E is given by:

$$d_1^E([z,t],[z',t']) =$$
$$\left(\left(\frac{\bar{z}t + z\bar{t}}{\bar{t}t + z\bar{z}} - \frac{\bar{z}'t' + z\bar{t}'}{\bar{t}'t' + z'\bar{z}'}\right)^2 + \left(\frac{i(\bar{z}t - z\bar{t})}{\bar{t}t + z\bar{z}} - \frac{i(\bar{z}'t' - z'\bar{t}')}{\bar{t}'t' + z'\bar{z}'}\right)^2 + \left(\frac{-\bar{t}t + z\bar{z}}{\bar{t}t + z\bar{z}} - \frac{-\bar{t}'t' + z'\bar{z}'}{\bar{t}'t' + z'\bar{z}'}\right)^2\right)^{\frac{1}{2}}.$$

6.1.4 Tangent map of a rational map

Given an analytic map $f \colon \mathbf{P}^1(\mathbb{C}) \to \mathbf{P}^1(\mathbb{C})$ and a point $p = [z,t] \in \mathbf{P}^1(\mathbb{C})$, there is an induced map between the tangent spaces

$$T_p f \colon T_p(\mathbf{P}^1(\mathbb{C})) \to T_{f(p)}(\mathbf{P}^1(\mathbb{C})).$$

Taking the bases $\frac{\partial}{\partial x}$ if $|t| \geq |z|$ and $\frac{\partial}{\partial y}$ if $|t| < |z|$ of the complex tangent space and writing $f(p) = [z',t']$, we have four cases when giving the 1×1 Jacobian matrix of $T_p f$:

$$J_p^{x,x} = \left((x \circ f \circ x^{-1})'(z/t)\right), \quad \text{if } |t| \geq |z| \text{ and } |t'| \geq |z'|,$$

$$J_p^{y,x} = \left((y \circ f \circ x^{-1})'(z/t)\right), \quad \text{if } |t| \geq |z| \text{ and } |t'| < |z'|,$$

$$J_p^{x,y} = \left((x \circ f \circ y^{-1})'(t/z)\right), \quad \text{if } |t| < |z| \text{ and } |t'| \geq |z'|,$$

$$J_p^{y,y} = \left((y \circ f \circ y^{-1})'(t/z)\right), \quad \text{if } |t| < |z| \text{ and } |t'| < |z'|.$$

Notice that if f is a rational map induced by polynomials $A(z,t), B(z,t)$, then the coordinate representations of f with respect to the corresponding pairs of charts are given by

$$(x \circ f \circ x^{-1})(z) = A(z,1)/B(z,1),$$

$$(y \circ f \circ x^{-1})(z) = B(z,1)/A(z,1),$$

$$(x \circ f \circ y^{-1})(t) = A(1,t)/B(1,t),$$

$$(y \circ f \circ y^{-1})(t) = B(1,t)/A(1,t),$$

so that we can consider its corresponding derivatives:

$$\frac{d(A(z,1)/B(z,1))}{dz}, \quad \frac{d(B(z,1)/A(z,1))}{dz}, \quad \frac{d(A(1,t)/B(1,t))}{dt}, \quad \frac{d(B(1,t)/A(1,t))}{dt}.$$

Hence, the norm of the tangent function at p, taking into account the metrics of $T_p(\mathbf{P}^1(\mathbb{C}))$ and $T_{f(p)}(\mathbf{P}^1(\mathbb{C}))$, is given by the formula:

$$|J_p(f)| = \begin{cases} \frac{1+z\bar{z}}{1+z'\bar{z'}}\mathrm{Abs}((x\circ f\circ x^{-1})'(z)), & \text{if } t=1, t'=1, \\ \frac{1+z\bar{z}}{1+t'\bar{t'}}\mathrm{Abs}((y\circ f\circ x^{-1})'(z)), & \text{if } t=1, z'=1, \\ \frac{1+t\bar{t}}{1+z'\bar{z'}}\mathrm{Abs}((x\circ f\circ y^{-1})'(t)), & \text{if } z=1, t'=1, \\ \frac{1+t\bar{t}}{1+t'\bar{t'}}\mathrm{Abs}((y\circ f\circ y^{-1})'(t)), & \text{if } z=1, z'=1. \end{cases}$$

We remark that, in the case of a fixed point $p = f(p) = [z, t]$ and using normalized homogeneous coordinates, we only have two cases for the 1×1 Jacobian matrix:

$$\begin{array}{ll} J_p^{x,x} = \big((x\circ f\circ x^{-1})'(z)\big) & \text{if } t=1, \\ J_p^{y,y} = \big((y\circ f\circ y^{-1})'(t)\big) & \text{if } z=1. \end{array}$$

We can use the norm of the tangent map to give the following definition.

Definition 6.1.1. *Let $f\colon \mathbf{P}^1(\mathbb{C}) \to \mathbf{P}^1(\mathbb{C})$ be an analytic function and $p \in \mathbf{P}^1(\mathbb{C})$ a fixed point. Then, p is said to be a* **super-attracting**, **attracting**, **indifferent** *or* **repelling** *fixed point if the norm (absolute value) of the tangent map at that point is zero, lower than* 1*, equal to* 1 *or greater than* 1*, respectively.*

As a matter of fact, in order to know if a fixed point is super-attracting, attracting, indifferent or repelling, it suffices to check if

$$|J_p(f)| = \begin{cases} \mathrm{Abs}((x\circ f\circ x^{-1})'(z)), & \text{if } t=1, t'=1, \\ \mathrm{Abs}((y\circ f\circ y^{-1})'(t)), & \text{if } z=1, z'=1, \end{cases}$$

is zero, lower than, equal to or greater than 1.

Knowing if a fixed point is super-attracting, attracting indifferent or repelling will be helpful later. In general, the basins of attraction of super-attracting and attracting fixed points are "easily visible"; however, for repelling fixed points it may be necessary "to apply some zooms" on suitable local rectangles in order to see their basins of attraction.

6.1.5 Basins of end points induced by a rational function $f \neq \mathrm{Id}$ on $\mathbb{C} \cup \{\infty\}$

Let $z \in \mathbb{C}$ and consider a function $h\colon \mathbb{C} \to \mathbb{C}$, $h(z) \neq z$, of the form $h(z) = a\,\dfrac{P(z)}{Q(z)}$, where $a \in \mathbb{C}$, $a \neq 0$ and $P(z), Q(z)$ is a pair of irreducible polynomials of degree p, q, respectively. We have seen in subsection 6.1.2 that h induces a new map $f = h^+$ on $\mathbb{C} \cup \{\infty\}$, which gives to $\mathbb{C} \cup \{\infty\}$ the structure of a discrete semi-flow $\varphi\colon \mathbb{N} \times \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ by the formula $\varphi(n, p) = f^n(p)$. If we consider the metric discrete semi-flow $(\mathbb{C} \cup \{\infty\}, d_1^E, \varphi)$, we also have the canonical map

$$\omega_{d_1^E}\colon \mathbb{C} \cup \{\infty\} \to \Pi\left(\mathbb{C} \cup \{\infty\}, d_1^E\right)$$

given by

$$\omega_{d_1^E}(p) = [(\varphi(0, p), \varphi(1, p), \varphi(2, p), \dots)] = [(p, f(p), f^2(p), \dots)].$$

By means of this approach, we obtain a representation of the basins of attraction of the attracting fixed points of f on the Riemann sphere. If f is of degree greater than or equal to 2, as we can read in [31], its *Julia set* $J(f)$ is an uncountable compact set containing no isolated points and it is the boundary of the basin of attraction of each attracting fixed point of f, including ∞, and $J(f) = J(f^p)$ for each positive integer p; its corresponding *Fatou set* is precisely the complementary set of $J(f)$. Consequently, in that case, the Julia set of f would be formed by the points lying on the boundary of the basins of the attracting points –and hence, the rest of points would be in the Fatou set.

The Julia set of a rational function f of degree at least 2 can also be described as:

$$J(f) = \overline{\{x \in X \mid x \text{ is a periodic repelling point}\}}.$$

Intuitively, we can say that a point x belongs to the Fatou set if there exists an open neighborhood U of x such that $\omega(x) = \omega(y)$, $\forall y \in U$ –that is, if the basin of an end point associated with any point which is close to x is the same as the basin of the end point to which x belongs.

Now, we shall study a particular example of discrete semi-flow induced by a rational function. Consider $h(z) = P(z)/Q(z)$, where $P(z) = 1 + 4z^5$ and $Q(z) = 5z^4$. In this case, the induced map $f = h^+$ has six fixed points:

$$p_0 = \infty, \quad p_1 = -0.809017 - 0.587785i, \quad p_2 = -0.809017 + 0.587785i,$$
$$p_3 = 0.309017 - 0.951057i, \quad p_4 = 0.309017 + 0.951057i, \quad p_5 = 1.$$

Therefore, the space $X = \mathbb{C} \cup \{\infty\}$ is divided into seven regions:

$$X = (X \setminus D) \sqcup D_\infty \sqcup D_{p_1} \sqcup D_{p_2} \sqcup D_{p_3} \sqcup D_{p_4} \sqcup D_{p_5},$$

where

$$D = D_\infty \cup D_{p_1} \cup D_{p_2} \cup D_{p_3} \cup D_{p_4} \cup D_{p_5}.$$

We can associate each one with a different color, as we see in Table 6.1. It is shown, moreover, which kind of fixed point (super-attracting, attracting, indifferent, or repelling) corresponds to every basin of attraction.

In region $X \setminus D$, we can find points whose basin of attraction corresponds to an end point which is not associated with any fixed point (for example, end points associated with a 2-cycle) or even points such that, after doing a prefixed limited number of iterations, belong to a sequence that has not converged to any fixed point yet (modulo a determined precision fixed beforehand). The rest of colors correspond to points which belong to the basin of attraction of an end point (and they are associated with a certain fixed point).

Figure 6.1 illustrates the particular example that we have considered in this subsection. It is clear that the basins of the points $\{p_1, p_2, p_3, p_4, p_5\}$ corresponds to colors $\{2, 3, 4, 5, 6\}$. The properties of the points of regions of color 0 (red) and 1 (yellow) is a little more complicated. Note that the points colored in yellow belong to the basin of a repelling fixed point. In this case, since we have chosen neither a high precision nor a high maximum number of iterations, some of these yellow points are not actually in the basin of the repelling fixed point ∞. The red points correspond to basins of end points induced by cycles and points that, with the specified maximum number of iterations and the given precision, are not close enough to a fixed point yet.

Region	Color	Type of associated fixed point
$X \setminus D$	0	
$D_\infty = \omega^{-1}(\omega(p_0))$	1	Repelling
$D_{p_1} = \omega^{-1}(\omega(p_1))$	2	Super-attracting
$D_{p_2} = \omega^{-1}(\omega(p_2))$	3	Super-attracting
$D_{p_3} = \omega^{-1}(\omega(p_3))$	4	Super-attracting
$D_{p_4} = \omega^{-1}(\omega(p_4))$	5	Super-attracting
$D_{p_5} = \omega^{-1}(\omega(p_5))$	6	Super-attracting

Table 6.1: Connection between regions, colors and types of fixed points.

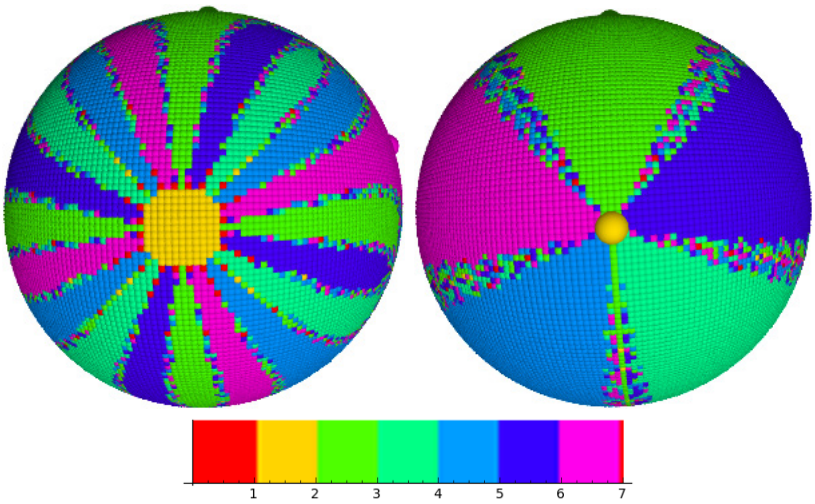


Figure 6.1: 3D fractal on sphere S^2. The image on the left represents those regions which are near the origin of coordinates (south pole of the sphere) and the image on the right shows regions which are close to the point at infinity (north pole of the sphere).

6.2 Description of the employed algorithms

Along previous subsections, we have introduced some mathematical techniques and developed basic theoretical aspects necessary to build computer programs with the ability of representing basins of attraction of end points associated with a determined rational function different from the identity. As usual in this work, a rational function f on $\mathbb{C} \cup \{\infty\}$ will be represented by a pair of homogeneous polynomials $A(z,t)$, $B(z,t)$ of the same degree –see subsection 6.1.2. We shall show in the next lines the algorithms which have been developed to study the basins induced by $f \neq \mathrm{Id}$.

6.2.1 Calculation of the fixed points of $f \neq \mathrm{Id}$

By Lemma 6.1.1, the set of roots

$$\{[z_1, t_1], \ldots, [z_{n+1}, t_{n+1}]\}$$

of $A(z,t)t - B(z,t)z = 0$ coincides with the set of fixed points of $f \neq \mathrm{Id}$.

- If $t = 0$ is a root of $B(1,t)$ and $z_1, \ldots, z_n$ are the roots of $A(z,1) - B(z,1)z = 0$, then the set of fixed points of f is $\{[1,0], [z_1,1], \ldots, [z_n,1]\}$.
- If $t = 0$ is not a root of $B(1,t)$ and $z_1, \ldots, z_n, z_{n+1}$ are the roots of $A(z,1) - B(z,1)z = 0$, $\{[z_1,1], \ldots, [z_n,1], [z_{n+1},1]\}$ is the set of fixed points of f.

A function called `fixedPointsZeros(A,B)` has been built in *Sage*. This function returns a list containing all the fixed points of a given rational function $f \neq \mathrm{Id}$ induced by A, B.

The following is an example of use: let

```
P.<z,t> = PolynomialRing(CC,2)
A=t**4+3*z**4
B=4*t*z**3
```

If we take as input

```
fixedPointsZeros(A,B),
```

the obtained output is

```
[(1, 0), (-1.000, 1), (1.000, 1), (-1.000*I, 1), (1.000*I, 1)].
```

Note that every point in the output is given in normalized homogeneous coordinates. In this case, the first point represents ∞.

The developed algorithm is:

```
def fixedPointsZeros (U, V) :
    U = U + 0*I
    V = V + 0*I
    if (expand(U(t=1) - V(t=1)*z)!=0):
        L = CC[z]
        solaux = (L(U(t=1) - V(t=1)*z)).roots()
        sol1 = [homogeneousNormalization((t[0], 1)) for t in solaux]
        con = V(z=1, t=0)
        if (con == 0):
            sol1.insert(0, (1, 0))
    else:
        sol1 = []
        print "There was a problem when solving equation f(x)=x: cannot solve
                equation 0=0"
    return sol1
```

Observe that the subroutine `homogeneousNormalization` is used within the algorithm. Its implementation in *Sage* is:

```
def homogeneousNormalization(twotuple):
    if abs(twotuple[0]) < abs(twotuple[1]):
        return (twotuple[0] / twotuple[1], 1)
    else:
        return (1, twotuple[1] / twotuple[0])
```

This subroutine allows us to obtain the normalized homogeneous coordinates of any point of $\mathbb{C} \cup \{\infty\}$.

Such algorithms have also been implemented in *Mathematica*:

```
FixedPointsZeros[A_, B_] := Module[{isInftyPole, solaux, le, sol, z},
   solaux = NSolve[A[{z, 1}] - B[{z, 1}] z == 0, z];
   le = Length[solaux];
   sol = Table[HomogeneousNormalization[{z /. solaux[[l]], 1}], {l, 1, le}];
   isInftyPole = B[{1, 0}];
   If[isInftyPole == 0., Prepend[sol, {1, 0}], sol]
   ];


HomogeneousNormalization[{z_, t_}] :=
   If[Abs[z] <= Abs[t], {N[z/t], 1}, {1, N[t/z]}];
```

6.2.2 Distance between two points

The chordal distance between any two points in $\mathbf{P}^1(\mathbb{C})$ can be obtained by using the bijection from $\mathbf{P}^1(\mathbb{C})$ to S^2 which appeared in subsection 6.1.3 and the Euclidean metric on S^2. To that end, the following functions were developed in *Sage* (an example of use is given):

```
sphereBijection((1,0))
(0,0,1)
chordalMetric((1,0),(1,1))
1.41421356237310
```

Function `chordalMetric` uses the Euclidean metric d^E to calculate distances between pairs of points in S^2, instead of the Riemannian metric. This is because its computational cost would be greater and it would be more inefficient if it employed the metric d^R, since in that case it would use inverse trigonometric functions to do the appropriate calculations. In fact, `chordalMetric` makes use of the map d_1^E defined in subsection 6.1.3.

An implementation in *Sage* of the functions that we have just presented is shown below:

```
def sphereBijection(twotuple):
    z = twotuple[0]
    t = twotuple[1]
    return ((conjugate(z)*t + conjugate(t)*z) /
            (conjugate(t)*t + conjugate(z)*z),
        (I*(conjugate(z)*t - conjugate(t)*z)) /
            (conjugate(t)*t + conjugate(z)*z),
        (-conjugate(t)*t + conjugate(z)*z) /
            (conjugate(t)*t + conjugate(z)*z))

def chordalMetric(twotuple, twotuple1):
    t1 = sphereBijection(twotuple)
    t2 = sphereBijection(twotuple1)
    m1 = Matrix([[t1[0], t1[1], t1[2]]])
    m2 = Matrix([[t2[0], t2[1], t2[2]]])
    return n(norm(m1-m2))
```

In *Mathematica*, the source code is the following:

```
SphereBijection[{z_,t_}] :=
   {Re[(Conjugate[z]*t+Conjugate[t]*z)/(Conjugate[t]*t+Conjugate[z]*z)],
   Re[(I*(Conjugate[z]*t-Conjugate[t]*z))/(Conjugate[t]*t+Conjugate[z]*z)],
   Re[(-Conjugate[t]*t+Conjugate[z]*z)/(Conjugate[t]*t+Conjugate[z]*z)]};

ChordalMetric[{a_, b_}, {c_, d_}] :=
   N[Norm[SphereBijection[{a, b}] - SphereBijection[{c, d}]]];
```

6.2.3 Iteration of the rational map f

With a view to find an end point associated with a point $x \in \mathbf{P}^1(\mathbb{C})$, the rational map $f \neq \mathrm{Id}$ must be iterated to obtain a finite sequence

$$(x, f(x), f^2(x), f^3(x), \ldots, f^{k-1}(x), f^k(x)).$$

In this context, remind that a maximum number of iterations l must be considered and a certain precision c_1 must be prefixed to determine when to stop the iterative process while programming the function which returns such a sequence. That is why we shall always work with sequences in which $k \leq l$.

After each iteration, there will be two possible cases:

(1) If the chordal distance from $f^k(x)$ to $f^{k+1}(x)$ is lower than 10^{-c_1}, then take as output the list $[f^{k+1}(x), k]$; otherwise, case 2) is applied.

(2) If $k < l$, a new iteration is done and case (1) is applied again; otherwise (if $k = l$), then the output $[f^{l+1}(x), l]$ is taken.

The following implementation, `newstep`, was developed in *Sage* according to what was intended (an example of use is given):

```
newstep(A,B,25,4,(0.5,1))
[(1, 0.999999999998902), 7]
```

The source code of the function is:

```
def newstep(U, V, iter, precision, pointinternumber):
    point = pointinternumber
    number = 0
    imagepoint = rationalFunction(U, V, point)
    while (chordalMetric(point, imagepoint) > 10.**(-precision))
            and (number < iter):
        point = imagepoint
        imagepoint = rationalFunction(U, V, point)
        number = number + 1
    return [imagepoint, number]
```

Observe that `newstep` uses the subroutine `rationalFunction` in order to calculate after each iteration the image by the given rational map of the corresponding point and to normalize its coordinates. The source code of the subroutine in *Sage* is shown below:

```
def rationalFunction(U, V, twotuple):
    c = twotuple[0]
    d = twotuple[1]
    return homogeneousNormalization((U(z=c, t=d), V(z=c, t=d)))
```

We have also developed these algorithms in *Mathematica*:

```
Newstep[{A_,B_},iter_,precision_][pointInterNumber_] :=
   Module[{point,number,imagePoint},
   point=pointInterNumber[[1]];
   number=pointInterNumber[[2]];
   imagePoint=RationalFunction[A,B][point];
   Which[number>=iter,{imagePoint,iter},
      ChordalMetric[point,imagePoint]<10.^-precision,{imagePoint,number},
      True,Newstep[{A,B},iter,precision][{imagePoint,number+1}]]
   ];

RationalFunction[A_, B_][{z_, t_}] :=
   HomogeneousNormalization[{A[{z, t}], B[{z, t}]}];
```

6.2.4 Determination of the fixed point to which an iteration sequence converges and number of iterations until convergence

Consider the ordered set of fixed points $\{x_1, x_2, \ldots, x_{n+1}\}$ associated with a rational map different from the identity. In the same way, given a point $x \in \mathbb{C} \cup \{\infty\}$, consider the iteration sequence $(x, f(x), \ldots, f^k(x))$. Fixed a precision c_2, if there exists $i \in \{1, \ldots, n+1\}$ such that the chordal distance from $f^k(x)$ to the fixed point x_i is lower than 10^{-c_2}, then the function `positionIterationNumber` described below must return $[i, k]$. Otherwise, $k = l$ and the output must be $[0, l]$, where l is the maximum number of iterations which was prefixed beforehand. Examples of use:

```
positionIterationNumber(A,B,fixedPointsZeros(A,B),25,4,2,(-0.1-0.1*i,1))
                                    [0,25]
positionIterationNumber(A,B,fixedPointsZeros(A,B),25,4,2,(-0.1-0.09*i,1))
                                    [5,23]
```

The implementation of `positionIterationNumber` in *Sage* is:

```
def positionIterationNumber(U, V, fixedpointlist, iter, precisionpoints,
        precisionroots, twotuple):
    result = newstep(U, V, iter, precisionpoints, twotuple)
    if (result[1] != iter):
        return [position(fixedpointlist, precisionroots, result[0]), result[1]]
    else:
        return [0, iter]
```

The parameter `fixedpointlist` is assigned to the list of fixed points associated with the given rational map, which will have been previously created within the function `fixedPointsZeros` and will be different from the identity, and parameters `precisionpoints` and `precisionroots` refer to precisions c_1 and c_2 previously defined, respectively.

The subroutine `position`, which appears in the subprogram above, returns the exact position within the list `fixedpointlist` where the fixed point to which the iteration sequence converges is found; in case that such a sequence does not converge to any fixed point, it returns 0. An implementation of this subroutine in *Sage* is shown in the next lines:

```
def position(fixedPointList, precision, twotuple):
    pos = -1; iter = 0; le = len(fixedPointList)
    while (iter < le) and (pos == -1):
        if (chordalMetric(twotuple, fixedPointList[iter]) < 10.**(-precision)):
            pos = iter
        else:
            iter = iter + 1
    else:
        return pos + 1
```

These subroutines have been implemented in *Mathematica*, too, as follows:

```
PositionIterationNumber[{A_,B_},fixedPointList_,iter_,precpoints_,precroots_]
   [{z_,t_}] := Module[{result},
   result=Newstep[{A,B},iter,precpoints][{{z,t},0}];
   {position[fixedPointList,precroots][result[[1]]],result[[2]]}
   ];

position[fixedPointList_, precision_][{z_, t_}] := Module[{len, PositionAux},
   len = Length[fixedPointList];
   PositionAux[{a_, b_}, i_] := If[i == len + 1, 0,
      If[ChordalMetric[{a, b}, fixedPointList[[i]]] < 10.^(-precision), i,
         PositionAux[{z, t}, i + 1]]];
   PositionAux[{z, t}, 1]
   ];
```

6.2.5 Derivative of a rational function at a fixed point

In order to know if a fixed point is super-attracting, attracting, indifferent or repelling (see subsection 6.1.4), the derivative of the corresponding rational function can be calculated by using the following algorithm (an example of use is given):

```
fixedPointsTangentMapNorm(A,B,(1,0))
((1,0),1.33333333333333)
```

Suppose that $A(z,t)$, $B(z,t)$ are homogeneous polynomials and $[z_0, t_0]$ is a fixed point represented in normalized homogeneous coordinates. The subprogram `fixedPointsTangentMapNorm` returns a list containing two elements: the considered fixed point $[z_0, t_0]$ and the absolute value of the derivative of the rational function at that point.

The implementation of the described algorithm, developed in *Sage*, is given by:

```
def fixedPointsTangentMapNorm(A, B, twotuple):
    a, b = var('a, b')
    nor = homogeneousNormalization(twotuple)
    if (nor[1] == 1):
        return (nor, abs(derivative(A(z=a, t=1) / B(z=a, t=1), a) (a=nor[0])))
    else:
        return (nor, abs(derivative(B(z=1, t=b) / A(z=1, t=b), b) (b=nor[1])))
```

In *Mathematica*, the source code for the same subroutine is the following:

```
FixedPointsTangentMapNorm[{A_, B_}][{z_, t_}] := Module[{result},
   result = TangentMap[{A, B}][{z, t}];
   {result[[1]], Abs[result[[2]]]}
   ];
```

```
TangentMap[{A_, B_}][{z_, t_}] := Module[{nor, imagePoint, a, b},
   nor = HomogeneousNormalization[{z, t}];
   imagePoint = HomogeneousNormalization[{A[{z, t}], B[{z, t}]}];
   If[nor[[2]] == 1, {nor, D[A[{a, 1}]/B[{a, 1}], a] /. {a -> nor[[1]]}},
      {nor, D[B[{1, b}]/A[{1, b}], b] /. {b -> nor[[2]]}}]
   ];
```

6.2.6 Fractal plotting

Next we shall show the algorithms and subroutines which have been developed in *Sage* and *Mathematica* with the aim of representing basins of end points associated with a rational function different from the identity on $\mathbb{C}\cup\{\infty\}$ and even on S^2, as well as the coloring strategies employed to build those subprograms which have been followed in order to plot the corresponding fractals.

Coloring strategies

We shall plot a fractal by using the pair given by the fixed point to which the iteration sequence converges (maybe such a point does not exist) and the number of iterations until convergence, together with one of the following strategies:

(1) Fixed point to which the iteration sequence converges: A color is assigned to each point x in $\mathbb{C}\cup\{\infty\}$ according to the fixed point to which the trajectory $(f^k(x))_{k\in\mathbb{N}}$ converges. That point is drawn with another different color if the trajectory has not converged yet after a determined number of iterations. In this way, basins of attraction can be distinguished by their colors. The algorithm which sets the color of each point in $\mathbb{C}\cup\{\infty\}$ developed in *Sage* is the following:

```
def onlyPosition(U, V, fixedpointlist, iter, precisionpoints,
        precisionroots, twotuple):
    return positionIterationNumber(U, V, fixedpointlist, iter,
        precisionpoints, precisionroots, twotuple)[0]
```

In *Mathematica*, the algorithm is written as follows:

```
OnlyPosition[{A_, B_}, fixedpointlist_, iter_, precpoints_,
   precroots_][{z_, t_}] := Module[{pair},
   pair = PositionIterationNumber[{A, B}, fixedpointlist, iter,
      precpoints, precroots][{z, t}];
   N[pair[[1]]]
   ];
```

(2) Number of iterations until convergence: Instead of assigning a color to each point taking into account the reached fixed point, that color is assigned in accordance with the number of needed iterations until convergence, given a prefixed precision. Eye-catching drawings may be generated with this strategy, too. The subroutine which allows us to find the number of iterations associated with a given point in *Sage* is:

```
def onlyConvergence(U, V, fixedpointlist, iter, precisionpoints,
        precisionroots, twotuple):
    return positionIterationNumber(U, V, fixedpointlist, iter,
        precisionpoints, precisionroots, twotuple)[1]
```

The same subroutine in *Mathematica* has the source code shown below:

```
OnlyConvergence[{A_, B_}, fixedpointlist_, iter_, precpoints_,
   precroots_][{z_, t_}] := Module[{pair},
   pair = PositionIterationNumber[{A, B}, fixedpointlist, iter,
      precpoints, precroots][{z, t}];
   pair[[2]]
   ];
```

(3) Combination of the both previous strategies: In this case, a color is assigned to each basin of attraction, but making it lighter or darker depending on the number of needed iterations until convergence. An implemented subprogram in *Sage* which satisfies this strategy is shown below:

```
def positionPlusConvergence(U, V, fixedpointlist, iter, precisionpoints,
        precisionroots, twotuple):
    pair = positionIterationNumber(U, V, fixedpointlist, iter,
        precisionpoints, precisionroots, twotuple)
    if(pair[0] == 0):
        away = 0
    else:
        away = pair[1]
    return n(pair[0] + away / iter * 3/4)
```

The same subprogram in *Mathematica* is given by the following source code:

```
PositionPlusConvergence[{A_, B_}, fixedpointlist_, iter_, precpoints_,
   precroots_][{z_, t_}] := Module[{pair, away},
   pair = PositionIterationNumber[{A, B}, fixedpointlist, iter,
      precpoints, precroots][{z, t}];
   away = If[pair[[2]] == iter, iter + 1, pair[[2]]];
   N[pair[[1]] + away/(iter + 2)]
   ];
```

Our programs use specific color palettes to plot fractals. If we are working with strategy (1) and the homogeneous polynomials which induce the rational function are of degree n, then a palette with $n+2$ colors will be used: the first color of the palette will be associated with points belonging to a basin of an end point which corresponds to no fixed points (for example, end points associated with 2-cycle points) or points whose induced trajectories have not converged to any fixed point yet, after a prefixed finite number of iterations; the other colors are related to points which are in the basin of attraction of an end point associated with a fixed point. If strategy (2) is considered, a palette with $l+1$ colors will be used (where l is the number of predefined maximum iterations), being the last color reserved for those points whose corresponding trajectory does not converge to any fixed point (because of any of the reasons explained above in this paragraph) and being the rest of colors associated with each one of the possible numbers of iterations $k \in \{0, 1, \ldots, l-1\}$. On the other hand, if strategy (3) is chosen, then a graduated palette constructed from a specified color map will be given. Figure 6.2 shows some examples of such color palettes.

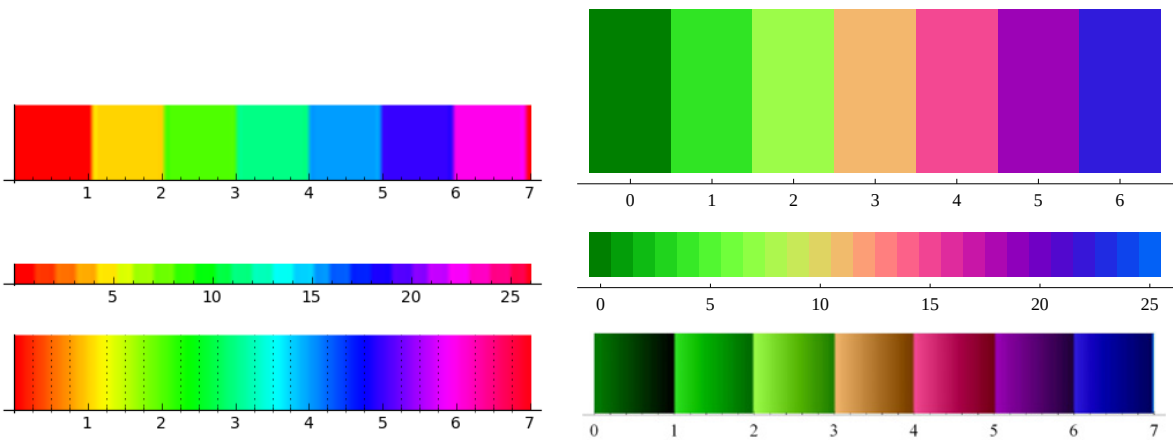


Figure 6.2: Different color palettes associated with the same color map in *Sage* (on the left) and *Mathematica* (on the right). The images correspond respectively to the strategies (1), (2) and (3) referenced.

Algorithms in *Sage*

The subprograms developed in *Sage* for plotting fractals related to basins of end points associated with rational functions different from the identity will be described in the following lines. They are `fractalPlotInsideOutside`, `fractalPlot`, `spherePlot` and `cubicSpherePlot`.

- Function `fractalPlotInsideOutside` returns two disks: one of them represents the intersection between the basins of attraction and the unit disk, and the other shows by means of the inversion method the intersection of those basins with the complementary of the unit disk on $\mathbb{C} \cup \{\infty\}$ –see Figure 6.3.

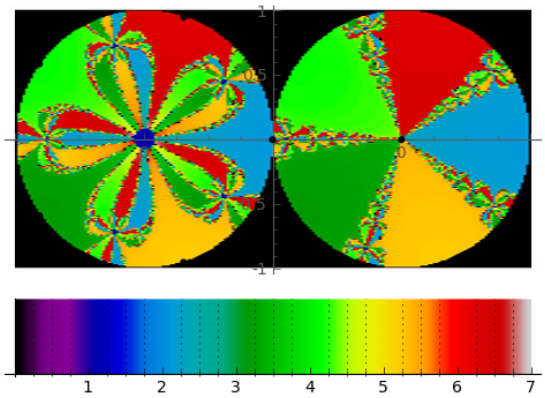


Figure 6.3: Fractal plotted by `fractalPlotInsideOutside` in *Sage*, obtained applying strategy (3).

- With function `fractalPlot`, a colored fractal in a rectangular region is obtained –see Figure 6.4.
- A 3D fractal in the unit sphere is obtained with `spherePlot`, showing all the fixed points in a bigger size than the others. An example of what we can get with this function was shown in Figure 6.1, and another one can be found in Figure 6.5.
- The subprogram `cubicSpherePlot` returns the same as `spherePlot`, but the sphere obtained with the former function is a bit different from the one returned by the latter, since its points are distributed all over its surface in a different way (by projecting the boundary of a subdivided cube onto the unit sphere). A comparison between both functions is established in Figure 6.6. In order to boost the efficiency of the algorithm and reduce its execution time, `cubicSpherePlot` reads the necessary data from a file attachment to plot the corresponding 3D fractal. The required data consists of two lists: those points on the sphere's surface which will be drawn in the fractal plot and their associated points in $\mathbf{P}^1(\mathbb{C})$. If the file attachment does not exist, the subprogram creates it and stores the data in it. The stored data will depend on the number of plot points.

In all cases above, a list containing the fixed points of the rational map, the absolute values of the derivative of the rational function at those fixed points (which allows us to know if every fixed point is super-attracting, attracting, repelling or indifferent) and a color palette associated with the basins of attraction are returned as well.

Moreover, there are other available subprograms which are used for plotting a certain single basin chosen by the user, such as `fractalPlotBasin` and `fractalPlotInsideOutsideBasin`. These algorithms return the same as `fractalPlotInsideOutside` and `fractalPlot`, but showing only one basin, which has to be specified as an input parameter. What is more, the algorithms `spherePlot` and `cubicSpherePlot` include an optional parameter that allows us to represent a single basin on the sphere's surface. Some examples of what we can obtain with these subprograms are shown in Figure 6.7.

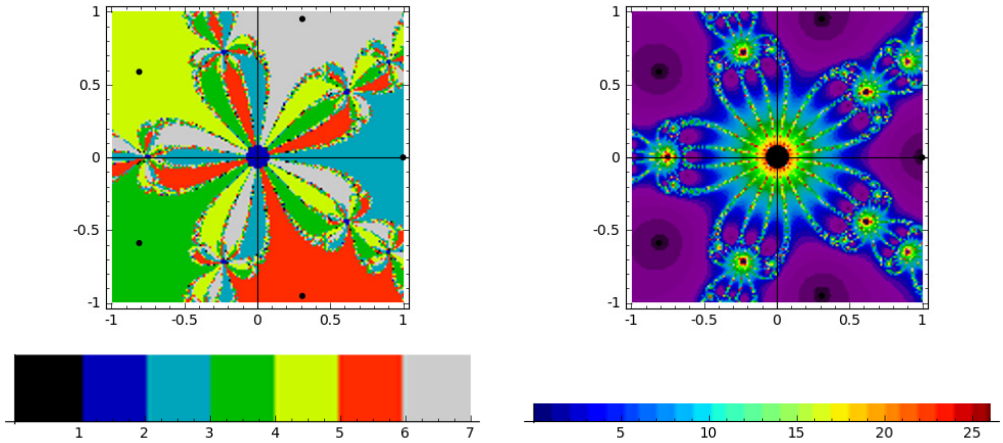


Figure 6.4: Fractals plotted by algorithm `fractalPlot` in *Sage*. The image on the left corresponds to strategy (1) and the image on the right was drawn applying strategy (2).

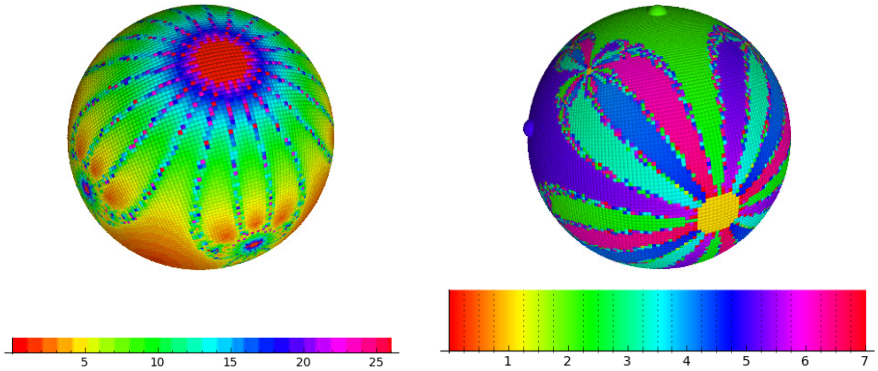


Figure 6.5: Fractals plotted by algorithm `spherePlot` in *Sage* by considering strategies (2) (on the left) and (3) (on the right), respectively. Both fractals were obtained from the same rational function.

`fractalPlotInsideOutside`, `spherePlot` and `cubicSpherePlot` have the numerator and denominator of a rational function as input parameters, whereas `fractalPlot` has in addition as compulsory input parameters the points which delimit the rectangular area where the fractal will be plotted. Furthermore, all the plotting algorithms described also have several other optional input parameters: precision, maximum number of iterations, plot points, coloring strategy, color map (except `spherePlot` and `cubicSpherePlot`) and number of compositions of the given rational function f with itself.

This latter parameter allows us to work with polynomials associated with f^2, f^3,... As an example of what it is useful for, note that the basins of attraction of the fixed points of $f^2 = f \circ f$ correspond, together, to basins of fixed points and 2-cycle points of f; that is, if two fixed points of f^2 form a 2-cycle, its associated basin of attraction is the union of the basins of these two fixed points. This fact can be generalized to any number of iterations.

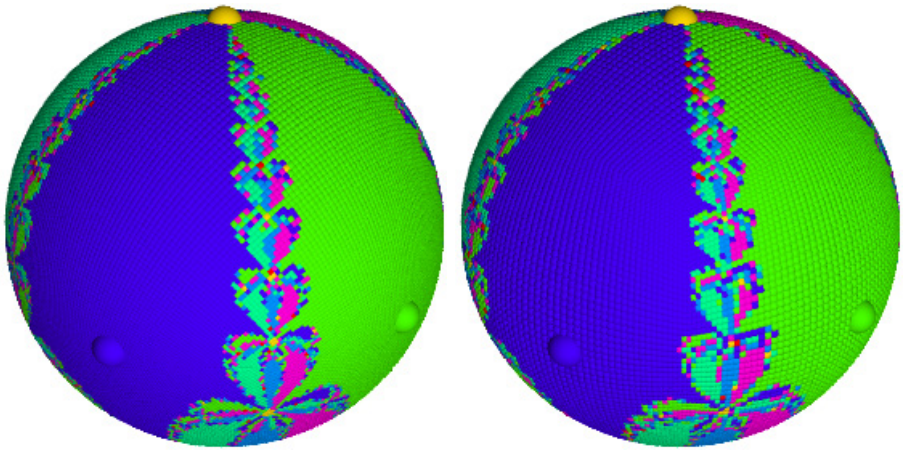


Figure 6.6: Comparison between output plots obtained with `spherePlot` (on the left) and `cubicSpherePlot` (on the right) in *Sage*. Algorithm `cubicSpherePlot` projects points belonging to the faces of a cube onto the unit sphere.

The subprogram responsible for composing n times a given rational function f/g with itself and obtaining a homogeneous rational map from it is:

`composeHomogenize(f,g,n)`.

An implementation in *Sage* of this subprogram is given:

```
def composeHomogenize(f, g, n):
    x = var('x')
    if(g.degree() == 0): comp = compose(f, n); homo = homogenize(comp, g)
    else:
        h = f/g; comp = compose(h, n)
        comp1 = (comp+0*I).simplify_rational('simple')
        S = P.fraction_field()
        comp2 = S(sage_eval(repr(comp1),locals=locals()))
        num = comp2.numerator(); den = comp2.denominator()
        homo = homogenize(num, den)
    A = homo[0]; B = homo[1]
    return [A, B]
```

Observe that the subroutines `compose(h,num)` and `homogenize(f,g)` are used in the subprogram above. The first of these subroutines composes a rational function h with itself a number of times indicated by the input parameter `num` (option `simplify_rational('simple')` is passed to preserve the composed rational function in the form of a quotient of polynomials) whereas the second one homogenizes a given rational function f/g. Their implementations developed in *Sage* are the following:

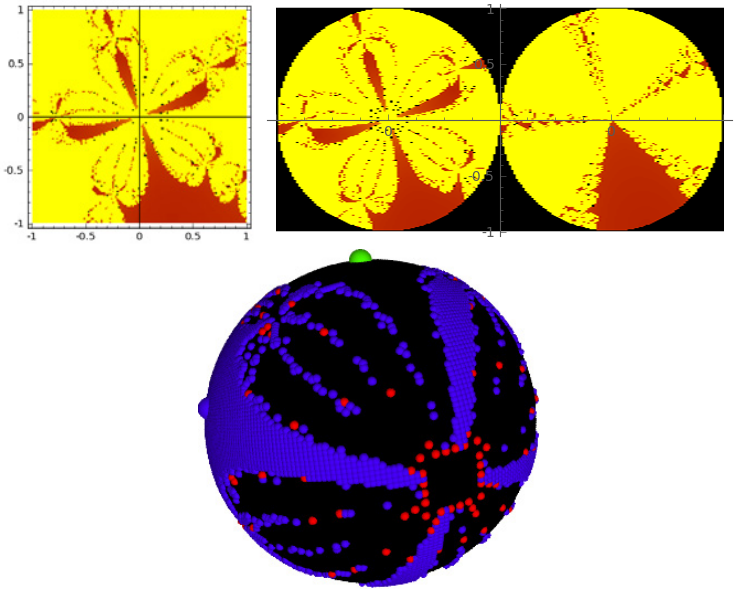


Figure 6.7: Representation of a single basin by means of the subprograms `fractalPlotInsideOutsideBasin` (on the top left), `fractalPlotBasin` (on the top right) and `spherePlot` (on the bottom) in *Sage*.

```
def compose(h, num):
    H = h
    if (num < 1): print("Rational function must be composed once at least")
    for cont in range(num-1): H = h.subs(x = H)
    return H

def homogenize(f, g):
    z = var('z')
    P.<x,t> = PolynomialRing(CC,2)
    fdeg = f.degree(); gdeg = g.degree()
    deg = max(fdeg, gdeg)
    f = f.homogenize('t'); g = g.homogenize('t')
    if (fdeg > gdeg):
        g = g*t**(fdeg - gdeg)
    else:
        f = f*t**(gdeg - fdeg)
    F = f.subs(x = z); G = g.subs(x = z)
    return [F, G]
```

To conclude, the source codes of the plotting algorithms we have just presented are shown below together with subprogram `basinOfFixedPoint`, which given a point on the Riemann sphere returns the fixed point to which it converges, if there is any.

```
from sage.plot.density_plot import DensityPlot
def fractalPlotInsideOutside(M,N,points=150,function=onlyPosition,ncomp=1,colorfunction='spectral',iter=25,
        precpoints=3,precroots=3,reflection=-1):
    ch=composeHomogenize(M,N,ncomp); A=ch[0]; B=ch[1]; p=fixedPointsZeros(A,B)
    if len(p)>0:
        grad=len(p)-1
        if function==onlyConvergence: range=iter+1; else: range=grad+1.99
        x,y=var('x,y'); disfp=[]
        if function!=onlyConvergence:
            colorpoints=[]
            def hidef(x,y):
                if x<=0:
                    if (x+1)**2+y**2>1: colorpoints.append(0)
                    else: colorpoints.append(function(A,B,p,iter,precpoints,precroots,(x+1+y*I,1)))
                else:
                    if (x-1)**2+y**2>1: colorpoints.append(0)
                    else: colorpoints.append(function(A,B,p,iter,precpoints,precroots,
                            (1,x-1+reflection*y*I)))
                return 0
            hideplot=density_plot(hidef,(x,-2,2),(y,-1,1),plot_points=points,aspect_ratio=1)
            if function!=positionPlusConvergence: colorpointsfloor=colorpoints
            else: colorpointsfloor=[floor(k) for k in colorpoints]
            posfp=0
            while posfp<grad+1:
                if posfp+1 not in colorpointsfloor: disfp.insert(0,posfp); p.append(p[posfp])
                posfp=posfp+1
            for fp in disfp:
                cp=0
                while cp<len(colorpoints):
                    if colorpoints[cp]>fp+1: colorpoints[cp]=colorpoints[cp]-1
                    cp=cp+1
                aux=p.pop(fp)
            count=[-1]
            def f(x,y):
                count[0]=count[0]+1
                return colorpoints[count[0]]
        else:
            def f(x,y):
                if x<=0:
                    if (x+1)**2+y**2>1: return 0
                    else: return onlyConvergence(A,B,p,iter,precpoints,precroots,(x+1+y*I,1))
                else:
                    if (x-1)**2+y**2>1: return 0
                    else: return onlyConvergence(A,B,p,iter,precpoints,precroots,(1,x-1+reflection*y*I))
        table=[fixedPointsTangentMapNorm(A,B,t) for t in p]
        from matplotlib import ticker
        L=density_plot(f,(x,-2,2),(y,-1,1),cmap=colorfunction,plot_points=points,aspect_ratio=1)
        xcoords=[]; ycoords=[]
        for fp in p:
            if fp[1]==1: xcoords.append(real(fp[0])-1); ycoords.append(imag(fp[0]))
            else: xcoords.append(real(fp[1])+1); ycoords.append(-imag(fp[1]))
        h2=list_plot(zip(xcoords,ycoords),rgbcolor='black',size=15); h=L+h2
        h[1].set_zorder(10); h.axes_color((0.35,0.35,0.35)); h.tick_label_color((0.35,0.35,0.35))
        if(function!=positionPlusConvergence):
            return h.show(tick_formatter=ticker.IndexFormatter([0,0])),
                density_plot(floor(x),(x,0,range-len(disfp)),(y,0,1),cmap=colorfunction,
                        plot_points=100,aspect_ratio=1).show(ticks=[None,[]]), table
        else:
            maxcp=max(colorpoints); if maxcp==0: maxcp=range
            return h.show(tick_formatter=ticker.IndexFormatter([0,0])),density_plot(x,(x,0,maxcp),(y,0,1),
                    cmap=colorfunction,plot_points=100,aspect_ratio=1).show(ticks=[None,[]],
                        gridlines=["minor",None]),table
```

```
def fractalPlot(M,N,xmin,xmax,ymin,ymax,points=100,function=onlyPosition,ncomp=1,
        colorfunction='spectral',iter=25,precpoints=3,precroots=3):
    ch=composeHomogenize(M,N,ncomp); A=ch[0]; B=ch[1]; p=fixedPointsZeros(A,B)
    if len(p)>0:
        grad=len(p)-1
        if function==onlyConvergence: range=iter+1
        else: range=grad+1.99
        x,y=var('x,y'); substract=0
        if function!=onlyConvergence:
            colorpoints=[]
            def hidef(x,y):
                colorpoints.append(function(A,B,p,iter,precpoints,precroots,(x+y*I,1)))
                return 0
            hideplot=density_plot(hidef,(x,xmin,xmax),(y,ymin,ymax),plot_points=points,
                aspect_ratio=1)
            if function!=positionPlusConvergence: colorpointsfloor=colorpoints
            else: colorpointsfloor=[floor(k) for k in colorpoints]
            disfp=[]; posfp=0
            while posfp<grad+1:
                if posfp+1 not in colorpointsfloor:
                    disfp.insert(0,posfp); p.append(p[posfp])
                posfp=posfp+1
            for fp in disfp:
                cp=0
                while cp<len(colorpoints):
                    if colorpoints[cp]>fp+1: colorpoints[cp]=colorpoints[cp]-1
                    cp=cp+1
                aux=p.pop(fp)
            count=[-1]
            def f(x,y):
                count[0]=count[0]+1; return colorpoints[count[0]]
            substract=len(disfp); aux=0
            if 0 not in colorpointsfloor:
                substract=substract+1; aux=1
        else:
            def f(x,y):
                return onlyConvergence(A,B,p,iter,precpoints,precroots,(x+y*I,1))
        table=[fixedPointsTangentMapNorm(A,B,t) for t in p]
        h1=density_plot(f,(x,xmin,xmax),(y,ymin,ymax),cmap=colorfunction,plot_points=points,
            aspect_ratio=1)
        if function!=positionPlusConvergence:
            colorPalette=density_plot(floor(x),(x,0,range-substract),(y,0,1),
                cmap=colorfunction,plot_points=100,aspect_ratio=1).show(ticks=[None,[]])
        else:
            colorPalette=density_plot(x,(x,0,max(colorpoints)-aux),(y,0,1),cmap=colorfunction,
                plot_points=100,aspect_ratio=1).show(ticks=[None,[]],gridlines=["minor", None])
        xcoords=[]; ycoords=[]
        for fp in p:
            if fp[1]!=0:
                re=real(fp[0]/fp[1]); im=imag(fp[0]/fp[1])
                if (xmin<=re<=xmax) and (ymin<=im<=ymax):
                    xcoords.append(re); ycoords.append(im)
        h2=list_plot(zip(xcoords,ycoords),rgbcolor='black',size=15);h=h1+h2;h[1].set_zorder(10)
        return h.show(frame=true),colorPalette,table
```

```
def spherePlot(M,N,function=onlyPosition,rotzoom=((0,0,0),1),points=100,ncomp=1,view='tachyon',
        basin=0,iter=25,precpoints=3,precroots=3):
    ch=composeHomogenize(M,N,ncomp)
    A=ch[0]; B=ch[1]
    p=fixedPointsZeros(A,B)
    if len(p)>0:
        grad=len(p)-1
        if function==onlyConvergence: ran=iter+1
        else: ran=grad+2
        x,y,twot=var('x,y,twot')
        def f(x,y,twot):
            if sphereBijection(twot)[2]>0:
                return function(A,B,p,iter,precpoints,precroots,(1,x-y*I))
            else: return function(A,B,p,iter,precpoints,precroots,(x+y*I,1))
        def g(x,y):
            if x**2+y**2<=1:
                colorpoint=f(x,y,(1,x-y*I))
                if 0<basin<=len(p):
                    if (floor(colorpoint)==basin) or (colorpoint==0):
                        return point3d(sphereBijection((1,x-y*I)),color=hue(colorpoint/ran))
                else: return point3d(sphereBijection((1,x-y*I)),color=hue(colorpoint/ran))
        def g1(x,y):
            if x**2+y**2<=1:
                colorpoint=f(x,y,(x+y*I,1))
                if 0<basin<=len(p):
                    if (floor(colorpoint)==basin) or (colorpoint==0):
                        return point3d(sphereBijection((x+y*I,1)),
                                color=hue(f(x,y,(x+y*I,1))/ran))
                else: return point3d(sphereBijection((x+y*I,1)),color=hue(colorpoint/ran))
        table=[fixedPointsTangentMapNorm(A,B,t) for t in p];
        l=[(n(k/(points//2))) for k in range(1,(points//2)+1)]
        for k in range((points//2)): l.append(n((-1)*l[k]))
        l.append(0)
        pl=[g(u,v) for u in l for v in l]
        for u in l:
            for v in l:
                pl.append(g1(u,v))
        if function!=onlyConvergence:
            for ind in range(ran-1):
                pl.append(point3d(sphereBijection(p[ind]),size=15,color=hue((ind+1)/ran)))
        if function!=positionPlusConvergence:
            return sum(pl,sphere(color='black')).rotateX(rotzoom[0][0])
                    .rotateY(rotzoom[0][1]).rotateZ(rotzoom[0][2])
                    .show(frame=False,viewer=view,aspect_ratio=[1,1,1],zoom=rotzoom[1]),
                density_plot(floor(x),(x,0,ran),(y,0,1),cmap='hsv',plot_points=100,
                    aspect_ratio=1).show(ticks=[None,[]]),
                table
        else:
            return sum(pl,sphere(color='black')).rotateX(rotzoom[0][0])
                    .rotateY(rotzoom[0][1]).rotateZ(rotzoom[0][2])
                    .show(frame=False,viewer=view,aspect_ratio=[1,1,1],zoom=rotzoom[1]),
                density_plot(x,(x,0,ran),(y,0,1),cmap='hsv',plot_points=100,aspect_ratio=1)
                    .show(ticks=[None,[]],gridlines=["minor", None]),
                table
```

```
import os.path; import csv
def cubicSpherePlot(M,N,function=onlyPosition,rotzoom=((0,0,0),1),numdiv=40,ncomp=1,view='tachyon',basin=0,
        iter=25,precpoints=3,precroots=3):
    ch=composeHomogenize(M,N,ncomp); A=ch[0]; B=ch[1]; p=fixedPointsZeros(A,B)
    if len(p)>0:
        grad=len(p)-1
        if function==onlyConvergence: ran=iter+1; else: ran=grad+2
        cubeSphereList=[]; sphereComplexList=[]; exist=False; sz=6*numdiv**2; pl=[]
        def sphereComplexProjLine(p):
            if p[2]==1:return (1,0);else:return homogeneousNormalization((p[0]/(1-p[2])+I*p[1]/(1-p[2]),1))
        if not os.path.isfile(DATA+'csp'+str(numdiv)+'.csv'):
            cube=[]
            for y1 in range(numdiv):
                for x1 in range(numdiv):
                    cube.append((((2*x1/numdiv-1)+(2*(x1+1)/numdiv-1))/2,
                            ((2*y1/numdiv-1)+(2*(y1+1)/numdiv-1))/2,-1))
                    cube.append((((2*x1/numdiv-1)+(2*(x1+1)/numdiv-1))/2,
                            ((2*y1/numdiv-1)+(2*(y1+1)/numdiv-1))/2,1))
                    cube.append((((2*x1/numdiv-1)+(2*(x1+1)/numdiv-1))/2,
                            -1,((2*y1/numdiv-1)+(2*(y1+1)/numdiv-1))/2))
                    cube.append((((2*x1/numdiv-1)+(2*(x1+1)/numdiv-1))/2,
                            1,((2*y1/numdiv-1)+(2*(y1+1)/numdiv-1))/2))
                    cube.append((-1,((2*x1/numdiv-1)+(2*(x1+1)/numdiv-1))/2,
                            ((2*y1/numdiv-1)+(2*(y1+1)/numdiv-1))/2))
                    cube.append((1,((2*x1/numdiv-1)+(2*(x1+1)/numdiv-1))/2,
                            ((2*y1/numdiv-1)+(2*(y1+1)/numdiv-1))/2))
            def cubeSphere(p):
                root=sqrt(p[0]**2+p[1]**2+p[2]**2); return (n(p[0]/root),n(p[1]/root),n(p[2]/root))
            for k in cube:
                spherePoint=cubeSphere(k); cubeSphereList.append(spherePoint)
                sphereComplexList.append(sphereComplexProjLine(spherePoint))
            with open(DATA+'csp'+str(numdiv)+'.csv','w') as f:
                writefile=csv.writer(f)
                for i in range(sz): writefile.writerow([cubeSphereList[i],sphereComplexList[i]])
        else:
            exist=True; with open(DATA+'csp'+str(numdiv)+'.csv','rU') as f: data=list(csv.reader(f))
            for i in data: cubeSphereList.append(i[0]); sphereComplexList.append(i[1])
        for j in range(sz):
            if exist: colorpoint=function(A,B,p,iter,precpoints,precroots,eval(sphereComplexList[j]))
            else: colorpoint=function(A,B,p,iter,precpoints,precroots,sphereComplexList[j])
            if 0<basin<=len(p):
                if (floor(colorpoint)==basin) or (colorpoint==0):
                    if exist: pl.append(point3d(eval(cubeSphereList[j]),color=hue(colorpoint/ran)))
                    else: pl.append(point3d(cubeSphereList[j],color=hue(colorpoint/ran)))
            else:
                if exist: pl.append(point3d(eval(cubeSphereList[j]),color=hue(colorpoint/ran)))
                else: pl.append(point3d(cubeSphereList[j],color=hue(colorpoint/ran)))
        if function!=onlyConvergence:
            for ind in range(ran-1):
                pl.append(point3d(sphereBijection(p[ind]),size=15,color=hue((ind+1)/ran)))
        table=[fixedPointsTangentMapNorm(A,B,t) for t in p]; x,y=var('x,y')
        if function!=positionPlusConvergence:
            return sum(pl,sphere(color='black')).rotateX(rotzoom[0][0])
                    .rotateY(rotzoom[0][1]).rotateZ(rotzoom[0][2])
                    .show(frame=False,viewer=view,aspect_ratio=[1,1,1],zoom=rotzoom[1]),
            density_plot(floor(x),(x,0,ran),(y,0,1),cmap='hsv',plot_points=100,aspect_ratio=1)
                    .show(ticks=[None,[]]),table
        else: return sum(pl,sphere(color='black')).rotateX(rotzoom[0][0])
                    .rotateY(rotzoom[0][1]).rotateZ(rotzoom[0][2])
                    .show(frame=False,viewer=view,aspect_ratio=[1,1,1],zoom=rotzoom[1]),
                density_plot(x,(x,0,ran),(y,0,1),cmap='hsv',plot_points=100,aspect_ratio=1)
                    .show(ticks=[None,[]],gridlines=["minor", None]), table
```

```
def fractalPlotInsideOutsideBasin(M,N,basin,points=150,ncomp=1,iter=25,
        precpoints=3,precroots=3,reflection=-1):
    ch=composeHomogenize(M,N,ncomp)
    A=ch[0]
    B=ch[1]
    p=fixedPointsZeros(A,B)
    if len(p)>0:
        if 0<basin<=len(p):
            x,y=var('x,y')
            def f(x,y):
                if x<=0:
                    if (x+1)**2+y**2>1: return 0
                    else: ncolor=positionPlusConvergence(A,B,p,iter,precpoints,
                            precroots,(x+1+y*I,1))
                else:
                    if (x-1)**2+y**2>1: return 0
                    else: ncolor=positionPlusConvergence(A,B,p,iter,precpoints,
                            precroots,(1,x-1+reflection*y*I))
                if ncolor==0: return 0
                elif floor(ncolor)!=basin: return 2
                else: return 1.0+ncolor-floor(ncolor)
            table=[fixedPointsTangentMapNorm(A,B,t) for t in p]
            from matplotlib import ticker
            L=density_plot(f,(x,-2,2),(y,-1,1),cmap='gnuplot',
                    plot_points=points,aspect_ratio=1)
            L.axes_color((0.35,0.35,0.35))
            L.tick_label_color((0.35,0.35,0.35))
            return L.show(tick_formatter=ticker.IndexFormatter([0,0])),table,
                    p[basin-1]
        else: print "Integer 'basin' must be greater than 0 and less or equal
                than the number of fixed points."
```

```
def fractalPlotBasin(M,N,xmin,xmax,ymin,ymax,basin,points=100,ncomp=1,iter=25,
        precpoints=3,precroots=3):
    ch=composeHomogenize(M,N,ncomp)
    A=ch[0]
    B=ch[1]
    p=fixedPointsZeros(A,B)
    if len(p)>0:
        if 0<basin<=len(p):
            x,y=var('x,y')
            def f(x,y):
                ncolor=positionPlusConvergence(A,B,p,iter,precpoints,precroots,
                        (x+y*I,1))
                if ncolor==0: return 0
                elif floor(ncolor)!=basin: return 2
                else: return 1.0+ncolor-floor(ncolor)
            table=[fixedPointsTangentMapNorm(A,B,t) for t in p]
            h=density_plot(f,(x,xmin,xmax),(y,ymin,ymax),cmap='gnuplot',
                    plot_points=points,aspect_ratio=1)
            return h.show(frame=true),table,p[basin-1]
        else: print "Integer 'basin' must be greater than 0 and less or equal
                than the number of fixed points."

def basinOfFixedPoint(M,N,point,ncomp=1,iter=25,precpoints=3,precroots=3):
    ch=composeHomogenize(M,N,ncomp)
    A=ch[0]
    B=ch[1]
    p=fixedPointsZeros(A,B)
    if len(p)>0:
        pos=onlyPosition(A,B,p,iter,precpoints,precroots,(point,1))
        if pos!=0:
            return p[pos-1]
        else:
            print "The point does not converge to any fixed point."
```

Algorithms in *Mathematica*

Part of the algorithms written in *Sage* and shown in the previous paragraph have also been developed in *Mathematica*; these algorithms are called `FractalPlotInsideOutside`, `FractalPlot`, `SpherePlot` and `SubdividedSpherePlot`.

- Function `FractalPlotInsideOutside` returns the same as `fractalPlotInsideOutside` in *Sage* –see Figure 6.8.

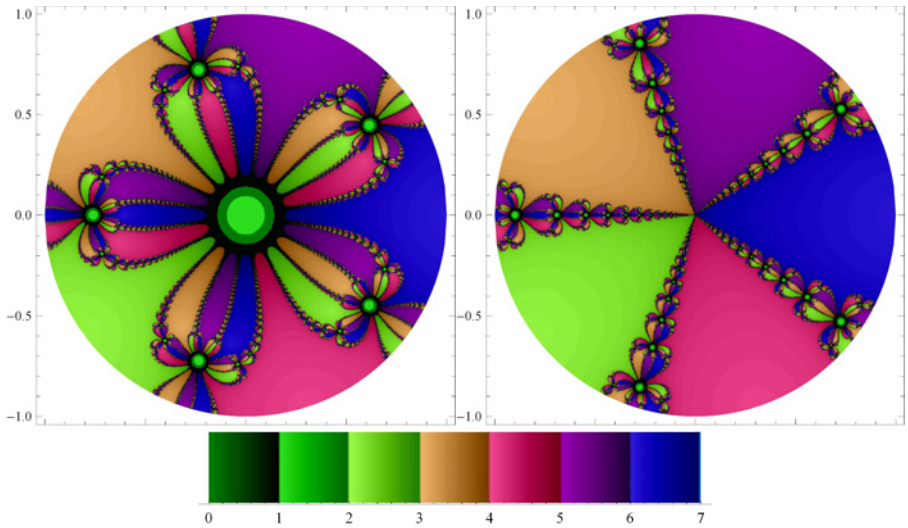


Figure 6.8: Fractal plotted by algorithm `FractalPlotInsideOutside` in *Mathematica*, obtained applying strategy (3).

- With function `FractalPlot`, one obtains the same as with `fractalPlot` in *Sage* –see Figure 6.9.
- A 3D fractal in the unit sphere is obtained with `SpherePlot`, showing the stereographic projection of the intersection between the basins of attraction and the unit complex disk onto the southern hemisphere and, by means of the inversion method, the stereographic projection onto the northern hemisphere of the intersection between those basins and the complementary of the unit complex disk. An example of what we can get with this function is shown in Figure 6.10.
- The subprogram `SubdividedSpherePlot` returns the same as `SpherePlot`, but the sphere obtained with the former function is a bit different from the one returned by the latter, since `SubdividedSpherePlot` actually shows a set of 3D points in the space and these points are distributed in a different way, by projecting the boundary of a subdivided cube onto the unit sphere. To make this subprogram more efficient, `SubdividedSpherePlot` follows a recursive algorithm by which the middle points of the subdivision cells formed in the boundary of the cube are part of the vertices in the next higher subdivision order. An example of this kind of fractal can be found in Figure 6.11.

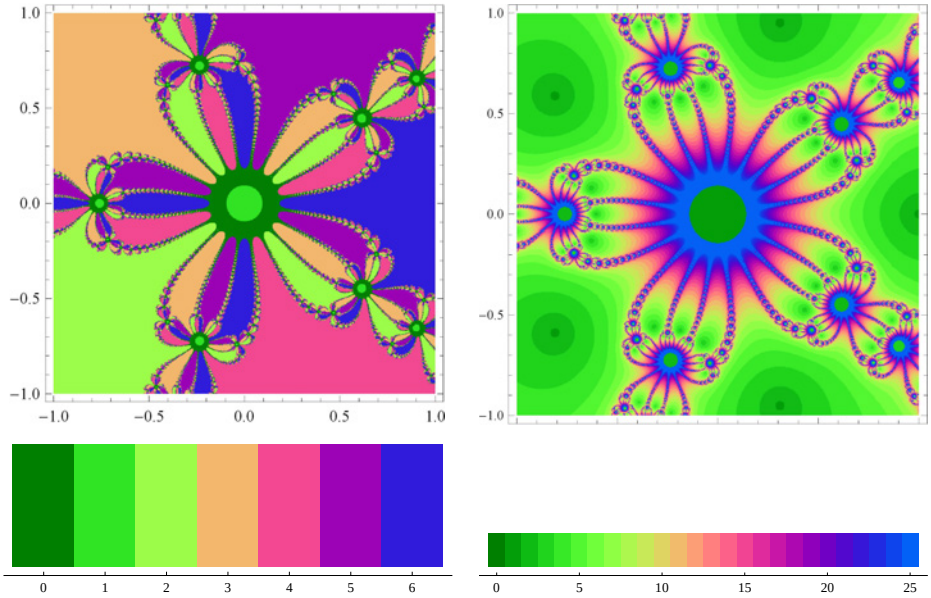


Figure 6.9: Fractals plotted by algorithm `FractalPlot` in *Mathematica*. The image on the left corresponds to strategy (1) and the image on the right was drawn applying strategy (2).

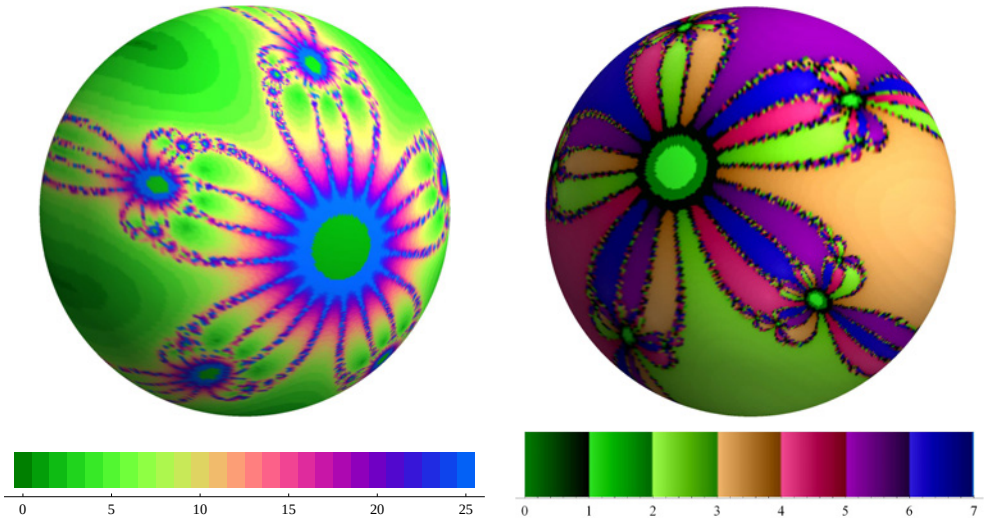


Figure 6.10: Fractals plotted by algorithm `SpherePlot` in *Mathematica* by considering strategies (2) (on the left) and (3) (on the right), respectively. Both fractals were obtained from the same rational function.

Again, a list containing the fixed points of the rational map, the absolute values of the derivative of the rational function at those fixed points and a color palette associated with the basins of attraction are returned in all the cases above.

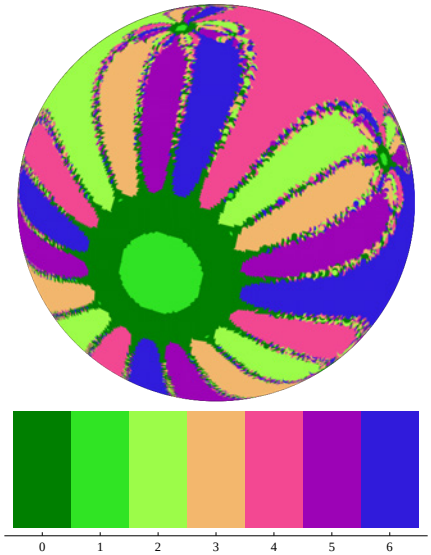


Figure 6.11: Output plot obtained with `SubdividedSpherePlot` in *Mathematica*, following strategy (1).

`FractalPlotInsideOutside`, `SpherePlot` and `SubdividedSpherePlot` have the numerator and denominator of a rational function as input parameters, whereas `FractalPlot` has in addition as compulsory input parameter the rectangular area in the complex plane where the fractal will be plotted. Furthermore, all the plotting algorithms described also have several other optional input parameters: precision, maximum number of iterations, plot points, coloring strategy, color map and number of compositions of the given rational function f with itself to work with polynomials associated with f^n and detect n-cyclic points, for every $n \in \mathbb{N}^*$.

The user could define any color map and introduce it as an input parameter. By way of example, we offer the user certain color maps based on some graphics directives of *Mathematica*, such as `Hue` or `CMYKColor`. These specific color maps, whose codes can be seen in the following lines, take into account the number of colors `degr` appearing in the corresponding palette to be used, and they are called `SpiralCMYKColor`, `CosCMYKColor`, `KnotCMYKColor` and `CosHueColor`. Figure 6.12 shows the palettes associated with them for a number of colors equal to 7.

```
SpiralCMYKColor[degr_][z_] := Module[{intpart, fracpart},
   intpart = IntegerPart[z];
   fracpart = FractionalPart[z];
   CMYKColor[0.5 + 0.5*Cos[-2*Pi*intpart/(degr)],
      0.5 + 0.5*Sin[-2*Pi*intpart/(degr)], 1 - (intpart/(degr)), 0.5*fracpart]
   ];

CosCMYKColor[degr_][z_] := CMYKColor[0.5 + 0.4*Cos[Pi*z/(degr^(1/3))],
   0.5 + 0.5*Cos[Pi*z/(degr^(2/3))], 0.5 + 0.5*Cos[Pi*z/degr],
   (0.7*FractionalPart[z])];
```

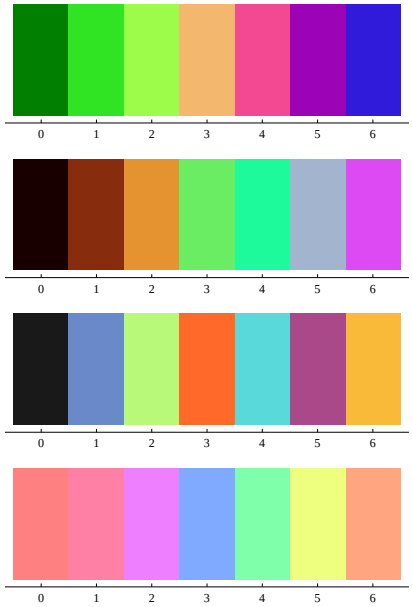


Figure 6.12: Different color palettes associated with the color maps `SpiralCMYKColor`, `CosCMYKColor`, `KnotCMYKColor` and `CosHueColor` in *Mathematica* for `degr`= 7. The first three color maps are based on the graphics directive `CMYKColor` and the last one is based on the graphics directive `Hue`.

```
KnotCMYKColor[degr_][z_] :=
   CMYKColor[(0.9 - FractionalPart[5*IntegerPart[z]/16]),
      (0.9 - FractionalPart[7*IntegerPart[z]/16]),
      (0.9 - FractionalPart[11*IntegerPart[z]/16]), (0.7*FractionalPart[z])];

CosHueColor[degr_][z_] := Hue[0.5 + 0.5*Cos[(Pi*IntegerPart[z]/(degr))], 1,
   0.8 + 0.2*(1 - FractionalPart[z]), 0.5 + 0.5*(FractionalPart[z])];
```

The following lines of code in *Mathematica* are employed within all the plotting algorithms to compose n times a given rational function f/g with itself and obtain a homogeneous rational map from it –parameters `A1` and `B1` represent the numerator and the denominator polynomials of the rational function introduced by the user, respectively, and parameter `ncomp` represents variable n (that is to say, the number of compositions):

```
h[z_] := A1[z]/B1[z];
rfh[{z_, t_}] := Together[Nest[h, z, ncomp] /. z -> z/t];
```

Besides, these algorithms use the formula given in Lemma 6.1.1 to compute the number of fixed points `deg` of the rational function composed with itself n (or `ncomp`) times:

```
deg = (Max[Exponent[A1[z], z], Exponent[B1[z], z]])^ncomp + 1;
```

We end showing the source codes of the plotting algorithms we have just presented.

```
FractalPlotInsideOutside[{A1_, B1_}, points_: 200, function_: OnlyPosition, ncomp_: 1,
   fractalcolorfunction_: SpiralCMYKColor, iter_: 25, precpoints_: 3, precroots_: 3,
   reflection_: -1] := Module[{a, b, c, d, h, z, t, rfh, deg, range, palette, A, B,
   fixedpointlist},
   h[z_] := A1[z]/B1[z];
   rfh[{z_, t_}] := Together[Nest[h, z, ncomp] /. z -> z/t];
   {A[{z_, t_}], B[{z_, t_}]} = {Numerator[rfh[{z, t}]], Denominator[rfh[{z, t}]]};
   deg = (Max[Exponent[A1[z], z], Exponent[B1[z], z]])^ncomp + 1;
   range = {0, If[SymbolName[function] == SymbolName[OnlyConvergence], iter + 1,
      deg + 1]};
   If[SymbolName[function] == SymbolName[PositionPlusConvergence], palette[x_] := x,
      palette[x_] := IntegerPart[x]
      ];
   fixedpointlist = FixedPointsZeros[A, B];
   a = DensityPlot[function[{A, B}, fixedpointlist, iter, precpoints, precroots]
         [{x + I*y, 1}], {x, -1, 1}, {y, -(1 - x^2)^(1/2), (1 - x^2)^(1/2)},
      PlotRange -> range, ColorFunction -> (fractalcolorfunction[range[[2]]][#] &),
      ColorFunctionScaling -> False, PlotPoints -> points, Mesh -> False
      ] // Timing;
   b = DensityPlot[function[{A, B}, fixedpointlist, iter, precpoints, precroots]
         [{1, x + reflection *I*y}], {x, -1, 1}, {y, -(1-x^2)^(1/2), (1-x^2)^(1/2)},
      PlotRange -> range, ColorFunction -> (fractalcolorfunction[range[[2]]][#] &),
      ColorFunctionScaling -> False, PlotPoints -> points, Mesh -> False
      ] // Timing;
   c = If[SymbolName[function] == SymbolName[PositionPlusConvergence],
      DensityPlot[palette[x], {x, 0, range[[2]]}, {y, 0, 1}, PlotRange -> range,
         ColorFunction -> (fractalcolorfunction[range[[2]]][#] &),
         ColorFunctionScaling -> False, PlotPoints -> 200, Mesh -> False,
         AspectRatio -> 1/range[[2]], Frame -> {{False, False}, {True, False}}
         ],
      ArrayPlot[Table[palette[y], {x, 0, 1}, {y, 0, range[[2]] - 1}],
         FrameTicks -> Automatic, DataRange -> {{0, range[[2]] - 1}, {0, 1}},
         ColorFunction -> (fractalcolorfunction[range[[2]]][#] &),
         ColorFunctionScaling -> False, Frame -> {{False, False}, {True, False}}
         ]
      ];
   d = Map[FixedPointsTangentMapNorm[{A, B}][#1] &, fixedpointlist];
   {a, b, c, d}
   ];
```

```
FractalPlot[{A1_, B1_}, rectangle_, points_: 200, function_: OnlyPosition, ncomp_: 1,
   fractalcolorfunction_: SpiralCMYKColor, iter_: 25, precpoints_: 3,
   precroots_: 3] := Module[{a, b, c, h, z, t, rfh, deg, range, palette, A, B,
   fixedpointlist},
   h[z_] := A1[z]/B1[z];
   rfh[{z_, t_}] := Together[Nest[h, z, ncomp] /. z -> z/t];
   {A[{z_, t_}], B[{z_, t_}]} = {Numerator[rfh[{z, t}]], Denominator[rfh[{z, t}]]};
   deg = (Max[Exponent[A1[z], z], Exponent[B1[z], z]])^ncomp + 1;
   range = {0, If[SymbolName[function] == SymbolName[OnlyConvergence], iter + 1,
      deg + 1]};
   If[SymbolName[function] == SymbolName[PositionPlusConvergence], palette[x_] := x,
      palette[x_] := IntegerPart[x]
      ];
   fixedpointlist = FixedPointsZeros[A, B];
   a = DensityPlot[function[{A, B}, fixedpointlist, iter, precpoints, precroots]
         [{x + I*y, 1}], {x, rectangle[[1]][[1]], rectangle[[1]][[2]]},
      {y, rectangle[[2]][[1]], rectangle[[2]][[2]]}, PlotRange -> range,
      ColorFunction -> (fractalcolorfunction[range[[2]]][#] &),
      ColorFunctionScaling -> False, PlotPoints -> points, Mesh -> False
      ];
   b = If[SymbolName[function] == SymbolName[PositionPlusConvergence],
      DensityPlot[palette[x], {x, 0, range[[2]]}, {y, 0, 1}, PlotRange -> range,
         ColorFunction -> (fractalcolorfunction[range[[2]]][#] &),
         ColorFunctionScaling -> False, PlotPoints -> 200, Mesh -> False,
         AspectRatio -> 1/range[[2]], Frame -> {{False, False}, {True, False}}
         ],
      ArrayPlot[Table[palette[y], {x, 0, 1}, {y, 0, range[[2]] - 1}],
         FrameTicks -> Automatic, DataRange -> {{0, range[[2]] - 1}, {0, 1}},
         ColorFunction -> (fractalcolorfunction[range[[2]]][#] &),
         ColorFunctionScaling -> False, Frame -> {{False, False}, {True, False}}
         ]
      ];
   c = Map[FixedPointsTangentMapNorm[{A, B}][#1] &, fixedpointlist];
   {a, b, c}
   ] // Timing;
```

```
SpherePlot[{A1_, B1_}, points_: 200, function_: OnlyPosition, ncomp_: 1,
   fractalcolorfunction_: SpiralCMYKColor, iter_: 25, precpoints_: 3,
   precroots_: 3] := Module[{a, b, c, h, z, t, rfh, deg, range, palette, A, B,
   fixedpointlist},
   h[z_] := A1[z]/B1[z];
   rfh[{z_, t_}] := Together[Nest[h, z, ncomp] /. z -> z/t];
   {A[{z_, t_}], B[{z_, t_}]} = {Numerator[rfh[{z, t}]], Denominator[rfh[{z, t}]]};
   deg = (Max[Exponent[A1[z], z], Exponent[B1[z], z]])^ncomp + 1;
   range = {0, If[SymbolName[function] == SymbolName[OnlyConvergence], iter + 1,
      deg + 1]};
   If[SymbolName[function] == SymbolName[PositionPlusConvergence], palette[x_] := x,
      palette[x_] := IntegerPart[x]
      ];
   fixedpointlist = FixedPointsZeros[A, B];
   a = ParametricPlot3D[{SphereBijection[{u+I*v, 1}], SphereBijection[{1, u-I*v}]},
      {u, -1, 1}, {v, -1, 1}, ColorFunction -> Function[{z, u, v},
         fractalcolorfunction[range[[2]]][function[{A, B}, fixedpointlist, iter,
            precpoints, precroots][If[z <= 0, {u + I*v, 1}, {1, u - I*v}]]
            ]
         ],
      ColorFunctionScaling -> False, PlotPoints -> points, Mesh -> False,
      Axes -> False, Boxed -> False
      ] // Timing;
   b = If[SymbolName[function] == SymbolName[PositionPlusConvergence],
      DensityPlot[palette[x], {x, 0, range[[2]]}, {y, 0, 1}, PlotRange -> range,
         ColorFunction -> (fractalcolorfunction[range[[2]]][#] &),
         ColorFunctionScaling -> False, PlotPoints -> 200, Mesh -> False,
         AspectRatio -> 1/range[[2]], Frame -> {{False, False}, {True, False}}
         ],
      ArrayPlot[Table[palette[y], {x, 0, 1}, {y, 0, range[[2]] - 1}],
         FrameTicks -> Automatic, DataRange -> {{0, range[[2]] - 1}, {0, 1}},
         ColorFunction -> (fractalcolorfunction[range[[2]]][#] &),
         ColorFunctionScaling -> False, Frame -> {{False, False}, {True, False}}
         ]
      ];
   c = Map[FixedPointsTangentMapNorm[{A, B}][#1] &, FixedPointsZeros[A, B]];
   {a, b, c}
   ];
```

```
SubdividedSpherePlot[{A1_, B1_}, subdivision_: 8, function_: OnlyPosition, ncomp_: 1,
   fractalcolorfunction_: SpiralCMYKColor, iter_: 25, precpoints_: 3, precroots_: 3] := Module[{a, b, c, h,
   z, t, rfh, deg, range, palette, A, B, fixedpointlist, points},
   h[z_] := A1[z]/B1[z];
   rfh[{z_, t_}] := Together[Nest[h, z, ncomp] /. z -> z/t];
   {A[{z_, t_}], B[{z_, t_}]} = {Numerator[rfh[{z, t}]], Denominator[rfh[{z, t}]]};
   deg = (Max[Exponent[A1[z], z], Exponent[B1[z], z]])^ncomp + 1;
   range = {0, If[SymbolName[function] == SymbolName[OnlyConvergence], iter + 1, deg + 1]};
   If[SymbolName[function] == SymbolName[PositionPlusConvergence], palette[x_] := x,
      palette[x_] := IntegerPart[x]
      ];
   fixedpointlist = FixedPointsZeros[A, B];
   points = 2^subdivision;
   a = ListPointPlot3D[Centers[subdivision], ColorFunction -> Function[{x, y, z},
         fractalcolorfunction[range[[2]]][function[{A, B}, fixedpointlist, iter, precpoints, precroots]
            [ComplexProjectiveLineBijection[{x, y, z}]]
            ]
         ],
         ColorFunctionScaling -> False, Axes -> False, Boxed -> False, BoxRatios -> {1, 1, 1}] // Timing;
   b = If[SymbolName[function] == SymbolName[PositionPlusConvergence], DensityPlot[palette[x],
         {x, 0, range[[2]]}, {y, 0, 1}, PlotRange -> range,
         ColorFunction -> (fractalcolorfunction[range[[2]]][#] &), ColorFunctionScaling -> False,
         PlotPoints -> 200, Mesh -> False, AspectRatio -> 1/range[[2]],
         Frame -> {{False, False}, {True, False}}
         ],
      ArrayPlot[Table[palette[y], {x, 0, 1}, {y, 0, range[[2]] - 1}], FrameTicks -> Automatic,
         DataRange -> {{0, range[[2]] - 1}, {0, 1}},
         ColorFunction -> (fractalcolorfunction[range[[2]]][#] &), ColorFunctionScaling -> False,
         Frame -> {{False, False}, {True, False}}
         ]
      ];
   c = Map[FixedPointsTangentMapNorm[{A, B}][#1] &, FixedPointsZeros[A, B]];
   {a, b, c}
   ];

ConstructCubeComplex[divisionNumber_] :=
   Module[{first, firstOp, second, secondOp, third, thirdOp, sphereComplex, subdivide},
   first = {{-1., -1., -1.}, {1., -1., -1.}, {1., 1., -1.}, {-1., 1., -1.}};
   firstOp = {{-1., 1., 1.}, {1., 1., 1.}, {1., -1., 1.}, {-1., -1., 1.}};
   second = {{-1., -1., -1.}, {-1., -1., 1.}, {1., -1., 1.}, {1., -1., -1.}};
   secondOp = {{-1., 1., -1.}, {1., 1., -1.}, {1., 1., 1.}, {-1., 1., 1.}};
   third = {{-1., -1., -1.}, {-1., 1., -1.}, {-1., 1., 1.}, {-1., -1., 1.}};
   thirdOp = {{1., -1., 1.}, {1., 1., 1.}, {1., 1., -1.}, {1., -1., -1.}};
   sphereComplex = {Map[Normalize[#] &, first], Map[Normalize[#] &, firstOp],
      Map[Normalize[#] &, second], Map[Normalize[#] &, secondOp],
      Map[Normalize[#] &, third], Map[Normalize[#] &, thirdOp]};
   subdivide[{a_, b_, c_, d_}] := Module[{ab, bc, cd, da, abcd},
      ab = (a + b) / 2; bc = (b + c) / 2; cd = (c + d) / 2; da = (d + a) / 2;
      abcd = (a + b + c + d) / 4;
      {Map[Normalize[#] &, {a, ab, abcd, da}], Map[Normalize[#] &, {ab, b, bc, abcd}],
         Map[Normalize[#] &, {abcd, bc, c, cd}], Map[Normalize[#] &, {da, abcd, cd, d}]}
      ];
   If[divisionNumber == 1, sphereComplex,
      Flatten[Map[subdivide[#1] &, ConstructCubeComplex[divisionNumber - 1]], 1]]
   ];

ComplexProjectiveLineBijection[{a_, b_, c_}] :=
   If[c == 1, {1, 0}, HomogeneousNormalization[{a + b*I, 1 - c}]];

Centers[subdivisionnumber_] := Module[{center},
   center[{a_, b_, c_, d_}] := Normalize[(a + b + c + d)/4];
   Map[center[#] &, ConstructCubeComplex[subdivisionnumber]]];
```

6.3 User manual

6.3.1 For *Sage*

The program is really easy to use. In order to plot the fractal associated with a given rational function, it suffices to specify the polynomials which form its numerator and denominator in variable x and execute one of the following subroutines, depending on what kind of fractal we want to draw: either `fractalPlotInsideOutside`, `fractalPlotInsideOutsideBasin`, `spherePlot`, `cubicSpherePlot`, `fractalPlot` or `fractalPlotBasin`; the same works for subroutine `basinOfFixedPoint` –see paragraph *Algorithms in Sage* in subsection 6.2.6. The rational function must be different from the identity; otherwise, one would obtain an infinite number of fixed points and basins.

For example, the fractal plotted in Figure 6.3, whose associated rational function is $\dfrac{4x^5+1}{5x^4}$, was obtained simply by typing and executing the following sequence in *Sage*:

```
P.<x,t> = PolynomialRing(CC,2)
M=4*x**5+1; N=5*x**4
fractalPlotInsideOutside(M,N,200,positionPlusConvergence,1,'spectral',50,4,2)
```

Next we show the input parameters of the plotting functions that are supported, as well as `basinOfFixedPoint`:

- **fractalPlotInsideOutside**(*M, N, points = 150, function = onlyPosition, ncomp = 1, colorfunction = 'spectral', iter = 25, precpoints = 3, precroots = 3, reflection = −1*)
  - `M,N` are the numerator and the denominator of the given rational function in variable x, respectively.
  - `points` is an integer (default: 150) that represents the number of points to plot in each direction of the grid.
  - `function` indicates the coloring strategy employed to plot the fractal: `onlyPosition` (which is set by default), `onlyConvergence` or `positionPlusConvergence`.
  - `ncomp` is an integer (default: 1) which represents the number of times that the rational function has to be composed with itself.
  - `colorfunction` is a colormap of *Sage* that is used to assign a color to each complex point. The colormap set by default is `spectral`.
  - `iter` is an integer l (default: 25) that represents the maximum number of iterations of the rational function.
  - `precpoints` is an integer c_1 (default: 3) such that, given a rational map f and a point p on the Riemann sphere, if the chordal distance between $f^k(p)$ and $f^{k+1}(p)$ is lower than 10^{-c_1}, $k < l$, then the developed algorithm stops the iteration of f.
  - `precroots` is an integer c_2 (default: 3) which satisfies that, given an iteration sequence $(p, f(p), f^2(p), \ldots, f^k(p))$, if the chordal distance between $f^k(p)$ and a certain fixed point p_0 is lower than 10^{-c_2}, then the developed algorithm considers that this iteration sequence converges to p_0.

- `reflection` is a number either equal to 1 or to -1 (default: -1) that indicates the sign of the reflection of the inversion method.

- **fractalPlot**(*M, N, xmin, xmax, ymin, ymax, points = 100, function = onlyPosition, ncomp = 1, colorfunction = 'spectral', iter = 25, precpoints = 3, precroots = 3*)
  - `M,N` are the numerator and the denominator of the given rational function in variable x, respectively.
  - The tuple given by `xmin,xmax,ymin,ymax` represents the vertices of the rectangle in which the fractal will be plotted.
  - `points` is an integer (default: 100) that represents the number of points to plot in each direction of the grid.
  - `function` indicates the coloring strategy employed to plot the fractal: `onlyPosition` (which is set by default), `onlyConvergence` or `positionPlusConvergence`.
  - `ncomp` is an integer (default: 1) which represents the number of times that the rational function has to be composed with itself.
  - `colorfunction` is a colormap of *Sage* that is used to assign a color to each complex point.
  - `iter` is an integer l (default: 25) that represents the maximum number of iterations of the rational function.
  - `precpoints` is an integer c_1 (default: 3) such that, given a rational map f and a point p on the Riemann sphere, if the chordal distance between $f^k(p)$ and $f^{k+1}(p)$ is lower than 10^{-c_1}, $k < l$, then the developed algorithm stops the iteration of f.
  - `precroots` is an integer c_2 (default: 3) which satisfies that, given an iteration sequence $(p, f(p), f^2(p), \ldots, f^k(p))$, if the chordal distance between $f^k(p)$ and a certain fixed point p_0 is lower than 10^{-c_2}, then the developed algorithm considers that this iteration sequence converges to p_0.

- **spherePlot**(*M, N, function = onlyPosition, rotzoom = ((0,0,0),1), points = 100, ncomp = 1, view = 'tachyon', basin = 0, iter = 25, precpoints = 3, precroots = 3*)
  - `M,N` are the numerator and the denominator of the given rational function in variable x, respectively.
  - `function` indicates the coloring strategy employed to plot the fractal: `onlyPosition` (which is set by default), `onlyConvergence` or `positionPlusConvergence`.
  - `rotzoom` is a tuple given by two elements: a 3-tuple $(xrot, yrot, zrot)$ and a positive real number z. Due to this parameter, the algorithm returns the sphere zoomed z times and self-rotated about the x-axis, y-axis and z-axis by the angles $xrot$, $yrot$ and $zrot$, respectively.
  - `points` is an integer (default: 100) that represents the number of points to plot in each direction of the grid.

- `ncomp` is an integer (default: 1) which represents the number of times that the rational function has to be composed with itself.
- `view` is a string that indicates the viewer which will be used in order to see the plot. Possible values: 'jmol', 'tachyon' (by default), 'java3d' and 'canvas3d'.
- `basin` is an integer (default: 0) which refers to the unique basin of a fixed point that will be plotted, according to the order of the fixed points in the list returned by this program, if its value is different from 0; otherwise, all the basins of attraction will be drawn.
- `iter` is an integer l (default: 25) that represents the maximum number of iterations of the rational function.
- `precpoints` is an integer c_1 (default: 3) such that, given a rational map f and a point p on the Riemann sphere, if the chordal distance between $f^k(p)$ and $f^{k+1}(p)$ is lower than 10^{-c_1}, $k < l$, then the developed algorithm stops the iteration of f.
- `precroots` is an integer c_2 (default: 3) which satisfies that, given an iteration sequence $(p, f(p), f^2(p), \ldots, f^k(p))$, if the chordal distance between $f^k(p)$ and a certain fixed point p_0 is lower than 10^{-c_2}, then the developed algorithm considers that this iteration sequence converges to p_0.

- **cubicSpherePlot**(*M, N, function = onlyPosition, rotzoom = ((0,0,0),1), numdiv = 40, ncomp = 1, view = 'tachyon', basin = 0, iter = 25, precpoints = 3, precroots = 3*)
  - `M,N` are the numerator and the denominator of the given rational function in variable x, respectively.
  - `function` indicates the coloring strategy employed to plot the fractal: `onlyPosition` (which is set by default), `onlyConvergence` or `positionPlusConvergence`.
  - `rotzoom` is a tuple given by two elements: a 3-tuple $(xrot, yrot, zrot)$ and a positive real number z. Due to this parameter, the algorithm returns the sphere zoomed z times and self-rotated about the x-axis, y-axis and z-axis by the angles $xrot$, $yrot$ and $zrot$, respectively.
  - `numdiv` is an integer (default: 40) that indicates the number of subdivisions of the faces of the cube which is projected upon the unit sphere.
  - `ncomp` is an integer (default: 1) which represents the number of times that the rational function has to be composed with itself.
  - `view` is a string that indicates the viewer which will be used in order to see the plot. Possible values: 'jmol', 'tachyon' (by default), 'java3d' and 'canvas3d'.
  - `basin` is an integer (default: 0) which refers to the unique basin of a fixed point that will be plotted, according to the order of the fixed points in the list returned by this program, if its value is different from 0; otherwise, all the basins of attraction will be drawn.
  - `iter` is an integer l (default: 25) that represents the maximum number of iterations of the rational function.

- `precpoints` is an integer c_1 (default: 3) such that, given a rational map f and a point p on the Riemann sphere, if the chordal distance between $f^k(p)$ and $f^{k+1}(p)$ is lower than 10^{-c_1}, $k < l$, then the developed algorithm stops the iteration of f.
- `precroots` is an integer c_2 (default: 3) which satisfies that, given an iteration sequence $(p, f(p), f^2(p), \ldots, f^k(p))$, if the chordal distance between $f^k(p)$ and a certain fixed point p_0 is lower than 10^{-c_2}, then the developed algorithm considers that this iteration sequence converges to p_0.

- **fractalPlotInsideOutsideBasin**(*M, N, basin, points = 150, ncomp = 1, iter = 25, precpoints = 3, precroots = 3, reflection = −1*)
  - `M,N` are the numerator and the denominator of the given rational function in variable x, respectively.
  - `basin` is an integer which refers to the unique basin of a fixed point that will be plotted, according to the order of the fixed points in the list returned by this program.
  - `points` is an integer (default: 150) that represents the number of points to plot in each direction of the grid.
  - `ncomp` is an integer (default: 1) which represents the number of times that the rational function has to be composed with itself.
  - `iter` is an integer l (default: 25) that represents the maximum number of iterations of the rational function.
  - `precpoints` is an integer c_1 (default: 3) such that, given a rational map f and a point p on the Riemann sphere, if the chordal distance between $f^k(p)$ and $f^{k+1}(p)$ is lower than 10^{-c_1}, $k < l$, then the developed algorithm stops the iteration of f.
  - `precroots` is an integer c_2 (default: 3) which satisfies that, given an iteration sequence $(p, f(p), f^2(p), \ldots, f^k(p))$, if the chordal distance between $f^k(p)$ and a certain fixed point p_0 is lower than 10^{-c_2}, then the developed algorithm considers that this iteration sequence converges to p_0.
  - `reflection` is a number either equal to 1 or to -1 (default: -1) that indicates the sign of the reflection of the inversion method.

- **fractalPlotBasin**(*M, N, xmin, xmax, ymin, ymax, basin, points = 100, ncomp = 1, iter = 25, precpoints = 3, precroots = 3*)
  - `M,N` are the numerator and the denominator of the given rational function in variable x, respectively.
  - The tuple given by `xmin,xmax,ymin,ymax` represents the vertices of the rectangle in which the fractal will be plotted.
  - `basin` is an integer which refers to the unique basin of a fixed point that will be plotted, according to the order of the fixed points in the list returned by this program.
  - `points` is an integer (default: 100) that represents the number of points to plot in each direction of the grid.

- `ncomp` is an integer (default: 1) which represents the number of times that the rational function has to be composed with itself.
- `iter` is an integer l (default: 25) that represents the maximum number of iterations of the rational function.
- `precpoints` is an integer c_1 (default: 3) such that, given a rational map f and a point p on the Riemann sphere, if the chordal distance between $f^k(p)$ and $f^{k+1}(p)$ is lower than 10^{-c_1}, $k < l$, then the developed algorithm stops the iteration of f.
- `precroots` is an integer c_2 (default: 3) which satisfies that, given an iteration sequence $(p, f(p), f^2(p), \ldots, f^k(p))$, if the chordal distance between $f^k(p)$ and a certain fixed point p_0 is lower than 10^{-c_2}, then the developed algorithm considers that this iteration sequence converges to p_0.

- **basinOfFixedPoint**(*M, N, point, ncomp = 1, iter = 25, precpoints = 3, precroots = 3*)
  - `M,N` are the numerator and the denominator of the given rational function in variable x, respectively.
  - `point` is the point in $\mathbb{C}$ that will be iterated in order to determine to which fixed point it converges.
  - `ncomp` is an integer (default: 1) which represents the number of times that the rational function has to be composed with itself.
  - `iter` is an integer l (default: 25) that represents the maximum number of iterations of the rational function.
  - `precpoints` is an integer c_1 (default: 3) such that, given a rational map f and a point p on the Riemann sphere, if the chordal distance between $f^k(p)$ and $f^{k+1}(p)$ is lower than 10^{-c_1}, $k < l$, then the developed algorithm stops the iteration of f.
  - `precroots` is an integer c_2 (default: 3) which satisfies that, given an iteration sequence $(p, f(p), f^2(p), \ldots, f^k(p))$, if the chordal distance between $f^k(p)$ and a certain fixed point p_0 is lower than 10^{-c_2}, then the developed algorithm considers that this iteration sequence converges to p_0.

6.3.2 For *Mathematica*

The level of difficulty to use this program in *Mathematica* is quite similar to that of *Sage*: low. In order to plot the fractal related to a given rational map, it is enough to specify the numerator and denominator polynomials in any variable and execute one of the following subroutines, depending on what kind of fractal we want to draw: either `FractalPlotInsideOutside`, `FractalPlot`, `SpherePlot` or `SubdividedSpherePlot` –see paragraph *Algorithms in Mathematica* in subsection 6.2.6. The rational function should be different from the identity.

For instance, the fractal plotted in Figure 6.8, whose associated rational map is again $\dfrac{4x^5+1}{5x^4}$, was obtained just by typing and executing the following sequence in *Mathematica*:

```
A[x_] := 4*x^5 + 1
B[x_] := 5*x^4
FractalPlotInsideOutside[{A, B}, 200, PositionPlusConvergence]
```

Next we show the input parameters of the plotting functions that are supported:

- **FractalPlotInsideOutside**[{*A1_, B1_*}, *points_: 200, function_: OnlyPosition, ncomp_: 1, fractalcolorfunction_: SpiralCMYKColor, iter_: 25, precpoints_: 3, precroots_: 3, reflection_: −1*]
  - `A1,B1` are the numerator and the denominator of the given rational function in any variable, respectively.
  - `points` is an integer (default: 200) that represents the number of points to plot in each direction of the grid.
  - `function` indicates the coloring strategy employed to plot the fractal: `OnlyPosition` (which is set by default), `OnlyConvergence` or `PositionPlusConvergence`.
  - `ncomp` is an integer (default: 1) which represents the number of times that the rational function has to be composed with itself.
  - `fractalcolorfunction` is a color map previously defined by the user in *Mathematica* that is used to assign a color to each complex point. The color map set by default, given by the developer, is `SpiralCMYKColor`. Other color maps that have already been predefined are `CosCMYKColor`, `KnotCMYKColor` and `CosHueColor`.
  - `iter` is an integer l (default: 25) that represents the maximum number of iterations of the rational function.
  - `precpoints` is an integer c_1 (default: 3) such that, given a rational map f and a point p on the Riemann sphere, if the chordal distance between $f^k(p)$ and $f^{k+1}(p)$ is lower than 10^{-c_1}, $k < l$, then the developed algorithm stops the iteration of f.
  - `precroots` is an integer c_2 (default: 3) which satisfies that, given an iteration sequence $(p, f(p), f^2(p), \ldots, f^k(p))$, if the chordal distance between $f^k(p)$ and a certain fixed point p_0 is lower than 10^{-c_2}, then the developed algorithm considers that this iteration sequence converges to p_0.
  - `reflection` is a number either equal to 1 or to -1 (default: -1) that indicates the sign of the reflection of the inversion method.
- **FractalPlot**[{*A1_, B1_*}, *rectangle_, points_: 200, function_: OnlyPosition, ncomp_: 1, fractalcolorfunction_: SpiralCMYKColor, iter_: 25, precpoints_: 3, precroots_: 3*]
  - `A1,B1` are the numerator and the denominator of the given rational function in any variable, respectively.
  - `rectangle` is a 2-tuple $\{\{x, x'\}, \{y, y'\}\}$ that represents the rectangle on the complex plane of vertices $x+iy$, $x'+iy$, $x'+iy'$ and $x+iy'$ in which the fractal will be plotted.
  - `points` is an integer (default: 200) that represents the number of points to plot in each direction of the grid.

- `function` indicates the coloring strategy employed to plot the fractal: `OnlyPosition` (which is set by default), `OnlyConvergence` or `PositionPlusConvergence`.
- `ncomp` is an integer (default: 1) which represents the number of times that the rational function has to be composed with itself.
- `fractalcolorfunction` is a color map previously defined by the user in *Mathematica* that is used to assign a color to each complex point. The color map set by default, given by the developer, is `SpiralCMYKColor`. Other color maps that have already been predefined are `CosCMYKColor`, `KnotCMYKColor` and `CosHueColor`.
- `iter` is an integer l (default: 25) that represents the maximum number of iterations of the rational function.
- `precpoints` is an integer c_1 (default: 3) such that, given a rational map f and a point p on the Riemann sphere, if the chordal distance between $f^k(p)$ and $f^{k+1}(p)$ is lower than 10^{-c_1}, $k < l$, then the developed algorithm stops the iteration of f.
- `precroots` is an integer c_2 (default: 3) which satisfies that, given an iteration sequence $(p, f(p), f^2(p), \ldots, f^k(p))$, if the chordal distance between $f^k(p)$ and a certain fixed point p_0 is lower than 10^{-c_2}, then the developed algorithm considers that this iteration sequence converges to p_0.

- **SpherePlot**[{*A1_, B1_*}, *points_: 200, function_: OnlyPosition, ncomp_: 1, fractalcolorfunction_: SpiralCMYKColor, iter_: 25, precpoints_: 3, precroots_: 3*]
  - `A1,B1` are the numerator and the denominator of the given rational function in any variable, respectively.
  - `points` is an integer (default: 200) that represents the number of points to plot in each direction of the grid.
  - `function` indicates the coloring strategy employed to plot the fractal: `OnlyPosition` (which is set by default), `OnlyConvergence` or `PositionPlusConvergence`.
  - `ncomp` is an integer (default: 1) which represents the number of times that the rational function has to be composed with itself.
  - `fractalcolorfunction` is a color map previously defined by the user in *Mathematica* that is used to assign a color to each complex point. The color map set by default, given by the developer, is `SpiralCMYKColor`. Other color maps that have already been predefined are `CosCMYKColor`, `KnotCMYKColor` and `CosHueColor`.
  - `iter` is an integer l (default: 25) that represents the maximum number of iterations of the rational function.
  - `precpoints` is an integer c_1 (default: 3) such that, given a rational map f and a point p on the Riemann sphere, if the chordal distance between $f^k(p)$ and $f^{k+1}(p)$ is lower than 10^{-c_1}, $k < l$, then the developed algorithm stops the iteration of f.
  - `precroots` is an integer c_2 (default: 3) which satisfies that, given an iteration sequence $(p, f(p), f^2(p), \ldots, f^k(p))$, if the chordal distance between $f^k(p)$ and a certain fixed point p_0 is lower than 10^{-c_2}, then the developed algorithm considers that this iteration sequence converges to p_0.

- **SubdividedSpherePlot**[{*A1_, B1_*}, *subdivision_: 8, function_: OnlyPosition, ncomp_: 1, fractalcolorfunction_: SpiralCMYKColor, iter_: 25, precpoints_: 3, precroots_: 3*]
  - `A1,B1` are the numerator and the denominator of the given rational function in any variable, respectively.
  - `subdivision` is an integer (default: 8) that indicates the number of times that the projection of the original cube's faces (spherical quadrilaterals) on the unit sphere are subdivided.
  - `function` indicates the coloring strategy employed to plot the fractal: `OnlyPosition` (which is set by default), `OnlyConvergence` or `PositionPlusConvergence`.
  - `ncomp` is an integer (default: 1) which represents the number of times that the rational function has to be composed with itself.
  - `fractalcolorfunction` is a color map previously defined by the user in *Mathematica* that is used to assign a color to each complex point. The color map set by default, given by the developer, is `SpiralCMYKColor`. Other color maps that have already been predefined are `CosCMYKColor`, `KnotCMYKColor` and `CosHueColor`.
  - `iter` is an integer l (default: 25) that represents the maximum number of iterations of the rational function.
  - `precpoints` is an integer c_1 (default: 3) such that, given a rational map f and a point p on the Riemann sphere, if the chordal distance between $f^k(p)$ and $f^{k+1}(p)$ is lower than 10^{-c_1}, $k < l$, then the developed algorithm stops the iteration of f.
  - `precroots` is an integer c_2 (default: 3) which satisfies that, given an iteration sequence $(p, f(p), f^2(p), \ldots, f^k(p))$, if the chordal distance between $f^k(p)$ and a certain fixed point p_0 is lower than 10^{-c_2}, then the developed algorithm considers that this iteration sequence converges to p_0.

Chapter 7

Computing areas on the Riemann sphere of basins of attraction

The single cellular procedure of measure construction that we considered in section 5.2 has got many interesting advantages, since the combinatorial nature of this method permits us, in some cases, to develop computational algorithms in order to obtain the measure of some subsets which are defined by iteration of finite construction methods. Incidentally, in this chapter, we develop and implement an algorithm for computing, up to a given precision, the measure of basins of attraction of rational maps different from the identity defined on the Riemann sphere. This algorithm is based on the subdivisions of a cubic decomposition of a sphere and it was made by using different computational environments.

As an application, we study the basins of attraction of the fixed points of the rational functions obtained when Newton's method is applied to a polynomial with two roots of multiplicities m and n. We focus our attention on the analysis of the influence of the multiplicities m and n on the measure of both basins of attraction. As a consequence of the numerical results given in this chapter, we conclude that, if $m > n$, then the probability of a point in the Riemann Sphere belonging to the basin of the root with multiplicity m is bigger than the probability associated with the root with multiplicity n. In addition, if n is fixed and m tends to infinity, then the probability of reaching the root with multiplicity n tends to zero.

Our study on the influence of the multiplicity of the roots on the area of the basins of attraction will be divided into three phases:

a) We will give two different computational algorithms to calculate the area of a basin of attraction of a fixed point of a rational function different from the identity on the Riemann sphere.

b) We will apply these algorithms and their implementations to compute the area of the basins of attraction of the rational functions obtained when Newton's method is applied to a non-constant polynomial.

c) In the particular case of a polynomial of the form $p(z) = (z-1)^m(z+1)^n$, we will quantify the influence of the multiplicities m, n on the measure of the corresponding basins.

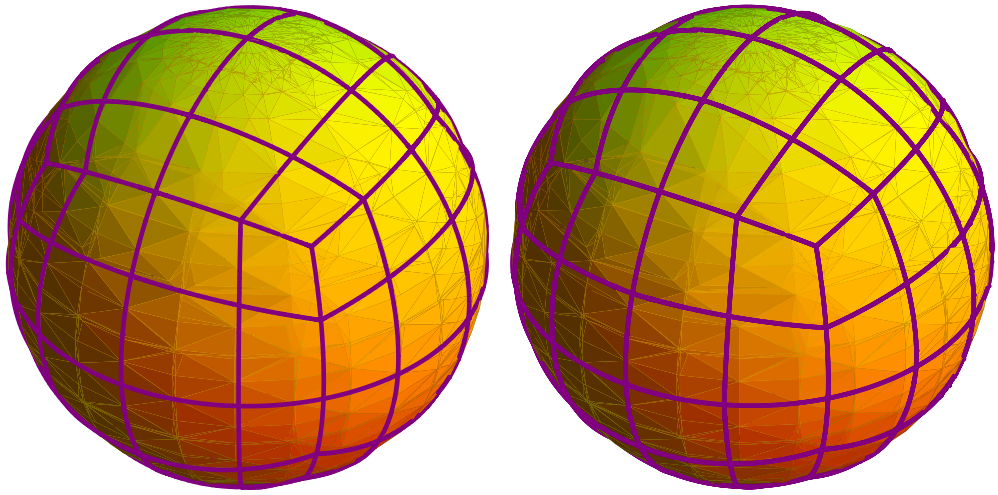
All above considered, the chapter will be organized as follows. The geometrical notions that were given along section 5.2 allowed us to introduce a usual measure on the 2-sphere in a computational way by using consecutive subdivisions of a cubic structure on the 2-sphere and the Girard's theorem, which gives the area of a spherical triangle in terms of the measure of its angles. In this process, the unit sphere was inscribed in the boundary of the 3-cube $[-1, 1]^3$ and the central projection gave a homeomorphism from the boundary of the 3-cube to the unit sphere. Remember that, in the referred section, the subdivision method that was considered consisted in the iterated subdivision of the projection of the boundary of the 3-cube; in section 7.1, a new canonical way of subdivision will be used: the projection of the iterated subdivisions of the cube. Both procedures induce the same standard measure on the unit 2-sphere; however, the former will give a more homogeneous distribution of the areas and, as a consequence, a faster algorithm, since fewer subdivisions will be needed to achieve a given precision.

In section 7.2, we will give a description of the algorithms that we use to compute the measure of the area of the basins of attraction of a rational map different from the identity on the Riemann sphere. In addition, some implementations of these algorithms using the computer programs *Mathematica* and *Sage* will be described in section 7.2.3. It is interesting to remark that the order of convergence plays an important role in the design of our algorithm, as we will show in subsection 7.2.1. To avoid some problems derived from the slow convergence in the case of linear order of convergence, we have to be careful when choosing precisions c_1 and c_2 –see subsections 6.2.3 and 6.2.4 for more details about such precisions. In the case of Newton's method, both precisions can be related with the formula $c_2 = c_1 - \log_{10}(m-1)$, where m is the maximum of the multiplicities of the roots.

In section 7.3, we will study the effects of the multiplicity of the roots on the basins of attraction of Newton's method applied to polynomials with two roots. In particular, we shall look for an answer to the following question: if we choose randomly a point in the Riemann sphere and we iterate the rational function given by Newton's method, what is the probability of the generated sequence converging to a given root? So, we can associate to each root a probability with the criterium above. In the case of polynomials with two simple roots (Cayley's problem), both roots have the same probability. However, the inspection of Figure 7.2 reveals that the probability related to the root with the biggest multiplicity is bigger than the other one. Although they could be many other ways to associate a probability to a root, in this work we have used the area (divided by the area of the sphere) of the corresponding basin of attraction in the Riemann sphere as the probability of the root. Section 7.3 also contains some numerical experiments with values of the probabilities associated with the fixed points 1 and -1 of the rational function $B_{m,n}$ introduced in equation (7.3) as a function of m and n, or more generally, probabilities of polynomials with two unitary opposite roots $\alpha_1 = -\alpha_2$ ($|\alpha_1| = |\alpha_2| = 1$).

7.1 Two different iterated subdivisions of the sphere

In this section, we describe another computational method to give a measure function on a sphere based once again on the excess of a spherical quadrilateral and a system of consecutive subdivisions of the sphere.



(a) The iterated subdivision $\tilde{\Gamma}^1_*(S^2)$. (b) The iterated subdivision $\Gamma^2_*(S^2)$.

Figure 7.1: Processes of consecutive cubic subdivisions of the sphere.

First of all, recall from section 5.2 that the unit 2-sphere can be inscribed in the boundary of the cube $[-1,1]^3$ and, in that case, observe that the Euclidean norm induces a canonical bijective projection

$$\eta\colon \partial([-1,1]^3) \to S^2, \quad \eta(x) = \frac{x}{\|x\|}.$$

Apart from this, remember the iterated subdivision $\Gamma^*_*([-1,1]^n)$ which was constructed in that section; setting $n = 3$ and taking the boundary of the resulting cube, we have a sequence of consecutive subdivisions for the boundary of the 3-cube that permits us to consider it as a measure space. These subdivision structures can be transformed by η into new subdivisions of the 2-sphere:

$$\tilde{\Gamma}^r_*(S^2) = \eta(\Gamma^r_*(\partial([-1,1]^3))).$$

For instance, for $r = 1$, one has the cubic structure of Figure 7.1(a). Besides, Figure 7.1(b) shows the iterated subdivision $\Gamma^2_*(S^2)$ on the 2-sphere considered in section 5.2.

Hence, we have described two different methods to obtain consecutive cubic subdivisions of the 2-sphere: $\tilde{\Gamma}^*_*(S^2) = \eta(\Gamma^*_*(\partial([-1,1]^3)))$ and $\Gamma^*_*(S^2)$. It would be nice if both iterative processes gave rise to the same measure on the 2-sphere. Remember, from the example about measures on S^2 we saw in subsection 5.2.2, that the 2-cellular-extension measure $\bar{\mu}\colon \mathcal{E}_{\mu^*_*}(S^2) \to [0,\infty]$ induced by the iterated subdivision $\Gamma^*_*(S^2)$ can be also considered as the solid angle measure which generalizes to dimension 2 the usual circle angle measure; therefore, any spherical quadrilateral, just like those formed by means of the iterated subdivision $\tilde{\Gamma}^*_*(S^2)$, is measurable with respect to the measure $\bar{\mu}$ –take into account that $\bar{\mu}$ is a Borel measure and any spherical quadrilateral is a closed subset of S^2. Given $\tilde{\gamma}^r \in \tilde{\Gamma}^r_*(S^2)$, consider the following subdivision operator:

$$\mathrm{Sd}_S(\tilde{\gamma}^r) = \{\eta(\gamma) \mid \gamma \in \mathrm{Sd}(\eta^{-1}(\tilde{\gamma}^r))\}.$$

Define the 2-cellular measure $\tilde{\mu}_*^* = \{\tilde{\mu}_*^r\}_{r\in\mathbb{N}}$ so that $\tilde{\mu}_2^r\colon \tilde{\Gamma}_2^r \to [0,\infty]$ is given by $\tilde{\mu}_2^r(\tilde{\gamma}^r) = \bar{\mu}(\mathring{\tilde{\gamma}}^r)$. Notice that $\tilde{\mu}_*^*$ has the subdivision invariance property shown in Definition 5.2.1:

$$\begin{aligned}\tilde{\mu}_2^{r+1}(\mathrm{Sd}_S(\tilde{\gamma}^r)) &= \sum_{\tilde{\beta}^{r+1}\in\mathrm{Sd}_S(\tilde{\gamma}^r)} \tilde{\mu}_2^{r+1}(\tilde{\beta}^{r+1}) = \sum_{\tilde{\beta}^{r+1}\in\mathrm{Sd}_S(\tilde{\gamma}^r)} \bar{\mu}\left(\mathring{\tilde{\beta}}^{r+1}\right) = \sum_{\gamma^{r+1}\in\mathrm{Sd}(\eta^{-1}(\tilde{\gamma}^r))} \bar{\mu}(\eta(\mathring{\gamma}^{r+1}))\\ &= \bar{\mu}(\eta(\eta^{-1}(\mathring{\tilde{\gamma}}^r))) = \bar{\mu}((\eta\circ\eta^{-1})(\mathring{\tilde{\gamma}}^r)) = \bar{\mu}(\mathring{\tilde{\gamma}}^r) = \tilde{\mu}_2^r(\tilde{\gamma}^r).\end{aligned}$$

Since $\tilde{\mu}_2^*$ has the subdivision invariance property, one has that $\tilde{\mu}_*^*$ is a 2-cellular measure for this iterated subdivision $\tilde{\Gamma}_*^*(S^2)$ on the 2-sphere. Then, applying Theorem 5.2.1 and Theorem 5.2.2, one has that the Borel σ-algebra $\sigma(\mathbf{t}_{S^2})$ is contained in the cellular-extension σ-algebra $\mathcal{E}_{\tilde{\mu}_*^*}(S^2)$ and we can consider the induced 2-cellular-extension measure $\bar{\tilde{\mu}}\colon \mathcal{E}_{\tilde{\mu}_*^*}(S^2) \to [0,\infty]$ of $\tilde{\mu}_*^*$, which is the usual measure of solid angles. As a result, since $\bar{\mu}\colon \mathcal{E}_{\mu_*^*}(S^2) \to [0,\infty]$ also agrees with the usual measure of solid angles, we have that $\bar{\mu} = \bar{\tilde{\mu}}$ and the iterated subdivisions $\Gamma_*^*(S^2)$ and $\tilde{\Gamma}_*^*(S^2)$ induce the same measure.

The probability measure of a measurable subset A belonging to the unit sphere is given by $P(A) = \dfrac{\bar{\mu}(A)}{4\pi} = \dfrac{\bar{\tilde{\mu}}(A)}{4\pi}$. In order to compute the measure of a basin, or probability associated with it, the author has developed algorithms implemented in *Sage* to construct the subdivisions $\tilde{\Gamma}_*^*(S^2)$ and other algorithms implemented in *Mathematica* to construct $\Gamma_*^*(S^2)$. This will allow us to compare different algorithms and different computational environments.

Remark 7.1.1. *A measure on the* 2*-sphere can be also introduced using a volume form, and the measure of many regions whose boundary is given by a smooth curve can be computed in many cases using suitable coordinates and by usual integration formulas. However, many problems appear when one wants to compute the area of a basin of attraction whose frontier is a Julia set with a fractal dimension between* 1 *and* 2*, and its length (if the Julia set can be considered as a measurable set) could be infinite.*

In the following sections, we will describe some computational algorithms to specifically give the area (or probability) of the basin of a fixed point of a rational function different from the identity on the Riemann sphere. As an application, we will compute, up to a given precision, the area of the basins of attraction of the rational function obtained by applying Newton's method to a certain non-constant polynomial, and we will also quantify the influence of the multiplicity for the case of a polynomial with two roots.

7.2 Multiplicities, algorithms and implementations

Previously, we have introduced some mathematical techniques and developed basic theoretical aspects necessary to build computer programs with the ability of representing basins of attraction of end points associated with a determined rational function different from the identity. We shall show in the next lines the algorithms which have been developed to study the basins induced by a rational function $f \neq \mathrm{Id}$ on the Riemann 2-sphere.

7.2.1 Adjusting the precisions c_1 and c_2 when considering Newton's method

When we apply Newton's method to a polynomial with non-simple roots, we have that the order of convergence is linear and, for this very reason, the considerations described below are convenient in order to obtain right plots and measure of basins. In fact, since the rational function $h(z) = \frac{4z^5+1}{5z^4}$ defined in subsection 6.1.5 is induced by applying Newton-Raphson's algorithm to the complex polynomial $p(z) = z^5 - 1$ (see (7.1)), all the illustrations of basins of end points of h that appear along chapter 6 –most of which correspond to the basins of attraction of the roots of p– have been drawn taking into account the following ideas.

Let $X = \mathbb{C} \cup \{\infty\}$ and $f\colon X \to X$ be a rational function on the Riemann sphere and suppose that we are working with a metric discrete semi-flow (X, d, φ), where $\varphi\colon \mathbb{N} \times X \to X$ is given by $\varphi(n, x) = f^n(x)$ and $d = d^E$ is the chordal distance recalled in subsection 6.1.3. If for a given point $x \in X$ one has that $(f^k(x))_{k\in\mathbb{N}}$ converges to a given fixed point y of f and

$$\lim_{k\to\infty} \left(\frac{d(f^k(x), y)}{d(f^{k-1}(x), y)} \right) = \lambda < 1,$$

then one has linear convergence. In this case, for large k, we have that

$$d(f^k(x), y) \sim \lambda d(f^{k-1}(x), y).$$

If we suppose that $(d(f^k(x), y))_{k\in\mathbb{N}}$ is a decreasing sequence, then one has that, for a large k,

$$\begin{aligned} d(f^k(x), f^{k-1}(x)) &\geq d(f^{k-1}(x), y) - d(f^k(x), y) \sim \frac{1}{\lambda} d(f^k(x), y) - d(f^k(x), y) \\ &= \left(\frac{1}{\lambda} - 1 \right) d(f^k(x), y); \end{aligned}$$

or equivalently,

$$d(f^k(x), y) \lesssim \frac{\lambda}{1-\lambda} d(f^k(x), f^{k-1}(x)).$$

For Newton's method, if we consider the rational map $f = N_p$, where p is a non-constant polynomial, one of the difficulties that arise when calculating numerically a multiple root of multiplicity m is that the convergence is slower than for the case of a simple root. For a root of multiplicity m, one has linear convergence with $\lambda = 1 - \dfrac{1}{m}$; hence, $\dfrac{\lambda}{1-\lambda} = m - 1$ and, therefore,

$$d(f^k(x), y) \lesssim (m-1) d(f^k(x), f^{k-1}(x)).$$

Then, for a large k, if $d(f^k(x), y) \sim (m-1) d(f^k(x), f^{k-1}(x))$, one could have that

$$d(f^{k-1}(x), y) \sim (m-1) d(f^k(x), f^{k-1}(x)).$$

This implies that, when the iterative process stops because

$$d(f^k(x), f^{k-1}(x)) \sim 10^{-c},$$

then one can have

$$d(f^k(x), y) \sim (m-1) 10^{-c} > 10^{-c} \quad (m > 2)$$

and the algorithm will give a wrong result, which is that x is not actually in the basin of y.

This problem can be avoided working with two precision numbers: 10^{-c_1} is the precision we use in order to stop the iteration process and a new precision 10^{-c_2} can be introduced to determine if the final point of the iteration is close to a given root (fixed point) of multiplicity $m > 2$. In order to compare both precision numbers, as a consequence of the effect of the multiplicity m, one has $10^{-c_2} \sim (m-1)10^{-c_1}$; thus, it suffices to take

$$c_2 = c_1 - \log_{10}(m-1).$$

In general, for a polynomial with at least one non-simple root and multiplicities $m_1, \ldots, m_s$, one can take

$$c_2 = c_1 - \log_{10}(\max\{m_1 - 1, \ldots, m_s - 1\})$$

to compute the basins of the corresponding rational function and their measure.

These precision numbers were kept in mind when designing the algorithms described along section 6.2, as reflected in the manuals shown in section 6.3: the parameter `precpoints` refers to the variable c_1 and the parameter `precroots` alludes to the variable c_2. Obviously, they have also been considered when developing algorithms for computing the area of basins of end points, as we will see in next subsections.

7.2.2 Algorithms to compute the area of basins of end points

Projection of subdivisions: $\tilde{\Gamma}^r_*(S^2) = \eta(\Gamma^r_*(\partial([-1,1]^3)))$

Let $X = \mathbb{C} \cup \{\infty\} \cong S^2$. Suppose that we are working with a metric exterior discrete semi-flow $(X, d^E, \varepsilon(X, \mathrm{Fix}(X)))$, where d^E is the chordal metric and $\mathrm{Fix}(X)$ is a finite subset, and consider the measure exterior discrete semi-flow $(X, \varepsilon(X, \mathrm{Fix}(X)), \mathbf{t}_{S^2}, \bar{\tilde{\mu}}|_{\sigma(\mathbf{t}_{S^2})})$, which is well-defined since $\bar{\tilde{\mu}}\colon \mathcal{E}_{\tilde{\mu}^*_*}(S^2) \to [0,\infty]$ is the 2-cellular-extension measure constructed in section 7.1 and the Borel σ-algebra $\sigma(\mathbf{t}_{S^2})$ is contained in the cellular-extension σ-algebra $\mathcal{E}_{\tilde{\mu}^*_*}(S^2)$.

For each 2-cube γ of $\Gamma^r_*(\partial([-1,1]^3))$, we compute its barycenter $\hat{\gamma}$ and the point $\eta(\hat{\gamma})$. After that, we use the algorithms given in subsections 6.2.3 and 6.2.4 to check whether $\eta(\hat{\gamma})$ is in the basin of one of the fixed points $x_1, \ldots, x_{n+1}$ (we remark that we have a prefixed maximal number of iterations l, as well as precisions 10^{-c_1} and 10^{-c_2}). We have two possibilities: if $\eta(\hat{\gamma})$ is in the basin of a fixed point $x_i \in \mathrm{Fix}(X)$, then the 2-cube γ is included in a list of 2-cubes $L_i(r)$ associated with the fixed point x_i; otherwise, the 2-cube γ is not included in any of the lists $L_1(r), \ldots, L_{n+1}(r)$.

Remember that the measure $\bar{\tilde{\mu}}$ satisfies $\bar{\tilde{\mu}}(\mathring{\gamma}) = \tilde{\mu}^*_*(\gamma)$, $\forall \gamma \in \tilde{\Gamma}^*_*(S^2)$, where $\bar{\tilde{\mu}}$ agrees with the usual measure of solid angles and $\tilde{\mu}^*_* = \{\tilde{\mu}^r_*\}_{r\in\mathbb{N}}$, $\tilde{\mu}^r_2\colon \tilde{\Gamma}^r_2 \to [0,\infty]$, is the 2-cellular measure defined in section 7.1. Thus, for each list $L_i(r)$, $i = 1, \ldots, n+1$, we can compute the finite sum of the areas of its spherical 2-cubes (quadrilaterals) $\bar{\tilde{\mu}}(L_i(r)) = \sum_{\gamma\in L_i(r)} \tilde{\mu}^r_2(\eta(\gamma))$, where the area $\tilde{\mu}^r_2(\eta(\gamma))$, $\gamma \in L_i(r)$, is given by computing the excess of the spherical quadrilateral $\eta(\gamma)$; see Remark 5.2.2. In order to calculate the probability $P_i(r)$ associated with the list of 2-cubes $L_i(r)$, it suffices to divide by the area 4π of the unit 2-sphere: $P_i(r) = \frac{\bar{\tilde{\mu}}(L_i(r))}{4\pi}$.

Should you want to obtain the area module a precision 10^{-c}, then the process above must be repeated for higher values of r until you find one such that $|\bar{\tilde{\mu}}(L_i(r)) - \bar{\tilde{\mu}}(L_i(r+1))| < 10^{-c}$. Now, take $\bar{\tilde{\mu}}(L_i(r+1))$ as the area of the basin of x_i.

Note the following fact: our algorithm states that, if the point $\eta(\hat{\gamma})$ is in the basin of a fixed point, then every point of $\eta(\gamma)$ is wrongly expected to be contained in that basin. Since we can work with very fine subdivisions, this is not actually a serious problem; however, this step of the algorithm could be improved by checking whether more points of $\eta(\gamma)$ are in the same basin (and finding which points they are, if there is any) or by using more information, namely the value of the derivative of an iterated function at $\eta(\hat{\gamma})$.

Subdivisions of the projection: $\Gamma_*^r(S^2)$

In this case, the algorithm is similar to the process above. The difference is that we consider consecutive subdivisions of the first projection $\eta(\partial([-1,1]^3))$ instead of projections of consecutive subdivisions; see the paragraph regarding measures on S^2 contained in subsection 5.2.2.

The main difference of this second process, $\Gamma_*^r(S^2)$, is that the areas of the 2-cubes of a given subdivision are more similar between one another than those of the 2-cubes obtained by the iterative method $\tilde{\Gamma}_*^r(S^2)$.

This implies that the election of the "middle points" is more homogeneous in this second method than in the first method and, as a matter of fact, we can obtain a better approximation of the area of a basin when considering the iterative subdivision $\Gamma_*^{r+1}(S^2)$ rather than when using $\tilde{\Gamma}_*^r(S^2)$ –notice that, given a subdivision order r, the cardinalities of the sets $\Gamma_2^{r+1}(S^2)$ and $\tilde{\Gamma}_2^r(S^2)$ are the same. The fact that the distributions of areas is more homogeneous can be seen just by looking the following matrices M_1 and M_2, which represent the areas of the "2-cubes" of one of the six faces corresponding to the boundary of the cube $[-1,1]^3$ when using respectively the first iterative subdivision method, $\tilde{\Gamma}_*^1(S^2)$, or the second one, $\Gamma_*^2(S^2)$, illustrated in Figures 7.1(a) and 7.1(b).

$$M_1 = \begin{pmatrix} 0.0814556 & 0.120393 & 0.120393 & 0.0814556 \\ 0.120393 & 0.201358 & 0.201358 & 0.120393 \\ 0.120393 & 0.201358 & 0.201358 & 0.120393 \\ 0.0814556 & 0.120393 & 0.120393 & 0.0814556 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0.109805 & 0.131197 & 0.131197 & 0.109805 \\ 0.131197 & 0.151401 & 0.151401 & 0.13119 \\ 0.131197 & 0.151401 & 0.151401 & 0.13119 \\ 0.109805 & 0.131197 & 0.131197 & 0.109805 \end{pmatrix}$$

Note that the distribution of the areas is more homogeneous in the second matrix; that is, the discrepancy among the areas is lower in the second case than in the first one.

7.2.3 Implementation of graphic algorithms in *Sage* and *Mathematica*

We have implemented in *Sage* the first algorithm (projection of subdivisions), whose name is `cubicProbabilityList`, to compute the probabilities of the different basins. The specification for this function is the following:

cubicProbabilityList(*M, N, numdiv=40, ncomp=1, iter=25, precpoints=3, precroots=3*)

- `M,N` are the numerator and the denominator of the given rational function in variable x, respectively.
- `numdiv` is an integer (default: 40) that indicates the number of subdivisions of the faces of the cube which is projected upon the unit sphere. The parameter `numdiv` and the variable r are related in the following way: `numdiv`$(r) = 2^{r+1}$.
- `ncomp` is an integer (default: 1) which represents the number of times that the rational function has to be composed with itself.
- `iter` is an integer l (default: 25) that represents the maximum number of iterations of the rational function.
- `precpoints` is an integer c_1 (default: 3) such that, given a rational map f and a point p on the Riemann sphere, if the chordal distance between $f^k(p)$ and $f^{k+1}(p)$ is lower than 10^{-c_1}, $k < l$, then the developed algorithm stops the iteration of f.
- `precroots` is an integer c_2 (default: 3) which satisfies that, given an iteration sequence $(p, f(p), f^2(p), \ldots, f^k(p))$, if the chordal distance between $f^k(p)$ and a certain fixed point p_0 is lower than 10^{-c_2}, then the developed algorithm considers that this iteration sequence converges to p_0.

The algorithm `cubicProbabilityList` works as follows. Firstly, it composes the rational function $f(z) = \frac{M(z)}{N(z)}$ $(z \in \mathbb{C} \cup \{\infty\})$ with itself the amount of times indicated in the parameter `ncomp` and homogenizes the resulting rational map in the variables z, t. Then, it computes the fixed points of the generated rational map. The vectors `cubeSphereList`, `areaSquareList` and `areasum` will contain the points on the complex projective line in normalized homogeneous coordinates that will be iterated (obtained from the barycenters of the original cubic structure), the areas of the quadrilaterals formed on the surface of the sphere by projecting the cubic subdivisions and the sums of the areas of the spherical quadrilaterals whose middle points converge to the same end point (that is, the approximations to the area of each basin of attraction), respectively.

If the algorithm has been previously run with a certain number of subdivisions, it retrieves the data related to the vectors `cubeSphereList` and `areaSquareList` from a file; otherwise, it computes this information from the barycenters of the 2-cells of the initial regular CW-complex –by calculating within this process the vertices of the spherical quadrilaterals on the surface of S^2– and stores the data creating a new file for future executions.

After that, it iterates the middle points contained in the vector `cubeSphereList` in order to see which fixed point they converge to and adds the area of the corresponding spherical quadrilateral to the appropriate element of the vector `areasum`. Finally, it divides the area associated with each basin of attraction by 4π (the area of the unit sphere) to get the probability related to the basins.

Next, we show the source code written in *Sage* of the function `cubicProbabilityList`.

```
import os.path; import csv
def cubicProbabilityList(M,N,numdiv=40,ncomp=1,iter=25,precpoints=3,precroots=3):
    ch=composeHomogenize(M,N,ncomp); A=ch[0]; B=ch[1]; p=fixedPointsZeros(A,B)
    if len(p)>0:
        grad=len(p)-1; ran=grad+2; areasum=[]
        for k in range(ran): areasum.append(0.0)
        cubeSphereList=[]; areaSquareList=[]; exist=False; sz=6*numdiv**2
        if not os.path.isfile(DATA+'cpl'+str(numdiv)+'.csv'):
            cube=[]
            for y1 in range(numdiv):
                for x1 in range(numdiv):
                    cube.append((((2*x1/numdiv-1)+(2*(x1+1)/numdiv-1))/2,
                            ((2*y1/numdiv-1)+(2*(y1+1)/numdiv-1))/2,-1))
                    cube.append((((2*x1/numdiv-1)+(2*(x1+1)/numdiv-1))/2,
                            ((2*y1/numdiv-1)+(2*(y1+1)/numdiv-1))/2,1))
                    cube.append((((2*x1/numdiv-1)+(2*(x1+1)/numdiv-1))/2,
                            -1,((2*y1/numdiv-1)+(2*(y1+1)/numdiv-1))/2))
                    cube.append((((2*x1/numdiv-1)+(2*(x1+1)/numdiv-1))/2,
                            1,((2*y1/numdiv-1)+(2*(y1+1)/numdiv-1))/2))
                    cube.append((-1,((2*x1/numdiv-1)+(2*(x1+1)/numdiv-1))/2,
                            ((2*y1/numdiv-1)+(2*(y1+1)/numdiv-1))/2))
                    cube.append((1,((2*x1/numdiv-1)+(2*(x1+1)/numdiv-1))/2,
                            ((2*y1/numdiv-1)+(2*(y1+1)/numdiv-1))/2))
            def cubeSphere(p):
                root=sqrt(p[0]**2+p[1]**2+p[2]**2)
                return (n(p[0]/root),n(p[1]/root),n(p[2]/root))
            def sphereComplexProjLine(p):
                if (p[2]==1): return (1,0)
                else: return homogeneousNormalization((p[0]/(1-p[2])+I*p[1]/(1-p[2]),1))
            def squareFromBarycenter(barycenter):
                if abs(barycenter[0])==1:
                    return ((barycenter[0],barycenter[1]-1/numdiv,barycenter[2]-1/numdiv),
                            (barycenter[0],barycenter[1]-1/numdiv,barycenter[2]+1/numdiv),
                            (barycenter[0],barycenter[1]+1/numdiv,barycenter[2]+1/numdiv),
                            (barycenter[0],barycenter[1]+1/numdiv,barycenter[2]-1/numdiv))
                elif abs(barycenter[1])==1:
                    return ((barycenter[0]-1/numdiv,barycenter[1],barycenter[2]-1/numdiv),
                            (barycenter[0]+1/numdiv,barycenter[1],barycenter[2]-1/numdiv),
                            (barycenter[0]+1/numdiv,barycenter[1],barycenter[2]+1/numdiv),
                            (barycenter[0]-1/numdiv,barycenter[1],barycenter[2]+1/numdiv))
                else:
                    return ((barycenter[0]-1/numdiv,barycenter[1]-1/numdiv,barycenter[2]),
                            (barycenter[0]+1/numdiv,barycenter[1]-1/numdiv,barycenter[2]),
                            (barycenter[0]+1/numdiv,barycenter[1]+1/numdiv,barycenter[2]),
                            (barycenter[0]-1/numdiv,barycenter[1]+1/numdiv,barycenter[2]))
            def areaSquare(square):
                v1=Matrix(square[0]); v2=Matrix(square[1])
                v3=Matrix(square[2]); v4=Matrix(square[3])
                return n(arccos(dotProduct(tangent(v1,v2),tangent(v1,v4)))+
                        arccos(dotProduct(tangent(v2,v3),tangent(v2,v1)))+
                        arccos(dotProduct(tangent(v3,v4),tangent(v3,v2)))+
                        arccos(dotProduct(tangent(v4,v1),tangent(v4,v3)))-2*pi)
            def dotProduct(v,w): return v[0][0]*w[0][0]+v[0][1]*w[0][1]+v[0][2]*w[0][2]
            def tangent(a,b): w=b-a-(dotProduct((b-a),a))*a; return w/norm(w)
```

```
        for k in cube:
            spherePoint=sphereComplexProjLine(cubeSphere(k));
            cubeSphereList.append(spherePoint);
            sq=squareFromBarycenter(k);
            spsq=(cubeSphere(sq[0]),cubeSphere(sq[1]),cubeSphere(sq[2]),cubeSphere(sq[3]))
            asq=areaSquare(spsq);
            areaSquareList.append(asq)
        with open(DATA+'cpl'+str(numdiv)+'.csv','w') as f:
            writefile=csv.writer(f)
            for i in range(sz): writefile.writerow([cubeSphereList[i],areaSquareList[i]])
    else:
        exist=True
        with open(DATA+'cpl'+str(numdiv)+'.csv','rU') as f: data=list(csv.reader(f))
        for i in range(sz):
            cubeSphereList.append(data[i][0])
            areaSquareList.append(eval(data[i][1]))
    for j in range(sz):
        if exist:
            colorpoint=onlyPosition(A,B,p,iter,precpoints,precroots,eval(cubeSphereList[j]))
        else: colorpoint=onlyPosition(A,B,p,iter,precpoints,precroots,cubeSphereList[j])
        areasum[colorpoint]=areasum[colorpoint]+areaSquareList[j]
    areasumprob=[n(ar/(4*pi)) for ar in areasum]
    return areasumprob,p
```

Besides, we have also implemented the second algorithm (subdivisions of the projection) using the programming environment of *Mathematica*. The function developed for this purpose is called `CubicAreaComplementAreaFixedPoints`, and it returns the area A_p of the basins of attraction related to each fixed point $p \in \mathrm{Fix}(\mathbb{C} \cup \{\infty\})$ on the surface of the unit sphere (to obtain the probabilities, just divide the resulting areas by 4π). The specification for this subroutine is the following:

CubicAreaComplementAreaFixedPoints[{*P_, Q_*},*iter_,precpoints_,precroots_, subdivisionBaryNumber_,ncomp_:1*]

- `P,Q` are the numerator and the denominator of the given rational function, respectively.
- `iter` is an integer l that represents the maximum number of iterations of the rational function.
- `precpoints` is an integer c_1 such that, given a rational map f and a point p on the Riemann sphere, if the chordal distance between $f^k(p)$ and $f^{k+1}(p)$ is lower than 10^{-c_1}, $k < l$, then the developed algorithm stops the iteration of f.
- `precroots` is an integer c_2 such that, given an iteration sequence $(p, f(p), f^2(p), \ldots, f^k(p))$, if the chordal distance between $f^k(p)$ and a certain fixed point p_0 is lower than 10^{-c_2}, then the developed algorithm considers that this iteration sequence converges to p_0.
- `subdivisionBaryNumber` is an integer that indicates the number of times that the projection of the original cube's faces (spherical quadrilaterals) on the unit sphere are subdivided. The parameter `subdivisionBaryNumber` and the variable r are related in the following way: `subdivisionBaryNumber`$(r) = r + 1$.

- `ncomp` is an integer (default: 1) which represents the number of times that the rational function has to be composed with itself.

The function `CubicAreaComplementAreaFixedPoints` works as follows. After forming the rational map $h(z) = \frac{P(z)}{Q(z)}$ ($z \in \mathbb{C} \cup \{\infty\}$), composing it with itself as many times as indicated in the parameter `ncomp` and computing its fixed points, it constructs recursively the iterated subdivision $\Gamma_*^r(S^2)$ on the surface of the 2-sphere, finds the set of middle points $M(\Gamma_2^r(S^2))$ of the 2-cells in this complex and calculates their areas. The algorithm makes this process more efficient by taking into account that, given a determined subdivision order r, the middle points of the cells in $\Gamma_2^r(S^2)$ are part of the vertices in $\Gamma_0^{r+1}(S^2)$. After that, it iterates the middle points to see which fixed point they converge to (if there is any) and adds the corresponding areas of the spherical quadrilaterals whose middle points converge to the same fixed point (if the iteration of a middle point did not reach any fixed point, the area of the spherical quadrilateral to which it belongs is included in the surface of the region $X \setminus D$, with $X = \mathbb{C} \cup \{\infty\}$). In order to do this last step, the program calculates the sum given by $A_p = \sum_{x \in M(\Gamma_2^r(S^2))} f_x(p)$ for every fixed point $p \in \mathrm{Fix}(X)$, where $f_x \colon \mathrm{Fix}(X) \to [0, 4\pi] \subset \mathbb{R}$ is defined as follows:

$$f_x(p) = \begin{cases} S_x & \text{if } p_x = p; \\ 0, & \text{if } p_x \neq p, \end{cases}$$

being S_x the area of the spherical square in $\Gamma_2^r(S^2)$ that contains the middle point $x \in M(\Gamma_2^r(S^2))$ and being $p_x \in \mathrm{Fix}(X)$ the fixed point to which x converges.

Before compiling the subroutine `CubicAreaComplementAreaFixedPoints`, it might be advisable to compute in advance the middle points and the areas of the 2-cells of the iterated subdivision $\Gamma_*^r(S^2)$ (storing the data in a vector called `centerAreaList`), as long as the number of subdivisions r is not too high (say, less than 8). This strategy could be certainly advantageous, especially when we intend to work repeatedly with a low value for r.

The algorithm `CubicAreaComplementAreaFixedPoints` written in *Mathematica* is presented in the next lines, together with the subprograms that are part of it –remember that the subroutines `ConstructCubeComplex` and `ComplexProjectiveLineBijection` were shown in subsection 6.2.6.

```
Tangent[a_, b_] := Normalize[b - a - ((b - a).a) a];

Area[{a_,b_,c_,d_}] := ArcCos[Tangent[a,b].Tangent[a,d]] + ArcCos[Tangent[b,c].Tangent[b,a]] +
   ArcCos[Tangent[c,d].Tangent[c,b]] + ArcCos[Tangent[d,a].Tangent[d,c]] - 2 Pi;

CentersArea[subdivisionNumber_] := Module[{center},
   center[{a_, b_, c_, d_}] := Normalize[(a + b + c + d) / 4];
   Map[{center[#], Area[#1]} &, ConstructCubeComplex[subdivisionNumber]]
   ];

centerAreaList = Table[CentersArea[k], {k, 1, 8}];
```

```
CubicAreaComplementAreaFixedPoints[{P_,Q_},iter_,precpoints_,precroots_,subdivisionBaryNumber_,
   ncomp_:1] :=
   Module[{h,rf,A,B,z,t,fixedPointList,listPointsAreas,listPointsAreas1,areaListAux,f},
   h[z_] := P[z]/Q[z];
   rf[{z_,t_}] := Together[Nest[h,z,ncomp] /. z -> z/t];
   {A[{z_,t_}],B[{z_,t_}]} = {Numerator[rf[{z,t}]],Denominator[rf[{z,t}]]};
   fixedPointList = FixedPointsZeros[A,B];
   listPointsAreas1 = If[subdivisionBaryNumber<=8,centerAreaList[[subdivisionBaryNumber]],
      CentersArea[subdivisionBaryNumber]];
   listPointsAreas = Transpose[{Map[ComplexProjectiveLineBijection[#] &,
      Transpose[listPointsAreas1][[1]]],Transpose[listPointsAreas1][[2]]}];
   areaListAux[point_,k_] := Module[{pos,niter},
      {pos,niter} =
         PositionIterationNumber[{A,B},fixedPointList,iter,precpoints,precroots][point];
      f[k][i_] := If[i==pos,listPointsAreas[[k]][[2]],0]
      ];
   Do[areaListAux[listPointsAreas[[i]][[1]],i],{i,1,Length[listPointsAreas]}];
   Table[Sum[f[i][j],{i,1,Length[listPointsAreas]}],{j,0,Length[fixedPointList]}]
   ];
```

In the next section, we shall see how these two different algorithms implemented in two different environments give similar results when one computes the measure of the basins of some rational function.

7.3 Quantifying the influence of the multiplicities with numerical experiments

Let $p : \mathbb{C} \to \mathbb{C}$ be a d-degree polynomial, with $d > 0$, whose roots are not necessarily distinct and let N_p be the iterating function of the well-known Newton's method:

$$z_{n+1} = N_p(z_n) = z_n - \frac{p(z_n)}{p'(z_n)}. \tag{7.1}$$

It is well-known [81, 5] that, if the previous sequence starts at an initial approximation z_0 close enough to a root of the polynomial p, then it converges to such a root. If the root is simple, then the order of convergence of the sequence (7.1) is quadratic; but if the root is multiple, the order of convergence is only linear. This fact reveals that the multiplicity of the roots plays an important role on the convergence of Newton's method.

There are other effects of the multiplicity of the roots on Newton's method. For instance, when Newton's method is applied to d-degree polynomials with simple roots, a rational function of degree d is obtained. However, for multiple roots, the degree of the corresponding rational function is strictly lower than d.

The classical Cayley's problem [68] is related to the basins of attraction of the roots of a quadratic polynomial when Newton's method is considered. To be more precise, Newton's method is applied to the polynomial $p(z) = (z - \alpha_1)(z - \alpha_2)$, where α_1 and α_2 are two distinct complex roots. Let us denote

$$\mathrm{Basin}(\alpha_i) = \{z_0 \in \mathbb{C} \mid N_p^k(z_0) \to \alpha_i,\ k \to +\infty\}, \quad i = 1, 2,$$

the i-th basin of attraction ($N_p^k = N_p \circ \cdots \circ N_p$, k-times). Cayley was able to show that, if $|z_0 - \alpha_1| < |z_0 - \alpha_2|$, then $z_0 \in \text{Basin}(\alpha_1)$ and that $z_0 \in \text{Basin}(\alpha_2)$ if $|z_0 - \alpha_2| < |z_0 - \alpha_1|$. The Julia set related to this problem ([7], for more details) is the frontier of both basins of attraction, that is, the Julia set for Newton's method (7.1) applied to the quadratic polynomial $p(z) = (z - \alpha_1)(z - \alpha_2)$ is the locus of points equidistant from the two roots. This problem has been profusely studied in the mathematical literature, not only for Newton's method but also for other different iterative procedures for solving nonlinear equations [2].

Furthermore, when Newton's method is applied to polynomials of the form

$$p(z) = (z - \alpha_1)^m (z - \alpha_2)^m, \quad m \geq 1,$$

it is also known [89] that the Julia set is again the perpendicular bisector of the segment joining the roots α_1 and α_2. That is, when the roots have the same multiplicity, the Julia set is a line. However, when the roots α_1 and α_2 have different multiplicity, the structure of the Julia set is more intricate. This fact was pointed out by Gilbert in [42] for the cubic polynomial

$$p(z) = (z - \alpha_1)^2 (z - \alpha_2).$$

In this chapter, we go a step beyond and analyze the influence of the multiplicities of the roots on the corresponding basins of attraction when Newton's method (7.1) is applied to polynomials of the form:

$$p(z) = (z - \alpha_1)^m (z - \alpha_2)^n, \quad \alpha_1, \alpha_2 \in \mathbb{C},\ \alpha_1 \neq \alpha_2,\ m, n \in \mathbb{N}^*. \tag{7.2}$$

For each polynomial $p(z)$ in (7.2), the application of Newton's method (7.1) gives a rational function $B_{m,n}(z)$ whose coefficients depend on the multiplicities m, n:

$$B_{m,n}(z) = \frac{(m+n-1)z^2 + (\alpha_1 + \alpha_2 - \alpha_1 n - \alpha_2 m)z - \alpha_1 \alpha_2}{(m+n)z - (\alpha_1 n + \alpha_2 m)}. \tag{7.3}$$

Notice that the rational function $B_{m,n}(z)$ has quadratic degree, regardless of the values of the multiplicities m and n.

In our work, we are interested in the study of the influence of the multiplicity on the geometrical properties of the basins (and Julia sets) associated with the rational function $B_{m,n}(z)$. This implies that we can only modify the rational functions of this type using transformations preserving the geometrical properties of the basins that we are studying.

As a starting point in our study, we consider the roots $\alpha_1 = 1$, $\alpha_2 = -1$. A first graphical inspection (see Figure 7.2) shows that there exists a clear influence of the multiplicities of the roots on the shapes of the basins of attraction. We can also observe that the basin of the root with higher multiplicity invades the basin of the root with lower multiplicity. In Figure 7.2, we have considered the rectangle $[-3, 1] \times [-2.5, 2.5]$ as a framework. One can see in these images that the area of the corresponding basin (in grey) of the root $z = 1$ increases when multiplicity m grows ($m = 2, 3, 4$). Since the entire areas of the basins on the complex plane $\mathbb{C}$ are not finite, this property cannot be easily extended to compare the basins.

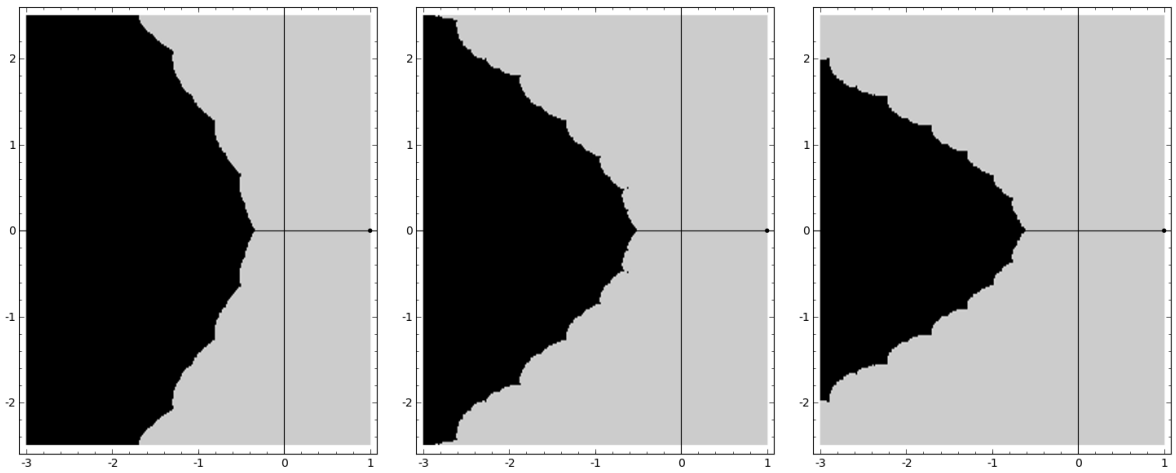


Figure 7.2: Basins of attraction of Newton's method applied to the polynomials $p(z) = (z-1)^m(z+1)$ for $m = 2, 3, 4$. We see how the basin of the multiple root (in grey) "invades" the other basin (in black) when m increases.

However, this difficulty can be avoided using the canonical bijection between the extended complex plane and the unit sphere $\mathbb{C} \cup \{\infty\} \cong S^2$ –see subsection 6.1.1. Using the usual measure of S^2, we can calculate a new measure of the area of the basins on S^2, with the important property that now these areas are finite. In this way, we can compare the area of different basins.

Since the structure of the boundary of any basin of attraction (the Julia set) is, in general, very complicated, the standard methods based on a good election of coordinates and integration theory using a volume (area) form cannot be easily applied –see Remark 7.1.1. We have sorted out this problem by developing some combinatorial methods and using the excess of a spherical triangle to compute, module a given precision, the area of a basin on the Riemann sphere.

In this section, we use the algorithms seen in section 7.2 to analyze the influence of the multiplicities m and n on the basins of attraction of the fixed points of a rational function of the form $B_{m,n}$ given in (7.3). In particular, for polynomials

$$p(z) = (z-1)^m(z+1)^n, \quad m, n \in \mathbb{N}^*, \tag{7.4}$$

$B_{m,n}$ can be written as follows:

$$B_{m,n}(z) = \frac{(m+n-1)z^2 + (m-n)z + 1}{(m+n)z + (m-n)}. \tag{7.5}$$

Subsequently, our experiment allows us to show the influence of the multiplicities on the probabilities related to the roots 1 and -1 of the polynomial (7.4), as explained above. Taking into account the criterium which was introduced there, for each $m, n \in \mathbb{N}$ we denote by $P_1(m,n)$ the probability of the root $z = 1$ and by $P_{-1}(m,n)$ the probability of the root $z = -1$; that is,

$$P_{-1}(m,n) = \frac{A_{-1}}{4\pi}, \quad P_1(m,n) = \frac{A_1}{4\pi},$$

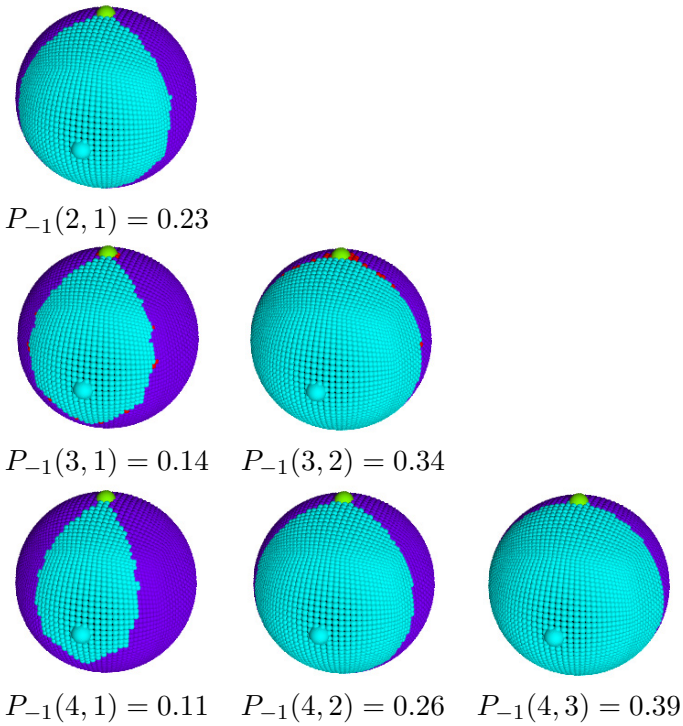


Figure 7.3: Basins on the Riemann sphere of the roots $z = -1$ (light blue) and $z = 1$ (dark blue) as fixed points of the rational function (7.5), together with the probability $P_{-1}(m, n)$ for different multiplicities m and n.

where A_{-1} and A_1 are respectively the areas of the basins of attraction on the Riemann sphere of the roots -1 and 1, considered as fixed points of (7.5).

7.3.1 A graphic approach plotted with *Sage*

In Figure 7.3, we have plotted the basins of attraction of the roots $z = -1$ and $z = 1$ for different values of m and n ($m > n$). We can see how the basin of the root $z = -1$ is smaller than the other basin. In addition, we have quantified this graphical fact by calculating the probability related with the two roots. In fact, we only have included the values of $P_{-1}(m, n)$, since $P_1(m, n) = 1 - P_{-1}(m, n)$. In order to obtain these images, we have used the projection of a subdivision with order $r = 5$.

Our numerical experiments reveal the following facts:

(1) It is necessary to increase the number of maximum iterations when multiplicities m and n go up.

(2) For a fixed m, the function $P_{-1}(m, n)$ is increasing on the variable n.

(3) For a fixed n, the function $P_{-1}(m, n)$ is decreasing on the variable m.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$m=1$	0.5	0.7659	0.8513	0.8883	0.9080	0.9285	0.9342	0.9466
$m=2$	0.2341	0.5	0.6539	0.7402	0.7930	0.8283	0.8536	0.8715
$m=3$	0.1487	0.3461	0.5	0.6054	0.6810	0.7310	0.7684	0.7966
$m=4$	0.1117	0.2598	0.3946	0.5	0.5831	0.6430	0.6926	0.7298
$m=5$	0.0920	0.2070	0.3190	0.4169	0.5	0.5641	0.6195	0.6620
$m=6$	0.0715	0.1717	0.2690	0.3570	0.4359	0.5	0.5565	0.6014
$m=7$	0.0658	0.1464	0.2317	0.3074	0.3805	0.4435	0.5	0.5448
$m=8$	0.0534	0.1285	0.2034	0.2702	0.3380	0.3986	0.4552	0.5

Table 7.1: Probabilities $P_{-1}^{\mathrm{M5}}(m,n)$ (related to the basin of $z=-1$) for different multiplicities m and n calculated with *Mathematica* by taking the iterated subdivision $\Gamma_*^5(S^2)$.

(4) For a fixed (m,n), $m \geq n$, the function $P_{-1}(m+s, n+s)$ is increasing on the integer variable $s \geq 0$.

7.3.2 Computing the precision obtained using two consecutive subdivisions

In Table 7.1, we have gathered the values of probabilities $P_{-1}^{\mathrm{M5}}(m,n)$ for $1 \leq m \leq 8$ and $1 \leq n \leq 8$ when taking the iterated subdivision $\Gamma_*^5(S^2)$ and a maximum number of iterations equal to 100. These probabilities have been calculated using the process described in subsection 5.2.2 implemented in *Mathematica*.

Note that one has

$$P_{-1}(m,n) + P_{-1}(n,m) = 1.$$

From the results obtained by means of this simulation, one also obtains that the Julia set in S^2 has probability measure equal to zero and the same happens for the basins associated with periodic points, so that $P_{-1}(m,n) + P_1(m,n) = 1$. Then, the resulting table for $P_1(m,n)$ can be also obtained by transposing rows and columns in the table above:

$$P_1(m,n) = P_{-1}(n,m).$$

Taking the iterated subdivision $\Gamma_*^6(S^2)$ and a maximum number of iterations equal to 100, the values of probabilities $P_{-1}^{\mathrm{M6}}(m,n)$ have been computed with *Mathematica* in Table 7.2. When we compare with the iterated subdivision $\Gamma_*^5(S^2)$, we have:

$$\left|P_{-1}^{\mathrm{M5}}(m,n) - P_{-1}^{\mathrm{M6}}(m,n)\right| < 4.8 \cdot 10^{-3}, \quad 1 \leq m,n \leq 8.$$

So one can easily check that a precision of two decimal places is obtained.

Remark 7.3.1. *The level of accuracy of these algorithms can be improved using finer subdivisions. For instance, for $m=2$ and $n=1$, taking a new iterated subdivision $\Gamma_*^7(S^2)$, we obtain a better precision:*

$$\left|P_{-1}^{\mathrm{M6}}(2,1) - P_{-1}^{\mathrm{M7}}(2,1)\right| < 1.2 \cdot 10^{-4}.$$

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$m=1$	0.5	0.7665	0.8512	0.8898	0.9127	0.9259	0.9357	0.9447
$m=2$	0.2335	0.5	0.6527	0.7402	0.7928	0.8287	0.8532	0.8711
$m=3$	0.1488	0.3473	0.5	0.6060	0.6801	0.7325	0.7712	0.7989
$m=4$	0.1102	0.2598	0.3940	0.5	0.5818	0.6445	0.6920	0.7284
$m=5$	0.0873	0.2072	0.3199	0.4182	0.5	0.5667	0.6196	0.6632
$m=6$	0.0741	0.1713	0.2675	0.3555	0.4333	0.5	0.5543	0.6018
$m=7$	0.0643	0.1468	0.2288	0.3080	0.3804	0.4457	0.5	0.5480
$m=8$	0.0553	0.1289	0.2011	0.2716	0.3368	0.3982	0.4520	0.5

Table 7.2: Probabilities $P_{-1}^{\mathrm{M6}}(m,n)$ (related to the basin of $z=-1$) for different multiplicities m and n calculated with *Mathematica* by taking the iterated subdivision $\Gamma_*^6(S^2)$.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$m=1$	0.5	0.7660	0.8502	0.8893	0.9114	0.9259	0.9360	0.9438
$m=2$	0.2340	0.5	0.6538	0.7397	0.7927	0.8284	0.8526	0.8711
$m=3$	0.1498	0.3462	0.5	0.6067	0.6805	0.7327	0.7702	0.7989
$m=4$	0.1107	0.2603	0.3933	0.5	0.5814	0.6446	0.6921	0.7285
$m=5$	0.0886	0.2073	0.3195	0.4186	0.5	0.5659	0.6193	0.6627
$m=6$	0.0741	0.1716	0.2673	0.3554	0.4341	0.5	0.5563	0.6028
$m=7$	0.0640	0.1474	0.2298	0.3079	0.3807	0.4437	0.5	0.5478
$m=8$	0.0562	0.1289	0.2011	0.2715	0.3373	0.3972	0.4522	0.5

Table 7.3: Probabilities $P_{-1}^{\mathrm{S5}}(m,n)$ (related to the basin of $z=-1$) for different multiplicities m and n calculated with *Sage* by taking the iterated subdivision $\tilde{\Gamma}_*^5(S^2)$.

However, in our study, we focus on the analysis of the influence of multiplicity more than on the observation of the level of accuracy in the calculation of the probability.

7.3.3 Comparing two different algorithms implemented in different computational environments

In Table 7.3, we have written the probability $P_{-1}^{\mathrm{S5}}(m,n)$ obtained by the implementation in *Sage* of the algorithm referred in subsection 7.2.2 (projection of the subdivisions) taking the iterated subdivision $\tilde{\Gamma}_*^5(S^2)$.

Remember that the consecutive subdivisions $\Gamma_*^{r+1}(S^2)$ and $\tilde{\Gamma}_*^r(S^2)$ give rise to the same cardinality, so $r=6$ for $\Gamma_*^r(S^2)$ and $r'=5$ for $\tilde{\Gamma}_*^{r'}(S^2)$ are exactly the subdivision orders that we must take into consideration to properly compare both procedures. Although different strategies and programs have been used to estimate the probabilities $P_{-1}^{\mathrm{M6}}(m,n)$ and $P_{-1}^{\mathrm{S5}}(m,n)$,

$$\left|P_{-1}^{\mathrm{S5}}(m,n)-P_{-1}^{\mathrm{M6}}(m,n)\right|<2.1\cdot 10^{-3},\quad 1\le m,n\le 8;$$

that is, both processes provide similar results.

n	$P_{-1}^{\mathrm{M6}}(m,n)$	R^2
1	$0.4872m^{-1.0547}$	0.9987
2	$1.0099m^{-0.9876}$	0.9987
3	$1.4223m^{-0.9358}$	0.9979
4	$1.7165m^{-0.883}$	0.9986
5	$1.9384m^{-0.8391}$	0.9989

Table 7.4: Potential approach to $P_{-1}^{\mathrm{M6}}(m,n)$ for $m \geq n$ and a fixed n.

m	$P_{-1}^{\mathrm{M6}}(m,n)$	R^2
8	$-0.0022n^2 + 0.084n - 0.0287$	0.9999
7	$-0.0028n^2 + 0.0961n - 0.0315$	0.9998
6	$-0.004n^2 + 0.1141n - 0.0377$	0.9999
5	$-0.0065n^2 + 0.1425n - 0.0497$	0.9999
4	$-0,0109n^2 + 0,1849n - 0,0644$	0.9999

Table 7.5: Polynomic approach to $P_{-1}^{\mathrm{M6}}(m,n)$ for $m \geq n$ and a fixed m.

7.3.4 Quantification of the influence of multiplicity with potential and polynomial functions

In order to obtain a more precise quantification of the influence of multiplicities m and n on the probability related to the roots of the polynomial (7.4), we show in Table 7.4 and Table 7.5 functions that are rather good approximations of $P_{-1}(m,n)$, in the first case for a fixed n and in the second case for a fixed m. These approximations have been calculated by means of the method of least squares. In both tables, the value R^2 represents the coefficient of determination, which measures the goodness-of-fit.

For each fixed $n \in [1,5]$, a graphic of the original probabilities $P_{-1}^{\mathrm{M6}}(m,n)$ and the corresponding potential fitting functions can be seen in Figure 7.4, and another plot of these probabilities for $4 \leq m \leq 8$ and their relative polynomic fitting functions is shown in Figure 7.5.

Remark 7.3.2. *Note that Newton's method can also be applied when the multiplicities m, n are not integers. Even more, the algorithms described in this paper also work in this case, and the potential and polynomic functions given in Table 7.4 and Table 7.5 give a good approximation to the probability when one of the two multiplicities is an integer. For instance, let us consider the case $n = 2$. The potential function obtained from Table 7.4 (see also Figure 7.4) for $n = 2$ is denoted by $g_2(m) = 1.0099m^{-0.9876}$.*

The following pairs give the values of $P_{-1}^{\mathrm{M6}}(m,2)$ and its approximations by $g_2(m)$ for different not necessarily integer values of m:

$P_{-1}^{\mathrm{M6}}(5,2) = 0.2072$, $g_2(5) = 0.2060$; $P_{-1}^{\mathrm{M6}}(5.2,2) = 0.1988$, $g_2(5.2) = 0.1982$;
$P_{-1}^{\mathrm{M6}}(5.4,2) = 0.1916$, $g_2(5.4) = 0.1909$; $P_{-1}^{\mathrm{M6}}(5.6,2) = 0.1853$, $g_2(5.6) = 0.1842$;
$P_{-1}^{\mathrm{M6}}(5.8,2) = 0.1777$, $g_2(5.8) = 0.1779$; $P_{-1}^{\mathrm{M6}}(6,2) = 0.1713$, $g_2(6) = 0.1720$.

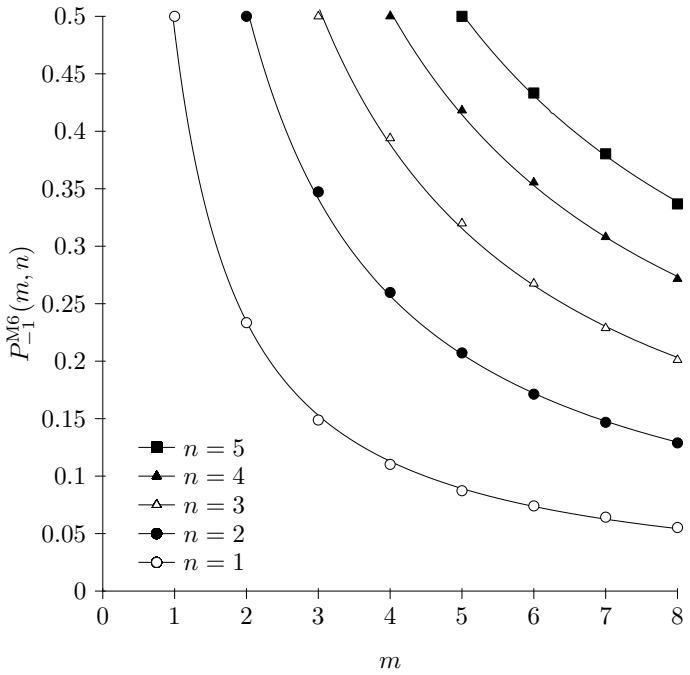


Figure 7.4: Values of the probabilities $P_{-1}^{\mathrm{M6}}(m,n)$ when the lowest multiplicity m is fixed.

Figure 7.5: Values of the probabilities $P_{-1}^{\mathrm{M6}}(m,n)$ when the highest multiplicity m is fixed.

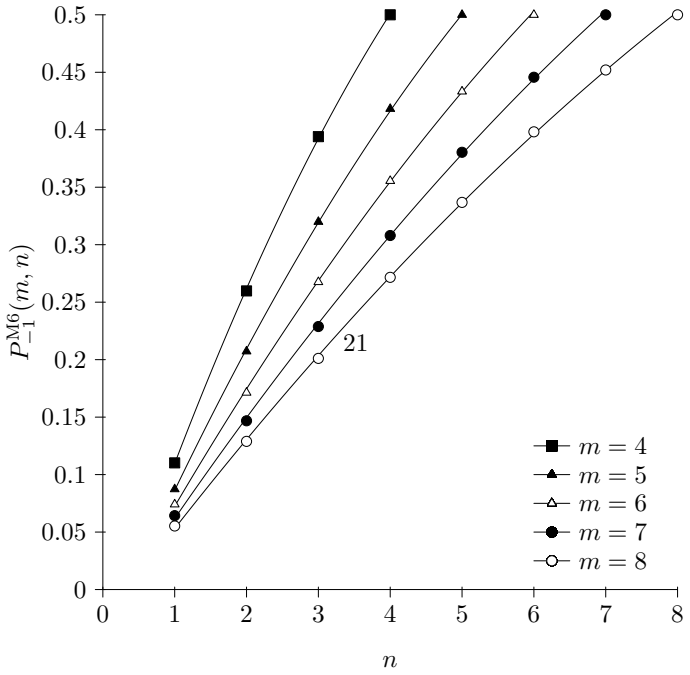


Figure 7.5: Values of the probabilities $P_{-1}^{\mathrm{M6}}(m,n)$ when the highest multiplicity m is fixed.

Similar observations can be done for the functions of Table 7.5 plotted in Figure 7.5.

Remark 7.3.3. *The graphics shown in Figure 7.4 suggest that, for a fixed n, the probability $P_{-1}^{\mathrm{M6}}(m,n) \to 0$ when $m \to +\infty$. Note that, for multiple roots, the rational functions given by Newton's method have always quadratic degree. For this reason, in this case we can avoid the problem of working with rational functions of high degree that appears when Newton's method is applied to a high order polynomial with simple roots. We have checked the goodness of fit of the calculations performed for $n = 2$ and higher values of m; in particular, $m = 10, 20, 30$. In these cases, the values of the potential function given in Table 7.4 for $n = 2$ (that is to say, $g_2(m) = 1.0099m^{-0.9876}$) are*

$$g_2(10) = 0.103915, \quad g_2(20) = 0.052406, \quad g_2(30) = 0.0351134,$$

whereas the values of the probabilities obtained by our algorithms are

$$P_{-1}^{\mathrm{M6}}(10,2) = 0.104219, \quad P_{-1}^{\mathrm{M6}}(20,2) = 0.0532655, \quad P_{-1}^{\mathrm{M6}}(30,2) = 0.0375119.$$

As we can see, the corresponding values are quite similar. This fact suggests that the potential functions also give a good approximation when m tends to infinity. Consequently, the probability of reaching the root with lower multiplicity decreases to 0 when the multiplicity of the other root goes to infinity.

Besides, note that the functions defined in Table 7.5, whose graphics are given in Figure 7.5, are polynomials of degree 2. From these graphics, one could infer that in general, for a fixed m and for $1 \le n \le m$, the discrete function $P_{-1}^{\mathrm{M6}}(m,n)$ can be approached by quadratic polynomials. In Figure 7.5, we also observe that, for $m = n$, one obtains $P_{-1}^{\mathrm{M6}}(m,n) = 0.5$, as we have already mentioned. This idea is supported by the result previously obtained in [89].

To summarize, the discrete function $P_{-1}^{\mathrm{M6}}(m,n)$, whose values are numerically given by our algorithms, can be approached, for a fixed n, by potential functions with (negative) non-integer exponents. In the same way, for a fixed m, the corresponding discrete function with domain $\{1,\ldots,m\}$ is approached by a polynomial of degree 2.

Remark 7.3.4. *One interesting question is to study how the probability of reaching a root changes when we consider different numerical methods. For instance, we have also made some experiments when the relaxed Newton's method,*

$$z_{n+1} = R_{p,\lambda}(z_n) = z_n - \lambda \frac{p(z_n)}{p'(z_n)},$$

is applied to the polynomial $p(z) = (z-1)^3(z+1)^2$. In particular, we have chosen the values $\lambda = 2$ and $\lambda = 3$ for the relaxing parameter. Let us consider the root $z = -1$ and let us denote $P_{-1}^{R_{p,3}}(3,2)$ and $P_{-1}^{R_{p,2}}(3,2)$ the probabilities of reaching the root $z = -1$ by the relaxed Newton's method for $\lambda = 3$ and $\lambda = 2$, respectively. $P_{-1}^{N_p}(3,2)$ denotes the probability obtained by Newton's method (given in Table 7.2). Then, we have

$$P_{-1}^{R_{p,3}}(3,2) = 0.2842 < P_{-1}^{R_{p,2}}(3,2) = 0.3250 < P_{-1}^{N_p}(3,2) = 0.3473.$$

In our opinion, the behavior of the probability of reaching a given root when using different generalized Newton's methods should be analyzed in more detail in future works by using more numerical calculi and new simulations. We are aware that, for generalized Newton's methods, there are other factors that must be taken into account to analyze the structure of the related Julia sets, such as the existence of extraneous fixed points.

Conclusions

Main ideas and techniques developed

The central idea of the first three chapters is that the use of Brown-Grossman, Steenrod and Borsuk-Čech invariants in dimension zero is a nice tool to study the basins of attraction of a discrete semi-flow.

Instead of employing all the end points of the sets $\pi_0^{\mathrm{BG}}(X)$, $\pi_0^{\mathrm{S}}(X)$ and $\check{\pi}_0(X)$, it is advisable to consider only ω-representable end points to obtain the sets ${}^{\omega}\pi_0^{\mathrm{BG}}(X)$, ${}^{\omega}\pi_0^{\mathrm{S}}(X)$ and ${}^{\omega}\check{\pi}_0(X)$ associated with an exterior discrete semi-flow X. The existence of the map $\omega\colon D(X) \to {}^{\omega}\pi_0^{\mathrm{BG}}(X)$ allows us to decompose an exterior discrete semi-flow X into the following disjoint union:

$$X = (X \setminus D(X)) \sqcup \left(\bigsqcup_{a \in {}^{\omega}\pi_0^{\mathrm{BG}}(X)} \omega^{-1}(a) \right).$$

Another important technique consists in the use of intrinsic topologies and intrinsic paths to study local stability properties of exterior discrete semi-flows. Through intrinsic topologies and paths, one can define new 0-dimensional invariant sets ${}^{\Omega}\pi_0^{\mathrm{BG}}(X)$, ${}^{\Omega}\pi_0^{\mathrm{S}}(X)$ and ${}^{\Omega}\check{\pi}_0(X)$, as well as natural transformations ${}^{\Omega}\pi_0^{\mathrm{BG}}(X) \to {}^{\omega}\pi_0^{\mathrm{BG}}(X)$, ${}^{\Omega}\pi_0^{\mathrm{S}}(X) \to {}^{\omega}\pi_0^{\mathrm{S}}(X)$ and ${}^{\Omega}\check{\pi}_0(X) \to {}^{\omega}\check{\pi}_0(X)$. When an exterior discrete semi-flow has good local stability properties, then these natural transformations become isomorphisms.

Moreover, we have discussed the notion of end point for metric discrete semi-flows; this allows us to consider the connections between end points of exterior semi-flows and end points associated with metrics, as well as how the basins of both types of end point are related, when having externologies given by open neighborhoods of a finite set of periodic points of the metric discrete semi-flow.

Being mindful of how the common iterative numerical methods induce rational functions, we have seen how to apply the algorithms described in chapter 6 to such processes (like Newton-Raphson or Tchebychev methods), which are used in the search of complex polynomial roots: in general, the roots of a complex polynomial are fixed points of the rational function obtained with the most usual iterative numerical methods. As indicated in subsection 5.1.1, each fixed point of a rational function can be directly regarded as an end point. Thus, every basin of attraction of a complex polynomial root is just the basin of attraction of its associated end point.

If p is a root of the original complex polynomial, the fact that a point x is in the basin of attraction corresponding to the fixed point p means that, when applying the rational function associated with the considered numerical method to the point x, the trajectory is approaching the root p of the given polynomial.

We have also analyzed the structure of a CW-complex which is provided with a system of subdivisions. This structure permits to construct algorithms to give graphical representations of basins of attraction associated with the iteration of a rational map defined on the Riemann sphere. In addition, the designed programs are able to indicate the number of iterations until convergence up to a given tolerance, which will be useful to know the speed of convergence of a point that belongs to the basin of a determined root.

This work is also closely connected to some aspects of fractal geometry, since the programs that we have developed and implemented along chapter 6 make possible the visualization of Julia sets of rational functions of degree greater than or equal to 2 on the sphere S^2. What is more, the cubic subdivision techniques that we have devised allow us to approximate the Julia set by means of finite cubical complexes, and we think that some portions of the algorithms regarding these iterative subdivisions may be useful to make a detailed study of the Julia sets of the obtained fractals by calculating their fractal dimension and Betti numbers, which would allow us to know the number of connected components, topological holes, etc.

Besides, the structure of a CW-complex and its consecutive subdivisions can be employed to give a measure map on a CW-complex. Furthermore, in the case of the Riemann sphere, one can give an algorithm to compute the area of basins of attraction. In this work, we have developed two programs to measure such basins: one of them has been implemented with *Sage* and the other one with *Mathematica*.

As an application, despite the difficulty of measuring, due to the fractal structure of its boundary, the area of a basin of attraction of a fixed point induced by a rational function obtained when Newton's method is applied to a polynomial with two roots $\alpha_1 = -1$ and $\alpha_2 = 1$ of multiplicities n and m, respectively, the techniques and algorithms introduced have allowed us to approximate the probability $P_{\alpha_i}(m, n)$ for a point on the Riemann sphere to belong to the basin of attraction of α_i, $i = 1, 2$. With this, we have been able to quantify the influence of the multiplicity of the roots on the size of the corresponding basins of attraction.

Trying to find a representative sample on the sphere in order to study probability phenomena is a well-known problem. This question is strongly related to the study of astronomical data distributed on the entire sky. In some cases, it is only necessary to cover a partial area corresponding to a galactic object. We think that the use of cubic structures of the sphere and measure procedures developed in this dissertation could be a very useful technique in Astronomy and Astrophysics. The satellite observation of the cosmos can be organized using cellular and cubic decompositions of the 2-sphere, and some modifications of the algorithms developed in this work can be useful for the study of astronomical data. Similar conclusions can be obtained when one tries to analyze geological data given by observation satellites, specifically designed to observe the Earth from orbit for uses such as environmental monitoring, meteorology, map making etc. There are other possible application fields as, for instance, the study of spherical viruses or spherical fullerene structures [26, 4], et cetera.

Further work

Following the results achieved, a lot of questions and problems whose consideration and solution are reason for further research have been contemplated. Regarding exterior discrete semi-flows, we will try to tackle the objectives shown below:

a) To study, in higher dimensions, diverse sequences of homotopy groups of ω-representable and Ω-representable end points.

b) To analyze continuous semi-flows by using results concerning exterior discrete semi-flows.

c) To obtain possible completions of exterior discrete semi-flows.

Furthermore, in our study, we have been working with intrinsic topologies associated with an exterior discrete semi-flow; however, other types of intrinsic topologies had been used before for the study of attractors of continuous flows by employing techniques related to shape invariants and the Conley index. In that sense, we think that it could be interesting to develop further research in order to establish a comparison between intrinsic topologies associated with exterior semi-flows and flows and the intrinsic topologies considered in [76].

Also, when a semi-flow is induced by a rational map f of degree d defined on the Riemann sphere, the associated Julia set $J(f)$ is right-invariant and the restriction map $f|_{J(f)}\colon J(f) \to J(f)$ has d sheets and it admits an overlay structure. Our algorithms provide an inverse system of cubic complexes approaching $J(f)$, and the shape invariants of the Julia space could be used to study the overlay structure of the map $f|_{J(f)}$ as a future work. For the classification and properties of overlays and shape invariants, the following references can be taken into account: see [63, 65, 21, 66] and [11, 20, 18, 12, 19].

Moreover, we will carry on with the design of new algorithms and their implementation in some programming environments (such as *Sage*, *Mathematica*, *Julia*, et cetera) for:

d) The graphic representation of basins on the torus.

e) The graphic representation of Julia sets.

f) The computation of the fractal dimension.

Besides, one of the aims of this doctoral thesis has been to connect to different theories such as the numerical solution of nonlinear equations and the dynamics of some topological spaces. We are aware that there are many other questions that could be taken into account in further works. For instance, we think that it would be worthy to complete our initial numerical experiments with, at least, the following aspects:

g) Analysis of the probabilities $P_{\alpha_i}(m,n)$ and their properties for other choices of the roots, and not only for two opposite unitary roots $-\alpha_1 = \alpha_2 = 1$.

h) Generalization to polynomials with three or more roots and their corresponding multiplicities.

i) Comparison of the influence of multiplicity on the measure of basins of attraction when other numerical methods are considered: relaxed Newton's method, Tchebychev's method, Halley's method, etc. In this way, it would be interesting to take into account previous results obtained for these methods in the references [85] or [87], for instance.

j) Application of our algorithms for the iteration of not necessarily rational functions, such as complex exponential functions, trigonometric functions, et cetera. The references [88] and [86] could be very helpful in that way.

k) Development of some algorithms to evaluate the length of the Julia set on the Riemann sphere.

l) Search and implementation of new measure algorithms based on a spherical subdivision in quadrilaterals with the same area. In our opinion, this fact would improve the efficiency of the algorithms.

m) Development of new measure algorithms based on the subdivision of other regular polyhedra; for instance, the icosahedron or other spherical subdivisions with adequate dispersion and discrepancy –see [43, 90].

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