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Discrete Harmonic Analysis Associated with Jacobi Expansions

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Prof. Juan Luis Varona Malumbres

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Análisis armónico discreto asociado a desarrollos de Jacobi

por Edgar Labarga Varona

Directores de tesis: Dr. D. Óscar Ciaurri Ramírez y
Dr. D. Juan Luis Varona Malumbres

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My heroes are and were my parents,
I cannot see having anyone else as my heroes.

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CONTENTS

ABSTRACT

In this work we consider the operator $\mathcal{J}^{(\alpha,\beta)}$ associated with the three-term recurrence relation for the Jacobi polynomials and we study some classical operators in Harmonic Analysis in this context. Particularly, we are interested in the heat and Poisson semigroups and in the maximal operators related to them, in the Riesz transforms, and in the Littlewood-Paley-Stein g_k-functions. We obtain weighted $\ell^p(\mathbb{N})$-inequalities for the heat and Poisson maximal operators and for the Riesz transforms when $1 < p < \infty$ and $\alpha, \beta \geq -1/2$, and weighted weak inequalities in the case $p = 1$ and $\alpha, \beta \geq -1/2$. We give weighted $\ell^p(\mathbb{N})$-estimates for the g_k-functions when $1 < p < \infty$ and $\alpha, \beta \geq -1/2$.

The method to prove these inequalities is based on the vector-valued Calderón-Zygmund theory in spaces of homogeneous type.

RESUMEN

En esta memoria consideramos el operador $\mathcal{J}^{(\alpha,\beta)}$ asociado a la relación de recurrencia a tres términos de los polinomios de Jacobi y estudiamos varios operadores clásicos del análisis armónico en este contexto. En concreto estamos interesados en los semigrupos del calor y de Poisson, que dan lugar al operador maximal del calor y al operador maximal de Poisson, respectivamente, en las transformadas de Riesz y en las g_k-funciones de Littlewood-Paley-Stein. Para los operadores maximales y las transformadas de Riesz se obtienen acotaciones de tipo fuerte en espacios $\ell^p(\mathbb{N})$ con peso, $1 < p < \infty$ y $\alpha, \beta \geq -1/2$, y acotaciones de tipo débil con peso cuando $p = 1$ y $\alpha, \beta \geq -1/2$. En el caso de las g_k-funciones se presentan acotaciones de tipo fuerte con peso en estos espacios para $1 < p < \infty$ y $\alpha, \beta \geq -1/2$.

El método para probar estas acotaciones se basa en la teoría de Calderón-Zygmund en espacios de tipo homogéneo para operadores con valores vectoriales.

CHAPTER I

INTRODUCTION

Harmonic Analysis is a very active and fruitful branch of Mathematics with many applications to other branches and fields such as Partial Differential Equations, Ergodic Theory, Group Theory, Number Theory, Probability Theory, Signal Processing, and Quantum Physics. Historically, classical Harmonic Analysis was intimately connected with Fourier Analysis. The beginnings of the story go back to some research carried out by D. Bernoulli, J.-B. le Rond D'Alembert, J.-L. Lagrange, and L. Euler while trying to solve the vibrating string problem during the mid and late 17th century. However, the landmark is the work *Théorie analytique de la chaleur* [23] by J.-B. J. Fourier. There, Fourier formulates and solves the heat initial value problem involving the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2},$$

which is a second order partial differential equation that describes the variation of temperature in a region over time. To do so, he introduced an original technique called Fourier's method or the method of separation of variables where the key step in the process was to assume that a periodic function f (the initial data) of period say 1 can be represented by a trigonometric series of the form

$$\sum_{k\in\mathbb{Z}} \hat{f}(k)e^{2\pi ikx}. \tag{I.1}$$

The series in (I.1) is called the Fourier series (in its complex form) of the function f and the coefficients $\hat{f}(k)$ are the so-called Fourier coefficients of f given by

$$\hat{f}(k) = \int_0^1 f(t)e^{-2\pi ikt}\,dt, \quad k\in\mathbb{Z}.$$

These notions are easily reformulated when the function f is no longer periodic. In that case, we have the so-called Fourier integral

$$\int_{\mathbb{R}^d} \hat{f}(\xi)e^{2\pi ix\cdot\xi}\,d\xi,$$

where now the Fourier transform of f is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi ix\cdot\xi}\,dx.$$

The central problem in Fourier Analysis is to study when and in what sense the Fourier series (equivalently, the Fourier integral) of a function converges to the function which represents. Most of the efforts in classical Harmonic Analysis focused on investigating this representation and led the research on this topic until the mid-19th century.

Fourier Analysis was initially in close connection with Complex Analysis and boundary values of analytic functions. Some operators can be studied naturally in this context. Concerning Fourier series we have the conjugate function

$$\tilde{f}(x) = \sum_{k=-\infty}^{\infty} -i \operatorname{sgn}(k) \hat{f}(k) e^{2\pi i k x}.$$

The analogue operator in the one-dimensional Fourier integral setting is the Hilbert transform

$$Hf(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} \, dy.$$

Due to the singularity the integral must be interpreted in the principal value sense, that is,

$$Hf(x) = \frac{1}{\pi} \, \text{p. v.} \int_{\mathbb{R}} \frac{f(y)}{x-y} \, dy. \tag{I.2}$$

The interest lay in obtaining L^p bounds for these operators. An example of the applications of this is that the L^p boundedness of the conjugate function (equivalently, the Hilbert transform of a function) implies the L^p-norm convergence of the Fourier series (equivalently, Fourier integral) of that function.

In dimension one, the first proofs [52] of the boundedness for both conjugate function and Hilbert transform used methods of Complex Analysis and this was an obstacle to study higher dimensions. This problem originated a second era in Harmonic Analysis in the mid-past century. The first step to deal with the multidimensional case was due to A. P. Calderón and A. Zygmund and appeared in their seminal paper [14]. They considered singular integral operators of convolution type of the form

$$Tf(x) = \text{p. v.} \int_{\mathbb{R}^d} f(y) K(x-y) \, dy$$

and obtained mapping properties for them under certain decay conditions on the kernel K which presents a singularity at the diagonal $x = y$. The main ingredient was the so-called Calderón-Zygmund decomposition. The work of Calderón and Zygmund initiated a new line of research in Harmonic Analysis known as Calderón-Zygmund theory.

The above discussion shows the huge development that underwent Harmonic Analysis on Euclidean spaces. Later on, B. Muckenhoupt and E. M. Stein investigated Harmonic Analysis associated with ultraspherical expansions in [41]. The idea was to study the analogue of the conjugate function in this context. The former author continued the analysis for other orthogonal families of polynomials in [38] and [39]. These papers are the starting point of the so-called Harmonic Analysis associated with classical orthogonal expansions which we outline below.

Let $\{\phi_n(x)\}_{n\in\mathbb{N}}$ be a complete orthonormal system on $L^2(X, d\mu)$, $X \subset \mathbb{R}$. Then, we have that

$$\langle \phi_n, \phi_m \rangle_{d\mu} = \int_X \phi_n(x) \overline{\phi_m(x)} \, d\mu(x) = \delta_{nm},$$

where δ_{nm} is the Kronecker's delta function. In analogy with classical Fourier theory, we form the Fourier expansion of a function f with respect to the system $\{\phi_n\}_{n\in\mathbb{N}}$ by means of the expression

$$\sum_{n=0}^{\infty} c_n(f)\phi_n(x), \tag{I.3}$$

where now the Fourier coefficients $c_n(f)$ are given by

$$c_n(f) = \langle f, \phi_n \rangle_{d\mu} = \int_X f(y)\overline{\phi_n(y)}\, d\mu(y).$$

One can ask for the same representation problem as in the classical Fourier theory, that is, when and in what sense the series (I.3) converges to the function f.

The next stage in the development of Harmonic Analysis is the seminal work [59] by E. M. Stein in 1970. The main goal was to carry out some topics in classical Harmonic Analysis such as Littlewood-Paley theory, Riesz transforms (generalisations of the Hilbert transform (I.2)), and maximal operators to the abstract setting of symmetric diffusion semigroups and Lie groups.

Harmonic Analysis has also been analysed from a discrete point of view. Let us focus on the one-dimensional case. It turns out that the mapping $f \mapsto \{\hat{f}(n)\}_{n\in\mathbb{N}}$ is an isometry from $L^2(0,1)$ to $\ell^2(\mathbb{Z})$. Therefore, for an appropriate sequence $f = \{f(n)\}_{n\in\mathbb{Z}}$ we can define the discrete Fourier transform by

$$\mathcal{F}f(x) = \sum_{n\in\mathbb{Z}} f(n)e^{2\pi inx}, \quad 0 \le x < 1,$$

and try to recover the sequence f from the inverse of the discrete Fourier transform

$$\mathcal{F}^{-1}g(n) = \int_0^1 g(x)e^{-2\pi inx}\, dx,$$

by the formula $\mathcal{F}^{-1}\mathcal{F}f(n)$.

This idea is the basis of discrete Fourier Analysis, the discrete counterpart of Fourier Analysis.

It may be say that this theory began with the work [52] by M. Riesz (see also [71]). He observed that the L^p-bound of the Hilbert transform (I.2) implies that of the discrete analogue $\widetilde{H}_{\mathrm{d}}$ given by

$$\widetilde{H}_{\mathrm{d}}f(n) = \frac{1}{\pi}\sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} \frac{f(m)}{n-m},$$

on $\ell^p(\mathbb{N})$. The same procedure is found in [14] were ℓ^p-bounds are obtained for discrete singular integrals from the L^p-bounds in the continuous setting. Moreover, one can apply this technique to other classical operators such as the discrete maximal function. However, there are some cases were it is not possible such an immediate conclusion (see for example [49]). A proper study of the latter was initiated with the work of J. Bourgain in the late eighties (see [12] and [13]) and was continued in several papers, for example, [61, 62, 63, 64], and [36].

Our work here aims to be a generalisation of the ones in [17] and [10]. The former is a study of discrete Harmonic Analysis associated with the discrete Laplacian

$$\Delta_{\mathrm{d}} f(n) = f(n-1) - 2f(n) + f(n+1), \qquad n \in \mathbb{Z}.$$

In particular, the authors study the heat semigroup related to Δ_{d} and mapping properties of the heat maximal operator. The same inequalities are obtained for the Poisson maximal operator by subordination. In addition, they analise the Littlewood-Paley-Stein g-functions and give weighted inequalities for them. The main tools to prove these results are semigroup theory, vector-valued Calderón-Zygmund theory in spaces of homogeneous type, and certain properties of the modified Bessel functions. They also consider the fractional Laplacian $(-\Delta_{\mathrm{d}})^{\sigma}$, $0 < \sigma < 1$, and prove maximum and comparison principles for it. Finally, the authors deal with the Riesz transforms. The definition is based on a factorization of Δ_{d} as a composition of first order difference operators. However, the usual construction of the Riesz transforms is not well defined in this setting and a limiting procedure is carried out. These operators turn out to be bounded on $\ell^p(\mathbb{N}, w)$ spaces and reduce to convolution operators with $\{1/(\pi(n+1/2))\}_{n\in\mathbb{Z}}$ or $\{1/(\pi(n-1/2))\}_{n\in\mathbb{Z}}$. One of them is precisely the discrete Hilbert transform (see [71] and Footnote 1 on page 40).

In [10], a study of discrete Harmonic Analysis associated with ultraspherical expansions is presented. They consider the discrete λ-Laplacian

$$\Delta_{\lambda} f(n) = a_{n-1}^{\lambda} f(n-1) - 2f(n) + a_n^{\lambda} f(n+1), \qquad n \in \mathbb{N}, \quad \lambda \geq 0,$$

where $f(-1) = 0$ and the elements of the sequence $\{a_n^{\lambda}\}_{n\in\mathbb{N}}$ are the ones involved in the three-term recurrence relation for the ultraspherical polynomials. In this way, the identity $\Delta_0 = \Delta_{\mathrm{d}}$ holds. That paper develops discrete vector-valued local Calderón-Zygmund theory which plays a fundamental role in proving mapping properties for the heat and Poisson maximal operators associated with Δ_{λ}, a transplantation result for ultraspherical coefficients, and the boundedness on $\ell^p(\mathbb{N}, w)$ of the Littlewood-Paley-Stein g_k-functions.

In this dissertation we consider a discrete Laplacian related to the three-term recurrence relation for Jacobi polynomials and we analise some classical operators in Harmonic Analysis for it. We are especially concern in the heat and Poisson maximal operators, the Riesz transforms, and Littlewood-Paley-Stein g_k-functions that arise in this discrete Jacobi setting. We have tried to make the present chapter independent of the rest of the dissertation. In this way, we will consider again and recall some topics or definitions that have already appeared in this chapter. The rest of the work is organised as follows:

In Chapter II we set some notation and we review some concepts and results of Jacobi polynomials that we will use throughout the next chapters. We reserve the final part of the chapter to include some aspects of the discrete vector-valued local Calderón-Zygmund theory of [10, Section 2]. All of this material is known and nothing original is found in it. Nevertheless, we believe that it is convenient to recall some of it in order to facilitate the reading.

In Chapter III we define the discrete Laplacian $\mathcal{J}^{(\alpha,\beta)}$ associated with the three-term recurrence for Jacobi polynomials that will be the main object in the dissertation. We analyse the initial-value problem for the heat equation and the heat semigroup associated with $\mathcal{J}^{(\alpha,\beta)}$ and we obtain some necessary conditions for the positivity of the heat semigroup $\{W_t^{(\alpha,\beta)}\}_{t\geq 0}$. We also note that it is possible to carry out some of these notions to the Jacobi matrix setting. Finally, we prove weighted ℓ^p inequalities for the heat and Poisson maximal operators, $1 < p < \infty$ and weak estimates when $p = 1$. The boundedness of the former is obtained by using the discrete vector-valued local Calderón-Zygmund theory of Chapter II. The latter is readily obtained by subordination.

In Chapter IV, we study the Riesz transforms. As in [17], we find an appropriate factorization of the operator $\mathcal{J}^{(\alpha,\beta)}$. Again, a limiting procedure is used to define the Riesz transforms. In the last part of the chapter we include mapping properties of the Riesz transforms in weighted $\ell^p(\mathbb{N}, w)$-spaces, $1 \leq p < \infty$ (weak estimates when $p = 1$). In this occasion, we apply the discrete local Calderón-Zygmund theory in the scalar case in the proof.

In Chapter V, we consider the Littlewood-Paley-Stein g_k-functions associated with both the heat and Poisson semigroups. By means of a duality argument, a transplantation theorem for Jacobi coefficients, and the classical vector-valued Calderón-Zygmund theory in spaces of homogeneous type we get weighted $\ell^p(\mathbb{N}, w)$-inequalities for the former, $1 < p < \infty$. The bound for the latter is deduced by expressing the g_k-functions associated with the Poisson semigroup in terms of the g_k-functions associated with the heat semigroup. As a consequence of these results, we obtain some corollaries about Laplace type multipliers and then, for imaginary powers of $\mathcal{J}^{(\alpha,\beta)}$.

At the end of the work it is possible to see the main conclusions and some further work as well as the publications related to the dissertation and the bibliography.

CHAPTER II

PRELIMINARIES

This chapter must be conceived as preparatory and it contains some notions, facts and results that we will use frequently in the forthcoming chapters. Most of them are well known and can be found in the wide specific literature dedicated to them (see, for example, [54, 20, 24, 67], [45, Section 6], and [10, Section 2]). We distinguish three main sections. First, we set the main spaces which we are interested in. Second, we review some well-known facts concerning Jacobi polynomials. Finally, the remaining section is devoted to discrete vector-valued local Calderón-Zygmund theory.

II.1 Notation and basic spaces

Let (X, Σ, μ) (also (X, μ) or simply X) be a positive measure space and p a positive real number. We define the classical Lebesgue spaces $L^p(X, d\mu)$ to be the class of all measurable functions f on X for which

$$\|f\|_{L^p(X,d\mu)} = \left(\int_X |f(x)|^p \, d\mu(x)\right)^{1/p} \tag{II.1}$$

is finite. As usual, we identify functions that are equal almost everywhere in X so that the elements of $L^p(X, d\mu)$ are in fact equivalence classes of functions with (II.1) finite. If $p = \infty$, we say that a measurable function f on X is essentially bounded on X if

$$\mu(\{x \in X : |f(x)| > C\}) = 0 \tag{II.2}$$

for some constant $C > 0$. We denote by $\|f\|_{L^\infty(X,d\mu)}$ the infimum of the constants C satisfying (II.2) and $L^\infty(X, d\mu)$ the class of all measurable functions f on X with $\|f\|_{L^\infty(X,d\mu)}$ finite.

In addition, when $p = 1$, we define the weak version of (II.1) by

$$\|f\|_{L^{1,\infty}(X,d\mu)} = \sup_{t>0} t\, \mu(\{x \in X : |f(x)| > t\})$$

and we define $L^{1,\infty}(X, d\mu)$ in a similar way to $L^1(X, d\mu)$ replacing $\|\cdot\|_{L^1(X,d\mu)}$ by $\|\cdot\|_{L^{1,\infty}(X,d\mu)}$.

Given $1 \le p \le \infty$, the conjugate exponent of p is denoted by p' and satisfies $1/p + 1/p' = 1$. Two important inequalities very useful for our purposes are Hölder's and Minkowski's inequalities.

Theorem II.1.1 (Hölder's inequality)**.** *Let (X, μ) be a positive measure space and $1 \le p \le \infty$. Let $f \in L^p(X, d\mu)$ and $g \in L^{p'}(X, d\mu)$. Then, $fg \in L^1(X, d\mu)$ and*

$$\|fg\|_{L^1(X,d\mu)} \le \|f\|_{L^p(X,d\mu)} \|g\|_{L^{p'}(X,d\mu)}.$$

Theorem II.1.2 (Minkowski's inequality)**.** *Let (X,μ) be a positive measure space and $1 \le p \le \infty$. Let $f \in L^p(X,d\mu)$ and $g \in L^p(X,d\mu)$. Then, $f+g \in L^p(X,d\mu)$ and*

$$\|f+g\|_{L^p(X,d\mu)} \le \|f\|_{L^p(X,d\mu)} + \|g\|_{L^p(X,d\mu)}.$$

Theorem II.1.2 can be proved via Theorem II.1.1 (see [54, Chapter 3]). It is now clear that the functional $\|\cdot\|_{L^p(X,d\mu)}$ is a norm on $L^p(X,d\mu)$, $1 \le p \le \infty$. Furthermore, we have that $L^p(X,d\mu)$ is a Banach space for $1 \le p \le \infty$.

Another important inequality is the so-called Minkowski's inequality for integrals.

Theorem II.1.3 (Minkowski's inequality for integrals)**.** *Let (X,μ) and (Y,ν) be measures spaces with σ-finite positive measures and $f(x,y)$ a measurable function on the σ-finite product measure space $X \times Y$. Then,*

$$\left(\int_Y \left|\int_X f(x,y)\,d\mu(x)\right|^p d\nu(y)\right)^{1/p} \le \int_X \left(\int_Y |f(x,y)|^p\,d\nu(y)\right)^{1/p} d\mu(x).$$

When the underlying measure μ is the counting measure on a countable set A, it is common to write $\ell^p(A)$ instead of $L^p(A,d\mu)$. In this particular case the elements of $\ell^p(A)$ are sequences $f = \{f(n)\}_{n\in A}$ satisfying

$$\|f\|_{\ell^p(A)} = \left(\sum_{n\in A} |f(n)|^p\right)^{1/p} < \infty, \qquad 1 \le p < \infty,$$

and

$$\|f\|_{\ell^\infty(A)} = \sup_{n\in A} |f(n)| < \infty.$$

Set $A = \mathbb{N} = \{0,1,2,3,\dots\}$. A weight on $\mathbb{N}$ is a strictly positive sequence $w = \{w(n)\}_{n\in\mathbb{N}} = \{w(n)\}_{n\ge 0}$. We define the weighted spaces $\ell^p(\mathbb{N},w)$ by

$$\ell^p(\mathbb{N},w) = \left\{ f = \{f(n)\}_{n\ge 0} : \|f\|_{\ell^p(\mathbb{N},w)} := \left(\sum_{m=0}^{\infty} |f(m)|^p w(m)\right)^{1/p} < \infty \right\},$$

$1 \le p < \infty$, and the weighted weak space $\ell^{1,\infty}(\mathbb{N},w)$ by

$$\ell^{1,\infty}(\mathbb{N},w) = \left\{ f = \{f(n)\}_{n\ge 0} : \|f\|_{\ell^{1,\infty}(\mathbb{N},w)} := \sup_{t>0} t \sum_{\{m\in\mathbb{N}:|f(m)|>t\}} w(m) < \infty \right\},$$

and we simply write $\ell^p(\mathbb{N})$ and $\ell^{1,\infty}(\mathbb{N})$ when $w(n) = 1$ for all $n \in \mathbb{N}$.

In this context, we say that a weight $w = \{w(n)\}_{n\ge 0}$ belongs to the discrete Muckenhoupt $A_p(\mathbb{N})$ class provided

$$\sup_{\substack{0\le n\le m\\ n,m\in\mathbb{N}}} \frac{1}{(m-n+1)^p} \left(\sum_{k=n}^{m} w(k)\right) \left(\sum_{k=n}^{m} w(k)^{-1/(p-1)}\right)^{p-1} < \infty,$$

for $1 < p < \infty$, and

$$\sup_{\substack{0\le n\le m\\ n,m\in\mathbb{N}}} \frac{1}{m-n+1} \left(\sum_{k=n}^{m} w(k)\right) \max_{n\le k\le m} w(k)^{-1} < \infty,$$

for $p = 1$.

II.2 Jacobi polynomials

Jacobi polynomials were introduced for the first time by C. G. J. Jacobi in *Untersuchungen über die Differentialgleichung der hypergeometrischen Reihe* [32] in 1859. He arrived at such a definition while studying Gauss's hypergeometric differential equation.

Given α and β real numbers and $n \in \mathbb{N}$, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ may be defined by means of Rodrigues' formula (see [67, p. 67, eq. (4.3.1)]), by

$$(\text{II.3}) \qquad P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n\, n!\,(1-x)^\alpha(1+x)^\beta}\frac{d^n}{dx^n}\Big((1-x)^{\alpha+n}(1+x)^{\beta+n}\Big).$$

Note that we can apply Leibniz' rule to calculate the nth derivative in this identity and then,

$$P_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n}\binom{n+\alpha}{n-j}\binom{n+\beta}{j}\left(\frac{x-1}{2}\right)^j\left(\frac{x+1}{2}\right)^{n-j},$$

where the symbol $\binom{a}{b}$ stands for the binomial coefficient. From this expression we readily obtain the normalisation

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}.$$

For $\alpha, \beta > -1$, Jacobi polynomials are orthogonal on the interval $[-1,1]$ with respect to the measure

$$d\mu_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta\, dx.$$

By using Rodrigues' formula and integrating by parts n times it is not difficult to see that

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) = (w_n^{(\alpha,\beta)})^{-2}\delta_{nm},$$

where

$$\begin{aligned}(w_n^{(\alpha,\beta)})^{-2} &= \|P_n^{(\alpha,\beta)}\|^2_{L^2([-1,1],d\mu_{\alpha,\beta})}\\ &= \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}\end{aligned}$$

(if $n=0$, we replace the product $(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)$ by $\Gamma(n+\alpha+\beta+2)$) and we have denoted Kronecker's delta by δ_{nm} (that is, 0 if $n \neq m$ and 1 if $n = m$).

Therefore, the family $\{p_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$, given by $p_n^{(\alpha,\beta)}(x) = w_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(x)$ is a complete orthonormal system in the space $L^2([-1,1], d\mu_{\alpha,\beta})$. We will adopt the notation $P_n^{(\alpha,\beta)}$ (capital letter) for the (orthogonal) Jacobi polynomials and $p_n^{(\alpha,\beta)}$ (small letter) for the orthonormalised Jacobi polynomials in the rest of this dissertation.

It turns out that there exists a three-term recurrence relation for the orthogonal Jacobi polynomials, namely [67, eq. (4.5.1), p. 71]

(II.4)

$$\begin{aligned}&2n(n+\alpha+\beta)(2n+\alpha+\beta-2)P_n^{(\alpha,\beta)}(x)\\ &\quad= (2n+\alpha+\beta-1)\Big((2n+\alpha+\beta)(2n+\alpha+\beta-2)x+\alpha^2-\beta^2\Big)P_{n-1}^{(\alpha,\beta)}(x)\\ &\qquad - 2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)P_{n-2}^{(\alpha,\beta)}(x)\end{aligned}$$

for $n \geq 2$, with

$$P_0^{(\alpha,\beta)}(x) = 1, \quad \text{and} \quad P_1^{(\alpha,\beta)}(x) = \frac{\alpha+\beta+2}{2}x + \frac{\alpha-\beta}{2}.$$

An important fact is that the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy the linear homogeneous differential equation of second order

$$(1-x^2)\frac{d^2y}{dx^2} + (\beta-\alpha-(\alpha+\beta+2)x)\frac{dy}{dx} + n(n+\alpha+\beta+1)y = 0.$$

Thus, we define the differential operator $L^{\alpha,\beta}$ by

$$L^{\alpha,\beta} = -(1-x^2)\frac{d^2}{dx^2} - (\beta-\alpha-(\alpha+\beta+2)x)\frac{d}{dx}.$$

Clearly, the Jacobi polynomials are eigenfunctions of $L^{\alpha,\beta}$ with eigenvalue $\lambda_n^{(\alpha,\beta)} = n(n+\alpha+\beta+1)$, i.e.,

$$L^{\alpha,\beta}p_n^{(\alpha,\beta)} = \lambda_n^{(\alpha,\beta)}p_n^{(\alpha,\beta)}.$$

It is well known that $L^{\alpha,\beta}$ is a symmetric operator on the domain $C_c^2(-1,1) \subset L^2([-1,1], d\mu_{\alpha,\beta})$ and for some interval $[r,s] \subset [-1,1]$, $r<s$, it is satisfied that

$$\text{(II.5)} \qquad \int_r^s f(x)L^{\alpha,\beta}g(x)\,d\mu_{\alpha,\beta}(x) = U_{\alpha,\beta}(f,g)(x)\Big|_{x=r}^{x=s} + \int_r^s g(x)L^{\alpha,\beta}f(x)\,d\mu_{\alpha,\beta}(x),$$

with

$$U_{\alpha,\beta}(f,g)(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1}\Big(g(x)\frac{df}{dx}(x) - f(x)\frac{dg}{dx}(x)\Big).$$

To prove this identity put

$$L^{\alpha,\beta}g(x) = \frac{-1}{(1-x)^\alpha(1+x)^\beta}\frac{d}{dx}\left((1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{dg}{dx}(x)\right)$$

and then integrate by parts twice to get

$$\begin{aligned}\int_r^s f(x)L^{\alpha,\beta}g(x)\,d\mu_{\alpha,\beta}(x) &= -\int_r^s f(x)\frac{d}{dx}\left((1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{dg}{dx}(x)\right)dx \\ &= U_{\alpha,\beta}(f,g)(x)\Big|_{x=r}^{x=s} + \int_r^s g(x)L^{\alpha,\beta}f(x)\,d\mu_{\alpha,\beta}(x).\end{aligned}$$

Moreover, we have some connection formulas between $L^{\alpha,\beta}$. More precisely,

(II.6)

$$\begin{aligned}L^{\alpha,\beta}(h_1h_2)(x) = h_2(x)L^{\alpha+1,\beta}h_1(x) - (1+x)h_2(x)\frac{dh_1}{dx}(x) - 2(1-x^2)\frac{dh_1}{dx}(x)\frac{dh_2}{dx}(x) \\ - (1-x^2)h_1(x)\frac{d^2h_2}{dx^2}(x) - (\beta-\alpha-(\alpha+\beta+2)x)h_1(x)\frac{dh_2}{dx}(x)\end{aligned}$$

and

(II.7)
$$L^{\alpha+1,\beta}(h_1h_2)(x) = h_2(x)L^{\alpha,\beta}h_1(x) + (1+x)h_2(x)\frac{dh_1}{dx}(x) - 2(1-x^2)\frac{dh_1}{dx}(x)\frac{dh_2}{dx}(x)$$
$$- (1-x^2)h_1(x)\frac{d^2h_2}{dx^2}(x) - (\beta-\alpha-1-(\alpha+\beta+3)x)h_1(x)\frac{dh_2}{dx}(x).$$

It is very useful to have identities and estimates for the Jacobi polynomials and its derivatives. Here we include some of them which we will use later.

First, note that

$$\text{(II.8)} \qquad w_n^{(\alpha,\beta)} \simeq n^{1/2}, \qquad \alpha,\beta \geq -\frac{1}{2}, \quad n > 0,$$

and $w_0^{(\alpha,\beta)} = C$.

For the Jacobi polynomials we have the following identities (see [67, p. 94, eq. (4.10.1)] or [48, p. 446, eq. 18.9.16] and [67, p. 71, eq. (4.5.4)]):

$$\text{(II.9)} \quad \frac{d}{dx}\Big((1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)\Big) = -2(n+1)(1-x)^{\alpha-1}(1+x)^{\beta-1}P_{n+1}^{(\alpha-1,\beta-1)}(x)$$

and

$$\text{(II.10)} \quad \frac{2n+\alpha+\beta+2}{2}(1-x)P_n^{(\alpha+1,\beta)}(x) = (n+\alpha+1)P_n^{(\alpha,\beta)}(x) - (n+1)P_{n+1}^{(\alpha,\beta)}(x),$$

where $\alpha,\beta > -1$. In addition, from [48, 18.9.6] it is possible to conclude that

(II.11)
$$-\frac{2n+\alpha+\beta+2}{2}(1-x)P_n^{(\alpha+1,\beta)}(x) + \alpha P_n^{(\alpha,\beta)}(x) = (n+1)(P_{n+1}^{(\alpha,\beta)}(x) - P_n^{(\alpha,\beta)}(x)),$$

where $\alpha,\beta > -1$.

There is also a formula for the derivatives of the Jacobi polynomials (see [48, 18.9.15] or [67, p. 63, eq. (4.21.7)])

$$\text{(II.12)} \qquad \frac{dP_n^{(a,b)}}{dx}(x) = \frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x), \qquad n > 0.$$

Finally, the estimates of the Jacobi polynomials are given by (see [40, eq. (2.6) and (2.7)])

(II.13)
$$|p_n^{(\alpha,\beta)}(x)| \leq C\begin{cases}(n+1)^{\alpha+1/2}, & 1-1/(n+1)^2 < x < 1,\\ (1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}, & -1+1/(n+1)^2 \leq x \leq 1-1/(n+1)^2,\\ (n+1)^{\beta+1/2}, & -1 < x < -1+1/(n+1)^2,\end{cases}$$

where C is a constant independent of n and x. Note that for $\alpha,\beta \geq -1/2$ the previous bound can be replaced by the simpler one (see [67, eq. 7.32.6])

$$\text{(II.14)} \qquad |p_n^{(\alpha,\beta)}(x)| \leq C(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}, \qquad -1 \leq x \leq 1.$$

Since $L^2(X, d\mu)$ is a Hilbert space with the usual inner product, we also have some facts about Fourier theory that can be adapted to the Jacobi setting. As it is well known, for each function $f \in L^2([-1,1], d\mu_{\alpha,\beta})$ its Fourier-Jacobi coefficients are given by

$$c_m^{(\alpha,\beta)}(f) = \int_{-1}^{1} f(x) p_m^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x)$$

and

$$f(x) = \sum_{m=0}^{\infty} c_m^{(\alpha,\beta)}(f) p_m^{(\alpha,\beta)}(x),$$

where the equality holds in $L^2([-1,1], d\mu_{\alpha,\beta})$. Moreover, $\{c_m^{(\alpha,\beta)}(f)\}_{m\geq 0}$ is a sequence in $\ell^2(\mathbb{N})$. Conversely, for each sequence $f \in \ell^2(\mathbb{N})$, the function

$$F_{\alpha,\beta}(x) = \sum_{m=0}^{\infty} f(m) p_m^{(\alpha,\beta)}(x) \tag{II.15}$$

belongs to $L^2([-1,1], d\mu_{\alpha,\beta})$ and Parseval's identity

$$\|f\|_{\ell^2(\mathbb{N})} = \|F_{\alpha,\beta}\|_{L^2([-1,1],d\mu_{\alpha,\beta})} \tag{II.16}$$

holds. Moreover, $c_m^{(\alpha,\beta)}(F_{\alpha,\beta}) = f(m)$.

Note that an obvious consequence of (II.16) is the useful polarisation type identity

$$\sum_{m=0}^{\infty} f(m)\overline{g(m)} = \int_{-1}^{1} F_{\alpha,\beta}(x)\overline{G_{\alpha,\beta}(x)}\, d\mu_{\alpha,\beta}(x), \qquad f, g \in \ell^2(\mathbb{N}), \tag{II.17}$$

where $F_{\alpha,\beta}$ is given by (II.15) and $G_{\alpha,\beta}$ is defined in a similar way.

The Jacobi polynomials form one of the so-called classical orthogonal families of polynomials for which there is a vast theory (see for example [67] and [15]). To conclude this section we mention some specific cases of the Jacobi polynomials which are of interest in this dissertation. For $\alpha = \beta = \lambda - 1/2$, $\lambda > -1/2$ we obtain the ultraspherical (or Gegenbauer) polynomials $C_n^\lambda(x)$. For $\alpha = \beta = -1/2$ we have the Chebyshov polynomials of the first kind, i.e.,

$$p_n^{(-1/2,-1/2)}(x) = \sqrt{\frac{2}{\pi}} T_n(x) = \sqrt{\frac{2}{\pi}} \cos(n\theta), \tag{II.18}$$

for $n \neq 0$ and where $x = \cos\theta$, $0 \leq \theta \leq \pi$, and $p_0^{(-1/2,-1/2)}(x) = 1/\sqrt{\pi} T_0(x) = 1/\sqrt{\pi}$. Finally, the Legendre polynomials P_n appear when $\alpha = \beta = 0$.

II.3 Discrete vector-valued local Calderón-Zygmund theory

In the 1950s, A. P. Calderón and A. Zygmund [14] developed real-variable techniques to obtain boundedness properties of the so-called Calderón-Zygmund operators (of convolution type) on $\mathbb{R}^d$. The subsequent years brought further research on the topic and consequently a better understanding of it.

All of this was mainly achieved on the Euclidean space $\mathbb{R}^d$ for complex-valued functions. A. Benedek, A. P. Calderón, and R. Panzone [9] considered a more general setting consisting on Banach space valued Calderón-Zygmund operators of convolution type on $\mathbb{R}^d$ but applied the results essentially to Hilbert spaces. In [53] the authors revisited the theory in [9] and presented an extent account of the Calderón-Zygmund theory in a vector-valued setting which allows the use of weights. Later on, two of those authors [55] investigated vector-valued singular integrals on spaces of homogeneous type in the sense of R. Coifman and G. Weiss [19]. In this spirit, see also the paper by L. Grafakos, L. Liu, and D. Yang [28].

In [45], A. Nowak and K. Stempak introduced the notion of local Calderón-Zygmund operator in dimension one to obtain weighted mapping properties for the Hankel transform transplantation operator. Some aspects of the local Calderón-Zygmund theory of [45, Section 4] have their discrete counterpart in [10, Section 2] and have been stated in a vector-valued setting. In this section we are interested in this discrete vector-valued local Calderón-Zygmund theory and we believe that it is more convenient for the reader to recall briefly some of the notions and results of this theory below.

Suppose that $\mathbb{B}_1$ and $\mathbb{B}_2$ are Banach spaces. We denote by $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$ the space of bounded linear operators from $\mathbb{B}_1$ into $\mathbb{B}_2$. Let us suppose that

$$K : (\mathbb{N} \times \mathbb{N}) \setminus D \longrightarrow \mathcal{L}(\mathbb{B}_1, \mathbb{B}_2),$$

where $D := \{(n,n) : n \in \mathbb{N}\}$, is (strongly) measurable (see [28] or [20, p. 105]) and that for certain positive constant C and for each n, $m \in \mathbb{N}$, the following conditions hold:

(a) The size condition:

$$\|K(n,m)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)} \leq \frac{C}{|n-m|};$$

(b) The regularity properties:

(b1) for $|n-m| > 2|m-l|$ and $2m/3, 2l/3 \leq n \leq 3m/2, 3l/2$,

$$\|K(n,m) - K(n,l)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)} \leq C\frac{|m-l|}{|n-m|^2},$$

(b2) for $|n-m| > 2|n-s|$ and $2n/3, 2s/3 \leq m \leq 3n/2, 3s/2$,

$$\|K(n,m) - K(s,m)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)} \leq C\frac{|n-s|}{|n-m|^2}.$$

We say that a kernel K is a semi-local $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$-standard kernel if it satisfies conditions (a) and (b). A local $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$-standard kernel $K(n,m)$ will be a semi-local $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$-standard kernel supported in the region $2n/3 \leq m \leq 3n/2$. Local $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$-standard kernels are discrete vector-valued analogues of the local Calderón-Zygmund kernels introduced in [45, Definition 4.1]. Here we have followed the terminology used in [45] and we have added the prefix “semi” to refer to the kernels in [10].

These kind of kernels define Calderón-Zygmund operators in a natural way. Let us denote by $\mathbb{B}_0^{\mathbb{N}}$ the space of $\mathbb{B}$-valued sequences $f = \{f(n)\}_{n\geq 0}$ such that there exists $j \in \mathbb{N}$ such that $f(n) = 0$ for all $n > j$. By a (discrete) semi-local Calderón-Zygmund operator we mean a linear and bounded operator T from $\ell^r_{\mathbb{B}_1}(\mathbb{N})$ into $\ell^r_{\mathbb{B}_2}(\mathbb{N})$, for some $1 < r < \infty$, and such that there exists a semi-local $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$-standard kernel K such that, for every sequence $f \in (\mathbb{B}_1)_0^{\mathbb{N}}$,

$$Tf(n) = \sum_{m=0}^{\infty} K(n,m) \cdot f(m),$$

for every $n \in \mathbb{N}$ such that $f(n) = 0$.

A (discrete) local Calderón-Zygmund operator is a linear and bounded operator T from $\ell^r_{\mathbb{B}_1}(\mathbb{N})$ into $\ell^r_{\mathbb{B}_2}(\mathbb{N})$, for some $1 < r < \infty$, and such that there exists a local $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$-standard kernel K such that, for every sequence $f \in (\mathbb{B}_1)_0^{\mathbb{N}}$,

$$Tf(n) = \sum_{\substack{m\in\mathbb{N} \\ 2n/3 \leq m \leq 3n/2}} K(n,m) \cdot f(m),$$

for every $n \in \mathbb{N}$ such that $f(n) = 0$.

Here we are only interested in semi-local Calderón-Zygmund operators. To see some applications of local Calderón-Zygmund operators see [6] and [4].

For a Banach space $\mathbb{B}$ and a weight $w = \{w(n)\}_{n\geq 0}$, we consider the space

$$\ell^p_{\mathbb{B}}(\mathbb{N}, w) = \{\mathbb{B}\text{-valued sequences } f = \{f(n)\}_{n\geq 0} : \{\|f(n)\|_{\mathbb{B}}\}_{n\geq 0} \in \ell^p(\mathbb{N}, w)\}$$

for $1 \leq p < \infty$, and

$$\ell^{1,\infty}_{\mathbb{B}}(\mathbb{N}, w) = \left\{\mathbb{B}\text{-valued sequences } f = \{f(n)\}_{n\geq 0} : \{\|f(n)\|_{\mathbb{B}}\}_{n\geq 0} \in \ell^{1,\infty}(\mathbb{N}, w)\right\};$$

in both cases, and in what follows, we tacitly assume that the sequences f are (strongly) measurable. As usual, we simply write $\ell^p_{\mathbb{B}}(\mathbb{N})$ and $\ell^{1,\infty}_{\mathbb{B}}(\mathbb{N})$ when $w(n) = 1$ for all $n \in \mathbb{N}$.

We conclude this section stating the main result concerning discrete vector-valued local Calderón-Zygmund theory. In forthcoming sections we will always apply it in the case when $r = 2$ in the definition of semi-local Calderón-Zygmund operator but we present it here in full generality. In addition, we give the proof to make the dissertation as self-contained as possible.

Theorem II.3.1 (Theorem 2.1 in [10]). *Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be Banach spaces and T a semi-local Calderón-Zygmund operator. Then,*

(i) for every $1 < p < \infty$ and $w \in A_p(\mathbb{N})$ the operator T can be extended from $\ell^r_{\mathbb{B}_1}(\mathbb{N}) \cap \ell^p_{\mathbb{B}_1}(\mathbb{N}, w)$ to $\ell^p_{\mathbb{B}_1}(\mathbb{N}, w)$ as a bounded operator from $\ell^p_{\mathbb{B}_1}(\mathbb{N}, w)$ into $\ell^p_{\mathbb{B}_2}(\mathbb{N}, w)$.

(ii) for every $w \in A_1(\mathbb{N})$ the operator T can be extended from $\ell^r_{\mathbb{B}_1}(\mathbb{N}) \cap \ell^1_{\mathbb{B}_1}(\mathbb{N}, w)$ to $\ell^1_{\mathbb{B}_1}(\mathbb{N}, w)$ as a bounded operator from $\ell^1_{\mathbb{B}_1}(\mathbb{N}, w)$ into $\ell^{1,\infty}_{\mathbb{B}_2}(\mathbb{N}, w)$.

Proof. First, we split the operator in the local and global version of it by defining

$$T_{\text{glob}}f(n) = T(\chi_{\mathbb{N}\setminus W_n} f)(n), \qquad n \in \mathbb{N},$$

and

$$T_{\text{loc}}f(n) = Tf(n) - T_{\text{glob}}f(n), \qquad n \in \mathbb{N},$$

for all $f = \{f(n)\}_{n\in\mathbb{N}} \in (\mathbb{B}_1)_0^{\mathbb{N}}$ and where

$$W_n = \{j \in \mathbb{N} : 2n/3 \le j \le 3n/2\}.$$

We first deal with the global part. Let $f \in (\mathbb{B}_1)_0^{\mathbb{N}}$ and note that

$$T_{\text{glob}}f(n) = \sum_{m\in\mathbb{N}\setminus W_n} K(n,m)\cdot f(m)$$

since $\chi_{\mathbb{N}\setminus W_n}(n) = 0$ for all $n \in \mathbb{N}$. By the size condition (a) we have that (we adopt the usual convention for empty sums)

$$\begin{aligned}\|T_{\text{glob}}f(n)\|_{\mathbb{B}_2} \le C \sum_{m\in\mathbb{N}\setminus W_n} \frac{\|f(m)\|_{\mathbb{B}_1}}{|n-m|} &\le \frac{C}{n+1}\sum_{\substack{m\in\mathbb{N}\\ m<2n/3}} \|f(m)\|_{\mathbb{B}_1} + C\sum_{\substack{m\in\mathbb{N}\\ m>3n/2}} \frac{\|f(m)\|_{\mathbb{B}_1}}{m+1}\\ &\le C\left(P(\|f\|_{\mathbb{B}_1})(n) + Q(\|f\|_{\mathbb{B}_1})(n)\right),\end{aligned}$$

where $\|f\|_{\mathbb{B}_1} = \{\|f(m)\|_{\mathbb{B}_1}\}_{m\in\mathbb{N}}$ and P and Q are the discrete Hardy operators defined by

$$Pg(n) = \frac{1}{n+1}\sum_{m=0}^{n} g(m)$$

and

$$Qg(n) = \sum_{m=n}^{\infty} \frac{g(m)}{m+1},$$

for $g = \{g(m)\}_{m\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$. These operators are bounded on $\ell^p(\mathbb{N},w)$ (see [1]) so we have that T_{glob} can be extended to $\ell^p_{\mathbb{B}_1}(\mathbb{N},w)$ as a bounded operator from $\ell^p_{\mathbb{B}_1}(\mathbb{N},w)$ into $\ell^p_{\mathbb{B}_2}(\mathbb{N},w)$ if $1<p<\infty$ and $w \in A_p(\mathbb{N})$, and to $\ell^1_{\mathbb{B}_1}(\mathbb{N},w)$ as a bounded operator from $\ell^1_{\mathbb{B}_1}(\mathbb{N},w)$ into $\ell^{1,\infty}_{\mathbb{B}_2}(\mathbb{N},w)$.

We analyse now the local operator T_{loc}. Given $f \in (\mathbb{B}_1)_0^{\mathbb{N}}$, for all $n \in \mathbb{N}$ such that $f(n) = 0$ we have

$$T_{\text{loc}}f(n) = \sum_{m\in W_n} K(n,m)\cdot f(m).$$

For $n,m \in \mathbb{N}$, $n \ne m$, define the local kernel $\widetilde{K}(n,m) = \chi_{W_n}(m)K(n,m)$. Note that $\chi_{W_n}(m) = \chi_{W_m}(n)$.

It turns out that $\widetilde{K}$ satisfies certain Hörmander type conditions that are natural discrete (vector-valued) analogues of (4.4) and (4.5) in [45]. More precisely, if $a,b \in \mathbb{N}$, $a \le b$, an interval in $\mathbb{N}$ is given by $I = [a,b]\cap\mathbb{N}$ and we set $I = \emptyset$ if $b < a$. Furthermore, we will denote by $2I$ the interval

$$2I = \left[a - \frac{b-a}{2}, b + \frac{b-a}{2}\right] \cap \mathbb{N} = \left[\frac{3a-b}{2}, \frac{3b-a}{2}\right] \cap \mathbb{N}.$$

In this way, we have that

$$\sum_{n\in\mathbb{N}\setminus 2I}\|\tilde{K}(n,m)-\tilde{K}(n,l)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)}\|f(n)\|_{\mathbb{B}_1}\leq C\mathcal{M}(\|f\|_{\mathbb{B}_1})(m),\qquad m,l\in I, \tag{II.19}$$

and

$$\sum_{m\in\mathbb{N}\setminus 2I}\|\tilde{K}(n,m)-\tilde{K}(s,m)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)}\|f(m)\|_{\mathbb{B}_1}\leq C\mathcal{M}(\|f\|_{\mathbb{B}_1})(n),\qquad n,s\in I, \tag{II.20}$$

for all intervals I in $\mathbb{N}$, $f\in(\mathbb{B}_1)_0^{\mathbb{N}}$, and where $\mathcal{M}$ denotes the discrete Hardy-Littlewood maximal function given by

$$\mathcal{M}g(n)=\sup_{\substack{I\text{ interval}\\ n\in I}}\frac{1}{\#(I)}\sum_{m\in I}g(m),\qquad n\in\mathbb{N}.$$

The proof of (II.19) and (II.20) follows essentially the same ideas of [45, Proposition 4.1]. Moreover, the proofs of both Hörmander type conditions are similar so we only prove (II.19).

Let $a,b\in\mathbb{N}$, $a<b$, $I=[a,b]\cap\mathbb{N}$, and $f\in(\mathbb{B}_1)_0^{\mathbb{N}}$. We assume that $m,l\in I$ and $m<l$. The case $l<m$ is similar.

First, observe that we have that

$$\frac{|m-n|}{3}<|l-n|<3|m-n| \tag{II.21}$$

if $n\in\mathbb{N}\setminus I$. For a proof of this fact see [10, eq. (20)]. Now we split the sum in (II.19) into three terms:

$$\begin{aligned}\sum_{n\in\mathbb{N}\setminus 2I}&\|\tilde{K}(n,m)-\tilde{K}(n,l)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)}\|f(n)\|_{\mathbb{B}_1}\\ &=\sum_{\substack{n\in\mathbb{N}\setminus 2I\\ n\in W_m\cap W_l}}\|K(n,m)-K(n,l)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)}\|f(n)\|_{\mathbb{B}_1}\\ &\quad+\sum_{\substack{n\in\mathbb{N}\setminus 2I\\ n\in W_m\setminus W_l}}\|K(n,m)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)}\|f(n)\|_{\mathbb{B}_1}\\ &\quad+\sum_{\substack{n\in\mathbb{N}\setminus 2I\\ n\in W_l\setminus W_m}}\|K(n,l)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)}\|f(n)\|_{\mathbb{B}_1}\\ &=:S_1(m,l)+S_2(m,l)+S_3(m,l).\end{aligned} \tag{II.22}$$

We consider two cases. If $9m<4l$, we have that $W_m\cap W_l=\emptyset$ and then $S_1(m,l)=0$. In addition, by using (a) and (II.21) we obtain that

$$S_2(m,l)+S_3(m,l)\leq\sum_{\substack{n\in\mathbb{N}\setminus 2I\\ n\in W_m}}\frac{\|f(n)\|_{\mathbb{B}_1}}{|n-m|}+\sum_{\substack{n\in\mathbb{N}\setminus 2I\\ n\in W_l}}\frac{\|f(n)\|_{\mathbb{B}_1}}{|n-m|}.$$

Note that if $n\in\mathbb{N}\setminus 2I$, $|n-m|>(b-a)/2\geq(l-m)/2>5l/18$ and then,

$$S_2(m,l)+S_3(m,l)\leq\frac{C}{l}\sum_{\substack{n\in\mathbb{N}\setminus 2I\\ n\in W_m\cup W_l}}\|f(n)\|_{\mathbb{B}_1}\leq\frac{C}{l}\sum_{n\in J}\|f(n)\|_{\mathbb{B}_1},$$

where $J = [2m/3, 3l/2] \cap \mathbb{N}$. Since $3l/2 \geq 3l/2 - 2m/3 > 65l/54$,

$$S_2(m,l) + S_3(m,l) \leq C\mathcal{M}(\|f\|_{\mathbb{B}_1})(m). \tag{II.23}$$

Now we suppose that $9m \geq 4l$. In this case $m \neq 0$ because $m < l$. We have that

$$W_m \cap W_l = \left[\frac{2l}{3}, \frac{3m}{2}\right] \cap \mathbb{N},$$
$$W_m \setminus W_l = \left[\frac{2m}{3}, \frac{2l}{3}\right] \cap \mathbb{N},$$

and

$$W_l \setminus W_m = \left[\frac{3m}{2}, \frac{3l}{2}\right] \cap \mathbb{N}.$$

Clearly, $l/3 < l - n$ if $n \in W_m \setminus W_l$ and $m/2 < m - n$ if $n \in W_l \setminus W_m$. So, by (a) and (II.21),
(II.24)

$$\begin{aligned} S_2(m,l) + S_3(m,l) &\leq C \sum_{\substack{n\in\mathbb{N}\setminus 2I \\ 2m/3\leq n<2l/3}} \frac{\|f(n)\|_{\mathbb{B}_1}}{|l-n|} + C \sum_{\substack{n\in\mathbb{N}\setminus 2I \\ 3m/2<n\leq 3l/2}} \frac{\|f(n)\|_{\mathbb{B}_1}}{|m-n|} \\ &\leq \frac{C}{l} \sum_{\substack{n\in\mathbb{N} \\ 1\leq n\leq l}} \|f(n)\|_{\mathbb{B}_1} + \frac{C}{m} \sum_{\substack{n\in\mathbb{N} \\ m\leq n\leq 4m}} \|f(n)\|_{\mathbb{B}_1} \leq C\mathcal{M}(\|f\|_{\mathbb{B}_1})(m). \end{aligned}$$

To study S_1 we decompose the sum into two terms:

$$\begin{aligned} S_1(m,l) = &\sum_{\substack{n\in\mathbb{N}\setminus 2I \\ 2l/3\leq n\leq 3m/2 \\ |n-m|\leq 2|m-l|}} \|K(n,m) - K(n,l)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)} \|f(n)\|_{\mathbb{B}_1} \\ &+ \sum_{\substack{n\in\mathbb{N}\setminus 2I \\ 2l/3\leq n\leq 3m/2 \\ |n-m|>2|m-l|}} \|K(n,m) - K(n,l)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)} \|f(n)\|_{\mathbb{B}_1} \\ =:& \, S_{1,1}(m,l) + S_{1,2}(m,l). \end{aligned} \tag{II.25}$$

To estimate $S_{1,1}$ we use (a) and (II.21) to get

$$S_{1,1}(m,l) \leq C \sum_{n\in\mathbb{N}\setminus 2I} \frac{|m-l|}{|n-m|^2} \|f(n)\|_{\mathbb{B}_1}$$

and the same estimate holds for $S_{1,2}$ by using (b1). Therefore,
(II.26)

$$\begin{aligned} S_{1,1}(m,l) + S_{1,2}(m,l) &\leq C \sum_{k=1}^{\infty} \sum_{n\in 2^{k+1}I\setminus 2^k I} \frac{|m-l|}{|n-m|^2} \|f(n)\|_{\mathbb{B}_1} \\ &\leq C \sum_{k=1}^{\infty} \frac{\#(I)}{2^{2k}(\#(I))^2} \sum_{n\in 2^{k+1}I} \|f(n)\|_{\mathbb{B}_1} \leq C\mathcal{M}(\|f\|_{\mathbb{B}_1})(m). \end{aligned}$$

The Hörmander type condition (II.19) follows from (II.22), (II.23), (II.24), (II.25), and (II.26).

Now, it is clear that T_{loc} is bounded from $\ell^r_{\mathbb{B}_1}(\mathbb{N})$ into $\ell^r_{\mathbb{B}_2}(\mathbb{N})$ because so are T and T_{glob}. By Theorem 1.1 in [28] (note that $(\mathbb{N}, \mu_{\mathrm{c}}, |\cdot|)$, where μ_{c} is the counting measure and $|\cdot|$ is the usual metric on $\mathbb{N}$, is a space of homogeneous type), (II.19), and (II.20), for $1 < p < \infty$, T_{loc} can be extended from $\ell^r_{\mathbb{B}_1}(\mathbb{N}) \cap \ell^p_{\mathbb{B}_1}(\mathbb{N})$ to $\ell^p_{\mathbb{B}_1}(\mathbb{N})$ as a bounded operator from $\ell^p_{\mathbb{B}_1}(\mathbb{N})$ into $\ell^p_{\mathbb{B}_2}(\mathbb{N})$ and T_{loc} can be extended from $\ell^r_{\mathbb{B}_1}(\mathbb{N}) \cap \ell^1_{\mathbb{B}_1}(\mathbb{N})$ to $\ell^1_{\mathbb{B}_1}(\mathbb{N})$ as a bounded operator from $\ell^1_{\mathbb{B}_1}(\mathbb{N})$ into $\ell^{1,\infty}_{\mathbb{B}_2}(\mathbb{N})$. Moreover, these properties also hold for T because T_{glob} also verifies them.

Finally, by adapting the arguments in Lemmas 5.15, 7.9, and 7.10, and Theorems 7.11 and 7.12 in [20] to vector-valued homogeneous settings, we conclude that T_{loc}, and therefore T, can be extended from $\ell^r_{\mathbb{B}_1}(\mathbb{N}) \cap \ell^p_{\mathbb{B}_1}(\mathbb{N}, w)$ to $\ell^p_{\mathbb{B}_1}(\mathbb{N}, w)$ as bounded operators from $\ell^p_{\mathbb{B}_1}(\mathbb{N}, w)$ into $\ell^p_{\mathbb{B}_2}(\mathbb{N}, w)$, for $1 < p < \infty$ and $w \in A_p(\mathbb{N})$, and from $\ell^r_{\mathbb{B}_1}(\mathbb{N}) \cap \ell^1_{\mathbb{B}_1}(\mathbb{N}, w)$ to $\ell^1_{\mathbb{B}_1}(\mathbb{N}, w)$ as bounded operators from $\ell^1_{\mathbb{B}_1}(\mathbb{N}, w)$ into $\ell^{1,\infty}_{\mathbb{B}_2}(\mathbb{N}, w)$, for every $w \in A_1(\mathbb{N})$. □

CHAPTER III

THE HEAT SEMIGROUP

One of the most fundamental partial differential equations in Mathematics is the so-called heat equation. Its classical form is

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t)$$

where the spatial variable x is in (an open set U of) $\mathbb{R}^d$ and the time variable t is a non-negative real number. Here,

$$\begin{array}{rccc} u: & \mathbb{R}^d \times [0,+\infty) & \longrightarrow & \mathbb{R} \\ & (x,t) & \longmapsto & u(x,t) \end{array}$$

and the Laplacian is taken with respect to x. Physically speaking, the heat equation describes the variation in temperature in a given region over time. In this way, $u(x,t)$ is the temperature at position x at time t.

One typically considers the initial-value (or Cauchy's) problem

$$\text{(III.1)} \qquad \begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta u(x,t), \\ u(x,0) = f(x). \end{cases}$$

A solution, the heat semigroup, is given by the convolution of the initial data f and the so-called fundamental solution (or heat kernel)

$$\Phi(x,t) = \frac{1}{(4\pi t)^{d/2}} e^{\frac{-|x|^2}{4t}}, \qquad x \in \mathbb{R}^d, \quad t > 0.$$

More precisely,

$$u(x,t) = e^{t\Delta} f(x) = \int_{\mathbb{R}^d} f(y)\Phi(x-y,t)\,dy.$$

The heat equation may be regarded as the beginning of Harmonic Analysis. Indeed, the theory of classical Fourier Analysis emerged from the necessity of explaining the distribution of heat along a region over time. It was J.-B. J. Fourier who solved (III.1) introducing the method of separation of variables in [23].

As we have mentioned in Chapter I, the study of Harmonic Analysis associated with orthogonal expansions was initiated in the seminal paper by B. Muckenhoupt and E. M. Stein [41]. Briefly, let (X, Σ, μ) be a positive measurable space, $X \subset \mathbb{R}$, and $\{\phi_n\}_{n\geq 0}$ be a complete orthonormal system in $L^2(X, d\mu)$, that is,

$$\overline{\operatorname{span}}\{\phi_n\}_{n\geq 0} = L^2(X, d\mu) \qquad \text{(closure in } L^2(X, d\mu))$$

and

$$\langle \phi_n, \phi_m \rangle_{d\mu} = \int_X \phi_n(x)\overline{\phi_m(x)}\,d\mu(x) = \delta_{nm},$$

where we have denoted by "span" the set of all finite linear combinations of ϕ_n. In this setting, we construct the Fourier expansion of a suitable function f with respect to this system by the expression

$$\sum_{n=0}^{\infty} c_n(f)\phi_n(x),$$

where the Fourier coefficients of f are

$$c_n(f) = \langle f, \phi_n \rangle_{d\mu} = \int_X f(y)\overline{\phi_n(y)}\, d\mu(y).$$

The Jacobi setting was revisited using the semigroup theory and vector-valued Calderón-Zygmund methods in [42] (see also [43]). The Jacobi differential operator $\mathcal{L}^{(\alpha,\beta)}$ is given by (see Section II.2)

$$\mathcal{L}^{(\alpha,\beta)} = (x^2-1)\frac{d^2}{dx^2} + (\alpha - \beta + (\alpha+\beta+2)x)\frac{d}{dx} + \left(\frac{\alpha+\beta+1}{2}\right)^2.$$

We have that $\mathcal{L}^{(\alpha,\beta)} p_n^{(\alpha,\beta)} = (\widetilde{\lambda}^{(\alpha,\beta)})^2 p_n^{(\alpha,\beta)}$, with $\widetilde{\lambda}^{(\alpha,\beta)} = n + (\alpha+\beta+1)/2$. In this setting, the natural semigroup to study is the Poisson semigroup given by

$$\mathcal{P}_t^{(\alpha,\beta)}(x)f = e^{-t\sqrt{\mathcal{L}^{(\alpha,\beta)}}} f(x) = \sum_{n=0}^{\infty} e^{-t|\widetilde{\lambda}_n^{(\alpha,\beta)}|} \langle f, p_n^{(\alpha,\beta)} \rangle_{d\mu_{\alpha,\beta}} p_n^{(\alpha,\beta)}(x), \qquad -1 \le x \le 1.$$

Between a great variety of operators considered in [42], the authors analyse the Poisson maximal operator

$$\mathcal{P}_*^{(\alpha,\beta)} f(x) = \sup_{t\ge 0} |\mathcal{P}_t^{(\alpha,\beta)} f(x)|$$

obtaining mapping properties in weighted L^p spaces.

There exists a discrete analogue of the classical Laplacian operator, the discrete Laplacian

$$\Delta_{\mathrm{d}} f(n) = f(n-1) - 2f(n) + f(n+1), \qquad n \in \mathbb{Z},$$

where now $f = \{f(n)\}_{n\in\mathbb{Z}}$ is an appropriate sequence on $\mathbb{Z}$. A natural question is to study discrete harmonic analysis associated with Δ_{d}. In this context, the initial-value problem (III.1) can be reformulated to obtain

$$\begin{cases} \dfrac{\partial u(n,t)}{\partial t} = \Delta_{\mathrm{d}} u(n,t), \\ u(n,0) = f(n). \end{cases}$$

It is known [29, 30] that the heat semigroup related to Δ_{d} is

$$W_t^{\mathrm{d}} f(n) = e^{t\Delta_{\mathrm{d}}} f(n) = \sum_{m\in\mathbb{Z}} e^{-2t} I_{n-m}(2t) f(m),$$

where I_ν denotes the modified Bessel function of order ν. The study of ℓ^p-mapping properties for the maximal heat semigroup in this context was carried out in [17].

Shortly after, a generalization of this problem was investigated in [10]. The authors consider the ultraspherical context and define the discrete λ-Laplacian by

$$\Delta_\lambda f(n) = a_{n-1}^{\lambda} f(n-1) - 2f(n) + a_n^{\lambda} f(n+1), \qquad \lambda \geq 0,$$

for sequences $f = \{f(n)\}_{n\geq 0} \in \mathbb{C}^{\mathbb{N}}$ and where the elements of the sequence $\{a_n^\lambda\}_{n\geq 0}$ are the ones involved in the three-term recurrence relation for the ultraspherical polynomials. As a consequence, $\Delta_0 = \Delta_{\mathrm{d}}$. Again, ℓ^p-mapping properties are derived for the heat maximal semigroup.

In this chapter we pursue a generalization of the aforementioned results both [17] and [10]. More precisely, we consider the Jacobi setting and we construct an operator that generalises in a natural way both Δ_{d} and Δ_λ. The aim of this chapter is to study the heat semigroup related to this operator.

III.1 The discrete heat semigroup for Jacobi expansions

We consider the sequences $\{a_n^{(\alpha,\beta)}\}_{n\geq 0}$ and $\{b_n^{(\alpha,\beta)}\}_{n\geq 0}$, $\alpha, \beta > -1$, given by

$$a_n^{(\alpha,\beta)} = \frac{2}{2n+\alpha+\beta+2}\sqrt{\frac{(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+3)}}, \quad n \geq 0,$$

and

$$b_n^{(\alpha,\beta)} = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \quad n \geq 0$$

(we assume the natural interpretation when $n = 0$). Note that these sequences are the ones involved in the three-term recurrence relation for the normalised Jacobi polynomials $p_n^{(\alpha,\beta)}(x)$, $-1 \leq x \leq 1$ (see (II.4)). For any given sequence $f = \{f(n)\}_{n\geq 0}$ we define $\{J^{(\alpha,\beta)} f(n)\}_{n\geq 0}$ by the relations

$$J^{(\alpha,\beta)} f(n) = a_{n-1}^{(\alpha,\beta)} f(n-1) + b_n^{(\alpha,\beta)} f(n) + a_n^{(\alpha,\beta)} f(n+1), \qquad n \geq 1,$$

and $J^{(\alpha,\beta)} f(0) = b_0^{(\alpha,\beta)} f(0) + a_0^{(\alpha,\beta)} f(1)$.

By definition, it is immediate that

$$J^{(\alpha,\beta)} p_n^{(\alpha,\beta)}(x) = x p_n^{(\alpha,\beta)}(x), \qquad x \in [-1,1].$$

It is convenient for us to work with the operator

$$\mathcal{J}^{(\alpha,\beta)} f(n) = (J^{(\alpha,\beta)} - I) f(n),$$

rather than working with $J^{(\alpha,\beta)}$ (here I denotes the identity operator), since the translated operator $-\mathcal{J}^{(\alpha,\beta)}$ is non-negative. In fact, the spectrum of $J^{(\alpha,\beta)}$ is the interval $[-1,1]$, so that the spectrum of $-\mathcal{J}^{(\alpha,\beta)}$ is $[0,2]$. Observe that one could also get a positive operator by defining $\tilde{\mathcal{J}}^{(\alpha,\beta)} f(n) = (J^{(\alpha,\beta)} + I) f(n)$, where in this case the spectrum would be the interval $[0,2]$ and similar results would be attained.

Our main goal in this section is to give a precise and simple expression of the heat semigroup associated with $\mathcal{J}^{(\alpha,\beta)}$.

In this setting, for $n \geq 0$, α, $\beta > -1$, and $t \geq 0$, the heat equation is given by

$$\frac{\partial u(n,t)}{\partial t} = \mathcal{J}^{(\alpha,\beta)} u(n,t)$$

and for each sequence $\{f(n)\}_{n\geq 0}$ the corresponding initial-value problem is

$$\begin{cases} \dfrac{\partial u(n,t)}{\partial t} = \mathcal{J}^{(\alpha,\beta)} u(n,t), \\ u(n,0) = f(n). \end{cases}$$

Let us begin checking that $W_t^{(\alpha,\beta)} f(n)$, where

$$W_t^{(\alpha,\beta)} f(n) = \sum_{m=0}^{\infty} f(m) K_t^{(\alpha,\beta)}(m,n) \tag{III.2}$$

and the kernel has the form

$$K_t^{(\alpha,\beta)}(m,n) = \int_{-1}^{1} e^{-(1-x)t} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x),$$

is a solution of the initial-value problem. First, note that $W_t^{(\alpha,\beta)} f$ is well defined for each sequence $f \in \ell^2(\mathbb{N})$. Indeed,

$$|W_t^{(\alpha,\beta)} f(n)| \leq \|f\|_{\ell^2(\mathbb{N})} \|K_t^{(\alpha,\beta)}(\cdot,n)\|_{\ell^2(\mathbb{N})}$$

and taking into account that

$$K_t^{(\alpha,\beta)}(m,n) = c_m^{(\alpha,\beta)}(e^{-(1-(\cdot))t} p_n^{(\alpha,\beta)}),$$

by Parseval's identity (II.16) we have

$$\|K_t^{(\alpha,\beta)}(\cdot,n)\|_{\ell^2(\mathbb{N})} = \|e^{-(1-(\cdot))t} p_n^{(\alpha,\beta)}\|_{L^2([-1,1],d\mu_{\alpha,\beta})} \leq \|p_n^{(\alpha,\beta)}\|_{L^2([-1,1],d\mu_{\alpha,\beta})} = 1.$$

Following a similar argument we have that $\frac{\partial}{\partial t} W_t^{(\alpha,\beta)} f(n)$ is well defined and

$$\frac{\partial}{\partial t} W_t^{(\alpha,\beta)} f(n) = -\sum_{m=0}^{\infty} f(m) \int_{-1}^{1} (1-x) e^{-(1-x)t} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x).$$

Then, using that

$$\mathcal{J}^{(\alpha,\beta)} p_n^{(\alpha,\beta)}(x) = -(1-x) p_n^{(\alpha,\beta)}(x), \qquad x \in [-1,1],$$

we get $\frac{\partial}{\partial t} W_t^{(\alpha,\beta)} f(n) = \mathcal{J}^{(\alpha,\beta)} W_t^{(\alpha,\beta)} f(n)$. In addition, by means of the identity

$$K_0^{(\alpha,\beta)}(m,n) = \delta_{mn},$$

it is immediate to see that $W_0^{(\alpha,\beta)} f(n) = f(n)$.

By construction, $W_t^{(\alpha,\beta)} f = e^{t\mathcal{J}^{(\alpha,\beta)}} f$, where

$$e^{t\mathcal{J}^{(\alpha,\beta)}} f = \int_{-1}^{1} e^{-t(1-\lambda)}\, dE_{J^{(\alpha,\beta)}}(\lambda) f$$

and E_J is the spectral measure associated with $J^{(\alpha,\beta)}$. Therefore, from the general theory (see [72]), it turns out that the family of operators $\{W_t^{(\alpha,\beta)}\}_{t\geq 0}$ is a is a strongly continuous semigroup of operators on $\ell^2(\mathbb{N})$, i.e., we have the following theorem:

Theorem III.1.1. *Let $W_t^{(\alpha,\beta)}$ be the operator defined by* (III.2). *Then, for each sequence $f \in \ell^2(\mathbb{N})$ we have that*

$$\|W_t^{(\alpha,\beta)} f\|_{\ell^2(\mathbb{N})} \leq \|f\|_{\ell^2(\mathbb{N})}. \tag{III.3}$$

Moreover,

(a) $W_{t_1}^{(\alpha,\beta)} W_{t_2}^{(\alpha,\beta)} f(n) = W_{t_1+t_2}^{(\alpha,\beta)} f(n)$*, for* $t_1, t_2 \geq 0$*,* $n \in \mathbb{N}$*,*

(b) $W_0^{(\alpha,\beta)} f(n) = f(n)$*,* $n \in \mathbb{N}$*, and*

(c) $\lim_{t\to 0^+} \|W_t^{(\alpha,\beta)} f - f\|_{\ell^2(\mathbb{N})} = 0$.

From item (c), we have the pointwise convergence of the heat semigroup, that is,

$$\lim_{t\to 0^+} W_t^{(\alpha,\beta)} f(n) = f(n), \qquad n \geq 0,$$

for all sequence $f \in \ell^2(\mathbb{N})$.

Now, for each $f \in \ell^2(\mathbb{N})$, bring in the heat maximal operator

$$W_*^{(\alpha,\beta)} f(n) = \sup_{t\geq 0} |W_t^{(\alpha,\beta)} f(n)|. \tag{III.4}$$

Using the ideas in [59, Chapter III] and the estimate (III.3) we can conclude the ℓ^2-boundedness of $W_*^{(\alpha,\beta)}$, that is,

$$\|W_*^{(\alpha,\beta)} f\|_{\ell^2(\mathbb{N})} \leq C\|f\|_{\ell^2(\mathbb{N})}. \tag{III.5}$$

The sketch of the proof of this inequality is as follows. Following [59, p. 74 and 75],

$$W_*^{(\alpha,\beta)} f(n) \leq M^{(\alpha,\beta)} f(n) + g^{(\alpha,\beta)}(f)(n),$$

where $g^{(\alpha,\beta)}(f)(n)$ denotes the Littlewood-Paley-Stein function and $M^{(\alpha,\beta)} f(n)$ is the supremum of the averages of the heat semigroup in this context. They are respectively given by

$$g^{(\alpha,\beta)}(f)(n) = \left(\int_0^\infty \left| t\frac{\partial}{\partial t} W_t^{(\alpha,\beta)} f(n)\right|^2 \frac{dt}{t}\right)^{1/2}$$

and

$$M^{(\alpha,\beta)} f(n) = \sup_{s>0} \left|\frac{1}{s}\int_0^s W_t^{(\alpha,\beta)} f(n)\, dt\right|.$$

Both operators are bounded from $\ell^2(\mathbb{N})$ into itself, that is,

$$\|g^{(\alpha,\beta)}(f)\|_{\ell^2(\mathbb{N})} \leq C\|f\|_{\ell^2(\mathbb{N})} \quad \text{and} \quad \|M^{(\alpha,\beta)} f\|_{\ell^2(\mathbb{N})} \leq C\|f\|_{\ell^2(\mathbb{N})},$$

so (III.5) follows directly. The proof for the bound (in fact, an equality is attained for some constant) for $g^{(\alpha,\beta)}(f)$ is given in Chapter V, Lemma V.2.1. On its behalf, the bound for $M^{(\alpha,\beta)} f$ is the discrete analogous of the continuous one presented in [58, Corollary 2] and it can be proved in a similar way. The contractivity of the semigroup $\{W_t^{(\alpha,\beta)}\}_{t\geq 0}$ given in (III.3) is a requirement there.

The heat semigroup for Jacobi matrices

All the results in the previous section carry over to a more general framework. We briefly describe the details below. An infinite tridiagonal matrix

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \cdots \\ 0 & 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with $a_n > 0$ and $b_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, is called a Jacobi matrix. We suppose that the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are bounded so that J defines a bounded self-adjoint linear operator on $\ell^2(\mathbb{N})$ (that we still denote by J). In this situation, Favard's theorem (see [22] and [15, Chapter 1, Theorem 4.4]) states that each Jacobi matrix corresponds to a spectral measure μ with a compact support X having an infinite number of points. Moreover, there exists a family of polynomials $\{p_n\}_{n\geq 0}$ orthonormalised in $L^2(X, d\mu)$, i.e.,

$$\int_X p_n(x)p_m(x)\, d\mu(x) = \delta_{nm},$$

satisfying the three-term recurrence relation

$$xp_n(x) = a_{n-1}p_{n-1}(x) + b_n p_n(x) + a_n p_{n+1}(x), \quad x \in X,$$

with $p_{-1}(x) = 0$ and where the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are the entries of the Jacobi matrix J associated with the measure μ.

It is known that the measure μ related to a Jacobi matrix may not be unique (see [66, § 56]). However, if it is unique, then the family of orthonormal polynomials $\{p_n\}_{n\geq 0}$ is dense in $L^2(X, d\mu)$ (see [56, Theorem 2.14]). Hence, the Fourier series of an arbitrary function in terms of the polynomials $\{p_n\}_{n\geq 0}$ is convergent in the space $L^2(X, d\mu)$, that is, for each $f \in L^2(X, d\mu)$, with the Fourier coefficients given by

$$c_m(f) = \int_X f(t)p_m(t)\, d\mu(t),$$

the identity

$$f(x) = \sum_{m=0}^{\infty} c_m(f)p_m(x),$$

holds in $L^2(X, d\mu)$. In order to guarantee the uniqueness of the measure μ, we suppose that $a_n \to a$ and $b_n \to b$ (with both a and b finite), so X is bounded with at most countably many points outside the interval $[b-2a, b+2a]$, with $b\pm 2a$ the limit points of X (see [15, Chapter 2, Theorem 5.6]).

Conversely, for each sequence in $f \in \ell^2(\mathbb{N})$ there is a function $F \in L^2(X, d\mu)$ such that

$$F(x) = \sum_{m=0}^{\infty} f(m)p_m(x), \tag{III.6}$$

where the convergence holds in the $L^2(X, d\mu)$ sense and we have that $c_m(F) = f(m)$. Furthermore, Parseval's identity

$$\|F\|_{L^2(X,d\mu)} = \|f\|_{\ell^2(\mathbb{N})}$$

holds. Obviously, given two sequences $f_1, f_2 \in \ell^2(\mathbb{N})$, the polarisation type identity

$$\int_X F_1(x)\overline{F_2(x)}\, d\mu(x) = \sum_{m=0}^{\infty} f_1(m)\overline{f_2(m)},$$

where the functions F_1 and F_2 are defined as in (III.6), holds.

For each Jacobi matrix J, let s be the maximum of the support of the measure μ and $s^+ = \max\{s, 0\}$, and define the operator

$$\mathcal{J} = J - s^+ I,$$

where I is the infinite identity matrix. Observe that now

$$\mathcal{J} p_n(x) = (x - s^+)p_n(x), \quad x \in X.$$

Then, for $n \geq 0$ and $t \geq 0$, and each appropriate sequence $\{f(n)\}_{n\geq 0}$, we consider the initial-value problem corresponding to the heat equation associated with the operator $\mathcal{J}$ given by

$$\begin{cases} \dfrac{\partial u(n,t)}{\partial t} = \mathcal{J} u(n,t), \\ u(n,0) = f(n). \end{cases}$$

Define W_t by

$$W_t f(n) = \sum_{m=0}^{\infty} f(m) K_t(m,n),$$

where

$$K_t(m,n) = \int_X e^{(x-s^+)t} p_m(x) p_n(x)\, d\mu(x).$$

Again,

$$W_t f(n) = e^{t\mathcal{J}} f(n) = \int_X e^{-t(s^+-\lambda)}\, dE_J(\lambda) f,$$

where E_J is the spectral measure associated with J, is a solution of the initial-value problem and we have an analogue of Theorem III.1.1 in this setting.

III.2 The positivity of the heat semigroup $W_t^{(\alpha,\beta)}$

In this section we study the positivity of the heat semigroup $W_t^{(\alpha,\beta)}$. In other words, we are interested in proving that $W_t^{(\alpha,\beta)} f$ is non-negative provided f is a non-negative sequence in $\ell^\infty(\mathbb{N})$. It is not possible to us to deal with the same question in the general setting of Jacobi matrices because we need extra information of the associated family of orthogonal polynomials.

The first step we take in our goal is to extend the definition of $W_t^{(\alpha,\beta)}$ to the space $\ell^\infty(\mathbb{N})$. This is attained in view of the following lemma.

Lemma III.2.1. *Let $\alpha, \beta \geq -1/2$ and $n \neq m$. Then,*

$$|K_t^{(\alpha,\beta)}(m,n)| \leq C \frac{t^{1/2}}{|m-n|^2}.$$

The proof of this result is quite technical and we postpone it to the last section of this chapter.

The main tool to prove the positivity of $W_t^{(\alpha,\beta)}$ is a well-known linearisation formula for the product of two Jacobi polynomials due to G. Gasper. For the normalised polynomials $p_n^{(\alpha,\beta)}$, $n \geq 0$, α, $\beta > -1$, it reads as follows

$$p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x) = \sum_{k=|m-n|}^{m+n} c(k,n,m,\alpha,\beta) p_k^{(\alpha,\beta)}(x), \tag{III.7}$$

where all the coefficients $c(k,n,m,\alpha,\beta)$ are non-negative if and only if $(\alpha,\beta) \in V$. Here we say that (α,β) belongs to the set V if $\alpha, \beta > -1$, $\alpha \geq \beta$ and

$$(\alpha+\beta+1)(\alpha+\beta+4)^2(\alpha+\beta+6) \geq (\alpha-\beta)^2((\alpha+\beta+1)^2 - 7(\alpha+\beta+1) - 24).$$

The previous general result was given in [27]. In [26], the positivity of the coefficients $c(k,n,m,\alpha,\beta)$ was established under the simpler (but less general) conditions $\alpha \geq \beta$ and $\alpha + \beta \geq -1$.

Now we can state the main result of this section.

Theorem III.2.2. *Let $\alpha \geq \beta \geq -1/2$ and $t \geq 0$. Then for each non-negative sequence $f \in \ell^\infty(\mathbb{N})$, the heat operator $W_t^{(\alpha,\beta)} f$ is non-negative.*

Proof. The operator $W_t^{(\alpha,\beta)}$ is well defined for sequences in $\ell^\infty(\mathbb{N})$ by Lemma III.2.1. In order to prove the result it suffices to see that the kernel $K_t^{(\alpha,\beta)}$ is non-negative. By using the linearisation formula (III.7), we can express the kernel in the following way:

$$K_t^{(\alpha,\beta)}(m,n) = \sum_{k=|m-n|}^{m+n} c(k,n,m,\alpha,\beta) \int_{-1}^{1} e^{-(1-x)t} p_k^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x), \tag{III.8}$$

with $c(k,n,m,\alpha,\beta) \geq 0$. So, by using that

$$h_t^{(\alpha,\beta)}(k) := \int_{-1}^{1} e^{-(1-x)t} p_k^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) = e^{-t} w_k^{(\alpha,\beta)} \int_{-1}^{1} e^{xt} P_k^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x)$$

the proof reduces to show the non-negativity of the last integral, but it is almost immediate from Rodrigues' formula (II.3) and integration by parts k times. Indeed,

$$\begin{aligned} \int_{-1}^{1} e^{xt} P_k^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) &= \frac{(-1)^k}{2^k k!} \int_{-1}^{1} e^{xt} \frac{d^k}{dx^k} \left((1-x)^{\alpha+k} (1+x)^{\beta+k} \right) dx \\ &= \frac{t^k}{2^k k!} \int_{-1}^{1} e^{xt} (1-x)^{\alpha+k} (1+x)^{\beta+k}\, dx, \end{aligned}$$

which is clearly non-negative. □

It is worth pointing out that by means of the linearisation (III.7) it is possible to define a positive convolution operator in $\ell^1(\mathbb{N})$ [27, Corollary 1]. Actually, this procedure is extensible to other orthogonal polynomials and it has been widely studied for example in [68], where general orthogonal polynomials are considered. This convolution structure is used in [10, eq. 13] to study several issues related to the heat semigroup in the ultraspherical setting. Regarding the Jacobi case, for any two sequences $f = \{f(n)\}_{n\geq 0}$ and $g = \{g(n)\}_{n\geq 0}$ the convolution operator is given by

$$(f * g)(n) = \sum_{m=0}^{\infty} f(m)\tau_n^{(\alpha,\beta)} g(m), \quad n \in \mathbb{N},$$

where $\tau_n^{(\alpha,\beta)} g(m)$ denotes the translation operator

$$\tau_n^{(\alpha,\beta)} g(m) = \sum_{k=|m-n|}^{m+n} c(k, n, m, \alpha, \beta) g(k).$$

Rewriting the equation (III.8) in terms of the translation operator as $K_t^{(\alpha,\beta)}(m, n) = \tau_n^{(\alpha,\beta)} h_t^{(\alpha,\beta)}(m)$ it is straightforward to give an expression for the heat semigroup as a convolution by

$$W_t^{(\alpha,\beta)} f(n) = (f * h_t^{(\alpha,\beta)})(n).$$

However, we will not follow this approach to analyse the heat semigroup.

III.3 Weighted inequalities for the heat and Poisson maximal operators

In this section, we consider the heat maximal operator $W_*^{(\alpha,\beta)}$ and the Poisson maximal operator $P_*^{(\alpha,\beta)}$ (see below for the definition) and we prove weighted inequalities for them when $\alpha, \beta \geq -1/2$. We use the discrete vector-valued local Calderón-Zygmund theory of Section II.3 as an indispensable tool.

The next theorem includes mapping properties in weighted ℓ^p-spaces of $W_*^{(\alpha,\beta)}$.

Theorem III.3.1. *Let $\alpha, \beta \geq -1/2$ and consider the maximal operator $W_*^{(\alpha,\beta)}$ defined by* (III.4)*.*

(a) If $1 < p < \infty$ and $w \in A_p(\mathbb{N})$, then

$$\|W_*^{(\alpha,\beta)} f\|_{\ell^p(\mathbb{N},w)} \leq C\|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w),$$

where C is a constant independent of f. Consequently, the operator $W_*^{(\alpha,\beta)}$ extends uniquely to a bounded operator from $\ell^p(\mathbb{N}, w)$ into itself.

(b) If $w \in A_1(\mathbb{N})$, then

$$\|W_*^{(\alpha,\beta)} f\|_{\ell^{1,\infty}(\mathbb{N},w)} \leq C\|f\|_{\ell^1(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^1(\mathbb{N}, w),$$

where C is a constant independent of f. Consequently, the operator $W_*^{(\alpha,\beta)}$ extends uniquely to a bounded operator from $\ell^1(\mathbb{N}, w)$ into $\ell^{1,\infty}(\mathbb{N}, w)$.

The Poisson maximal operator $P_*^{(\alpha,\beta)}$ is defined by

$$P_*^{(\alpha,\beta)} f(n) = \sup_{t\geq 0} |P_t^{(\alpha,\beta)} f(n)|, \tag{III.9}$$

where the Poisson semigroup $\{P_t^{(\alpha,\beta)}\}_{t\geq 0}$ is given by subordination by the identity

$$P_t^{(\alpha,\beta)} f(n) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} W_{t^2/(4u)}^{(\alpha,\beta)} f(n)\, du, \quad t \geq 0, \tag{III.10}$$

for sequences $f \in \ell^\infty(\mathbb{N})$. Note that it is well defined by Lemma III.2.1.

As an immediate consequence of Theorem III.3.1 and the pointwise domination

$$P_*^{(\alpha,\beta)} f(n) \leq W_*^{(\alpha,\beta)} f(n), \quad n \geq 0, \quad \alpha,\beta > -1,$$

which follows from the expression (III.10), we deduce the following result for $P_*^{(\alpha,\beta)}$.

Corollary III.3.2. *Let $\alpha,\beta \geq -1/2$ and consider the maximal operator $P_*^{(\alpha,\beta)}$ defined by* (III.9)*.*

(a) If $1 < p < \infty$ and $w \in A_p(\mathbb{N})$, then

$$\|P_*^{(\alpha,\beta)} f\|_{\ell^p(\mathbb{N},w)} \leq C \|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N},w),$$

where C is a constant independent of f. Consequently, the operator $P_*^{(\alpha,\beta)}$ extends uniquely to a bounded operator from $\ell^p(\mathbb{N},w)$ into itself.

(b) If $w \in A_1(\mathbb{N})$, then

$$\|P_*^{(\alpha,\beta)} f\|_{\ell^{1,\infty}(\mathbb{N},w)} \leq C \|f\|_{\ell^1(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^1(\mathbb{N},w),$$

where C is a constant independent of f. Consequently, the operator $P_*^{(\alpha,\beta)}$ extends uniquely to a bounded operator from $\ell^1(\mathbb{N},w)$ into $\ell^{1,\infty}(\mathbb{N},w)$.

Noteworthy, the previous corollary can be stated for other subordinated semigroups. Due to (III.10), it is clear that the Poisson semigroup is subordinated of the heat one, so the properties related to weighted norm inequalities for the latter can be transferred to the former (see for instance [59, Theorem 1', p. 46]). Unsurprisingly, this transfer property does not only work for the Poisson semigroup, but also for other subordinated semigroups of the heat one. As it is explained in [72, Chapter IX, Section 11], one possible way to construct subordinated semigroups of $W_t^{(\alpha,\beta)}$ is essentially by means of the positive powers of the infinitesimal generator $\mathcal{J}^{(\alpha,\beta)}$. To be more specific, the semigroup with infinitesimal generator $-(-\mathcal{J}^{(\alpha,\beta)})^\sigma$, where $0 < \sigma < 1$, is subordinated of $W_t^{(\alpha,\beta)}$. In the particular case of the Poisson semigroup, its infinitesimal generator is given by $-\sqrt{-\mathcal{J}^{(\alpha,\beta)}}$.

Now, in order to prove Theorem III.3.1 we use the discrete vector-valued local Calderón-Zygmund theory of Section II.3. Set $\mathbb{B} = L^\infty(0,\infty)$. We observe first that the operator

$$\begin{array}{rcccl} T & : & \ell^2(\mathbb{N}) & \longrightarrow & \ell^2_{\mathbb{B}}(\mathbb{N}) \\ & & f & \longmapsto & Tf(n,t) := W_t^{(\alpha,\beta)} f(n), \end{array}$$

is bounded from $\ell^2(\mathbb{N})$ into $\ell^2_{\mathbb{B}}(\mathbb{N})$. Indeed, it is a consequence of the ℓ^2-boundedness of the heat maximal operator (see (III.5)).

To obtain the regularity properties (b1) and (b2) for the kernel $K_t^{(\alpha,\beta)}$ it suffices to prove the inequalities

$$\|K_t^{(\alpha,\beta)}(n,m+1) - K_t^{(\alpha,\beta)}(n,m)\|_{L^\infty(0,\infty)} \leq \frac{C}{|n-m|^2}, \tag{III.11}$$

for $n,m \in \mathbb{N}$, $n \neq m$, and $2m/3 \leq n \leq 3m/2$, and

$$\|K_t^{(\alpha,\beta)}(n+1,m) - K_t^{(\alpha,\beta)}(n,m)\|_{L^\infty(0,\infty)} \leq \frac{C}{|n-m|^2}, \tag{III.12}$$

for $n,m \in \mathbb{N}$, $n \neq m$, and $2n/3 \leq m \leq 3n/2$. The proof of this fact is based on the ideas of [10, p. 17 and 18] and it actually works when the norm comes from a general Banach space.

Let us see that (III.12) implies (b2) (the proof that (III.11) implies (b1) is analogous in a general context but note that in our case $K_t^{(\alpha,\beta)}(n,m) = K_t^{(\alpha,\beta)}(m,n)$). If $n = l$ the conclusion follows readily. Let us suppose that $n < s$. By the triangle inequality, we obtain

$$\begin{aligned}\|K_t^{(\alpha,\beta)}(n,m) &- K_t^{(\alpha,\beta)}(s,m)\|_{L^\infty(0,\infty)} \\ &\leq \sum_{j=0}^{s-n-1} \|K_t^{(\alpha,\beta)}(n+j,m) - K_t^{(\alpha,\beta)}(n+j+1,m)\|_{L^\infty(0,\infty)}.\end{aligned}$$

If $n > m$ we apply (III.12) to get the desired estimate. When $n < m$ we apply (III.12) and then use that $|n-m| > 2|n-s|$ so the result follows. The case $n > s$ is similar and we omit the details.

Lemma III.3.3. *Let $n,m \in \mathbb{N}$, $n \neq m$, $\alpha,\beta \geq -1/2$, and $t \geq 0$. Then,*

$$\|K_t^{(\alpha,\beta)}(n,m)\|_{L^\infty(0,\infty)} \leq \frac{C}{|n-m|}. \tag{III.13}$$

Moreover,

$$\|K_t^{(\alpha,\beta)}(n,n)\|_{L^\infty(0,\infty)} \leq C. \tag{III.14}$$

Lemma III.3.4. *Let $n,m \in \mathbb{N}$, $n \neq m$, $2m/3 \leq n \leq 3m/2$, $\alpha,\beta \geq -1/2$ and $t \geq 0$. Then,*

$$\|K_t^{(\alpha,\beta)}(n+1,m) - K_t^{(\alpha,\beta)}(n,m)\|_{L^\infty(0,\infty)} \leq \frac{C}{|n-m|^2}.$$

We postpone the proofs of both lemmas to the next section. Therefore, we have that

$$\begin{aligned}W_*^{(\alpha,\beta)}f(n) &\leq \left\| \sum_{\substack{m=0\\ m\neq n}}^{\infty} f(m)K_t^{(\alpha,\beta)}(m,n) \right\|_{L^\infty(0,\infty)} + \|f(n)K_t^{(\alpha,\beta)}(n,n)\|_{L^\infty(0,\infty)} \\ &=: T_1 f(n) + T_2 f(n).\end{aligned}$$

By using Theorem II.3.1, we obtain that

$$\|T_1 f\|_{\ell^p(\mathbb{N},w)} \leq C\|f\|_{\ell^p(\mathbb{N},w)}$$

and the corresponding weak inequality for $p = 1$. The bound

$$\|T_2 f\|_{\ell^p(\mathbb{N},w)} \leq C\|f\|_{\ell^p(\mathbb{N},w)},$$

$1 \leq p < \infty$, becomes clear by using (III.14). Thus, Theorem III.3.1 is proved.

III.4 Technical results

Throughout this section we adopt the following notation

$$\mathfrak{I}_t^{(a,b,A,B,\alpha,\beta)}(n,m) = \int_{-1}^{1} e^{-t(1-x)} P_n^{(a,b)}(x) P_m^{(A,B)}(x)(1-x)^{\alpha}(1+x)^{\beta}\, dx,$$

where $n, m \in \mathbb{N}$, $a, b, A, B, \alpha, \beta > -1$, and $t \geq 0$.

We begin this section by giving a technical lemma related to the family of integrals $\mathfrak{I}_t^{(a,b,A,B,\alpha,\beta)}(n,m)$ which we will use to prove Lemma III.2.1 and the required Calderón-Zygmund estimates contained in Lemmas III.3.3 and III.3.4.

Lemma III.4.1. *Let n, $m \in \mathbb{N}$ and a, b, A, B, α, $\beta > -1$ such that $n+a+b+1 \neq 0$, $m + A + B + 1 \neq 0$, and $n(n + a + b + 1) \neq m(m + A + B + 1)$.*

(a) If n, $m \neq 0$, we have that

$$\begin{aligned}\mathfrak{I}_t^{(a,b,A,B,\alpha,\beta)}(n,m) = {} & \frac{(n+a+b+1)(m+A+B+1)}{2(n(n+a+b+1) - m(m+A+B+1))} \\ & \times \Bigg(\frac{t}{m+A+B+1} \mathfrak{I}_t^{(a+1,b+1A,B,\alpha+1,\beta+1)}(n-1,m) \\ & - \frac{\alpha - a}{m+A+B+1} \mathfrak{I}_t^{(a+1,b+1,A,B,\alpha,\beta+1)}(n-1,m) \\ & + \frac{\beta - b}{m+A+B+1} \mathfrak{I}_t^{(a+1,b+1,A,B,\alpha+1,\beta)}(n-1,m) \\ & - \frac{t}{n+a+b+1} \mathfrak{I}_t^{(a,b,A+1,B+1,\alpha+1,\beta+1)}(n,m-1) \\ & + \frac{\alpha - A}{n+a+b+1} \mathfrak{I}_t^{(a,b,A+1,B+1,\alpha,\beta+1)}(n,m-1) \\ & - \frac{\beta - B}{n+a+b+1} \mathfrak{I}_t^{(a,b,A+1,B+1,\alpha+1,\beta)}(n,m-1) \Bigg).\end{aligned}$$

(b) If $n = 0$ and $m \in \mathbb{N}$,

$$\begin{aligned}\mathfrak{I}_t^{(a,b,A,B,\alpha,\beta)}(0,m) = {} & \frac{t}{2m} \mathfrak{I}_t^{(a,b,A+1,B+1,\alpha+1,\beta+1)}(0,m-1) \\ & - \frac{\alpha - A}{2m} \mathfrak{I}_t^{(a,b,A+1,B+1,\alpha,\beta+1)}(0,m-1) \\ & + \frac{\beta - B}{2m} \mathfrak{I}_t^{(a,b,A+1,B+1,\alpha+1,\beta)}(0,m-1).\end{aligned}$$

(c) If $n \in \mathbb{N}$ and $m = 0$,

$$\begin{aligned}\mathfrak{I}_t^{(a,b,A,B,\alpha,\beta)}(n,0) = {}& \frac{t}{2n}\mathfrak{I}_t^{(a+1,b+1,A,B,\alpha+1,\beta+1)}(n-1,0) \\ & - \frac{\alpha - a}{2n}\mathfrak{I}_t^{(a+1,b+1,A,B,\alpha,\beta+1)}(n-1,0) \\ & + \frac{\beta - b}{2n}\mathfrak{I}_t^{(a+1,b+1,A,B,\alpha+1,\beta)}(n-1,0).\end{aligned}$$

Proof. First, we prove case (a). By using the identities (II.12) and (II.9) and applying integration by parts we have

$$\begin{aligned}&\mathfrak{I}_t^{(a,b,A,B,\alpha,\beta)}(n,m) \\ &\quad = \frac{-1}{2n}\int_{-1}^{1} e^{-t(1-x)}\frac{d}{dx}\left(P_{n-1}^{(a+1,b+1)}(x)(1-x)^{a+1}(1+x)^{b+1}\right)P_m^{(A,B)}(x) \\ &\qquad \times (1-x)^{\alpha-a}(1+x)^{\beta-b}\,dx \\ &\quad = \frac{t}{2n}\mathfrak{I}_t^{(a+1,b+1,A,B,\alpha+1,\beta+1)}(n-1,m) \\ &\qquad + \frac{m+A+B+1}{4n}\mathfrak{I}_t^{(a+1,b+1,A+1,B+1,\alpha+1,\beta+1)}(n-1,m-1) \\ &\qquad - \frac{\alpha-a}{2n}\mathfrak{I}_t^{(a+1,b+1,A,B,\alpha,\beta+1)}(n-1,m) \\ &\qquad + \frac{\beta-b}{2n}\mathfrak{I}_t^{(a+1,b+1,A,B,\alpha+1,\beta)}(n-1,m).\end{aligned}$$

In a similar way, we obtain that

$$\begin{aligned}&\mathfrak{I}_t^{(a+1,b+1,A+1,B+1,\alpha+1,\beta+1)}(n-1,m-1) \\ &\quad = \frac{2}{n+a+b+1}\int_{-1}^{1} e^{-t(1-x)}\frac{d}{dx}\left(P_n^{(a,b)}(x)\right) \\ &\qquad \times \left(P_{m-1}^{(A+1,B+1)}(x)(1-x)^{A+1}(1+x)^{B+1}\right)(1-x)^{\alpha-A}(1+x)^{\beta-B}\,dx \\ &\quad = \frac{-2t}{n+a+b+1}\mathfrak{I}_t^{(a,b,A+1,B+1,\alpha+1,\beta+1)}(n,m-1) \\ &\qquad + \frac{4m}{n+a+b+1}\mathfrak{I}_t^{(a,b,A,B,\alpha,\beta)}(n,m) \\ &\qquad + \frac{2(\alpha-A)}{n+a+b+1}\mathfrak{I}_t^{(a,b,A+1,B+1,\alpha,\beta+1)}(n,m-1) \\ &\qquad - \frac{2(\beta-B)}{n+a+b+1}\mathfrak{I}_t^{(a,b,A+1,B+1,\alpha+1,\beta)}(n,m-1)\end{aligned}$$

and the result follows.

For cases (b) and (c) write

$$\begin{aligned}\mathfrak{I}_t^{(a,b,A,B,\alpha,\beta)}(0,m) = {}& -\frac{1}{2m}\int_{-1}^{1} e^{-t(1-x)}\frac{d}{dx}\left((1-x)^{A+1}(1+x)^{B+1}P_{m-1}^{(A+1,B+1)}(x)\right) \\ & \times (1-x)^{\alpha-A}(1+x)^{\beta-B}\,dx\end{aligned}$$

in the former and

$$\mathfrak{I}_t^{(a,b,A,B,\alpha,\beta)}(n,0) = -\frac{1}{2n}\int_{-1}^{1} e^{-t(1-x)}\frac{d}{dx}\left((1-x)^{a+1}(1+x)^{b+1}P_{n-1}^{(a+1,b+1)}(x)\right) \times (1-x)^{\alpha-a}(1+x)^{\beta-b}\,dx$$

in the latter and integrate by parts. □

Proof of Lemma III.2.1. We only check the cases $n,m \geq 2$, with $n \neq m$, by using (a) in Lemma III.4.1. The remaining cases can be obtained from (b) and (c) in the same lemma. Taking $a = A = \alpha$ and $b = B = \beta$ in Lemma III.4.1 case (a) and noting that

$$n(n+\alpha+\beta+1) - m(m+\alpha+\beta+1) = (n-m)(n+m+\alpha+\beta+1)$$

we have

$$|K_t^{(\alpha,\beta)}(n,m)| \leq \frac{Ct}{|n-m|} w_n^{(\alpha,\beta)} w_m^{(\alpha,\beta)} \left(\left|\mathfrak{I}_t^{(\alpha+1,\beta+1,\alpha,\beta,\alpha+1,\beta+1)}(n-1,m)\right| + \left|\mathfrak{I}_t^{(\alpha,\beta,\alpha+1,\beta+1,\alpha+1,\beta+1)}(n,m-1)\right| \right). \tag{III.15}$$

Lemma III.4.1 gives that

$$\left|\mathfrak{I}_t^{(\alpha+1,\beta+1,\alpha,\beta,\alpha+1,\beta+1)}(n-1,m)\right| \leq \frac{C}{|n-m|}\Big(t|\mathfrak{I}^{(\alpha+2,\beta+2,\alpha,\beta,\alpha+2,\beta+2)}(n-2,m)| + t|\mathfrak{I}^{(\alpha+1,\beta+1,\alpha+1,\beta+1,\alpha+2,\beta+2)}(n-1,m-1)| + |\mathfrak{I}^{(\alpha+1,\beta+1,\alpha+1,\beta+1,\alpha+1,\beta+2)}(n-1,m-1)| + |\mathfrak{I}^{(\alpha+1,\beta+1,\alpha+1,\beta+1,\alpha+2,\beta+1)}(n-1,m-1)| \Big).$$

Now, with the uniform bound (II.14), taking into account the asymptotic behaviour (II.8) and the bound

$$\int_{-1}^{0} e^{-t(1-x)}(1-x)^{-1/2}\,dx = \int_0^1 e^{-t(1+x)}(1+x)^{-1/2}\,dx \leq \int_0^1 e^{-t(1-x)}(1-x)^{-1/2}\,dx,$$

we obtain that

$$\begin{aligned}
&\left|\mathfrak{I}_t^{(\alpha+1,\beta+1,\alpha,\beta,\alpha+1,\beta+1)}(n-1,m)\right| \\
&\quad\leq \frac{C}{\sqrt{nm}|n-m|}\left(t\int_{-1}^{1} e^{-t(1-x)}(1-x)^{1/2}\,dx + \int_0^1 e^{-t(1-x)}(1-x)^{-1/2}\,dx\right) \\
&\quad\leq \frac{C}{\sqrt{nm}|n-m|}\left(t^{1/2}\int_{-1}^{1} e^{-t(1-x)/2}\,dx + t^{-1/2}\int_0^\infty e^{-s}s^{-1/2}\,ds\right) \\
&\quad\leq \frac{Ct^{-1/2}}{\sqrt{nm}|n-m|}.
\end{aligned}$$

Following the same procedure, we deduce that

$$\left|\mathfrak{I}_t^{(\alpha,\beta,\alpha+1,\beta+1,\alpha+1,\beta+1)}(n,m-1)\right| \le \frac{Ct^{-1/2}}{\sqrt{nm}|n-m|}.$$

Then, from (III.15), the result follows. □

Proof of Lemma III.3.3. For the cases $n,m\ge 1$, with $n\neq m$, the result follows from (III.15), (II.14), and (II.8). The estimate (III.13) for the remaining cases is a consequence of (b) and (c) in Lemma III.4.1. The bound (III.14) is obvious. □

Proof of Lemma III.3.4. Note that the conditions $n\neq m$ and $2m/3\le n\le 3m/2$ imply that $n,m\ge 2$.

We begin by using the relation (II.10) to get

$$\begin{aligned} p_n^{(\alpha,\beta)}(x)-p_{n+1}^{(\alpha,\beta)}(x) &= \left(1-\frac{w_{n+1}^{(\alpha,\beta)}}{w_n^{(\alpha,\beta)}}\right)p_n^{(\alpha,\beta)}(x)+w_{n+1}^{(\alpha,\beta)}\left(\frac{p_n^{(\alpha,\beta)}(x)}{w_n^{(\alpha,\beta)}}-\frac{p_{n+1}^{(\alpha,\beta)}(x)}{w_{n+1}^{(\alpha,\beta)}}\right)\\ &= \left(1-\frac{w_{n+1}^{(\alpha,\beta)}}{w_n^{(\alpha,\beta)}}\right)p_n^{(\alpha,\beta)}(x)-\frac{\alpha}{n+1}\frac{w_{n+1}^{(\alpha,\beta)}}{w_n^{(\alpha,\beta)}}p_n^{(\alpha,\beta)}(x)\\ &\qquad+\frac{2n+\alpha+\beta+2}{2(n+1)}\frac{w_{n+1}^{(\alpha,\beta)}}{w_n^{(\alpha+1,\beta)}}(1-x)p_n^{(\alpha+1,\beta)}(x). \end{aligned}$$

Therefore,

$$\begin{aligned} &K_t^{(\alpha,\beta)}(n,m)-K_t^{(\alpha,\beta)}(n+1,m)\\ &\qquad= \left(1-\frac{w_{n+1}^{(\alpha,\beta)}}{w_n^{(\alpha,\beta)}}\right)K_t^{(\alpha,\beta)}(n,m)-\frac{\alpha}{n+1}\frac{w_{n+1}^{(\alpha,\beta)}}{w_n^{(\alpha,\beta)}}K_t^{(\alpha,\beta)}(n,m)\\ &\qquad\qquad+\frac{2n+\alpha+\beta+2}{2(n+1)}\frac{w_{n+1}^{(\alpha,\beta)}}{w_n^{(\alpha+1,\beta)}}D_t^{(\alpha,\beta)}(n,m), \end{aligned}$$

with

$$D_t^{(\alpha,\beta)}(n,m)=w_n^{(\alpha+1,\beta)}w_m^{(\alpha,\beta)}\mathfrak{I}_t^{(\alpha+1,\beta,\alpha,\beta,\alpha+1,\beta)}(n,m).$$

Now, the limit

$$\lim_{n\to\infty} n\left(\frac{w_{n+1}^{(\alpha,\beta)}}{w_n^{(\alpha,\beta)}}-1\right)=\frac{1}{2},$$

which is a consequence of (II.8), gives us the estimate

$$\left|1-\frac{w_{n+1}^{(\alpha,\beta)}}{w_n^{(\alpha,\beta)}}\right|\le\frac{C}{n}.$$

The last inequality is used together with Lemma III.3.3 to obtain

$$\sup_{t\ge 0}\left|K_t^{(\alpha,\beta)}(n,m)-K_t^{(\alpha,\beta)}(n+1,m)\right|\le\frac{C}{|n-m|^2}+C\sup_{t\ge 0}\left|D_t^{(\alpha,\beta)}(n,m)\right|.$$

So the study reduces to prove that

$$\sup_{t\geq 0}\left|D_t^{(\alpha,\beta)}(n,m)\right| \leq \frac{C}{|n-m|^2}. \tag{III.16}$$

First, we apply Lemma III.4.1 to obtain that

$$\begin{aligned}\left|\mathfrak{I}_t^{(\alpha+1,\beta,\alpha,\beta,\alpha+1,\beta)}(n,m)\right| \leq \frac{C}{|n-m|}\Bigg(& t\left|\mathfrak{I}_t^{(\alpha+2,\beta+1,\alpha,\beta,\alpha+2,\beta+1)}(n-1,m)\right| \\ &+ t\left|\mathfrak{I}_t^{(\alpha+1,\beta,\alpha+1,\beta+1,\alpha+2,\beta+1)}(n,m-1)\right| \\ &+ \left|\mathfrak{I}_t^{(\alpha+1,\beta,\alpha+1,\beta+1,\alpha+1,\beta+1)}(n,m-1)\right|\Bigg).\end{aligned}$$

Now, applying again Lemma III.4.1 to each term on the right-hand side of the previous inequality we have

$$\begin{aligned}t\left|\mathfrak{I}_t^{(\alpha+2,\beta+1,\alpha,\beta,\alpha+2,\beta+1)}(n-1,m)\right| \leq{}& \frac{C}{|n-m|} \\ &\times\Bigg(t^2\left|\mathfrak{I}_t^{(\alpha+3,\beta+2,\alpha,\beta,\alpha+3,\beta+2)}(n-2,m)\right| \\ &+ t^2\left|\mathfrak{I}_t^{(\alpha+2,\beta+1,\alpha+1,\beta+1,\alpha+3,\beta+2)}(n-1,m-1)\right| \\ &+ t\left|\mathfrak{I}_t^{(\alpha+2,\beta+1,\alpha+1,\beta+1,\alpha+2,\beta+2)}(n-1,m-1)\right| \\ &+ t\left|\mathfrak{I}_t^{(\alpha+2,\beta+1,\alpha+1,\beta+1,\alpha+3,\beta+1)}(n-1,m-1)\right|\Bigg),\end{aligned}$$

$$\begin{aligned}t\left|\mathfrak{I}_t^{(\alpha+1,\beta,\alpha+1,\beta+1,\alpha+2,\beta+1)}(n,m-1)\right| \leq{}& \frac{C}{|n-m|} \\ &\times\Bigg(t^2\left|\mathfrak{I}_t^{(\alpha+2,\beta+1,\alpha+1,\beta+1,\alpha+3,\beta+2)}(n-1,m-1)\right| \\ &+ t^2\left|\mathfrak{I}_t^{(\alpha+1,\beta,\alpha+2,\beta+2,\alpha+3,\beta+2)}(n,m-2)\right| \\ &+ t\left|\mathfrak{I}_t^{(\alpha+2,\beta+1,\alpha+1,\beta+1,\alpha+2,\beta+2)}(n-1,m-1)\right| \\ &+ t\left|\mathfrak{I}_t^{(\alpha+1,\beta,\alpha+2,\beta+2,\alpha+2,\beta+2)}(n,m-2)\right| \\ &+ t\left|\mathfrak{I}_t^{(\alpha+2,\beta+1,\alpha+1,\beta+1,\alpha+3,\beta+1)}(n-1,m-1)\right|\Bigg),\end{aligned}$$

and

$$\begin{aligned}\left|\mathfrak{I}_t^{(\alpha+1,\beta,\alpha+1,\beta+1,\alpha+1,\beta+1)}(n,m-1)\right| \leq \frac{C}{|n-m|} \\ \times \Bigg(t\left|\mathfrak{I}_t^{(\alpha+2,\beta+1,\alpha+1,\beta+1,\alpha+2,\beta+2)}(n-1,m-1)\right| \\ + t\left|\mathfrak{I}_t^{(\alpha+1,\beta,\alpha+2,\beta+2,\alpha+2,\beta+2)}(n,m-2)\right| \\ + \left|\mathfrak{I}_t^{(\alpha+2,\beta+1,\alpha+1,\beta+1,\alpha+2,\beta+1)}(n-1,m-1)\right| \Bigg).\end{aligned}$$

Finally, by using the bound (II.14) and (II.8) we conclude that

$$\begin{aligned}\left|D_t^{(\alpha,\beta)}(n,m)\right| &\leq \frac{C}{|n-m|^2} \\ &\quad \times \Bigg(t^2 \int_{-1}^{1} e^{-t(1-x)}(1-x)(1+x)^{1/2}\,dx + t\int_{-1}^{1} e^{-t(1-x)}(1+x)^{1/2}\,dx \\ &\quad + t\int_{-1}^{1} e^{-t(1-x)}(1-x)(1+x)^{-1/2}\,dx + \int_{-1}^{1} e^{-t(1-x)}(1+x)^{-1/2}\,dx \Bigg) \\ &\leq \frac{C}{|n-m|^2}\end{aligned}$$

and the proof of (III.16) is completed. □

CHAPTER IV

THE RIESZ TRANSFORMS

This chapter is a continuation of the study of discrete Harmonic Analysis associated with the discrete Laplacian $\mathcal{J}^{(\alpha,\beta)}$ related to the three-term recurrence for Jacobi polynomials. In this occasion we focus on the Riesz transforms which arise in this setting.

The Riesz transforms are classical operators in Harmonic Analysis. In the Euclidean context, they are a straightforward generalization to higher dimensions of the Hilbert transform on the real line

$$Hf(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy,$$

defined for suitable functions f and where the integral is interpreted in the principal value sense. More precisely, the Riesz transforms are singular integral operators of the form

$$R_j f(x) = c_d \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\,dy, \qquad 1 \leq j \leq d,$$

with c_d a constant which depends only on the dimension d ($c_1 = 1/\pi$). Mapping properties for the Hilbert transform (and for the conjugate function) were obtained by M. Riesz in his celebrated paper [52]. The boundedness on L^p of a wide class of singular integrals in $\mathbb{R}^d$ was first studied by A. P. Calderón and A. Zygmund in the classical article [14].

In the non-trigonometric setting, the Riesz transforms have been studied in many situations. We recommend [44] to the interested reader and the references therein. These operators have also been treated in very abstract settings as for example Riemannian manifolds or compact Lie groups (see for example [18] and [21], respectively).

The chapter is organised as follows: the definition of the Riesz transforms is given in the first section. It is based on the Riesz potentials (also called fractional integrals) of $\mathcal{J}^{(\alpha,\beta)}$ which are also included in that section. The next section contains the main theorem of the chapter about ℓ^p-mapping properties of the Riesz transforms. The result generalises the one presented in [17] for the corresponding Riesz transforms associated with the discrete Laplacian Δ_{d}. The proof of the main theorem relies on discrete Calderón-Zygmund theory so the last section is devoted to show the estimates that are necessary to apply it.

IV.1 The Riesz transforms associated with Jacobi polynomials

In Chapter III we have seen that the infinitesimal generator of the heat semigroup $\{W_t^{(\alpha,\beta)}\}_{t\geq 0}$ is $\mathcal{J}^{(\alpha,\beta)}$. In order to define the Riesz transforms we follow a standard procedure (see [59, p. 57], [69], and [70]).

First, we decompose the operator $\mathcal{J}^{(\alpha,\beta)}$ by noting that the sequences $\{d_n\}_{n\geq 0}$ and $\{e_n\}_{n\geq 0}$ given by

$$d_n = \sqrt{\frac{2(n+\alpha+\beta+1)(n+\alpha+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}}$$

and

$$e_n = \sqrt{\frac{2(n+\beta+1)(n+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3))}},$$

for $n \geq 0$ (we assume the natural interpretation for d_0), satisfy the relations

$$\begin{aligned} a_n^{(\alpha,\beta)} &= d_n e_n, && n \geq 0, \\ b_n^{(\alpha,\beta)} &= 1 - d_n^2 - e_{n-1}^2, && n \geq 1, \end{aligned}$$

and $b_0^{(\alpha,\beta)} = 1 - d_0^2$.

In this way, we have that

$$\mathcal{J}^{(\alpha,\beta)} = -\delta^\star \delta,$$

where

$$\begin{aligned} \delta f(n) &= d_n f(n) - e_n f(n+1), && n \geq 0, \\ \delta^\star f(n) &= d_n f(n) - e_{n-1} f(n-1), && n \geq 1, \end{aligned}$$

and $\delta^\star f(0) = d_0 f(0)$. Note that both δ and $\delta^\star$ are adjoint operators in $\ell^2(\mathbb{N})$.

Second, we define the Riesz potentials (also fractional integrals) $(-\mathcal{J}^{(\alpha,\beta)})^\sigma$ following [60, Chapter 5]. So, by using the formula [51, Section 2.3.3, eq. 1]

$$\int_0^\infty t^{\sigma-1} e^{-rt}\, dt = \Gamma(\sigma) r^{-\sigma},$$

which holds for $\sigma, r > 0$, the Riesz potentials (fractional integrals) are given by

$$(-\mathcal{J}^{(\alpha,\beta)})^{-\sigma} f(n) = \frac{1}{\Gamma(\sigma)} \int_0^\infty \frac{W_t^{(\alpha,\beta)} f(n)}{t^{1-\sigma}}\, dt, \qquad \sigma > 0.$$

Finally, we formally define the Riesz transforms $\mathcal{R}^{(\alpha,\beta)}$ associated with the operator $\mathcal{J}^{(\alpha,\beta)}$ by the composition

$$\mathcal{R}^{(\alpha,\beta)} f(n) = \delta(-\mathcal{J}^{(\alpha,\beta)})^{-1/2} f(n).$$

Unfortunately, the next result shows that the operator $(-\mathcal{J}^{(\alpha,\beta)})^{-1/2}$ is not well defined for $\alpha, \beta \geq -1/2$, so we will need and alternative way to define the Riesz transforms.

Throughout the rest of the work we will use the standard notation c_{00} instead of the less usual $(\mathbb{C})_0^{\mathbb{N}}$.

Proposition IV.1.1. *Let $\alpha, \beta \geq -1/2$, $\sigma > 0$, and $f \in c_{00}$. Then $(-\mathcal{J}^{(\alpha,\beta)})^{-\sigma}$ is well defined if and only if $\sigma < 1/2$.*

Proof. First of all, we have that $W_t^{(\alpha,\beta)}f$ is well defined for $f \in c_{00}$ (it is well defined for sequences in $\ell^\infty(\mathbb{N})$ by Lemma III.2.1 and then for those in $c_{00} \subset \ell^\infty(\mathbb{N})$). Then, we will prove that $(-\mathcal{J}^{(\alpha,\beta)})^{-\sigma}$ is finite if and only if $0 < \sigma < 1/2$.

The argument for the sufficiency is as follows. It is clear that

$$\begin{aligned}\left|(-\mathcal{J}^{(\alpha,\beta)})^{-\sigma} f(n)\right| &\le \frac{1}{\Gamma(\sigma)} \int_0^\infty |W_t^{(\alpha,\beta)} f(n)| \frac{dt}{t^{1-\sigma}} \\ &\le \frac{1}{\Gamma(\sigma)} \left(\int_0^1 |W_t^{(\alpha,\beta)} f(n)| \frac{dt}{t^{1-\sigma}} + \int_1^\infty |W_t^{(\alpha,\beta)} f(n)| \frac{dt}{t^{1-\sigma}} \right) \\ &=: \frac{I_1 + I_2}{\Gamma(\sigma)}.\end{aligned}$$

For I_1 we use the estimate (see Lemma III.2.1 for the case $m \neq n$ and note that for $m = n$ is obvious)

$$|K_t^{(\alpha,\beta)}(m,n)| \le C \begin{cases} \dfrac{t^{1/2}}{|m-n|^2}, & m \neq n, \\ 1, & m = n, \end{cases}$$

to obtain that

$$I_1 \le C \left(\sum_{\substack{m=0 \\ m \neq n}}^{\infty} \frac{|f(m)|}{|m-n|^2} \int_0^1 \frac{dt}{t^{1/2-\sigma}} + |f(n)| \int_0^1 \frac{dt}{t^{1-\sigma}} \right)$$

and both terms are finite for $\sigma > 0$. To deduce the convergence of I_2, using that $f \in c_{00}$ and the bound (II.14), it is enough to show that

$$\int_1^\infty \int_{-1}^1 \frac{e^{-(1-x)t}}{\sqrt{1-x^2}} \, dx \frac{dt}{t^{1-\sigma}} < \infty.$$

Since

$$\int_{-1}^1 \frac{e^{-(1-x)t}}{\sqrt{1-x^2}} \, dx \le C \int_0^1 \frac{e^{-(1-x)t}}{\sqrt{1-x}} \, dx = \frac{C}{\sqrt{t}} \int_0^t \frac{e^{-s}}{\sqrt{s}} \, ds \simeq \frac{C}{\sqrt{t}},$$

we have

$$\int_1^\infty \int_{-1}^1 \frac{e^{-(1-x)t}}{\sqrt{1-x^2}} \, dx \frac{dt}{t^{1-\sigma}} \le C \int_1^\infty t^{\sigma - 3/2} \, dt \le C,$$

where we have used that $\sigma < 1/2$.

To show the necessity of the condition $\sigma < 1/2$, we will use the inequality

$$\int_{-1}^1 \frac{e^{-(1-x)t}}{\sqrt{1-x^2}} \, dx < \pi \liminf_{n \to \infty} \int_{-1}^1 e^{-(1-x)t} (p_n^{(\alpha,\beta)}(x))^2 \, d\mu_{\alpha,\beta}(x).$$

This is a particular case of a classical result due to A. Máté, P. Nevai, and V. Totik [37, Theorem 2]. From this fact, there exists $N \in \mathbb{N}$ such that for every $n \ge N$,

$$C \int_{-1}^1 \frac{e^{-(1-x)t}}{\sqrt{1-x^2}} \, dx < \int_{-1}^1 e^{-(1-x)t} (p_n^{(\alpha,\beta)}(x))^2 \, d\mu_{\alpha,\beta}(x).$$

Then, taking $j \in \mathbb{N}$ such that $j \geq N$ and the sequence $\{f_j(m) = \delta_{jm}\}_{m\geq 0}$ we have

$$\begin{aligned}(-\mathcal{J}^{(\alpha,\beta)})^{-\sigma} f_j(j) &\geq \int_1^\infty K_t^{(\alpha,\beta)}(j,j)\frac{dt}{t^{1-\sigma}} \\ &= \int_1^\infty \int_{-1}^1 e^{-(1-x)t}(p_j^{(\alpha,\beta)}(x))^2\, d\mu_{\alpha,\beta}(x)\frac{dt}{t^{1-\sigma}} \\ &> C\int_1^\infty \int_{-1}^1 \frac{e^{-(1-x)t}}{\sqrt{1-x^2}}\, dx \frac{dt}{t^{1-\sigma}}.\end{aligned} \tag{IV.1}$$

Now, using that $t > 1$, we obtain that

$$\int_{-1}^1 \frac{e^{-(1-x)t}}{\sqrt{1-x^2}}\, dx \geq \int_0^1 \frac{e^{-(1-x)t}}{\sqrt{1-x}}\, dx = \frac{C}{\sqrt{t}}\int_0^t \frac{e^{-s}}{\sqrt{s}}\, ds \simeq \frac{C}{\sqrt{t}}. \tag{IV.2}$$

Then, since $(-\mathcal{J}^{(\alpha,\beta)})^{-\sigma} f_j(j)$ is well defined, from (IV.1) and (IV.2) we deduce that $\sigma < 1/2$. □

Motivated by [17], we define the Riesz transforms $\mathcal{R}^{(\alpha,\beta)}$ by

$$\mathcal{R}^{(\alpha,\beta)} f(n) = \lim_{\sigma\to\frac{1}{2}^-} \delta(-\mathcal{J}^{(\alpha,\beta)})^{-\sigma} f(n). \tag{IV.3}$$

This is a natural way to proceed because in that paper it was shown that the Riesz transform associated with Δ_{d} turns out to be the discrete one-dimensional Hilbert transform[1]

$$H_{\mathrm{d}} f(n) = \frac{1}{\pi}\sum_{m\in\mathbb{Z}} \frac{f(m)}{n-m+1/2}, \qquad n \in \mathbb{Z},$$

defined for appropriate functions $f = \{f(n)\}_{n\in\mathbb{Z}}$.

IV.2 Mapping properties of the Riesz transforms

In this section, we are going to prove ℓ^p-estimates for the Riesz transforms $\mathcal{R}^{(\alpha,\beta)}$ for $\alpha, \beta \geq -1/2$. As in the proof of Theorem III.3.1, we use the discrete Calderón-Zymgund theory of Section II.3. The difference here is that we invoke it in the simpler scalar case.

To do so, we first express the Riesz transforms given in (IV.3) in the form of Theorem II.3.1. Taking into account Proposition IV.1.1, for $\alpha, \beta \geq -1/2$, $0 < \sigma <$

[1]This definition is the one given in [71]. There is another typical definition of the discrete one-dimensional Hilbert transform for sequences in the literature given by

$$\tilde{H}_{\mathrm{d}} f(n) = \frac{1}{\pi}\sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} \frac{f(m)}{n-m}, \qquad n \in \mathbb{Z}.$$

In this direction see [2, 3, 31] and [8].

$1/2$, and $f \in c_{00}$, Fubini's theorem gives

$$\begin{aligned}(-\mathcal{J}^{(\alpha,\beta)})^{-\sigma} f(n) &= \sum_{m=0}^{\infty} f(m) \frac{1}{\Gamma(\sigma)} \int_0^\infty K_t^{(\alpha,\beta)}(m,n)\, \frac{dt}{t^{1-\sigma}} \\ &= \frac{1}{\Gamma(\sigma)} \sum_{m=0}^{\infty} f(m) \int_{-1}^{1} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x) \int_0^\infty t^{\sigma-1} e^{-(1-x)t}\, dt\, d\mu_{\alpha,\beta}(x) \\ &= \sum_{m=0}^{\infty} f(m) \int_{-1}^{1} \frac{p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)}{(1-x)^\sigma}\, d\mu_{\alpha,\beta}(x).\end{aligned}$$

By [48, 18.9.6], it is easy to check that

$$\delta p_n^{(\alpha,\beta)}(x) = (1-x) p_n^{(\alpha+1,\beta)}(x),$$

and therefore, for each sequence in $f \in c_{00}$,

$$\begin{aligned}\mathcal{R}^{(\alpha,\beta)} f(n) &= \lim_{\sigma \to \frac{1}{2}^-} \delta(-\mathcal{J}^{(\alpha,\beta)})^{-\sigma} f(n) \\ &= \lim_{\sigma \to \frac{1}{2}^-} \sum_{m=0}^{\infty} f(m) \int_{-1}^{1} \frac{p_m^{(\alpha,\beta)}(x) p_n^{(\alpha+1,\beta)}(x)}{(1-x)^{\sigma-1}}\, d\mu_{\alpha,\beta}(x) \\ &= \sum_{m=0}^{\infty} f(m) R^{(\alpha,\beta)}(m,n),\end{aligned}$$

with

$$R^{(\alpha,\beta)}(m,n) = \int_{-1}^{1} (1-x)^{1/2} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha+1,\beta)}(x)\, d\mu_{\alpha,\beta}(x).$$

We now state the main theorem of this chapter.

Theorem IV.2.1. *Let $\alpha, \beta \geq -1/2$ and let $\mathcal{R}^{(\alpha,\beta)}$ be the Riesz transforms defined in* (IV.3).

(a) If $1 < p < \infty$ and $w \in A_p(\mathbb{N})$, then

$$\|\mathcal{R}^{(\alpha,\beta)} f\|_{\ell^p(\mathbb{N},w)} \leq C \|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N},w),$$

where C is a constant independent of f. Consequently, the operator $\mathcal{R}^{(\alpha,\beta)}$ extends uniquely to a bounded linear operator from $\ell^p(\mathbb{N},w)$ into itself.

(b) If $w \in A_1(\mathbb{N})$, then

$$\|\mathcal{R}^{(\alpha,\beta)} f\|_{\ell^{1,\infty}(\mathbb{N},w)} \leq C \|f\|_{\ell^1(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^1(\mathbb{N},w),$$

where C is a constant independent of f. Consequently, the operator $\mathcal{R}^{(\alpha,\beta)}$ extends uniquely to a bounded linear operator from $\ell^1(\mathbb{N},w)$ into $\ell^{1,\infty}(\mathbb{N},w)$.

As it was mentioned above, the proof of this result relies on the discrete Calderón-Zygmund theory of Section II.3 (in the scalar case). Below, we include the steps for the proof of the theorem.

First, we present an auxiliary result concerning $A_p(\mathbb{N})$ weights that we will use later.

Lemma IV.2.2. *Let* $1 \le p < \infty$ *and* $w \in A_p(\mathbb{N})$*. Then,* $w(n) \simeq w(n+1)$*.*

Proof. For $1 < p < \infty$ and $w \in A_p(\mathbb{N})$, it is clear that

$$[w]_{A_p(\mathbb{N})} \ge \frac{1}{2^p}(w(n) + w(n+1))(w(n)^{-1/(p-1)} + w(n+1)^{-1/(p-1)})^{p-1}, \qquad n \in \mathbb{N}.$$

Now, by means of the inequality $(a+b)^r \ge C_r(a^r + b^r)$, where $a, b, r > 0$ and $C_r = \min\{2^{r-1}, 1\}$, we have

$$[w]_{A_p(\mathbb{N})} \ge \frac{C_{p-1}}{2^p}(w(n) + w(n+1))(w(n)^{-1} + w(n+1)^{-1}) > \frac{C_{p-1}}{2^p} w(n)w(n+1)^{-1}$$

and, similarly,

$$[w]_{A_p(\mathbb{N})} > \frac{C_{p-1}}{2^p} w(n+1)w(n)^{-1}.$$

So,

$$\frac{C_{p-1}}{2^p[w]_{A_p(\mathbb{N})}} w(n) < w(n+1) < \frac{2^p[w]_{A_p(\mathbb{N})}}{C_{p-1}} w(n).$$

For $p = 1$, if we suppose first $w(n) \le w(n+1)$, then it is clear that

$$\begin{aligned}[w]_{A_1(\mathbb{N})} &\ge \frac{1}{2}(w(n) + w(n+1)) \max\{w(n)^{-1}, w(n+1)^{-1}\} \\ &= \frac{1}{2}(1 + w(n+1)w(n)^{-1}) > \frac{w(n+1)w(n)^{-1}}{2}\end{aligned}$$

and we obtain $w(n) \le w(n+1) < 2[w]_{A_1(\mathbb{N})} w(n)$. On the other hand, supposing $w(n+1) < w(n)$ the procedure is exactly the same. □

Let us see now that the operator $\mathcal{R}^{(\alpha,\beta)}$ is bounded from $\ell^2(\mathbb{N})$ into itself. Let $F_{\alpha,\beta}(x)$ be defined by (II.15), which belongs to $L^2([-1,1], d\mu_{\alpha,\beta})$ for each sequence f in $\ell^2(\mathbb{N})$. Recall that (see (II.16))

$$\|f\|_{\ell^2(\mathbb{N})} = \|F_{\alpha,\beta}\|_{L^2([-1,1], d\mu_{\alpha,\beta})}. \tag{IV.4}$$

Therefore, noting that

$$\begin{aligned}\mathcal{R}^{(\alpha,\beta)} f(n) &= \int_{-1}^{1} (1-x)^{1/2} p_n^{(\alpha+1,\beta)}(x) F_{\alpha,\beta}(x)\, d\mu_{\alpha,\beta}(x) \\ &= c_n^{(\alpha+1,\beta)}((1-\cdot)^{-1/2} F_{\alpha,\beta}),\end{aligned}$$

by (IV.4) we have that

$$\begin{aligned}\|\mathcal{R}^{(\alpha,\beta)} f\|_{\ell^2(\mathbb{N})} &= \|c_n^{(\alpha+1,\beta)}((1-\cdot)^{-1/2} F_{\alpha,\beta})\|_{\ell^2(\mathbb{N})} \\ &= \|(1-\cdot)^{-1/2} F_{\alpha,\beta}\|_{L^2([-1,1], d\mu_{\alpha+1,\beta})} = \|F_{\alpha,\beta}\|_{L^2([-1,1], d\mu_{\alpha,\beta})} = \|f\|_{\ell^2(\mathbb{N})}\end{aligned}$$

and then $\mathcal{R}^{(\alpha,\beta)}$ is a bounded operator from $\ell^2(\mathbb{N})$ into itself.

Next, we note that it is possible to split the m variable into even and odd parts, that is,

$$\mathcal{R}^{(\alpha,\beta)} f(n) = \sum_{m=0}^{\infty} f(2m) R^{(\alpha,\beta)}(2m, n) + \sum_{m=0}^{\infty} f(2m+1) R^{(\alpha,\beta)}(2m+1, n),$$

which motivates the following definitions

$$
\begin{aligned}
{}^{\mathrm{e,e}}\mathcal{R}^{(\alpha,\beta)}f(n) &= \sum_{m=0}^{\infty} f(m)\,{}^{\mathrm{e,e}}R^{(\alpha,\beta)}(m,n), \quad {}^{\mathrm{e,e}}R^{(\alpha,\beta)}(m,n) = R^{(\alpha,\beta)}(2m,2n),\\
{}^{\mathrm{e,o}}\mathcal{R}^{(\alpha,\beta)}f(n) &= \sum_{m=0}^{\infty} f(m)\,{}^{\mathrm{e,o}}R^{(\alpha,\beta)}(m,n), \quad {}^{\mathrm{e,o}}R^{(\alpha,\beta)}(m,n) = R^{(\alpha,\beta)}(2m,2n+1),\\
{}^{\mathrm{o,e}}\mathcal{R}^{(\alpha,\beta)}f(n) &= \sum_{m=0}^{\infty} f(m)\,{}^{\mathrm{o,e}}R^{(\alpha,\beta)}(m,n), \quad {}^{\mathrm{o,e}}R^{(\alpha,\beta)}(m,n) = R^{(\alpha,\beta)}(2m+1,2n),
\end{aligned}
$$

and

$$
{}^{\mathrm{o,o}}\mathcal{R}^{(\alpha,\beta)}f(n) = \sum_{m=0}^{\infty} f(m)\,{}^{\mathrm{o,o}}R^{(\alpha,\beta)}(m,n), \quad {}^{\mathrm{o,o}}R^{(\alpha,\beta)}(m,n) = R^{(\alpha,\beta)}(2m+1,2n+1).
$$

Hence, we obtain that

$$
\mathcal{R}^{(\alpha,\beta)}f(2n) = {}^{\mathrm{e,e}}\mathcal{R}^{(\alpha,\beta)}\tilde{f}(n) + {}^{\mathrm{o,e}}\mathcal{R}^{(\alpha,\beta)}\hat{f}(n)
$$

and

$$
\mathcal{R}^{(\alpha,\beta)}f(2n+1) = {}^{\mathrm{e,o}}\mathcal{R}^{(\alpha,\beta)}\tilde{f}(n) + {}^{\mathrm{o,o}}\mathcal{R}^{(\alpha,\beta)}\hat{f}(n),
$$

with $\tilde{f}(n) = f(2n)$ and $\hat{f}(n) = f(2n+1)$, $n \in \mathbb{N}$. In addition, note that ${}^{\mathrm{e,e}}\mathcal{R}^{(\alpha,\beta)}$, ${}^{\mathrm{o,e}}\mathcal{R}^{(\alpha,\beta)}$, ${}^{\mathrm{e,o}}\mathcal{R}^{(\alpha,\beta)}$, and ${}^{\mathrm{o,o}}\mathcal{R}^{(\alpha,\beta)}$ are bounded operators in $\ell^2(\mathbb{N})$ because so is $\mathcal{R}^{(\alpha,\beta)}$. Indeed, let us define the functions

$$
g(n) = f(n/2)\chi_{\mathcal{E}}(n) \qquad \text{and} \qquad h(n) = f((n-1)/2)\chi_{\mathcal{O}}(n),
$$

where $\mathcal{E}$ and $\mathcal{O}$ denotes the sets of even and odd numbers respectively. Then, we have that ${}^{\mathrm{e,e}}\mathcal{R}^{(\alpha,\beta)}f(n) = \mathcal{R}^{(\alpha,\beta)}g(2n)$, ${}^{\mathrm{o,e}}\mathcal{R}^{(\alpha,\beta)}f(n) = \mathcal{R}^{(\alpha,\beta)}h(2n)$, ${}^{\mathrm{e,o}}\mathcal{R}^{(\alpha,\beta)}f(n) = \mathcal{R}^{(\alpha,\beta)}g(2n+1)$, and ${}^{\mathrm{o,o}}\mathcal{R}^{(\alpha,\beta)}f(n) = \mathcal{R}^{(\alpha,\beta)}h(2n+1)$, so the boundedness on $\ell^2(\mathbb{N})$ of each operator follows immediately.

Therefore, it is enough to prove that the kernels ${}^{\mathrm{e,e}}R^{(\alpha,\beta)}$, ${}^{\mathrm{o,e}}R^{(\alpha,\beta)}$, ${}^{\mathrm{e,o}}R^{(\alpha,\beta)}$, and ${}^{\mathrm{o,o}}R^{(\alpha,\beta)}$ are semi-local $\mathcal{L}(\mathbb{C},\mathbb{C})$-standard kernels. This fact is an immediate consequence of the following propositions[2].

Proposition IV.2.3. *Let $n,m \in \mathbb{N}$, $n \neq m$, $\alpha,\beta \geq -1/2$. Then,*

$$
|R^{(\alpha,\beta)}(m,n)| \leq \frac{C}{|m-n|}. \tag{IV.5}
$$

Moreover,

$$
|R^{(\alpha,\beta)}(n,n)| \leq C. \tag{IV.6}
$$

[2]Note that in Proposition IV.2.4 we estimate the difference $R^{(\alpha,\beta)}(m+2,n) - R^{(\alpha,\beta)}(m,n)$ instead of $R^{(\alpha,\beta)}(m+1,n) - R^{(\alpha,\beta)}(m,n)$. The reason is that the former is more appropriate because $p_n^{(a,b)}(x) - p_{n+2}^{(a,b)}(x)$ behaves better than $p_n^{(a,b)}(x) - p_{n+1}^{(a,b)}(x)$ (see [50]).

Proposition IV.2.4. *Let $n, m \in \mathbb{N}$, $n \neq m$, $2m/3 \leq n \leq 3m/2$, $\alpha, \beta \geq -1/2$. Then,*

$$|R^{(\alpha,\beta)}(m+2,n) - R^{(\alpha,\beta)}(m,n)| \leq \frac{C}{|m-n|^2} \tag{IV.7}$$

and

$$|R^{(\alpha,\beta)}(m,n+2) - R^{(\alpha,\beta)}(m,n)| \leq \frac{C}{|m-n|^2}. \tag{IV.8}$$

The proofs of these two propositions are the most delicate points of the chapter and they are postponed to the next section.

In this way, by Theorem II.3.1 (invoked in the scalar setting), taking the weights $w_e(n) = w(2n)$ and $w_o(n) = w(2n+1)$ (note that both of them belongs to $A_p(\mathbb{N})$ because $w \in A_p(\mathbb{N})$), and applying the bound (IV.6) to control the diagonal terms, for $1 < p < \infty$, we have

$$\begin{aligned}
\|^{\mathrm{e,e}}\mathcal{R}^{(\alpha,\beta)}\tilde{f}\|_{\ell^p(\mathbb{N},w_e)} &\leq C\|\tilde{f}\|_{\ell^p(\mathbb{N},w_e)},\\
\|^{\mathrm{o,e}}\mathcal{R}^{(\alpha,\beta)}\hat{f}\|_{\ell^p(\mathbb{N},w_e)} &\leq C\|\hat{f}\|_{\ell^p(\mathbb{N},w_e)},\\
\|^{\mathrm{e,o}}\mathcal{R}^{(\alpha,\beta)}\tilde{f}\|_{\ell^p(\mathbb{N},w_o)} &\leq C\|\tilde{f}\|_{\ell^p(\mathbb{N},w_o)},\\
\|^{\mathrm{o,o}}\mathcal{R}^{(\alpha,\beta)}\hat{f}\|_{\ell^p(\mathbb{N},w_o)} &\leq C\|\hat{f}\|_{\ell^p(\mathbb{N},w_o)},
\end{aligned}$$

and the corresponding weak inequalities for $p = 1$. To complete the proof, it is enough to observe that, by Lemma IV.2.2,

$$\|\hat{f}\|_{\ell^p(\mathbb{N},w_e)} \leq C\|\hat{f}\|_{\ell^p(\mathbb{N},w_o)} \leq C\|f\|_{\ell^p(\mathbb{N},w)}$$

and

$$\|\tilde{f}\|_{\ell^p(\mathbb{N},w_o)} \leq C\|\tilde{f}\|_{\ell^p(\mathbb{N},w_e)} \leq C\|f\|_{\ell^p(\mathbb{N},w)}.$$

IV.3 Proofs of Propositions IV.2.3 and IV.2.4

Proof of Proposition IV.2.3. The proof of (IV.6) is obvious, so we will focus on the proof of (IV.5).

First, we suppose that $n > m$. We decompose $R^{(\alpha,\beta)}(m,n)$ according to the intervals $I_1 = (-1, -1 + 1/(n+1)^2)$, $I_2 = [-1 + 1/(n+1)^2, 1 - 1/(n+1)^2]$, and $I_3 = (1 - 1/(n+1)^2, 1)$ and denote the corresponding integrals by $R_1(m,n)$, $R_2(m,n)$, and $R_3(m,n)$. From (II.13), for $\alpha, \beta \geq -1/2$, we have

$$|R_1(m,n)| \leq C(n+1)^{\beta+1/2}(m+1)^{\beta+1/2}\int_{I_1}(1+x)^{\beta}\,dx \leq \frac{C}{n+1}$$

and

$$|R_3(m,n)| \leq C(n+1)^{\alpha+3/2}(m+1)^{\alpha+1/2}\int_{I_3}(1-x)^{\alpha+1/2}\,dx \leq \frac{C}{n+1},$$

and these estimates are enough to prove (IV.5).

Let us focus on $R_2(m,n)$. We consider the notation

$$J(m,n) = \int_{I_2} H_{\alpha,\beta}(x) p_n^{(\alpha+1,\beta)}(x) p_m^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x),$$

with

$$H_{\alpha,\beta}(x) = \frac{2\beta - 2\alpha + 1 - (2\alpha + 2\beta + 3)x}{4(1-x)^{1/2}} \tag{IV.9}$$

and

$$S(m,n) = U_{\alpha,\beta}((1-(\cdot))^{1/2} p_n^{(\alpha+1,\beta)}, p_m^{(\alpha,\beta)})(x)\Big|_{x=-1+1/(n+1)^2}^{x=1-1/(n+1)^2}.$$

To give a proper expression of the integral $R_2(m,n)$, we use (II.5), with $f(x) = (1-x)^{1/2} p_n^{(\alpha+1,\beta)}(x)$ and $g(x) = p_m^{(\alpha,\beta)}(x)$, and (II.6), with $h_1(x) = p_n^{(\alpha+1,\beta)}(x)$ and $h_2(x) = (1-x)^{1/2}$. Then, we get that

$$\begin{aligned}
\lambda_m^{(\alpha,\beta)} R_2(m,n) &= \int_{I_2} (1-x)^{1/2} p_n^{(\alpha+1,\beta)}(x) L^{\alpha,\beta} p_m^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \\
&= S(m,n) + \int_{I_2} L^{\alpha,\beta}((1-(\cdot))^{1/2} p_n^{(\alpha+1,\beta)})(x) p_m^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \\
&= S(m,n) + \lambda_n^{(\alpha+1,\beta)} R_2(m,n) + J(m,n).
\end{aligned}$$

Therefore, noting that $\lambda_m^{(\alpha,\beta)} \neq \lambda_n^{(\alpha+1,\beta)}$,

$$R_2(m,n) = \frac{S(m,n) + J(m,n)}{\lambda_m^{(\alpha,\beta)} - \lambda_n^{(\alpha+1,\beta)}}. \tag{IV.10}$$

Now, we use the identity (II.12) (if $n = 0$ then $dP_n^{(\alpha,\beta)}(x)/dx = 0$), the estimate (II.13), and the restrictions $\alpha, \beta \geq -1/2$ to obtain that

$$|S(m,n)| \leq C(n+1). \tag{IV.11}$$

In order to estimate the term $J(m,n)$ we decompose it according to the intervals $V_1 = [-1 + 1/(n+1)^2, -1 + 1/(m+1)^2)$, $V_2 = [-1 + 1/(m+1)^2, 1 - 1/(m+1)^2]$, and $V_3 = (1 - 1/(m+1)^2, 1 - 1/(n+1)^2]$. We denote the corresponding integrals by $J_1(m,n)$, $J_2(m,n)$, and $J_3(m,n)$. In this way, by using (II.13), the estimate $|H_{\alpha,\beta}(x)| \leq C(1-x)^{-1/2}$ for $-1 < x < 1$, and the condition $\alpha, \beta \geq -1/2$, we deduce the bounds

$$\begin{aligned}
|J_1(m,n)| &\leq C(m+1)^{\beta+1/2} \int_{V_1} (1+x)^{\beta/2 - 1/4}\, dx \\
&\leq C \int_{V_1} (1+x)^{-1/2}\, dx \leq C,
\end{aligned}$$

$$\begin{aligned}
|J_2(m,n)| &\leq C \int_{V_2} (1+x)^{-1/2} (1-x)^{-3/2}\, dx \\
&\leq C(m+1),
\end{aligned}$$

and

$$|J_3(m,n)| \le C(m+1)^{\alpha+1/2} \int_{V_3} (1-x)^{\alpha/2-5/4}\,dx$$
$$\le C \int_{V_3} (1-x)^{-3/2}\,dx \le C(n+1).$$

Then, we have

$$|J(m,n)| \le C(n+1), \tag{IV.12}$$

and, from (IV.10), (IV.11), and (IV.12), we obtain that $|R_2(m,n)| \le C|n-m|^{-1}$ and the estimate (IV.5) is proved for $n > m$.

The case $n < m$ follows from the above argument by interchanging the roles of n and m but we include some details for the sake of completeness.

We decompose $R^{(\alpha,\beta)}(m,n)$ according to the intervals $I_1' = (-1, -1+1/(m+1)^2)$, $I_2' = [-1+1/(m+1)^2, 1-1/(m+1)^2]$, and $I_3' = (1-1/(m+1)^2, 1)$ and denote the corresponding integrals by $R_1'(m,n)$, $R_2'(m,n)$, and $R_3'(m,n)$. By similar arguments than above we obtain that

$$|R_1'(m,n)| \le \frac{C}{m+1} \qquad \text{and} \qquad |R_3'(m,n)| \le \frac{C}{m+1}.$$

Now, for $R_2'(m,n)$, by using (II.7) and noting again that $\lambda_m^{(\alpha,\beta)} \neq \lambda_n^{(\alpha+1,\beta)}$, we deduce the identity

$$R_2'(m,n) = \frac{S'(m,n) - J'(m,n)}{\lambda_n^{(\alpha+1,\beta)} - \lambda_m^{(\alpha,\beta)}}, \tag{IV.13}$$

where

$$J'(m,n) = \int_{I_2'} H_{\alpha,\beta}(x) p_n^{(\alpha+1,\beta)}(x) p_m^{(\alpha,\beta)}(x)\,d\mu_{\alpha,\beta}(x),$$

with $H_{\alpha,\beta}$ as in (IV.9), and

$$S'(m,n) = U_{\alpha+1,\beta}((1-(\cdot))^{-1/2} p_m^{(\alpha,\beta)}, p_n^{(\alpha+1,\beta)})(x)\Big|_{x=-1+1/(m+1)^2}^{x=1-1/(m+1)^2}.$$

As in the previous case, we deduce the estimate

$$|S'(m,n)| \le (m+1). \tag{IV.14}$$

To analyze $J'(m,n)$ we decompose it according to the intervals $V_1' = [-1+1/(m+1)^2, -1+1/(n+1)^2)$, $V_2' = [-1+1/(n+1)^2, 1-1/(n+1)^2]$, and $V_3' = (1-1/(n+1)^2, 1-1/(m+1)^2]$. The corresponding integrals are denoted by $J_1'(m,n)$, $J_2'(m,n)$, and $J_3'(m,n)$, and we have

$$|J_1'(m,n)| \le C, \qquad |J_2'(m,n)| \le C(n+1), \qquad \text{and} \qquad |J_3'(m,n)| \le C(m+1).$$

Therefore

$$|J'(m,n)| \le C(m+1). \tag{IV.15}$$

Then (IV.5) is also proved for $n < m$ and the proof of the proposition is finished. □

In the proof of the Proposition IV.2.4 we will use the following lemmas.

Lemma IV.3.1. *Let $n \in \mathbb{N}$ and $a, b > -1$. Then,*

$$\begin{aligned} &|p_{n+2}^{(a,b)}(x) - p_n^{(a,b)}(x)| \\ &\quad \le C \begin{cases} (n+1)^{a-1/2}, & 1 - 1/(n+1)^2 < x < 1, \\ (1-x)^{-a/2+1/4}(1+x)^{-b/2+1/4}, & -1 + 1/(n+1)^2 \le x \le 1 - 1/(n+1)^2, \\ (n+1)^{b-1/2}, & -1 < x < -1 + 1/(n+1)^2. \end{cases} \end{aligned}$$

Proof of Lemma IV.3.1. First of all, note that it is enough to proof that

$$|p_{n+2}^{(a,b)}(x) - p_n^{(a,b)}(x)| \le C \begin{cases} (n+1)^{a-1/2}, & 1 - 1/(n+1)^2 < x < 1, \\ (1-x)^{-a/2+1/4}, & 0 \le x \le 1 - 1/(n+1)^2, \end{cases} \tag{IV.16}$$

because the bound for $-1 < x < 0$ is obtained immediately from latter by using the relation $P_n^{(a,b)}(-z) = (-1)^n P_n^{(b,a)}(z)$, $-1 < z < 1$.

It is straightforward to check that

$$p_{n+2}^{(a,b)}(x) - p_n^{(a,b)}(x) = \left(\frac{w_{n+2}^{(a,b)}}{w_n^{(a,b)}} - 1 \right) p_n^{(a,b)}(x) + w_{n+2}^{(a,b)}(P_{n+2}^{(a,b)}(x) - P_n^{(a,b)}(x)). \tag{IV.17}$$

From the estimate

$$\left| \frac{w_{n+2}^{(a,b)}}{w_n^{(a,b)}} - 1 \right| \le \frac{C}{n+1}$$

and the uniform estimate (II.13) (note that if $0 \le x < 1 - 1/(n+1)^2$, then $\frac{1}{n+1} \le (1-x)^{1/2}$), we conclude that

$$\left| \left(\frac{w_{n+2}^{(a,b)}}{w_n^{(a,b)}} - 1 \right) p_n^{(a,b)}(x) \right| \le C \begin{cases} (n+1)^{a-1/2}, & 1 - 1/(n+1)^2 < x < 1, \\ (1-x)^{-a/2+1/4}, & 0 \le x \le 1 - 1/(n+1)^2. \end{cases} \tag{IV.18}$$

Now we apply the identity (II.11) to deduce the estimate

$$\begin{aligned} &w_{n+2}^{(a,b)} |P_{n+1}^{(a,b)}(x) - P_n^{(a,b)}(x)| \\ &\qquad \le \frac{(2n+a+b+2)}{2(n+1)}(1-x) \frac{w_{n+2}^{(a,b)}}{w_n^{(a+1,b)}} |p_n^{(a+1,b)}(x)| + \frac{|a|}{n+1} \frac{w_{n+2}^{(a,b)}}{w_n^{(a,b)}} |p_n^{(a,b)}(x)|. \end{aligned}$$

Therefore, the uniform estimate (II.13) implies that

$$\begin{aligned} &w_{n+2}^{(a,b)} |P_{n+1}^{(a,b)}(x) - P_n^{(a,b)}(x)| \\ &\qquad \le C \begin{cases} (n+1)^{a-1/2}, & 1 - 1/(n+1)^2 < x < 1, \\ (1-x)^{-a/2+1/4}, & 0 \le x \le 1 - 1/(n+1)^2, \end{cases} \end{aligned} \tag{IV.19}$$

and the same bound holds for the term $w_{n+2}^{(a,b)} |P_{n+2}^{(a,b)}(x) - P_{n+1}^{(a,b)}(x)|$. Then, (IV.16) follows from (IV.17), (IV.18), and (IV.19). □

Lemma IV.3.2. *Let $n \in \mathbb{N}$, $a, b > -1$. Then,*

$$|(p_{n+2}^{(a,b)} - p_n^{(a,b)})'(x)| \le C(n+1) \times \begin{cases} (n+1)^{a+1/2}, & 1 - 1/(n+1)^2 < x < 1, \\ (1-x)^{-a/2-1/4}(1+x)^{-b/2-1/4}, & -1 + 1/(n+1)^2 \le x \le 1 - 1/(n+1)^2, \\ (n+1)^{b+1/2}, & -1 < x < -1 + 1/(n+1)^2. \end{cases}$$

Proof of Lemma IV.3.2. First, we assume that $n \neq 0$. By (II.12), it is easy to check that

$$(p_{n+2}^{(a,b)} - p_n^{(a,b)})'(x) = \frac{w_{n+2}^{(a,b)}}{w_{n+1}^{(a+1,b+1)}} \frac{n+a+b+3}{2} (p_{n+1}^{(a+1,b+1)}(x) - p_{n-1}^{(a+1,b+1)}(x)) + \left(\frac{w_{n+2}^{(a,b)}}{w_{n+1}^{(a+1,b+1)}} \frac{n+a+b+3}{2} - \frac{w_n^{(a,b)}}{w_{n-1}^{(a+1,b+1)}} \frac{n+a+b+1}{2} \right) p_{n-1}^{(a+1,b+1)}(x).$$

Then, using that

$$\left| \frac{w_{n+2}^{(a,b)}}{w_{n+1}^{(a+1,b+1)}} \frac{n+a+b+3}{2} - \frac{w_n^{(a,b)}}{w_{n-1}^{(a+1,b+1)}} \frac{n+a+b+1}{2} \right| \le C,$$

and the estimate (II.13) and Lemma IV.3.1, the result follows. If $n = 0$, we proceed in a similar way using that $dP_0^{(\alpha,\beta)}(x)/dx = 0$. □

Proof of Proposition IV.2.4. We will prove the estimate (IV.7) for $n > m$ and (IV.8) for $n < m$. The remaining two cases can be treated in a similar way and we omit the details.

In this way, we first assume that $n > m$ and prove (IV.7).

We decompose the difference $R^{(\alpha,\beta)}(m+2, n) - R^{(\alpha,\beta)}(m, n)$ into three integrals $\mathcal{R}_1(m,n)$, $\mathcal{R}_2(m,n)$, and $\mathcal{R}_3(m,n)$ over the intervals $I_1 = (-1, -1 + 1/(n+1)^2)$, $I_2 = [-1 + 1/(n+1)^2, 1 - 1/(n+1)^2]$, and $I_3 = (1 - 1/(n+1)^2, 1)$. From (II.13) and Lemma IV.3.1 (note that by hypothesis $2m/3 \le n \le 3m/2$), we have

$$|\mathcal{R}_1(m,n)| \le C(m+1)^{\beta-1/2}(n+1)^{\beta+1/2} \int_{I_1} (1+x)^\beta \, dx \le \frac{C}{(n+1)^2}$$

and

$$|\mathcal{R}_3(m,n)| \le C(n+1)^{\alpha+3/2}(m+1)^{\alpha-1/2} \int_{I_3} (1-x)^{\alpha+1/2} \, dx \le \frac{C}{(n+1)^2},$$

which are enough to prove (IV.7) in these cases.

We deal now with the most delicate integral $\mathcal{R}_2(m,n)$. We recover some notation from the proof of Proposition IV.2.3 and denote

$$\mathcal{J}(m,n) = \int_{I_2} H_{\alpha,\beta}(x) p_n^{(\alpha+1,\beta)}(x)(p_{m+2}^{(\alpha,\beta)}(x) - p_m^{(\alpha,\beta)}(x)) \, d\mu_{\alpha,\beta}(x)$$

and

$$\mathcal{S}(m,n) = U_{\alpha,\beta}((1-(\cdot))^{1/2} p_n^{(\alpha+1,\beta)}, p_{m+2}^{(\alpha,\beta)} - p_m^{(\alpha,\beta)})(x) \Big|_{x=-1+1/(n+1)^2}^{x=1-1/(n+1)^2}.$$

By (IV.10), using that $\lambda_{m+2}^{(\alpha,\beta)} \neq \lambda_n^{(\alpha+1,\beta)}$ and $\lambda_m^{(\alpha,\beta)} \neq \lambda_n^{(\alpha+1,\beta)}$, we obtain that

$$\text{(IV.20)}\quad \begin{aligned} \mathcal{R}_2(m,n) &= \frac{S(m+2,n)+J(m+2,n)}{\lambda_{m+2}^{(\alpha,\beta)} - \lambda_n^{(\alpha+1,\beta)}} - \frac{S(m,n)+J(m,n)}{\lambda_m^{(\alpha,\beta)} - \lambda_n^{(\alpha+1,\beta)}} \\ &= \frac{\mathcal{S}(m,n)+\mathcal{J}(m,n)}{\lambda_{m+2}^{(\alpha,\beta)} - \lambda_n^{(\alpha+1,\beta)}} - \frac{2(2m+\alpha+\beta+3)(S(m,n)+J(m,n))}{(\lambda_{m+2}^{(\alpha,\beta)} - \lambda_n^{(\alpha+1,\beta)})(\lambda_m^{(\alpha,\beta)} - \lambda_n^{(\alpha+1,\beta)})}. \end{aligned}$$

We use (IV.11) and (IV.12) to obtain that

$$\text{(IV.21)}\qquad \left| \frac{2(2m+\alpha+\beta+3)(S(m,n)+J(m,n))}{(\lambda_{m+2}^{(\alpha,\beta)} - \lambda_n^{(\alpha+1,\beta)})(\lambda_m^{(\alpha,\beta)} - \lambda_n^{(\alpha+1,\beta)})} \right| \leq \frac{C}{|n-m|^2}.$$

From (II.13), (II.12), Lemmas IV.3.1 and IV.3.2, we have

$$|\mathcal{S}(m,n)| \leq C$$

and hence

$$\text{(IV.22)}\qquad \left| \frac{\mathcal{S}(m,n)}{\lambda_{m+2}^{(\alpha,\beta)} - \lambda_n^{(\alpha+1,\beta)}} \right| \leq \frac{C}{|n-m|^2}.$$

Now, to analyse the term $\mathcal{J}(m,n)$ we will use (II.7). Therefore, taking the notation

$$\tilde{\mathcal{S}}(m,n) = U_{\alpha+1,\beta}\Big(\mathcal{H}_{\alpha,\beta}(p_{m+2}^{(\alpha,\beta)} - p_m^{(\alpha,\beta)}), p_n^{(\alpha+1,\beta)}\Big)(x)\Big|_{x=-1+1/(n+1)^2}^{x=1-1/(n+1)^2},$$

where

$$\mathcal{H}_{\alpha,\beta}(x) = \frac{H_{\alpha,\beta}(x)}{1-x},$$

$$\begin{aligned} T_1(m,n) = \int_{I_2} &((1+x)\mathcal{H}_{\alpha,\beta}(x) - 2(1-x^2)\mathcal{H}'_{\alpha,\beta}(x)) \\ &\times (p_{m+2}^{(\alpha,\beta)} - p_m^{(\alpha,\beta)})'(x) p_n^{(\alpha+1,\beta)}(x)\, d\mu_{\alpha+1,\beta}(x), \end{aligned}$$

and

$$\begin{aligned} T_2(m,n) = \int_{I_2} &((1-x^2)\mathcal{H}''_{\alpha,\beta}(x) + (\beta-\alpha-1-(\alpha+\beta+3)x)\mathcal{H}'_{\alpha,\beta}(x)) \\ &\times (p_{m+2}^{(\alpha,\beta)}(x) - p_m^{(\alpha,\beta)}(x)) p_n^{(\alpha+1,\beta)}(x)\, d\mu_{\alpha+1,\beta}(x), \end{aligned}$$

we have

$$\begin{aligned} &\lambda_n^{(\alpha+1,\beta)} \mathcal{J}(m,n) \\ &\quad = \int_{I_2} \mathcal{H}_{\alpha,\beta}(x)(p_{m+2}^{(\alpha,\beta)}(x) - p_m^{(\alpha,\beta)}(x)) L^{\alpha+1,\beta} p_n^{(\alpha+1,\beta)}(x)\, d\mu_{\alpha+1,\beta}(x) \\ &\quad = \tilde{\mathcal{S}}(m,n) + \int_{I_2} L^{\alpha+1,\beta}(\mathcal{H}_{\alpha,\beta}(p_{m+2}^{(\alpha,\beta)} - p_m^{(\alpha,\beta)}))(x) p_n^{(\alpha+1,\beta)}(x)\, d\mu_{\alpha+1,\beta}(x) \\ &\quad = \tilde{\mathcal{S}}(m,n) + \int_{I_2} H_{\alpha,\beta}(x) L^{\alpha,\beta}(p_{m+2}^{(\alpha,\beta)} - p_m^{(\alpha,\beta)}))(x) p_n^{(\alpha+1,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \\ &\qquad + T_1(m,n) - T_2(m,n). \end{aligned}$$

We use now the identity

$$L^{\alpha,\beta}(p_{m+2}^{(\alpha,\beta)}-p_m^{(\alpha,\beta)})(x)=\lambda_{m+2}^{(\alpha,\beta)}(p_{m+2}^{(\alpha,\beta)}(x)-p_m^{(\alpha,\beta)}(x))+(\lambda_{m+2}^{(\alpha,\beta)}-\lambda_m^{(\alpha,\beta)})p_m^{(\alpha,\beta)}(x) \tag{IV.23}$$

to deduce that

$$\begin{aligned}\lambda_n^{(\alpha+1,\beta)}\mathcal{J}(m,n)=\tilde{\mathcal{S}}(m,n)+\lambda_{m+2}^{(\alpha,\beta)}\mathcal{J}(m,n)\\ +2(2m+\alpha+\beta+3)J(m,n)+T_1(m,n)-T_2(m,n).\end{aligned}$$

In this way,
(IV.24)

$$\frac{\mathcal{J}(m,n)}{\lambda_{m+2}^{(\alpha,\beta)}-\lambda_n^{(\alpha+1,\beta)}}=\frac{-\tilde{\mathcal{S}}(m,n)-2(2m+\alpha+\beta+3)J(m,n)-T_1(m,n)+T_2(m,n)}{(\lambda_{m+2}^{(\alpha,\beta)}-\lambda_n^{(\alpha+1,\beta)})^2}.$$

From (IV.12), we deduce the estimate

$$\left|\frac{2(2m+\alpha+\beta+3)J(m,n)}{(\lambda_{m+2}^{(\alpha,\beta)}-\lambda_n^{(\alpha+1,\beta)})^2}\right|\le\frac{C}{|n-m|^2}.$$

Then, it suffices to show that

$$|\tilde{\mathcal{S}}(m,n)|+|T_1(m,n)|+|T_2(m,n)|\le C(n+1)^2 \tag{IV.25}$$

because using (IV.20), (IV.21), (IV.22), and (IV.24), the proof of (IV.7) for $n>m$ will be completed.

From (II.13), (II.12), Lemmas IV.3.1 and IV.3.2, and using the bounds $|\mathcal{H}_{\alpha,\beta}(x)|\le C(1-x)^{-3/2}$ and $|\mathcal{H}'_{\alpha,\beta}(x)|\le C(1-x)^{-5/2}$, for $-1<x<1$, we obtain the estimate

$$|\tilde{\mathcal{S}}(m,n)|\le C(n+1)^2.$$

Now we decompose $T_1(m,n)$ and $T_2(m,n)$ according the intervals $V_1=[-1+1/(n+1)^2,-1+1/(m+1)^2)$, $V_2=[-1+1/(m+1)^2,1-1/(m+1)^2]$, and $V_3=(1-1/(m+1)^2,1-1/(n+1)^2]$. Using (II.13), Lemma IV.3.2, and the estimate

$$|(1+x)\mathcal{H}_{\alpha,\beta}(x)-2(1-x^2)\mathcal{H}'_{\alpha,\beta}(x)|\le C(1+x)(1-x)^{-3/2},\qquad -1<x<1,$$

for $2m/3\le n\le 3m/2$ and $\alpha\ge -1/2$ we have

$$\begin{aligned}|T_1(m,n)|\le C\Bigg((m+1)^{\beta+3/2}\int_{V_1}(1+x)^{\beta/2+3/4}\,dx\\ +(m+1)\int_{V_2}(1+x)^{1/2}(1-x)^{-3/2}\,dx\\ +(m+1)^{\alpha+3/2}\int_{V_3}(1-x)^{\alpha/2-5/4}\,dx\Bigg)\le C(n+1)^2.\end{aligned}$$

Finally, by (II.13), Lemma IV.3.1, and the bound

$$|(1-x^2)\mathcal{H}''_{\alpha,\beta}(x)+2(\beta-\alpha-1-(\alpha+\beta+3)x)\mathcal{H}'_{\alpha,\beta}(x)|\le C(1-x)^{-5/2},\quad -1<x<1,$$

we can show that for $2m/3 \le n \le 3m/2$ and $\alpha \ge -1/2$,

$$|T_2(m,n)| \le C\bigg((m+1)^{\beta-1/2}\int_{V_1}(1+x)^{\beta/2-1/4}\,dx + \int_{V_2}(1-x)^{-2}\,dx \\ +(m+1)^{\alpha-1/2}\int_{V_3}(1-x)^{\alpha/2-9/4}\,dx\bigg) \le C(n+1)^2,$$

and the proof of (IV.25) is completed.

Now we will prove the estimate (IV.8) for $n<m$.

Again, we decompose the difference $R^{(\alpha,\beta)}(m,n+2) - R^{(\alpha,\beta)}(m,n)$ into three integrals $\mathcal{R}'_1(m,n)$, $\mathcal{R}'_2(m,n)$, and $\mathcal{R}'_3(m,n)$, over the intervals $I'_1 = (-1,-1+1/(m+1)^2)$, $I'_2 = [-1+1/(m+1)^2, 1-1/(m+1)^2]$, $I'_3 = (1-1/(m+1)^2, 1)$. We use (II.13) and Lemma IV.3.1 and we deduce the estimates

$$|\mathcal{R}'_1(m,n)| \le C(m+1)^{\beta+1/2}(n+1)^{\beta-1/2}\int_{I'_1}(1+x)^{\beta}\,dx \le \frac{C}{(m+1)^2}$$

and

$$|\mathcal{R}'_3(m,n)| \le C(m+1)^{\alpha+1/2}(n+1)^{\alpha+1/2}\int_{I'_3}(1-x)^{\alpha+1/2}\,dx \le \frac{C}{(m+1)^2}.$$

We analyse now the term $\mathcal{R}'_2(m,n)$. By (IV.13), using that $\lambda_{n+2}^{(\alpha+1,\beta)} \ne \lambda_m^{(\alpha,\beta)}$ and $\lambda_n^{(\alpha+1,\beta)} \ne \lambda_m^{(\alpha,\beta)}$, it is possible to prove the identity

$$\begin{aligned}\mathcal{R}'_2(m,n) &= \frac{S'(m,n+2)-J'(m,n+2)}{\lambda_{n+2}^{(\alpha+1,\beta)}-\lambda_m^{(\alpha,\beta)}} - \frac{S'(m,n)-J'(m,n)}{\lambda_{n}^{(\alpha+1,\beta)}-\lambda_m^{(\alpha,\beta)}}\\ &= \frac{\mathcal{S}'(m,n)-\mathcal{J}'(m,n)}{\lambda_{n+2}^{(\alpha+1,\beta)}-\lambda_m^{(\alpha,\beta)}} - \frac{2(2n+\alpha+\beta+4)(S'(m,n)-J'(m,n))}{(\lambda_{n+2}^{(\alpha+1,\beta)}-\lambda_m^{(\alpha,\beta)})(\lambda_{n}^{(\alpha+1,\beta)}-\lambda_m^{(\alpha,\beta)})},\end{aligned}$$

where

$$\mathcal{J}'(m,n) = \int_{I'_2} H_{\alpha,\beta}(x)p_m^{(\alpha,\beta)}(x)(p_{n+2}^{(\alpha+1,\beta)}(x) - p_n^{(\alpha+1,\beta)}(x))\,d\mu_{\alpha,\beta}(x)$$

and

$$\mathcal{S}'(m,n) = U_{\alpha+1,\beta}\Big((1-(\cdot))^{-1/2}p_m^{(\alpha,\beta)}, p_{n+2}^{(\alpha+1,\beta)} - p_n^{(\alpha+1,\beta)}\Big)(x)\bigg|_{x=-1+1/(m+1)^2}^{x=1-1/(m+1)^2}.$$

By (IV.14) and (IV.15) we obtain that

$$\left|\frac{2(2n+\alpha+\beta+4)(S'(m,n)-J'(m,n))}{(\lambda_{n+2}^{(\alpha+1,\beta)}-\lambda_m^{(\alpha,\beta)})(\lambda_{n}^{(\alpha+1,\beta)}-\lambda_m^{(\alpha,\beta)})}\right| \le \frac{C}{|n-m|^2}.$$

Now, from (II.13), (II.12), and Lemmas IV.3.1 and IV.3.2, we deduce the estimate

$$|\mathcal{S}'(m,n)| \le C$$

and therefore

$$\left|\frac{\mathcal{S}'(m,n)}{\lambda_{n+2}^{(\alpha+1,\beta)}-\lambda_m^{(\alpha,\beta)}}\right| \le \frac{C}{|n-m|^2}.$$

We deal now with the term $\mathcal{J}'(m,n)$. By using (II.6) we have that

$$\begin{aligned}
&\lambda_m^{(\alpha,\beta)}\mathcal{J}'(m,n)\\
&\quad= \int_{I_2'} H_{\alpha,\beta}(x)(p_{n+2}^{(\alpha+1,\beta)}(x) - p_n^{(\alpha+1,\beta)}(x))L^{\alpha,\beta}p_m^{(\alpha,\beta)}(x)\,d\mu_{\alpha,\beta}(x)\\
&\quad= \tilde{\mathcal{S}}'(m,n) + \int_{I_2'} L^{\alpha,\beta}(H_{\alpha,\beta}(p_{n+2}^{(\alpha+1,\beta)} - p_n^{(\alpha+1,\beta)}))(x)p_m^{(\alpha,\beta)}(x)\,d\mu_{\alpha,\beta}(x)\\
&\quad= \tilde{\mathcal{S}}'(m,n) + \int_{I_2'} H_{\alpha,\beta}(x)L^{\alpha+1,\beta}(p_{n+2}^{(\alpha+1,\beta)} - p_n^{(\alpha+1,\beta)}))(x)p_m^{(\alpha,\beta)}(x)\,d\mu_{\alpha,\beta}(x)\\
&\qquad - T_1'(m,n) - T_2'(m,n),
\end{aligned}$$

where

$$\tilde{\mathcal{S}}'(m,n) = U_{\alpha,\beta}\Big(H_{\alpha,\beta}(p_{n+2}^{(\alpha+1,\beta)} - p_n^{(\alpha+1,\beta)}), p_m^{(\alpha,\beta)}\Big)(x)\Big|_{x=-1+1/(m+1)^2}^{x=1-1/(m+1)^2},$$

$$\begin{aligned}
T_1'(m,n) = \int_{I_2'}&((1+x)H_{\alpha,\beta}(x) + 2(1-x^2)H_{\alpha,\beta}'(x))\\
&\times (p_{n+2}^{(\alpha+1,\beta)} - p_n^{(\alpha+1,\beta)})'(x)p_m^{(\alpha,\beta)}(x)\,d\mu_{\alpha,\beta}(x),
\end{aligned}$$

and

$$\begin{aligned}
T_2'(m,n) = \int_{I_2'}&((1-x^2)H_{\alpha,\beta}''(x) + (\beta-\alpha-(\alpha+\beta+2)x)H_{\alpha,\beta}'(x))\\
&\times (p_{n+2}^{(\alpha+1,\beta)}(x) - p_n^{(\alpha+1,\beta)}(x))p_m^{(\alpha,\beta)}(x)\,d\mu_{\alpha,\beta}(x).
\end{aligned}$$

Applying (IV.23) we get

$$\frac{\mathcal{J}'(m,n)}{\lambda_{n+2}^{(\alpha+1,\beta)} - \lambda_m^{(\alpha,\beta)}} = \frac{-\tilde{\mathcal{S}}'(m,n) - 2(2n+\alpha+\beta+4)J'(m,n) + T_1'(m,n) + T_2'(m,n)}{(\lambda_{n+2}^{(\alpha+1,\beta)} - \lambda_m^{(\alpha,\beta)})^2}.$$

From (IV.15), it is easy to show that

$$\left|\frac{2(2n+\alpha+\beta+4)J'(m,n)}{(\lambda_{n+2}^{(\alpha+1,\beta)} - \lambda_m^{(\alpha,\beta)})^2}\right| \le \frac{C}{|n-m|^2}.$$

To estimate the term $\tilde{\mathcal{S}}'(m,n)$ we use (II.13), (II.12), Lemmas IV.3.1 and IV.3.2, and the estimates $|H_{\alpha,\beta}(x)| \le C(1-x)^{-1/2}$, $|H_{\alpha,\beta}'(x)| \le C(1-x)^{-3/2}$, for $-1 < x < 1$. Then,

$$|\tilde{\mathcal{S}}'(m,n)| \le C(m+1)^2$$

and

$$\left|\frac{\tilde{\mathcal{S}}'(m,n)}{(\lambda_{n+2}^{(\alpha+1,\beta)} - \lambda_m^{(\alpha,\beta)})^2}\right| \le \frac{C}{|n-m|^2}.$$

Finally, we estimate the terms $T_1'(m,n)$ and $T_2'(m,n)$. We split both of them according to the intervals $V_1' = [-1+1/(m+1)^2, -1+1/(n+1)^2)$, $V_2' = [-1+1/(n+1)^2, 1-1/(n+$

$1)^2]$, and $V_3' = (1-1/(n+1)^2, 1-1/(m+1)^2]$. Thus, using (II.13), Lemma IV.3.2, and the estimate

$$|(1+x)H_{\alpha,\beta}(x) - 2(1-x^2)H'_{\alpha,\beta}(x)| \le C(1+x)(1-x)^{-1/2}, \qquad -1<x<1,$$

for $2m/3 \le n \le 3m/2$ and $\alpha \ge -1/2$ we have

$$\begin{aligned} |T_1'(m,n)| \le C \Bigg((n+1)^{\beta+3/2} &\int_{V_1'} (1+x)^{\beta/2+3/4}\,dx \\ &+ (n+1)\int_{V_2'}(1+x)^{1/2}(1-x)^{-3/2}\,dx \\ &+ (n+1)^{\alpha+5/2}\int_{V_3'}(1-x)^{\alpha/2-3/4}\,dx \Bigg) \le C(m+1)^2. \end{aligned}$$

Moreover, by (II.13), Lemma IV.3.1, and the estimate

$$|(1-x^2)H''_{\alpha,\beta}(x) + 2(\beta-\alpha-(\alpha+\beta+2)x)H'_{\alpha,\beta}(x)| \le C(1-x)^{-3/2}, \quad -1<x<1,$$

we conclude that for $2m/3 \le n \le 3m/2$ and $\alpha,\beta \ge -1/2$,

$$\begin{aligned} |T_2'(m,n)| \le C \Bigg((n+1)^{\beta-1/2} \int_{V_1'}(1+x)^{\beta/2-1/4}\,dx + \int_{V_2'}(1-x)^{-2}\,dx \\ + (n+1)^{\alpha+1/2}\int_{V_3'}(1-x)^{\alpha/2-7/4}\,dx \Bigg) \le C(m+1)^2 \end{aligned}$$

and the proof of the proposition is finished. □

CHAPTER V

THE LITTLEWOOD-PALEY-STEIN g_k-FUNCTIONS

This chapter concludes the discrete Harmonic Analysis carried out in this dissertation. After studying the heat semigroup and the Riesz transforms associated with discrete Jacobi expansions we are now concern in the study of other classical operators in Harmonic Analysis: the Littlewood-Paley-Stein g_k-functions.

The history of g_k-functions goes back to the seminal paper by J. E. Littlewood and R. E. A. C. Paley [35], published in 1936, where they introduced the g-function ($k = 1$) for the trigonometric Fourier series. The extension to the Fourier transform on $\mathbb{R}^d$ was given by E. M. Stein in [57] more than twenty years later. He himself treated the question in a very abstract setting in [59]. In the last few years, there has been a deep research on these operators in different contexts and considering weights. For example, for the Hankel transform they were studied in [11], for Jacobi expansions in [46], for Laguerre expansions in [47], for Hermite expansions in [65], and for Fourier-Bessel expansions in [16].

Our work in this chapter will generalise the ones in [17] and [10] for the discrete Laplacian Δ_{d} and for the λ-Laplacian Δ_λ. In the first case, the corresponding g_k-functions were analysed for $k = 1$ and in the second one, they considered the more general case $k \geq 1$.

Let us now introduce the g_k-functions associated with the discrete Laplacian $\mathcal{J}^{(\alpha,\beta)}$. The definition is given via the heat semigroup that we investigated in Chapter III (see (III.2)). The Littlewood-Paley-Stein $g_k^{(\alpha,\beta)}$-functions associated with $\mathcal{J}^{(\alpha,\beta)}$ are

$$g_k^{(\alpha,\beta)}(f)(n) = \left(\int_0^\infty t^{2k-1}\left|\frac{\partial^k}{\partial t^k}W_t^{(\alpha,\beta)}f(n)\right|^2 dt\right)^{1/2}, \qquad k \geq 1. \tag{V.1}$$

It is very common to define g_k-functions in terms of the Poisson semigroup instead of the heat semigroup. In our case the Poisson semigroup can be defined by subordination through the identity (III.10) and then we have the $\mathfrak{g}_k^{(\alpha,\beta)}$-functions

$$\mathfrak{g}_k^{(\alpha,\beta)}(f)(n) = \left(\int_0^\infty t^{2k-1}\left|\frac{\partial^k}{\partial t^k}P_t^{(\alpha,\beta)}f(n)\right|^2 dt\right)^{1/2}, \qquad k \geq 1.$$

The chapter is organised in the following way: In the first section we state the main theorem concerning mapping properties of the $g_k^{(\alpha,\beta)}$-functions on the spaces $\ell^p(\mathbb{N}, w)$, $1 < p < \infty$, and we present some corollaries such as the boundedness of $\mathfrak{g}_k^{(\alpha,\beta)}$ and a result about Laplace type multipliers. In the next section, we prove the main theorem. Later, we give the estimates that allow us to apply classical vector-valued Calderón-Zygmund theory in spaces of homogeneous type in the proof of the main theorem. Finally, the last sections contain the proofs of the corresponding corollaries.

V.1 Mapping properties of the $g_k^{(\alpha,\beta)}$-functions

The main result of this chapter states that the $g_k^{(\alpha,\beta)}$-functions defined by (V.1) are bounded from $\ell^p(\mathbb{N}, w)$ into itself for $\alpha, \beta \geq -1/2$. In fact, there exists an equivalence in those spaces between the norm of an appropriate sequence f and the norm of $g_k^{(\alpha,\beta)}$, that is:

Theorem V.1.1. *Let $\alpha, \beta \geq -1/2$, $1 < p < \infty$, and $w \in A_p(\mathbb{N})$. Then,*

$$C_1 \|f\|_{\ell^p(\mathbb{N},w)} \leq \|g_k^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)} \leq C_2 \|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w), \tag{V.2}$$

where C_1 and C_2 are positive constants independent of f.

We obtain some results that can be deduced from this theorem. The first one establishes the same ℓ^p-estimates for the $\mathfrak{g}_k^{(\alpha,\beta)}$-functions.

Corollary V.1.2. *Let $\alpha, \beta \geq -1/2$, $1 < p < \infty$, and $w \in A_p(\mathbb{N})$. Then,*

$$C_1 \|f\|_{\ell^p(\mathbb{N},w)} \leq \|\mathfrak{g}_k^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)} \leq C_2 \|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w),$$

where C_1 and C_2 are positive constants independent of f.

We also get a result about Laplace type multipliers as a consequence of Theorem V.1.1. Given a bounded function M defined on $[0, 2]$, the multiplier associated with M is the operator, initially defined on $\ell^2(\mathbb{N})$,

$$T_M f(n) = c_n^{(\alpha,\beta)}(M(1 - \cdot) F_{\alpha,\beta}),$$

where $F_{\alpha,\beta}$ is given by (II.15). We say that T_M is a Laplace type multiplier when

$$M(x) = x \int_0^\infty e^{-xt} a(t)\, dt,$$

with a being a bounded function. From a spectral point of view, $T_M = M(\mathcal{J}^{(\alpha,\beta)})$.

The Laplace type multipliers were introduced by Stein in [59, Chapter 2]. There, it is observed that they satisfy $|x^k M^{(k)}(x)| \leq C_k$ for $k = 0, 1, \dots,$ and then form a subclass of Marcinkiewicz multipliers. The result for the operators T_M is the following.

Theorem V.1.3. *Let $\alpha, \beta \geq -1/2$, $1 < p < \infty$, and $w \in A_p(\mathbb{N})$. Then,*

$$\|T_M f\|_{\ell^p(\mathbb{N},w)} \leq C \|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w),$$

where C is a constant independent of f.

In addition, from the identity

$$x^{i\gamma} = \frac{x}{\Gamma(1 - i\gamma)} \int_0^\infty e^{-xt} t^{-i\gamma}\, dt, \qquad \gamma \in \mathbb{R},$$

we deduce the following corollary.

Corollary V.1.4. *Let $\alpha, \beta \geq -1/2$, $1 < p < \infty$, and $w \in A_p(\mathbb{N})$. Then,*

$$\|(\mathcal{J}^{(\alpha,\beta)})^{i\gamma} f\|_{\ell^p(\mathbb{N},w)} \leq C\|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w),$$

where C is a constant independent of f.

As we mentioned in the introduction of the chapter, the proof of Theorem V.1.1 is postponed to the next section and the proofs of Corollary V.1.2 and Theorem V.1.3 are included in Sections V.4 and V.5, respectively.

V.2 Proof of Theorem V.1.1

We devote this section to prove Theorem V.1.1. First, we will see that the second inequality in (V.2) implies the first one. Later, two appropriate reductions will show that the former can be deduced from the case $(\alpha, \beta) = (-1/2, -1/2)$ and $k = 1$. This particular case will be obtained from classical vector-valued Calderón-Zygmund theory in spaces of homogeneous type (see [53] and [55]).

We consider the Banach space $\mathbb{B}_k = L^2\left((0,\infty), t^{2k-1}\, dt\right)$, with $k \geq 1$, and the operator

$$G_{t,k}^{(\alpha,\beta)} f(n) = \sum_{m=0}^{\infty} f(m) G_{t,k}^{(\alpha,\beta)}(m,n),$$

where

$$\begin{aligned} G_{t,k}^{(\alpha,\beta)}(m,n) &= \frac{\partial^k}{\partial t^k} K_t^{(\alpha,\beta)}(m,n) \\ &= (-1)^k \int_{-1}^{1} (1-x)^k e^{-t(1-x)} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x). \end{aligned}$$

Then, it is clear that

$$g_k^{(\alpha,\beta)}(f)(n) = \left\| G_{t,k}^{(\alpha,\beta)} f(n) \right\|_{\mathbb{B}_k}.$$

A first tool to prove Theorem V.1.1 is the following result about the ℓ^2-boundedness of the $g_k^{(\alpha,\beta)}$-functions.

Lemma V.2.1. *Let $\alpha, \beta \geq -1/2$ and $k \geq 1$. Then,*

$$\|g_k^{(\alpha,\beta)}(f)\|_{\ell^2(\mathbb{N})}^2 = \frac{\Gamma(2k)}{2^{2k}} \|f\|_{\ell^2(\mathbb{N})}^2. \tag{V.3}$$

Proof. For a sequence $f \in \ell^2(\mathbb{N})$, it is satisfied that

$$\begin{aligned} G_{t,k}^{(\alpha,\beta)} f(n) &= (-1)^k \int_{-1}^{1} (1-x)^k e^{-t(1-x)} F_{\alpha,\beta}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \\ &= (-1)^k c_n^{(\alpha,\beta)}((1-\cdot)^k e^{-t(1-\cdot)} F_{\alpha,\beta}), \end{aligned}$$

with $F_{\alpha,\beta}$ defined by (II.15). Then, by using Parseval's identity (II.16), we have

$$\begin{aligned}
\|g_k^{(\alpha,\beta)}(f)\|^2_{\ell^2(\mathbb{N})} &= \sum_{n=0}^{\infty}\int_0^\infty t^{2k-1}|c_n^{(\alpha,\beta)}((1-\cdot)^k e^{-t(1-\cdot)}F_{\alpha,\beta})|^2\,dt\\
&= \int_0^\infty t^{2k-1}\sum_{n=0}^{\infty}|c_n^{(\alpha,\beta)}((1-\cdot)^k e^{-t(1-\cdot)}F_{\alpha,\beta})|^2\,dt\\
&= \int_0^\infty t^{2k-1}\int_{-1}^1 (1-x)^{2k}e^{-2t(1-x)}|F_{\alpha,\beta}(x)|^2\,d\mu_{\alpha,\beta}(x)\,dt\\
&= \int_{-1}^1 (1-x)^{2k}|F_{\alpha,\beta}(x)|^2\int_0^\infty t^{2k-1}e^{-2t(1-x)}\,dt\,d\mu_{\alpha,\beta}(x)\\
&= \frac{\Gamma(2k)}{2^{2k}}\int_{-1}^1 |F_{\alpha,\beta}(x)|^2\,d\mu_{\alpha,\beta}(x)\\
&= \frac{\Gamma(2k)}{2^{2k}}\|f\|^2_{\ell^2(\mathbb{N})}
\end{aligned}$$

and the proof is completed. □

Now, let us see that

$$\|g_k^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)} \le C_2\|f\|_{\ell^p(\mathbb{N},w)} \tag{V.4}$$

implies the reverse inequality

$$\|f\|_{\ell^p(\mathbb{N},w)} \le C_1\|g_k^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)}. \tag{V.5}$$

Polarising the identity (V.3), we have

$$\sum_{n=0}^{\infty} f(n)\overline{h(n)} = \frac{2^{2k}}{\Gamma(2k)}\sum_{n=0}^{\infty}\int_0^\infty t^{2k-1}\left(\frac{\partial^k}{\partial t^k}W_t^{(\alpha,\beta)}f(n)\right)\overline{\left(\frac{\partial^k}{\partial t^k}W_t^{(\alpha,\beta)}h(n)\right)}\,dt$$

and, obviously,

$$\left|\sum_{n=0}^{\infty} f(n)\overline{h(n)}\right| \le C\sum_{n=0}^{\infty} g_k^{(\alpha,\beta)}(f)(n)g_k^{(\alpha,\beta)}(h)(n).$$

Taking $h(n) = w^{1/p}(n)f_1(n)$, we have

$$\begin{aligned}
\left|\sum_{n=0}^{\infty} f(n)w^{1/p}(n)\overline{f_1(n)}\right| &\le C\sum_{n=0}^{\infty} g_k^{(\alpha,\beta)}(f)(n)g_k^{(\alpha,\beta)}(w^{1/p}f_1)(n)\\
&= C\sum_{n=0}^{\infty} g_k^{(\alpha,\beta)}(f)(n)w^{1/p}(n)w^{-1/p}(n)g_k^{(\alpha,\beta)}(w^{1/p}f_1)(n)\\
&\le C\|g_k^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)}\|g_k^{(\alpha,\beta)}(w^{1/p}f_1)\|_{\ell^{p'}(\mathbb{N},w')},
\end{aligned}$$

where $w' = w^{-1/(p-1)}$. Note that $w \in A_p(\mathbb{N})$ implies $w' \in A_{p'}(\mathbb{N})$ and, by (V.4),

$$\|g_k^{(\alpha,\beta)}(w^{1/p}f_1)\|_{\ell^{p'}(\mathbb{N},w')} \le C\|w^{1/p}f_1\|_{\ell^{p'}(\mathbb{N},w')} = \|f_1\|_{\ell^{p'}(\mathbb{N})}.$$

So, we obtain that

$$\left|\sum_{n=0}^{\infty} f(n)w^{1/p}(n)\overline{f_1(n)}\right| \leq C\|g_k^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)}\|f_1\|_{\ell^{p'}(\mathbb{N})}$$

and taking the supremum over all $f_1 \in \ell^{p'}(\mathbb{N})$ such that $\|f_1\|_{\ell^{p'}(\mathbb{N})} \leq 1$, we conclude the inequality (V.5).

In this way, we have reduced the proof of Theorem V.1.1 to prove (V.4). Now, we proceed with two additional reductions. First, we are going to use a proper transplantation operator to deduce (V.4) from the case $(\alpha, \beta) = (-1/2, -1/2)$ for $k \geq 1$. Finally, we will see how to obtain (V.4) for $g_k^{(-1/2,-1/2)}$ with $k > 1$ from the case $k = 1$. These reductions in the proof are inspired by the work in [25] (see also [10, Section 4]).

For $f \in \ell^2(\mathbb{N})$ we define the transplantation operator

$$T_{\alpha,\beta}^{\gamma,\delta} f(n) = \sum_{m=0}^{\infty} f(m) K_{\alpha,\beta}^{\gamma,\delta}(n,m)$$

where

$$K_{\alpha,\beta}^{\gamma,\delta}(n,m) = \int_{-1}^{1} p_n^{(\gamma,\delta)}(x) p_m^{(\alpha,\beta)}(x)\, d\mu_{\gamma/2+\alpha/2,\delta/2+\beta/2}(x).$$

This operator was analysed in [5] in connection with a classical result by R. Askey [7]. The precise result in [5] is the following theorem.

Theorem V.2.2 (Theorem 1.1 in [5])**.** *Let $\alpha, \beta, \gamma, \delta \geq -1/2$, with $(\alpha, \beta) \neq (\gamma, \delta)$.*

(i) If $1 < p < \infty$ and $w \in A_p(\mathbb{N})$, then

$$\|T_{\alpha,\beta}^{\gamma,\delta} f\|_{\ell^p(\mathbb{N},w)} \leq C\|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w),$$

where C is a constant independent of f. Consequently, the operator $T_{\alpha,\beta}^{\gamma,\delta}$ extends uniquely to a bounded linear operator from $\ell^p(\mathbb{N}, w)$ into itself.

(ii) If $w \in A_1(\mathbb{N})$, then

$$\|T_{\alpha,\beta}^{\gamma,\delta} f\|_{\ell^{1,\infty}(\mathbb{N},w)} \leq C\|f\|_{\ell^1(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^1(\mathbb{N}, w),$$

where C is a constant independent of f. Consequently, the operator $T_{\alpha,\beta}^{\gamma,\delta}$ extends uniquely to a bounded linear operator from $\ell^1(\mathbb{N}, w)$ into $\ell^{1,\infty}(\mathbb{N}, w)$.

By a result due to J.-L. Krivine (see [34, Theorem 1.f.14]), it is possible to give, in an obvious way, a vector-valued extension of the transplantation operator to the space $\mathbb{B}_k$, denoted by $\widetilde{T}_{\alpha,\beta}^{\gamma,\delta}$, satisfying

$$\|\widetilde{T}_{\alpha,\beta}^{\gamma,\delta} f\|_{\ell^p_{\mathbb{B}_k}(\mathbb{N},w)} \leq C\|f\|_{\ell^p_{\mathbb{B}_k}(\mathbb{N},w)}, \qquad 1 < p < \infty,$$

with weights in $A_p(\mathbb{N})$.

In this way, we note that

$$G_{t,k}^{(\alpha,\beta)} f = \widetilde{T}_{-1/2,-1/2}^{\alpha,\beta} G_{t,k}^{(-1/2,-1/2)} T_{\alpha,\beta}^{-1/2,-1/2} f. \tag{V.6}$$

Indeed, we have

$$\begin{aligned} G_{t,k}^{(-1/2,-1/2)} T_{\alpha,\beta}^{-1/2,-1/2} f(n) &= \frac{\partial^k}{\partial t^k} W_t^{(-1/2,-1/2)} T_{\alpha,\beta}^{-1/2,-1/2} f(n) \\ &= \sum_{m=0}^{\infty} f(m) \sum_{j=0}^{\infty} G_{t,k}^{(-1/2,-1/2)}(j,n) K_{\alpha,\beta}^{-1/2,-1/2}(j,m) \end{aligned}$$

and, by using (II.17) and the identities

$$G_{t,k}^{(-1/2,-1/2)}(j,n) = (-1)^k c_j^{(-1/2,-1/2)}((1-\cdot)^k e^{-t(1-\cdot)} p_n^{(-1/2,-1/2)})$$

and

$$K_{\alpha,\beta}^{-1/2,-1/2}(j,m) = c_j^{(-1/2,-1/2)}((1-\cdot)^{\alpha/2+1/4}(1+\cdot)^{\beta/2+1/4} p_m^{(\alpha,\beta)}),$$

we deduce that

$$\begin{aligned} &\sum_{j=0}^{\infty} G_{t,k}^{(-1/2,-1/2)}(j,n) K_{\alpha,\beta}^{-1/2,-1/2}(j,m) \\ &\qquad = (-1)^k \int_{-1}^{1} (1-x)^k e^{-t(1-x)} p_n^{(-1/2,-1/2)}(x) p_m^{(\alpha,\beta)}(x)\, d\mu_{\alpha/2-1/4,\beta/2-1/4}(x). \end{aligned}$$

Applying a similar argument to the other composition the proof of (V.6) follows.

Now, let us see that it is enough to analyse the $g_1^{(-1/2,-1/2)}$-function. In fact, using induction we can deduce the boundedness of the $g_k^{(-1/2,-1/2)}$-functions for $k > 1$. Let us suppose that the operator $G_{t,k}^{(-1/2,-1/2)}$ is bounded from $\ell^p(\mathbb{N}, w)$ into $\ell^p_{\mathbb{B}_k}(\mathbb{N}, w)$. Taking $k = 1$ and applying again Krivine's theorem, we deduce that the operator $\widetilde{G}_{t,1}^{(-1/2,-1/2)} : \ell^p_{\mathbb{B}_k}(\mathbb{N}, w) \longrightarrow \ell^p_{\mathbb{B}_k \times \mathbb{B}_1}(\mathbb{N}, w)$, given by

$$\{f_s(n)\}_{s\geq 0} \longmapsto \{G_{t,1}^{(-1/2,-1/2)} f_s\}_{t,s\geq 0},$$

is bounded. Moreover, $\widetilde{G}_{t,1}^{(-1/2,-1/2)} \circ G_{s,k}^{(-1/2,-1/2)}$ is a bounded operator from $\ell^p(\mathbb{N}, w)$ into $\ell^p_{\mathbb{B}_k \times \mathbb{B}_1}(\mathbb{N}, w)$. Now, using the identity

$$\frac{\partial}{\partial t}\left(W_t^{(-1/2,-1/2)}\left(\frac{\partial^k}{\partial s^k} W_s^{(-1/2,-1/2)} f\right)\right) = \left.\frac{\partial^{k+1}}{\partial u^{k+1}} W_u^{(-1/2,-1/2)} f\right|_{u=s+t},$$

we have

$$\begin{aligned} &\left\|\widetilde{G}_{t,1}^{(-1/2,-1/2)} \circ G_{s,k}^{(-1/2,-1/2)} f\right\|^2_{\mathbb{B}_k \times \mathbb{B}_1} \\ &\qquad = \int_0^\infty \int_0^\infty t s^{2k-1} \left| \left.\frac{\partial^{k+1}}{\partial u^{k+1}} W_u^{(-1/2,-1/2)} f\right|_{u=s+t} \right|^2 ds\, dt \\ &\qquad = \int_0^\infty \int_t^\infty t (r-t)^{2k-1} \left| \left.\frac{\partial^{k+1}}{\partial u^{k+1}} W_u^{(-1/2,-1/2)} f\right|_{u=r} \right|^2 dr\, dt \\ &\qquad = \int_0^\infty \left| \frac{\partial^{k+1}}{\partial r^{k+1}} W_r^{(-1/2,-1/2)} f \right|^2 \int_0^r t(r-t)^{2k-1}\, dt\, dr \\ &\qquad = \frac{1}{(2k+1)(2k)} \int_0^\infty r^{2k+1} \left| \frac{\partial^{k+1}}{\partial r^{k+1}} W_r^{(-1/2,-1/2)} f \right|^2 dr \\ &\qquad = \frac{g_{k+1}^{(-1/2,-1/2)}(f)}{(2k+1)(2k)}. \end{aligned}$$

Finally, to complete the proof of Theorem V.1.1 we have to prove (V.4) for $(\alpha,\beta) = (-1/2,-1/2)$ and $k=1$. This fact will be a consequence of the following propositions.

Proposition V.2.3. *Let $n,m\in\mathbb{N}$ with $n\neq m$. Then,*

$$\|G_{t,1}^{(-1/2,-1/2)}(m,n)\|_{\mathbb{B}_1} \le \frac{C}{|n-m|}. \tag{V.7}$$

Moreover

$$\|G_{t,1}^{(-1/2,-1/2)}(n,n)\|_{\mathbb{B}_1} \le C. \tag{V.8}$$

Proposition V.2.4. *Let $n,m\in\mathbb{N}$ with $n\neq m$. Then,*

$$\|G_{t,1}^{(-1/2,-1/2)}(m+1,n) - G_{t,1}^{(-1/2,-1/2)}(m,n)\|_{\mathbb{B}_1} \le \frac{C}{|n-m|^2}.$$

The proofs of these propositions are found in the next section. In addition, note that we have only included one of the estimates related to the regularity properties of Calderón-Zygmund theory because, as in the case of the kernel $K_t^{(\alpha,\beta)}$ for the heat semigroup $W_t^{(\alpha,\beta)}$, we have that $G_{t,1}^{(-1/2,-1/2)}(m,n) = G_{t,1}^{(-1/2,-1/2)}(n,m)$.

Now, by using that

$$\begin{aligned} |g_1^{(-1/2,-1/2)}f(n)| &\le \left\| \sum_{\substack{m=0\\ m\neq n}}^{\infty} f(m) G_{t,1}^{(-1/2,-1/2)}(m,n) \right\|_{\mathbb{B}_1} + \left\| f(n) G_{t,1}^{(-1/2,-1/2)}(n,n)\right\|_{\mathbb{B}_1} \\ &=: T_1 f(n) + T_2 f(n), \end{aligned}$$

we can apply Lemma V.2.1, (V.7) of Proposition V.2.3, and Proposition V.2.4 to deduce, from the classical vector-valued Calderón-Zygmund theory in spaces of homogeneous type, the inequality

$$\|T_1 f\|_{\ell^p(\mathbb{N},w)} \le C \|f\|_{\ell^p(\mathbb{N},w)},$$

and (V.8) to obtain that

$$\|T_2 f\|_{\ell^p(\mathbb{N},w)} \le C \|f\|_{\ell^p(\mathbb{N},w)},$$

finishing the proof of Theorem V.1.1.

V.3 Proofs of Propositions V.2.3 and V.2.4

The identity (see [51, p. 456])

$$\frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(m\theta)\,d\theta = I_m(z), \qquad |\arg(z)|<\pi,$$

where I_m denotes the Bessel function of imaginary argument of order m, implies (see also (II.18))

$$K_t^{(-1/2,-1/2)}(m,n) = e^{-t}(I_{m+n}(t) + I_{n-m}(t)), \qquad n,m\neq 0, \tag{V.9}$$

(V.10)
$$K_t^{(-1/2,-1/2)}(m,0) = \sqrt{2}e^{-t}I_m(t), \qquad \text{and} \qquad K_t^{(-1/2,-1/2)}(0,n) = \sqrt{2}e^{-t}I_n(t).$$

To simplify notation, we set $K_t(n) = e^{-t}I_n(t)$.

We note that the proofs of Propositions V.2.3 and V.2.4 are similar to the one given in [17, Proposition 4] but we have included them for a self-contained exposition of the dissertation and to fix some details.

Proof of Proposition V.2.3. The identity [48, eq. 10.29.1]

$$2I'_m(t) = I_{m+1}(t) + I_{m-1}(t)$$

yields

$$\frac{\partial K_t(n)}{\partial t} = \frac{1}{2}(K_t(n+1) - 2K_t(n) + K_t(n-1)), \qquad n \geq 1, \tag{V.11}$$

and

$$\frac{\partial K_t(0)}{\partial t} = K_t(1) - K_t(0).$$

The next identity is known as Schläfli's integral representation of Poisson type for modified Bessel functions (see [33, eq. (5.10.22)]):

$$I_\nu(z) = \frac{z^\nu}{\sqrt{\pi}\, 2^\nu \Gamma(\nu+1/2)} \int_{-1}^{1} e^{-zs}(1-s^2)^{\nu-1/2}\, ds, \quad |\arg z| < \pi, \quad \nu > -\frac{1}{2}. \tag{V.12}$$

Integrating by parts once and twice in (V.12), we have, respectively, the identities

$$I_\nu(z) = -\frac{z^{\nu-1}}{\sqrt{\pi}\, 2^{\nu-1}\Gamma(\nu-1/2)} \int_{-1}^{1} e^{-zs}s(1-s^2)^{\nu-3/2}\, ds, \quad \nu > \frac{1}{2}, \tag{V.13}$$

and

$$I_\nu(z) = \frac{z^{\nu-2}}{\sqrt{\pi}\, 2^{\nu-2}\Gamma(\nu-3/2)} \int_{-1}^{1} e^{-zs}\frac{1+zs}{z}s(1-s^2)^{\nu-5/2}\, ds, \quad \nu > \frac{3}{2}. \tag{V.14}$$

Then from (V.11), using (V.12), (V.13), and (V.14) with $\nu = n-1$, $\nu = n$, and $\nu = n+1$, respectively, we deduce that, for $n \geq 1$,

$$\frac{\partial K_t(n)}{\partial t} = \frac{1}{2}\left(I_{1,t}(n) + I_{2,t}(n)\right),$$

where

$$I_{1,t}(n) = \frac{t^{n-2}}{\sqrt{\pi}2^{n-1}\Gamma(n-1/2)} \int_{-1}^{1} e^{-t(1+s)}s(1-s^2)^{n-3/2}\, ds$$

and

$$I_{2,t}(n) = \frac{t^{n-1}}{\sqrt{\pi}2^{n-1}\Gamma(n-1/2)} \int_{-1}^{1} e^{-t(1+s)}(1+s)^2(1-s^2)^{n-3/2}\, ds.$$

Now, for $n \geq 2$,

$$\begin{aligned}
&\pi(\Gamma(n-1/2))^2 \|I_{1,t}(n)\|^2_{\mathbb{B}_1} \\
&\quad = \frac{1}{2^{2n-2}} \int_0^\infty t^{2n-3} \int_{-1}^1 e^{-t(1+s)} s(1-s^2)^{n-3/2}\, ds \int_{-1}^1 e^{-t(1+r)} r(1-r^2)^{n-3/2}\, dr \\
&\quad = \frac{1}{2^{2n-2}} \int_{-1}^1 \int_{-1}^1 sr(1-s^2)^{n-3/2}(1-r^2)^{n-3/2} \int_0^\infty t^{2n-3} e^{-t(2+s+r)}\, dt\, ds\, dr \\
&\quad = \frac{\Gamma(2n-2)}{2^{2n-2}} \int_{-1}^1 \int_{-1}^1 \frac{sr(1-s^2)^{n-3/2}(1-r^2)^{n-3/2}}{(2-s-r)^{2n-2}}\, ds\, dr \\
&\quad = \Gamma(2n-2) \int_0^1 \int_0^1 \frac{(2u-1)(2v-1)(u(1-u))^{n-3/2}(v(1-v))^{n-3/2}}{(u+v)^{2n-2}}\, du\, dv,
\end{aligned}$$

where in the last step, we have applied the change of variables $s = 2u-1$ and $r = 2v-1$, and

$$\begin{aligned}
\|I_{1,t}(n)\|^2_{\mathbb{B}_1} &\leq C \frac{\Gamma(2n-2)}{(\Gamma(n-1/2))^2} \int_0^1 (v(1-v))^{n-3/2} \int_0^1 \frac{u^{n-3/2}}{(u+v)^{2n-2}}\, du\, dv \\
&= C \frac{\Gamma(2n-2)}{(\Gamma(n-1/2))^2} \int_0^1 (1-v)^{n-3/2} \int_0^{1/v} \frac{z^{n-3/2}}{(1+z)^{2n-2}}\, dz\, dv \\
&\leq C \frac{\Gamma(2n-2)}{(\Gamma(n-1/2))^2} \left(\int_0^1 (1-v)^{n-3/2}\, dv \right) \left(\int_0^\infty \frac{z^{n-3/2}}{(1+z)^{2n-2}}\, dz \right) \\
&= \frac{C}{(n-1/2)^2}.
\end{aligned}$$

In a similar way and again for $n \geq 2$, we obtain that

$$\|I_{2,t}(n)\|^2_{\mathbb{B}_1} = \frac{16\Gamma(2n)}{\pi(\Gamma(n-1/2))^2} \int_0^1 \int_0^1 \frac{(uv)^{n+1/2}((1-u)(1-v))^{n-3/2}}{(u+v)^{2n}}\, du\, dv$$

and

$$\begin{aligned}
\|I_{2,t}(n)\|^2_{\mathbb{B}_1} &\leq C \frac{\Gamma(2n)}{(\Gamma(n-1/2))^2} \left(\int_0^1 v^2(1-v)^{n-3/2}\, dv \right) \left(\int_0^\infty \frac{z^{n+1/2}}{(1+z)^{2n}}\, dz \right) \\
&= \frac{C}{(n+1/2)^2}.
\end{aligned}$$

Hence,

$$\left\| \frac{\partial K_t(n)}{\partial t} \right\|_{\mathbb{B}_1} \leq \frac{C}{n}, \qquad \text{for } n \geq 2. \tag{V.15}$$

Now, we prove that

$$\left\| \frac{\partial K_t(0)}{\partial t} \right\|_{\mathbb{B}_1} + \left\| \frac{\partial K_t(1)}{\partial t} \right\|_{\mathbb{B}_1} \leq C. \tag{V.16}$$

By Theorem II.1.3, it is clear that

$$\begin{aligned}\left\|\frac{\partial K_t(0)}{\partial t}\right\|_{\mathbb{B}_1} &= \frac{1}{\pi}\left\|\frac{\partial}{\partial t}\int_0^\pi e^{-t(1-\cos\theta)}\,d\theta\right\|_{\mathbb{B}_1}\\ &= \frac{1}{\pi}\left\|\int_0^\pi e^{-t(1-\cos\theta)}(1-\cos\theta)\,d\theta\right\|_{\mathbb{B}_1}\\ &\le \frac{1}{\pi}\int_0^\pi(1-\cos\theta)\left\|e^{-t(1-\cos\theta)}\right\|_{\mathbb{B}_1}\le C.\end{aligned}$$

Similarly, we obtain that $\left\|\frac{\partial K_t(1)}{\partial t}\right\|_{\mathbb{B}_1}\le C$ and the proof of (V.16) is finished.

Finally, using (V.9), (V.10), (V.15), (V.16), and the identity

$$I_{-n}(t)=I_n(t),$$

we conclude the proof of the proposition. □

Proof of the Proposition V.2.4. By using (V.9) and (V.10) the proof will follow from the estimate

$$\left\|\frac{\partial}{\partial t}(K_t(n+1)-K_t(n))\right\|_{\mathbb{B}_1}\le\frac{C}{n^2},\qquad \text{for } n\neq 0. \tag{V.17}$$

Using (V.11), we have

$$\frac{\partial}{\partial t}(K_t(n+1)-K_t(n))=\frac{1}{2}(K_t(n+2)-3K_t(n+1)+3K_t(n)-K_t(n-1)). \tag{V.18}$$

Integrating by parts three times in (V.12) gives

$$\begin{aligned}I_\nu(z) &= -\frac{z^{\nu-3}}{\sqrt{\pi}\,2^{\nu-3}\Gamma(\nu-5/2)}\\ &\quad\times\int_{-1}^1 e^{-zs}\,\frac{s(s^2z^2+3sz+3)}{z^2}(1-s^2)^{\nu-7/2}\,ds,\quad \nu>\frac{5}{2}.\end{aligned} \tag{V.19}$$

Then, using (V.12), (V.13), (V.14), and (V.19) with $\nu=n-1$, $\nu=n$, $\nu=n+1$, and $\nu=n+2$, respectively, (V.18) becomes

$$\frac{\partial}{\partial t}(K_t(n+1)-K_t(n))=\frac{-1}{2}\left(3J_{1,t}(n)+3J_{2,t}(n)+J_{3,t}(n)\right),$$

where

$$J_{1,t}(n)=\frac{t^{n-3}}{\sqrt{\pi}2^{n-1}\Gamma(n-1/2)}\int_{-1}^1 e^{-t(1+s)}s(1-s^2)^{n-3/2}\,ds,$$

$$J_{2,t}(n)=\frac{t^{n-2}}{\sqrt{\pi}2^{n-1}\Gamma(n-1/2)}\int_{-1}^1 e^{-t(1+s)}s(1+s)(1-s^2)^{n-3/2}\,ds,$$

and

$$J_{3,t}(n)=\frac{t^{n-1}}{\sqrt{\pi}2^{n-1}\Gamma(n-1/2)}\int_{-1}^1 e^{-t(1+s)}(1+s)^3(1-s^2)^{n-3/2}\,ds.$$

To estimate these inequalities we proceed as in the previous proposition. In fact, for $n \geq 4$,

$$\|J_{1,t}(n)\|^2_{\mathbb{B}_1} = \frac{4\Gamma(2n-4)}{\pi(\Gamma(n-1/2))^2} \times \int_0^1\int_0^1 \frac{(1-2u)(1-2v)(u(1-u))^{n-3/2}(v(1-v))^{n-3/2}}{(u+v)^{2n-4}}\,du\,dv,$$

and

$$\|J_{1,t}(n)\|^2_{\mathbb{B}_1} \leq C\frac{\Gamma(2n-4)}{(\Gamma(n-1/2))^2}\left(\int_0^1 v^2(1-v)^{n-3/2}\,dv\right) \times \left(\int_0^\infty \frac{z^{n-3/2}}{(1+z)^{2n-4}}\,dz\right) \leq \frac{C}{n^6};$$

$$\|J_{2,t}(n)\|^2_{\mathbb{B}_1} = \frac{4\Gamma(2n-2)}{\pi(\Gamma(n-1/2))^2} \times \int_0^1\int_0^1 \frac{(2u-1)(2v-1)(uv)^{n-1/2}((1-u)(1-v))^{n-3/2}}{(u+v)^{2n-2}}\,du\,dv,$$

and

$$\|J_{2,t}(n)\|^2_{\mathbb{B}_1} \leq C\frac{\Gamma(2n-2)}{(\Gamma(n-1/2))^2}\left(\int_0^1 v^2(1-v)^{n-3/2}\,dv\right) \times \left(\int_0^\infty \frac{z^{n-1/2}}{(1+z)^{2n-2}}\,dz\right) \leq \frac{C}{n^4};$$

and finally,

$$\|J_{3,t}(n)\|^2_{\mathbb{B}_1} = \frac{16\Gamma(2n)}{\pi(\Gamma(n-1/2))^2}\int_0^1\int_0^1 \frac{(uv)^{n+3/2}((1-u)(1-v))^{n-3/2}}{(u+v)^{2n}}\,du\,dv,$$

and

$$\|J_{3,t}(n)\|^2_{\mathbb{B}_1} \leq C\frac{\Gamma(2n)}{(\Gamma(n-1/2))^2}\left(\int_0^1 v^4(1-v)^{n-3/2}\,dv\right) \times \left(\int_0^\infty \frac{z^{n+3/2}}{(1+z)^{2n}}\,dz\right) \leq \frac{C}{n^4}.$$

We deduce (V.17) from the previous estimates for $n \geq 4$. The remainder cases can be proved as (V.16) in the previous proposition and then,

$$\left\|\frac{\partial}{\partial t}(K_t(n+1) - K_t(n))\right\|_{\mathbb{B}_1} \leq C, \qquad n = 1,2,3. \qquad \square$$

V.4 Proof of Corollary V.1.2

We are going to prove Corollary V.1.2. The Poisson semigroup $\{P_t^{(\alpha,\beta)}\}_{t\geq 0}$ was given by (III.10). Then, it is easy to check that

$$P_t^{(\alpha,\beta)}f(n) = \sum_{m=0}^{\infty} f(m)\mathcal{K}_t^{(\alpha,\beta)}(m,n),$$

where

$$\mathcal{K}_t^{(\alpha,\beta)}(m,n) = \int_{-1}^{1} e^{-t\sqrt{1-x}} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x).$$

Hence, we have the following result for the $\mathfrak{g}_k^{(\alpha,\beta)}$-functions which is the analogue of Lemma V.2.1.

Lemma V.4.1. *Let $\alpha, \beta \geq -1/2$ and $k \geq 1$. Then,*

$$\|\mathfrak{g}_k^{(\alpha,\beta)}(f)\|_{\ell^2(\mathbb{N})}^2 = \frac{\Gamma(2k)}{2^{2k}}\|f\|_{\ell^2(\mathbb{N})}^2.$$

This lemma can be proved following step by step the proof of Lemma V.2.1, so we omit the details. Now, using polarisation, we deduce the identity

$$\sum_{n=0}^{\infty} f(n)\overline{h(n)} = \frac{2^{2k}}{\Gamma(2k)} \sum_{n=0}^{\infty} \int_0^{\infty} t^{2k-1} \left(\frac{\partial^k}{\partial t^k} P_t^{(\alpha,\beta)} f(n)\right) \left(\overline{\frac{\partial^k}{\partial t^k} P_t^{(\alpha,\beta)} h(n)}\right) dt.$$

From this fact, we obtain the inequality

$$\|f\|_{\ell^p(\mathbb{N},w)} \leq C\|\mathfrak{g}_k^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)}$$

from the direct inequality

$$\|\mathfrak{g}_k^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)} \leq C\|f\|_{\ell^p(\mathbb{N},w)} \tag{V.20}$$

as we did in the proof of Theorem V.1.1. Finally, inequality (V.20) is an immediate consequence of the following lemma.

Lemma V.4.2. *Let $\alpha, \beta > -1$. Then*

$$\mathfrak{g}_k^{(\alpha,\beta)}(f)(n) \leq \sum_{j=0}^{[k/2]} A_j g_{k-j}^{(\alpha,\beta)}(f)(n),$$

where A_j are some constants and $[\cdot]$ denotes the floor function.

Proof. First, we observe that

$$\frac{\partial^k}{\partial t^k} h\left(\frac{t^2}{4u}\right) = \sum_{j=0}^{[k/2]} B_j \left.\frac{\partial^{k-j}}{\partial s^{k-j}} h(s)\right|_{s=\frac{t^2}{4u}} \frac{t^{k-2j}}{(4u)^{k-j}},$$

for some constants B_j. Then, from (III.10), we have

$$\frac{\partial^k}{\partial t^k} P_t^{(\alpha,\beta)} f(n) = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{[k/2]} B_j \int_0^{\infty} \frac{e^{-u}}{\sqrt{u}} \left(\left.\frac{\partial^{k-j}}{\partial s^{k-j}} W_s^{(\alpha,\beta)} f(n)\right|_{s=\frac{t^2}{4u}}\right) \frac{t^{k-2j}}{(4u)^{k-j}}\, du$$

and, by Theorem II.1.3,

$$\mathfrak{g}_k^{(\alpha,\beta)}(f)(n) \leq \sum_{j=0}^{[k/2]} B_j P_j(n),$$

where

$$P_j(n) \quad = \quad \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}(4u)^{k-j}} \left(\int_0^\infty t^{4k-4j-1} \left| \frac{\partial^{k-j}}{\partial s^{k-j}} W_s^{(\alpha,\beta)} f(n) \Big|_{s=\frac{t^2}{4u}} \right|^2 dt \right)^{1/2} du.$$

Now, by using an appropriate change of variables, we have

$$\begin{aligned} P_j(n) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \left(\int_0^\infty s^{2k-2j-1} \left| \frac{\partial^{k-j}}{\partial s^{k-j}} W_s^{(\alpha,\beta)} f(n) \right|^2 ds \right)^{1/2} du \\ &= \frac{1}{\sqrt{2}} g_{k-j}^{(\alpha,\beta)}(f)(n) \end{aligned}$$

and the result follows. □

V.5 Proof of Theorem V.1.3

In order to prove Theorem V.1.3, we need only prove that

$$g_1^{(\alpha,\beta)}(T_M f)(n) \leq C g_2^{(\alpha,\beta)}(f)(n), \tag{V.21}$$

since by Theorem V.1.1 we get that

$$\|T_M f\|_{\ell^p(\mathbb{N},w)} \leq C \|g_1^{(\alpha,\beta)}(T_M f)\|_{\ell^p(\mathbb{N},w)} \leq C \|g_2^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)} \leq C \|f\|_{\ell^p(\mathbb{N},w)}.$$

Moreover, it suffices to prove (V.21) for sequences in c_{00}. First, we have

$$T_M f(n) = -\int_0^\infty a(s) \frac{\partial}{\partial s} W_s^{(\alpha,\beta)} f(n)\, ds,$$

which is an elementary consequence of the relation

$$\begin{aligned} &\int_{-1}^1 M(1-x) p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \\ &\qquad = \int_0^\infty a(s) \int_{-1}^1 (1-x) e^{-s(1-x)} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x)\, ds \\ &\qquad = -\int_0^\infty a(s) \frac{\partial}{\partial s} \int_{-1}^1 e^{-s(1-x)} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x)\, ds. \end{aligned}$$

Then, applying the semigroup property of $W_t^{(\alpha,\beta)}$ we obtain that

$$W_t^{(\alpha,\beta)}(T_M f)(n) = -\int_0^\infty a(s) \frac{\partial}{\partial s} W_{s+t}^{(\alpha,\beta)} f(n)\, ds$$

and hence,

$$\frac{\partial}{\partial t}W_t^{(\alpha,\beta)}(T_M f)(n) = -\int_0^\infty a(s)\frac{\partial}{\partial t}\frac{\partial}{\partial s}W_{s+t}^{(\alpha,\beta)}f(n)\,ds$$
$$= -\int_0^\infty a(s)\frac{\partial^2}{\partial s^2}W_{s+t}^{(\alpha,\beta)}f(n)\,ds.$$

In this way,

$$\left|\frac{\partial}{\partial t}W_t^{(\alpha,\beta)}(T_M f)(n)\right| \le C\int_t^\infty s\left|\frac{\partial^2}{\partial s^2}W_s^{(\alpha,\beta)}f(n)\right|\frac{ds}{s}$$
$$\le Ct^{-1/2}\left(\int_t^\infty s^2\left|\frac{\partial^2}{\partial s^2}W_s^{(\alpha,\beta)}f(n)\right|^2 ds\right)^{1/2}.$$

Finally,

$$(g_1^{(\alpha,\beta)}(T_M f)(n))^2 = \int_0^\infty t\left|\frac{\partial}{\partial t}W_t^{(\alpha,\beta)}(T_M f)(n)\right|^2 dt$$
$$\le C\int_0^\infty\int_t^\infty s^2\left|\frac{\partial^2}{\partial s^2}W_s^{(\alpha,\beta)}f(n)\right|^2 ds\,dt$$
$$= C\int_0^\infty s^3\left|\frac{\partial^2}{\partial s^2}W_s^{(\alpha,\beta)}f(n)\right|^2 ds = C(g_2^{(\alpha,\beta)}f(n))^2$$

and the proof of (V.21) is completed.

CONCLUSIONS AND FURTHER WORK

The main goal of this dissertation has been to study some classical operators in Harmonic Analysis in a discrete setting for Jacobi expansions. To do so, the central object to consider has been the discrete Laplacian $\mathcal{J}^{(\alpha,\beta)}$ related to the three-term recurrence relation for Jacobi polynomials.

After some premilinaries, in Chapter III we investigated the heat initial-value problem associated with $\mathcal{J}^{(\alpha,\beta)}$ and we showed that a solution of it is the heat semigroup $\{W_t^{(\alpha,\beta)}\}_{t\geq 0}$. Some conditions on the parameters α and β imply the positivity of the semigroup. It is interesting to ask if those conditions on α and β are also sufficient to obtain a characterization of the positivity of $\{W_t^{(\alpha,\beta)}\}_{t\geq 0}$. Finally, we proved that the heat maximal operator is bounded from $\ell^p(\mathbb{N}, w)$ into itself, $1 < p < \infty$, and a weak type inequality in the case $p = 1$. As a consequence, similar bounds were given for the Poisson maximal operator $P_*^{(\alpha,\beta)}$. The proofs rely on some size and regularity estimates to apply discrete vector-valued local Calderón-Zygmund theory.

The Riesz transforms are the main objects in Chapter IV. In order to define them we had to give the Riesz potentials (or fractional integrals) $(-\mathcal{J}^{(\alpha,\beta)})^{\sigma}$ and turn to a limit process. By using the Calderón-Zygmund theory (only the scalar case was needed) we provided weighted ℓ^p-estimates for these operators, $1 < p < \infty$, and a weighted weak type inequality for $p = 1$. A deep analysis of potential operators, namely fractional derivatives and Riesz potentials (or fractional integrals), could be carried out in the context of Jacobi matrices as we did for the heat semigroup.

Finally, we dealt with Littlewood-Paley-Stein g_k-functions in Chapter V. The main theorem of this chapter contains mapping properties for the $g_k^{(\alpha,\beta)}$-functions associated with the heat semigroup $W_t^{(\alpha,\beta)}$. In this case, classical vector-valued Calderón-Zygmund theory in spaces of homogeneous type plays a fundamental role in the proof and we also used a transplantation result. The combination of both ingredients in the proof excludes the weak type inequality. It would be interesting to study the boundedness when $p = 1$. Some corollaries were derived from this result. The first one is about mapping properties for the $\mathfrak{g}_k^{(\alpha,\beta)}$-functions associated with the Poisson semigroup $\{P_t^{(\alpha,\beta)}\}_{t\geq 0}$, $1 < p < \infty$. We also gave a proof of the boundedness of Laplace type multipliers which appear naturally in this context. This allowed us to show the same bounds for the imaginary powers of $\mathcal{J}^{(\alpha,\beta)}$.

It is worth pointing out that the above-mentioned results were stated for $\alpha, \beta \geq -1/2$. The main reason is the use of estimates for the Jacobi polynomials which hold for $\alpha, \beta \geq -1/2$. It would be desirable to extend this range to $\alpha, \beta > -1$ which is the natural one in the Jacobi setting. In addition, an open problem is to develop the whole analysis in a more symmetric context. Again, the formulas available for Jacobi polynomials give somehow more importance to the parameter α over β and then, the operator $\mathcal{J}^{(\alpha,\beta)}$ losses the symmetry which does have the operator Δ_{d}.

Also, we suggest the study of other families of classical orthogonal polynomials such as Laguerre and Hermite polynomials. The case of the Jacobi matrices and the higher dimensional setting seem to be much more abstract and complex.

CONCLUSIONES Y TRABAJO FUTURO

El principal objetivo de esta memoria era estudiar algunos operadores del análisis armónico en un contexto discreto para desarrollos de Jacobi. Para ello se ha considerado el laplaciano discreto $\mathcal{J}^{(\alpha,\beta)}$ asociado a la relación de recurrencia a tres términos de los polinomios de Jacobi.

Tras algunos preliminares, en el Capítulo III se analizó el problema de valor inicial para la ecuación del calor asociada al operador $\mathcal{J}^{(\alpha,\beta)}$ y se obtuvo una solución en términos del semigrupo del calor $\{W_t^{(\alpha,\beta)}\}_{t\geq 0}$. Vimos que ciertas condiciones sobre los parámetros α y β implican la positividad del semigrupo. Es interesante preguntarse si esas condiciones sobre α y β caracterizan la positividad de $\{W_t^{(\alpha,\beta)}\}_{t\geq 0}$. Por último, probamos que el operador maximal del calor está acotado de $\ell^p(\mathbb{N}, w)$ en sí mismo cuando $1 < p < \infty$ y obtuvimos una desigualdad de tipo débil en el caso en que $p = 1$. Como consecuencia se dieron las mismas acotaciones para el operador maximal de Poisson $P_*^{(\alpha,\beta)}$. Las pruebas se basan en demostrar estimaciones de tamaño y regularidad para poder aplicar la teoría local y discreta de Calderón-Zygmund para operadores con valores vectoriales.

El objeto principal del Capítulo IV son las transformadas de Riesz. Para dar una definición tuvimos que presentar los potenciales de Riesz (también llamados integrales fraccionarias) $(-\mathcal{J}^{(\alpha,\beta)})^{\sigma}$ y utilizar un proceso de límite. Gracias a la teoría discreta de Calderón-Zygmund (en este caso era suficiente el caso escalar) pudimos probar acotaciones en espacios ℓ^p con peso para esos operadores, $1 < p < \infty$, y acotaciones de tipo débil cuando $p = 1$. Podría plantearse un estudio con más detalle de los operadores potenciales, es decir, las derivadas fraccionarias y los potenciales de Riesz (o integrales fraccionarias), en el contexto de las matrices de Jacobi, algo que ya hicimos para el semigrupo del calor.

Finalmente tratamos las g_k-funciones de Littlewood-Paley-Stein en el Capítulo V. El teorema principal aquí muestra la acotación de las $g_k^{(\alpha,\beta)}$-funciones asociadas al semigrupo del calor $W_t^{(\alpha,\beta)}$. En este caso, la pieza fundamental en la prueba era la teoría clásica de Calderón-Zygmund para operadores con valores vectoriales en espacios de tipo homogéneo, aunque también se hacía uso de un resultado de transplantación. La combinación de ambos impide obtener desigualdades de tipo débil. Sería de interés estudiar la acotación cuando $p = 1$. De este resultado obtuvimos varios corolarios. El primero muestra la acotación de las $\mathfrak{g}_k^{(\alpha,\beta)}$-funciones asocidas al semigrupo de Poisson $\{P_t^{(\alpha,\beta)}\}_{t\geq 0}$, $1 < p < \infty$. También se dio la misma acotación para multiplicadores de tipo Laplace que aparecen de forma natural en este contexto. Esto implicaba de forma inmediata la acotación de las potencias imaginarias de $\mathcal{J}^{(\alpha,\beta)}$.

Es importante resaltar que los resultados anteriores se obtuvieron para $\alpha, \beta \geq -1/2$. Esto se debe a que usamos estimaciones para los polinomios de Jacobi válidas para $\alpha, \beta \geq -1/2$. Sería interesante ampliar este rango a $\alpha, \beta > -1$ pues ese es el natural en el contexto de los polinomios de Jacobi. Además, queda como problema abierto desarrollar todo este análisis en un contexto más simétrico, porque de algún modo las fórmulas que hemos utilizado priorizan el parámetro α provocando una pérdida de simetría en el operador $\mathcal{J}^{(\alpha,\beta)}$ que no ocurre en el caso de Δ_{d}.

También sugerimos considerar otras familias de polinomios ortogonales clásicos como las familias de los polinomios de Laguerre o Hermite. El análisis completo en el contexto de las matrices de Jacobi y el estudio de dimensiones superiores parecen ser mucho más abstractos y complejos.

PUBLICATIONS

This dissertation is based on some papers. We enumerate here a list containing all of them.

1. A. Arenas, Ó. Ciaurri, and E. Labarga, Discrete Harmonic Analysis Associated with Jacobi Expansions I: The Heat Semigroup, submitted for publication. Available on arXiv: 1806.00056 (2018).

2. A. Arenas, Ó. Ciaurri, and E. Labarga, Discrete Harmonic Analysis Associated with Jacobi Expansions II: The Riesz Transform, submitted for publication. Available on arXiv:1902.01761 (2019).

3. A. Arenas, Ó. Ciaurri, and E. Labarga, Discrete Harmonic Analysis Associated with Jacobi Expansions III: The Littlewood-Paley-Stein g_k-functions and the Laplace Type Multipliers, submitted for publication. Available on arXiv:1906.07999 (2019).

The third chapter of this dissertation is based on the first work on the list. There, we studied the heat initial-value problem associated with $\mathcal{J}^{(\alpha,\beta)}$ and we obtain an expression for the heat semigroup $\{W_t^{(\alpha,\beta)}\}_{t\geq 0}$. We also showed the positivity of this semigroup by assuming some conditions on the parameters α and β. Finally, we proved weighted inequalities for the heat maximal operator ($\alpha, \beta \geq -1/2$) by applying discrete vector-valued local Calderón-Zygmund theory and derived similar ones for the Poisson maximal operator by subordination.

The fourth chapter of this dissertation is based on the second work on the list. We defined the Riesz transforms associated with $\mathcal{J}^{(\alpha,\beta)}$ by a standard limit argument and we proved weighted inequalities for them ($\alpha, \beta \geq -1/2$) by using discrete local Calderón-Zygmund theory.

The fifth chapter is devoted to investigate the Littlewood-Paley-Stein $g_k^{(\alpha,\beta)}$-functions associated with $\mathcal{J}^{(\alpha,\beta)}$. It is based on the third item on the list. In this work one can find weighted inequalities for $g_k^{(\alpha,\beta)}$ and the similar ones for $\mathfrak{g}_k^{(\alpha,\beta)}$, the g_k-functions associated with the Poisson semigroup $\{P_t^{(\alpha,\beta)}\}_{t\geq 0}$. From these results we obtain some corollaries about the boundedness of the Laplace type multipliers and imaginary powers of $\mathcal{J}^{(\alpha,\beta)}$. The proof of the main theorem for $g_k^{(\alpha,\beta)}$ uses classical vector-valued Calderón-Zygmund theory in spaces of homogeneous type and a transplantation result presented in [5].

During the PhD period I have had the opportunity to produce (in collaboration) other research papers and preprints:

1. A. Arenas, Ó. Ciaurri, and E. Labarga, Weighted Inequalities for the Riesz Potential on the Sphere, *Integral Transforms Spec. Funct.*, **27** (2016), 511–522.

2. A. Arenas, Ó. Ciaurri, and E. Labarga, A Hardy Inequality for Ultraspherical Expansions with an Application to the Sphere, *J. Fourier Anal. Appl.*, **24** (2018), 416–430.

3. A. Arenas, E. Labarga, and A. Nowak, Exotic Multiplicity Functions and Heat Maximal Operators in Certain Dunkl Settings, *Integral Transforms Spec. Funct.*, **29** (2018), 771–793.

4. A. Arenas, Ó. Ciaurri, and E. Labarga, A Weighted Transplantation Theorem for Jacobi Coefficients, submitted for publication. Available on arXiv: 1812.08422 (2018).

5. A. Arenas, Ó. Ciaurri, and E. Labarga, The Convergence of Discrete Fourier-Jacobi Series, submitted for publication. Available on arXiv:1906.08004 (2019).

6. A. Arenas, Ó. Ciaurri, and E. Labarga, Weighted Transplantation for Laguerre Coefficients, preprint (2019).

PUBLICACIONES

El contenido de esta memoria se basa en varios artículos de investigación que pasamos a enumerar a continuación:

1. A. Arenas, Ó. Ciaurri y E. Labarga, Discrete Harmonic Analysis Associated with Jacobi Expansions I: The Heat Semigroup, enviado para publicación. Disponible en arXiv: 1806.00056 (2018).

2. A. Arenas, Ó. Ciaurri y E. Labarga, Discrete Harmonic Analysis Associated with Jacobi Expansions II: The Riesz Transform, enviado para publicación. Disponible en arXiv:1902.01761 (2019).

3. A. Arenas, Ó. Ciaurri y E. Labarga, Discrete Harmonic Analysis Associated with Jacobi Expansions III: The Littlewood-Paley-Stein g_k-functions and the Laplace Type Multipliers, enviado para publicación. Disponible en arXiv:1906.07999 (2019).

El tercer capítulo de esta memoria se basa en el primero de estos trabajos. Allí estudiamos el problema de valor inicial para la ecuación del calor asociada al operador $\mathcal{J}^{(\alpha,\beta)}$ y obtuvimos una expresión para el semigrupo del calor $\{W_t^{(\alpha,\beta)}\}_{t\geq 0}$. Además mostramos la positividad del semigrupo bajo ciertas condiciones sobre los parámetros α y β. Finalmente dimos acotaciones con peso para el operador maximal del calor (con $\alpha, \beta \geq -1/2$) por medio de la teoría local y discreta de Calderón-Zygmund para operadores vector valuados y dedujimos acotaciones análogas para el operador maximal de Poisson por un proceso de subordinación.

El cuarto capítulo de la memoria se basa en el segundo trabajo de la lista donde definimos las transformadas de Riesz asociadas a $\mathcal{J}^{(\alpha,\beta)}$ mediante un proceso de límite y probamos acotaciones con peso para éstas ($\alpha, \beta \geq -1/2$) utilizando la teoría local y discreta de Calderón-Zygmund.

El quinto capítulo se dedicó al estudio de las $g_k^{(\alpha,\beta)}$-funciones de Littlewood-Paley-Stein asociadas a $\mathcal{J}^{(\alpha,\beta)}$ y se basa en el tercer artículo de la lista. En ese trabajo se pueden encontrar estimacinoes con peso para los operadores $g_k^{(\alpha,\beta)}$ y las correspondientes para $\mathfrak{g}_k^{(\alpha,\beta)}$, las g_k-funciones asociadas al semigrupo de Poisson $\{P_t^{(\alpha,\beta)}\}_{t\geq 0}$. De estos resultados obtuvimos otros sobre la acotación de multiplicadores de tipo Laplace y potencias imaginarias de $\mathcal{J}^{(\alpha,\beta)}$. La prueba del teorema principal para las $g_k^{(\alpha,\beta)}$-funciones requiere de la teoría clásica de Calderón-Zygmund en espacios de tipo homogéneo para operadores con valores vectoriales y un resultado de transplantación presentado en [5].

Durante mi etapa como estudiante de doctorado tuve la oportunidad de elaborar (junto con diferentes autores) otros artículos de investigación:

1. A. Arenas, Ó. Ciaurri y E. Labarga, Weighted Inequalities for the Riesz Potential on the Sphere, *Integral Transforms Spec. Funct.*, **27** (2016), 511–522.

2. A. Arenas, Ó. Ciaurri y E. Labarga, A Hardy Inequality for Ultraspherical Expansions with an Application to the Sphere, *J. Fourier Anal. Appl.*, **24** (2018), 416–430.

3. A. Arenas, E. Labarga y A. Nowak, Exotic Multiplicity Functions and Heat Maximal Operators in Certain Dunkl Settings, *Integral Transforms Spec. Funct.*, **29** (2018), 771–793.

4. A. Arenas, Ó. Ciaurri y E. Labarga, A Weighted Transplantation Theorem for Jacobi Coefficients, enviado para publicación. Disponible en arXiv:1812.08422 (2018).

5. A. Arenas, Ó. Ciaurri y E. Labarga, The Convergence of Discrete Fourier-Jacobi Series, submitted for publication. Disponible en arXiv:1906.08004 (2019).

6. A. Arenas, Ó. Ciaurri y E. Labarga, Weighted Transplantation for Laguerre Coefficients, en preparación (2019).

BIBLIOGRAPHY

[1] H. Aimar and R. Macías, Weighted Norm Inequalities for the Hardy-Littlewood Maximal Operator on Spaces of Homogeneous Type, *Proc. Amer. Math. Soc.* **91** (1984), 213–216.

[2] K. F. Andersen, Discrete Hilbert Transforms and Rearrangement Invariant Sequence Spaces, *Appl. Anal.* **5** (1976), 193–200.

[3] K. F. Andersen, Inequalities with Weights for Discrete Hilbert Transforms, *Canad. Math. Bull.* **20** (1977), 9–16.

[4] A. Arenas, *Discrete Harmonic Analysis Related to Classical Orthogonal Polynomials*, PhD Dissertation, University of La Rioja, Logroño, 2019.

[5] A. Arenas, Ó. Ciaurri, and E. Labarga, A Weighted Transplantation Theorem for Jacobi Coefficients, preprint, arXiv:1812.08422 (2018).

[6] A. Arenas, Ó. Ciaurri, and E. Labarga, Weighted Transplantation for Laguerre Coefficients, preprint.

[7] R. Askey, A Transplantation Theorem for Jacobi Coefficients, *Pacific J. Math.* **21** (1967), 393–404.

[8] R. Bañuelos and M. Kwaśnicki, On the ℓ^p-norm of the Discrete Hilbert Transform, *Duke Math. J.* **168** (2019), 471–504.

[9] A. Benedek, A. P. Calderón, and R. Panzone, Convolution Operators on Banach Space Valued Functions, *Proc. Nat. Acad. Sci. U.S.A.* **48** (1962), 356–365.

[10] J. J. Betancor, A. J. Castro, J. C. Fariña, and L. Rodríguez-Mesa, Discrete Harmonic Analysis Associated with Ultraspherical Expansions, *Potential Anal.* to appear, `https://doi.org/10.1007/s11118-019-09777-9`.

[11] J. J. Betancor, A. J. Castro, and A. Nowak, Calderón-Zygmund Operators in the Bessel Setting, *Monatsh Math.* **167** (2012), 375–403.

[12] J. Bourgain, On the Maximal Ergodic Theorem for Certain Subsets of the Integers, *Israel J. Math.* **61** (1988), 39–72.

[13] J. Bourgain, On the Pointwise Ergodic Theorem on L^p for Arithmetic sets, *Israel J. Math.* **61** (1988), 73–84.

[14] A. P. Calderón and A. Zygmund, On the Existence of Certain Singular Integrals, *Acta Math.* **88** (1952), 85–139.

[15] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, 1978 (reprinted, Dover, 2011).

[16] Ó. Ciaurri and L. Roncal, Littlewood-Paley-Stein g_k-functions for Fourier-Bessel Expansions, *J. Funct. Anal.* **258** (2010), 2173–2204.

[17] Ó. Ciaurri, T. A. Gillespie, L. Roncal, J. L. Torrea, and J. L. Varona, Harmonic Analysis Associated with a Discrete Laplacian, *J. Anal. Math.* **132** (2017), 109–131.

[18] Ó. Ciaurri, L. Roncal, P. R. Stinga, Riesz Transforms on Compact Riemannian Symmetric Spaces of Rank One, *Milan J. Math.* **83** (2015), 345–370.

[19] R. Coifman and G. Weiss, *Analyse Harmonique Non-commutative sur Certains Espaces Homogènes*, Lecture Notes in Mathematics 242, Springer-Berlin-New York, 1971.

[20] J. Duoandikoetxea, *Fourier Analysis*, Graduate studies in Mathematics, **29**, American Mathematical Society, Providence, RI, 2001.

[21] A. F. M. ter Elst, D. W. Robinson, and A. Sikora, Heat Kernels and Riesz Transforms on Nilpotent Lie Groups, *Colloq. Math.* **74** (1997), 191–218.

[22] J. Favard, Sur les Polynomes de Tchebisheff, *C. R. Acad. Sci. Paris* **200** (1935), 2052–2053.

[23] J.-B. J. Fourier, *Théorie Analytique de la Chaleur*, facsimile copy in Éditions Jacques Gabay, 1988.

[24] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Mathematics Studies **116**, Notas de Matemática (104), North-Holland Pub. Co., Amsterdam-New York-Oxford, 1985.

[25] G. Garrigós, E. Harboure, T. Signes, J. L. Torrea, and B. Viviani, A Sharp Weighted Transplantation Theorem for Laguerre Function Expansions, *J. Funct. Anal.* **244** (2007), 247–276.

[26] G. Gasper, Linearization of the Product of Jacobi Polynomials. I, *Canad. J. Math.* **22** (1970), 171–175.

[27] G. Gasper, Linearization of the Product of Jacobi Polynomials. II, *Canad. J. Math.* **22** (1970), 582–593.

[28] L. Grafakos, L. Liu, and D. Yang, Vector-valued Singular Integrals and Maximal Functions on Spaces of Homogeneous Type, *Math. Scand.* **104** (2009), 296–310.

[29] F. A. Grünbaum, The Bispectral Problem: An Overview, in *Special Functions 2000: Current Perspective and Future Directions (Eds.: J. Bustoz, M. E. H. Ismail, S. K. Suslov)*, 129–140, Kluwer Acad. Publ., Dordrecht, 2001.

[30] F. A. Grünbaum and P. Iliev, Heat Kernel Expansions on the Integers, *Math. Phys. Anal. Geom.* **5** (2002), 183–200.

[31] R. Hunt, B. Muckenhoupt, and R. Wheeden, Weighted Norm Inequalities for the Conjugate Function and Hilbert Transform, *Trans. Amer. Math. Soc.* **176** (1973), 227–251.

[32] C. G. J. Jacobi, Untersuchungen über die Differentialgleichung der Hypergeometrischen Reihe, *J. Reine Angew. Math.* **56** (1859), 149–165.

[33] N. N. Lebedev, *Special Functions and Its Applications*, Dover, New York, 1972.

[34] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer-Verlag, Berlin, 1979.

[35] J. E. Littlewood and R. E. A. C. Paley, Theorems on Fourier Series and Power Series (II), *Proc. London Math. Soc. (2)* **42** (1936), 52–89.

[36] A. Magyar, E. M. Stein and S. Wainger, Discrete Analogues in Harmonic Analysis: Spherical Averages, *Ann. of Math. (2)* **155** (2002), 189–208.

[37] A. Máté, P. Nevai, and V. Totik, Necessary Conditions for Weighted Mean Convergence of Fourier Series in Orthogonal Polynomials, *J. Approx. Theory* **46** (1986), 314–322.

[38] B. Muckenhoupt, Poisson Integrals for Hermite and Laguerre Expansions, *Trans. Amer. Math. Soc.* **139** (1969), 231–242.

[39] B. Muckenhoupt, Hermite Conjugate Expansions, *Trans. Amer. Math. Soc.* **139** (1969), 243–260.

[40] B. Muckenhoupt, Transplantation Theorems and Multiplier Theorems for Jacobi Series, *Mem. Amer. Math. Soc.* **64** (1986).

[41] B. Muckenhoupt and E. M. Stein, Classical Expansions and Their Relation to Conjugate Harmonic Functions, *Trans. Amer. Math. Soc.* **118** (1965), 17–92.

[42] A. Nowak and P. Sjögren, Calderón-Zygmund Operators Related to Jacobi Expansions, *J. Fourier Anal. Appl.* **18** (2012), 717–749.

[43] A. Nowak and P. Sjögren, Sharp Estimates of the Jacobi Heat Kernel, *Studia Math.* **218** (2013), 219–244.

[44] A. Nowak and K. Stempak, L^2-theory of Riesz Transforms for Orthogonal Expansions, *J. Fourier Anal. Appl.* **12** (2006), 675–711.

[45] A. Nowak and K. Stempak, Weighted Estimates for the Hankel Transform Transplantation Operator, *Tohoku Math. J. (2)* **58** (2006), 277–301.

[46] A. Nowak, P. Sjögren, and T. S. Szarek, Analysis Related to All Admissible Type Parameters in the Jacobi Setting, *Constr. Approx.* **41** (2015), 185–218.

[47] A. Nowak, K. Stempak, and T. Z. Szarek, On Harmonic Analysis Operators in Laguerre-Dunkl and Laguerre-symmetrized Settings, *SIGMA Symmetry Integrability Geom. Methods Appl.* **12** (2016), Paper No. 096, 39 pp.

[48] F. W. J. Olver (editor-in-chief), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010.

[49] L. B. Pierce, *Discrete Analogues in Harmonic Analysis*, PhD Dissertation, Princeton University, Princeton, 2009.

[50] H. Pollard, The Mean Convergence of Orthogonal Series. I, *Trans. Amer. Math. Soc.* **62** (1947), 387–403.

[51] A. P. Prudnikov, A. Y. Breychkov, and O. I. Marichev, *Integrals and Series. Vol. 1. Elementary Functions*, Gordon and Breach Science Publishers, New York, 1986.

[52] M. Riesz, Sur les Fonctions Conjuguées, *Math. Zeit.* **27** (1928), 218–244.

[53] J. L. Rubio de Francia, F. J. Ruiz, and J. L. Torrea, Calderón-Zygmund Theory for Operator-valued Kernels, *Adv. Math.* **62** (1986), 7–48.

[54] W. Rudin, *Real and Complex Analysis*, Third edition, McGraw-Hill, New York, 1987.

[55] F. J. Ruiz, and J. L. Torrea, Vector-valued Calderón-Zygmund Theory and Carleson Measures on Spaces of Homogeneous Nature, *Studia Math.* **88** (1988), 221–243.

[56] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments*, Mathematical Surveys **1**, American Mathematica Society, 1943.

[57] E. M. Stein, On the Functions of Littlewood-Paley, Lusin, and Marcinkiewicz, *Trans. Amer. Math. Soc.* **88** (1958), 430–466.

[58] E. M. Stein, On the Maximal Ergodic Theorem, *Proc. Nat. Acad. Sci. U.S.A.* **47** (1961), 1894–1897.

[59] E. M. Stein, *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory*, Annals of Mathematics Studies **63**, Princeton Univ. Press, Princeton, NJ, 1970.

[60] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, NJ, 1970.

[61] E. M. Stein and S. Wainger, Discrete Analogues of Singular Radon Transforms, *Bull. Amer. Math. Soc. (N.S.)* **23** (1990), 537–544.

[62] E. M. Stein and S. Wainger, Discrete Analogues in Harmonic Analysis I: ℓ^2 Estimates for Singular Radon Transforms, *Amer. J. Math.* **121** (1999), 1291–1336.

[63] E. M. Stein and S. Wainger, Discrete Analogues in Harmonic Analysis II: Fractional Integration, *J. Anal. Math.* **80** (2000), 335–355.

[64] E. M. Stein and S. Wainger, Two Discrete Fractional Integral Operators Revisited, *J. Anal. Math.* **87** (2002), 451–479.

[65] K. Stempak and J. L. Torrea, On g-functions for Hermite Function Expansions, *Acta Math. Hungar.* **109** (2005), 99–125.

[66] T. J. Stieltjes, Recherches sur les Fractions Continues, *Ann. Fac. Sci. Tolouse* **8** (1894), J. 1–122, **9** (1895), A. 1–47. Also in Œuvres complètes de Thomas Jan Stieltjes, tome II, 402–566, P. Noordhoff, Groningen, 1918.

[67] G. Szegő, *Orthogonal Polynomials*, American Mathematical Society Providence, Rhode Island, 1975.

[68] R. Szwarc, Convolution Structures Associated with Orthogonal Polynomials, *J. Math. Anal. Appl.* **170** (1992), 158–170.

[69] S. Thangavelu, *Lectures on Hermite and Laguerre Expansions*, Princeton University Press, Princeton, New Jersey, 1993.

[70] S. Thangavelu, On conjugate Poisson integrals and Riesz transforms for Hermite expansions, *Colloq. Math.* **64** (1993), 103–113.

[71] E. C. Titchmarsh, Reciprocal formulae involving series and integrals, *Math. Z.* **25** (1926), 321–347.

[72] K. Yosida, *Functional Analysis* (sixth edition), Grundlehren der Mathematischen Wissenschaften **123**, Springer-Verlag, Berlin-New York, 1980.