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Graph-type biharmonic surfaces in $\mathbb{R}^3$

Superfícies bi-harmônicas de tipo gráfico em $\mathbb{R}^3$

Carlos M. C. Riveros, Armando M. V. Corro and Raquel P. de Araújo

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Abstract
In this work, we study biharmonic surfaces that are parameterized by biharmonic coordinate functions. We study a class of biharmonic surfaces called graph-type biharmonic surfaces. Also, we define a class of surfaces associated to two harmonic functions (FH2A-surfaces), these surfaces satisfy a relation between the Gaussian curvature, the projection of the Gauss map on a fixed plane and two harmonic functions. We show that a particular class of graph-type biharmonic surfaces are FH2A-surfaces. Finally, we classify the FH2A-surfaces of rotation.
Keywords . Biharmonic surfaces, Harmonic surfaces, Gaussian curvature

Resumen
Neste artigo, estudamos superfícies bi-harmônicas que são parametrizadas por funções coordenadas bi-harmônicas. Estudamos uma classe de superfícies bi-harmônicas, chamadas superfícies bi-harmônicas de tipo gráfico. Também, definimos uma classe de superfícies associadas a duas funções harmônicas (FH2A-surfaces), essas superfícies satisfazem uma relação entre a curvatura Gaussiana, a projeção da aplicação de Gauss sobre um plano fixo e duas funções harmônicas. Mostramos que uma classe particular de superfícies bi-harmônicas de tipo gráfico são FH2A-surfaces. Finalmente, classificamos as FH2A-surfaces de rotação.
Palabras clave. Superfícies bi-harmônicas, Superfícies harmônicas, Curvatura Gaussiana

1. Introduction. One special class of surfaces is the minimal surface, which has been widely studied by several authors. We also know that harmonic immersions have as their particular case the minimal surfaces. In [1], the author studies the geometric properties of harmonic immersions and in [2], the authors present a study on the class of graph-type harmonic surfaces.

Let $M \subset \mathbb{R}^3$ be a regular surface with Gauss map $N : M \to \mathbb{S}^2$, we say that M is a *graph-type surface* if $N(M)$it is contained in an open spherical cap. Let $\sigma \in \mathbb{R}^3$ be a plane, we define the *projection application* $\pi_\sigma : \mathbb{R}^3 \to \sigma$ as the orthogonal projection of p into σ.

In [3], the authors motivated by the work on Laguerre minimal surfaces, introduce the class of *Laguerre type surface* as being a graph-type surface M in the Euclidean space with Gauss map N, mean curvature H and Gaussian curvature K, such that there exists a plane σ passing through the origin, such that

$$\triangle_L \left(\frac{h_\sigma H}{K} \right) = 0,$$

where L is the metric in σ induced by the application $\pi_\sigma|_M$, $N(M)$ is contained in an open spherical cap determined by σ and $h_\sigma = \pi_\sigma \circ N$.

*Departamento de Matemática, Universidade de Brasília, 70910-900, Brasília-DF, Brazil (carlos@mat.unb.br).
†Instituto de Matemática e Estatística, Universidade Federal de Goiâs, 74001-970, Goiânia-GO, Brazil, (avcorro@gmail.com).
‡Instituto de Matemática e Estatística, Universidade Federal de Goiâs, 74001-970, Goiânia-GO, Brazil, (raquelaraujo06@hotmail.com).

Motivated by the work of Laguerre type surfaces [3], we introduce the class of surfaces associated to two harmonic functions (FH2A-surfaces) defined in Euclidean space, with Gaussian curvature K and a plane $\sigma \subset \mathbb{R}^3$, such that

$$\frac{K}{(d_\sigma)^4} = \mu^2 - e^\lambda,$$

where $\Delta_L \lambda = 0$, $\Delta_L \mu = 0$, L is the metric in σ induced by the application $\pi_\sigma \mid M$, $N(M)$ is contained in an open spherical cap determined by σ and $d_\sigma = \pi_\sigma \circ N$.

By analyzing the relationship between harmonic applications and holomorphic functions we can represent the harmonic surfaces using holomorphic functions, since harmonic surfaces are a subclass of biharmonic surfaces, we will use this same method to represent biharmonic surfaces. Thus, we will use holomorphic functions to characterize biharmonic surfaces.

In this work, we study biharmonic surfaces that are parameterized by biharmonic coordinate functions. We study a class of biharmonic surfaces called graph-type biharmonic surfaces. Also, we define a class of surfaces associated to two harmonic functions (FH2A-surfaces), these surfaces satisfy a relation between the Gaussian curvature, the projection of the Gauss map on a fixed plane and two harmonic functions. We show that a particular class of graph-type biharmonic surfaces are FH2A-surfaces. Finally, we classify the FH2A-surfaces of rotation. We note that all figures in this paper were made with the Mathematica software(version 7).

2. Preliminaries. In this section, we fix some notation on local classical differential geometry on surfaces. Let $X(x,y)$ be a regular parameterized surface defined in the open set $U \subset \mathbb{R}^2 \cong \mathbb{C}$, $(x,y) \in U$. Let N denote the unit normal vector field to X given by

$$N = \frac{X_x \wedge X_y}{|X_x \wedge X_y|},\ X_x = \frac{\partial X}{\partial x},\ X_y = \frac{\partial X}{\partial y},$$

where $\wedge$ stands the cross product of $\mathbb{R}^3$.
In each tangent plane, the induced metric $\langle , \rangle$ determines the first fundamental form

$$I = \langle dX, dX \rangle = Edx^2 + 2Fdxdy + Gdy^2,$$

with differentiable coefficients

$$E = \langle X_x, X_x \rangle,\ F = \langle X_x, X_y \rangle,\ G = \langle X_y, X_y \rangle.$$

The second fundamental form of X is given by

$$II = -\langle dN, dX \rangle = edx^2 + 2fdxdy + gdy^2,$$

with

$$e = \langle N, X_{xx} \rangle,\ f = \langle N, X_{xy} \rangle,\ g = \langle N, X_{yy} \rangle.$$

The mean curvature H and the Gauss curvature K of X have the classical expressions

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)},\ \ K = \frac{eg - f^2}{EG - F^2}.$$

Definition 2.1. Let $X(x,y) = (\varphi_1(x,y), \varphi_2(x,y), \varphi_3(x,y))$ be a regular parameterized surface defined in U. X is *harmonic* if $\Delta\varphi_i = 0$, for $i = 1, 2, 3$.

Observation 2.1. In this paper we consider the inner product

$$\begin{array}{cccc} \langle , \rangle : & \mathbb{C} \times \mathbb{C} & \longrightarrow & \mathbb{R} \\ & (u,v) & \longmapsto & \langle u, v \rangle = u_1 v_1 + u_2 v_2, \end{array}$$

where $u = u_1 + iu_2$, $v = v_1 + iv_2$.
Now, let $f, g : \mathbb{C} \to \mathbb{C}$, holomorphic functions given by $f = f_1 + if_2$, $g = g_1 + ig_2$, where $f_1, f_2, g_1, g_2 : \mathbb{C} \to \mathbb{R}$.

We define the differentiable function

$$\langle f(z), g(z) \rangle = f_1(z) g_1(z) + f_2(z) g_2(z).$$

In the computation we use the following properties:(see [2]) If f, g are holomorphic functions, $z = x + iy \in \mathbb{C}$ then

$$\langle f, g\rangle_x = \langle f', g\rangle + \langle f, g'\rangle, \tag{2.1}$$

$$\langle f, g\rangle_y = \langle if', g\rangle + \langle f, ig'\rangle, \tag{2.2}$$

$$f = \langle 1, f\rangle + i\langle i, f\rangle, \tag{2.3}$$

$$\langle f, gh\rangle = \langle f\bar{g}, h\rangle, \tag{2.4}$$

$$f = \langle 1, f\rangle - i\langle 1, if\rangle, \tag{2.5}$$

$$\bar{f} = \langle 1, f\rangle - i\langle i, f\rangle = \langle 1, f\rangle + i\langle 1, if\rangle, \tag{2.6}$$

$$\langle 1, f\rangle\langle i, f\rangle = \frac{1}{2}\langle i, f^2\rangle. \tag{2.7}$$

In [2] was obtained the following result.

Theorem 2.1. *Let M be a harmonic regular parameterized surface defined in $U \subset \mathbb{C}$ open and simply-connected. Then M is locally reparameterized by*

$$X(x, y) = (\langle 1, f_1(x, y)\rangle, \langle 1, f_2(x, y)\rangle, \langle 1, f_3(x, y)\rangle), \tag{2.8}$$

where f_i, $i = 1, 2, 3$, are holomorphic functions.

Definition 2.2. Let X be a harmonic surface defined in $U \subset \mathbb{C}$ open and simply-connected, given by $X(x, y) = (\langle 1, f_1(x, y)\rangle, \langle 1, f_2(x, y), \langle 1, f_3(x, y)\rangle)$. If $f_2 = -if_1$ then $X(x, y) = (f_1(x, y), \langle 1, f_3(x, y)\rangle)$ and the surface X is called *graph-type harmonic surface*.

Definition 2.3. Let $X(x, y) = (\varphi_1(x, y), \varphi_2(x, y), \varphi_3(x, y))$ be a regular parameterized surface. The surface X is *Biharmonic* if $\Delta^2\varphi_i = 0$, for $i = 1, 2, 3$.

Observation 2.2. Let X be a biharmonic surface defined in $U \subset \mathbb{C}$ open and simply-connected given by

$$X(x, y) = (\langle 1, f_1(x, y)\rangle + \langle z, g_1\rangle, \langle 1, f_2(x, y)\rangle + \langle z, g_2\rangle, \langle 1, f_3(x, y)\rangle + \langle z, g_3\rangle), \tag{2.9}$$

where f_i and g_i, $i = 1, 2, 3$ are holomorphic functions.

3. Main results. Let X be a regular parameterized surface given by $X(x, y) = (\langle 1, f_1(x, y)\rangle, \langle 1, f_2(x, y)\rangle, h(x, y))$, where f_j, $j = 1, 2$ are holomorphic functions and $h : U \longrightarrow \mathbb{R}$ a differentiable function, U open set of $\mathbb{R}^2$. If $f_2 = -if_1$ then X can be write as $X(x, y) = (f_1(x, y), h(x, y))$.

Theorem 3.1. *Let X be a regular parameterized surface given by $X(x, y) = (f_1(x, y), h(x, y))$ where f_1 is a holomorphic function and $h : U \subset \mathbb{C} \longrightarrow \mathbb{R}$, a differentiable function in U. Then the Gaussian curvature K and the mean curvature H of X are given by*

$$K = \frac{\left\langle \overline{\left(\frac{f_1''}{f_1'}\right)}\nabla h, \omega \right\rangle - \left\| \overline{\left(\frac{f_1''}{f_1'}\right)}\nabla h \right\|^2 + \frac{1}{4}((\Delta h)^2 - \|\omega\|^2)}{\left(\sqrt{\|\nabla h\|^2 + \|f_1'\|^2}\right)^4}, \tag{3.1}$$

$$H = \frac{\|\nabla h\|^2 \left(2\left\langle \overline{\left(\frac{f_1''}{f_1'}\right)}, \nabla h \right\rangle + \Delta h\right) + 2\|f_1'\|^2 \Delta h - \langle (\nabla h)^2, \omega\rangle}{4\|f_1'\| \left(\sqrt{\|\nabla h\|^2 + \|f_1'\|^2}\right)^3}, \tag{3.2}$$

where $\omega = h_{xx} - h_{yy} + 2ih_{xy}$.

Proof: Given the surface $X(x, y) = (f_1(x, y), h(x, y))$, from (2.1) and (2.2) we get

$$\begin{aligned} X_x &= (f_1', h_x), \\ X_y &= (if_1', h_y). \end{aligned} \tag{3.3}$$

By (3.3) the coefficients of the first fundamental form are given by

$$\begin{aligned} E &= \|f_1'\|^2 + \|h_x\|^2, \\ F &= h_x h_y, \\ G &= \|f_1'\|^2 + \|h_y\|^2. \end{aligned} \tag{3.4}$$

Also, from (3.3) we obtain

$$X_x \times X_y = \left(\langle i, f_1'\rangle h_y - \langle i, if_1'\rangle h_x, \langle 1, if_1'\rangle h_x - \langle 1, f_1'\rangle h_y, \langle 1, f_1'\rangle \langle i, if_1'\rangle - \langle 1, if_1'\rangle \langle i, f_1'\rangle\right). \tag{3.5}$$

Using (2.5) and (2.6) the coordinates of $N(x,y)$ are given by

$$\langle i, f_1'\rangle h_y - \langle i, if_1'\rangle h_x = -\left[\langle\langle 1, f_1'\rangle - i\langle i, f_1'\rangle, (h_x, h_y)\rangle\right] = -\left\langle \overline{f_1'}, \nabla h\right\rangle, \tag{3.6}$$

$$\langle 1, if_i'\rangle h_x - \langle 1, f_1'\rangle h_y = \langle -i\left(\langle 1, f_1'\rangle - i\langle i, f_1'\rangle\right), \nabla h\rangle = -\left\langle i\overline{f_1'}, \nabla h\right\rangle, \tag{3.7}$$

$$\langle 1, f_1'\rangle \langle i, if_1'\rangle - \langle 1, if_1'\rangle \langle i, f_1'\rangle = \|f_1'\|^2. \tag{3.8}$$

Substituting (3.6), (3.7) and (3.8) in (3.5),

$$X_x \times X_y = \left(-\left\langle \overline{f_1'}, \nabla h\right\rangle, -\left\langle i\overline{f_1'}, \nabla h\right\rangle, \|f_1'\|^2\right). \tag{3.9}$$

Using (3.9) we obtain that the normal vector of X is given by

$$N = \frac{(-f_1'\nabla h, \|f_1'\|^2)}{\|v\|}, \tag{3.10}$$

where $\|v\| = \|f_1'\|\sqrt{\|\nabla h\|^2 + \|f_1'\|^2}$.

From (2.1) and (2.2) we get

$$\begin{aligned} X_{xx} &= (f_1'', h_{xx}),\\ X_{xy} &= (if_1'', h_{xy}),\\ X_{yy} &= (-f_1'', h_{yy}). \end{aligned} \tag{3.11}$$

Also, the coefficients of the second fundamental form are given by

$$e = \langle X_{xx}, N\rangle = \frac{1}{\|v\|}\left[-\left\langle f_1''\overline{f_1'}, \nabla h\right\rangle + \|f_1'\|^2 h_{xx}\right] = \frac{\|f_1'\|^2}{\|v\|}\left[h_{xx} - \left\langle \frac{f_1''}{f_1'}, \nabla h\right\rangle\right],$$

$$f = \langle X_{xy}, N\rangle = \frac{1}{\|v\|}\left[-\langle if_1'', f_1'\nabla h\rangle + \|f_1'\|^2 h_{xy}\right] = \frac{\|f_1'\|^2}{\|v\|}\left[h_{xy} - \left\langle i\frac{f_1''}{f_1'}, \nabla h\right\rangle\right],$$

$$g = \langle X_{yy}, N\rangle = \frac{1}{\|v\|}\left[\langle f_1'', f_1'\nabla h\rangle + \|f_1'\|^2 h_{yy}\right] = \frac{\|f_1'\|^2}{\|v\|}\left[h_{yy} + \left\langle \frac{f_1''}{f_1'}, \nabla h\right\rangle\right].$$

On the other hand,

$$\begin{aligned} eg - f^2 &= \frac{\|f_1'\|^4}{\|v\|^2}\left\{-\left\langle \frac{f_1''}{f_1'}, \nabla h\right\rangle^2 + \left\langle \frac{f_1''}{f_1'}, \nabla h\right\rangle h_{xx} - \left\langle \frac{f_1''}{f_1'}, \nabla h\right\rangle h_{yy} + h_{xx}h_{yy}\right.\\ &\quad \left. - \left\langle i\frac{f_1''}{f_1'}, \nabla h\right\rangle^2 + 2\left\langle \frac{f_1''}{f_1'}, \nabla h\right\rangle h_{xy} - h_{xy}^2\right\},\\ &= \frac{\|f_1'\|^4}{\|v\|^2}\left\{\left\langle \frac{f_1''}{f_1'}, \nabla h\right\rangle (h_{xx} - h_{yy}) + 2\left\langle i\frac{f_1''}{f_1'}, \nabla h\right\rangle h_{xy} - \left\|\overline{\left(\frac{f_1''}{f_1'}\right)}\nabla h\right\|^2\right.\\ &\quad \left. + h_{xx}h_{yy} - h_{xy}^2\right\}. \end{aligned}$$

Putting

$$\omega = h_{xx} - h_{yy} + 2ih_{xy}. \tag{3.12}$$

Utilizing (2.4), (3.12), (3.13) and the fact that $EG - F^2 = \|v\|^2$ we have that the Gaussian curvature K is given by

$$K = \frac{\|f_1'\|^4}{\|v\|^4}\left\{\left\langle \left(\overline{\frac{f_1''}{f_1'}}\right)\nabla h, \omega\right\rangle - \left\|\left(\overline{\frac{f_1''}{f_1'}}\right)\nabla h\right\|^2 + h_{xx}h_{yy} - h_{xy}^2\right\}. \tag{3.13}$$

Note that,

$$\|\omega\|^2 = h_{xx}^2 - 2h_{xx}h_{yy} + h_{yy}^2 + 4h_{xy}^2 = (h_{xx} + h_{yy})^2 - 4(h_{xx}h_{yy} - h_{xy}^2) \tag{3.14}$$

From (3.14) we have

$$h_{xx}h_{yy} - h_{xy}^2 = \frac{1}{4}\left[(\triangle h)^2 - \|\omega\|^2\right]. \tag{3.15}$$

Thus, using (3.15) in (3.13), we obtain (3.1).
Also,

$$\begin{aligned} eG - 2fF + gE = \frac{\|f_1'\|^2}{\|v\|}\Big\{&\left\langle \frac{f_1''}{f_1'}, \nabla h\right\rangle (h_x^2 - h_y^2) + 2\left\langle i\frac{f_1''}{f_1'}, \nabla h\right\rangle h_x h_y \\ &+ \|f_1'\|^2 (h_{xx} + h_{yy}) + h_{xx}h_y^2 + h_{yy}h_x^2 - 2h_{xy}h_xh_y\Big\}. \end{aligned} \tag{3.16}$$

Since, $(\nabla h)^2 = h_x^2 - h_y^2 + 2ih_xh_y$, we get in (3.16)

$$\begin{aligned} eG - 2fF + gE = \frac{\|f_1'\|^2}{\|v\|}\Big\{&\left\langle \left(\overline{\frac{f_1''}{f_1'}}\right)\nabla h, (\nabla h)^2\right\rangle + \|f_1'\|^2 \Delta h \\ &+ h_{xx}h_y^2 + h_{yy}h_x^2 - 2h_{xy}h_xh_y\Big\}. \end{aligned} \tag{3.17}$$

From (3.12) we have

$$\begin{aligned} \left\langle (\nabla h)^2, \omega\right\rangle &= h_x^2h_{xx} - h_x^2h_{yy} - h_y^2h_{xx} + h_y^2h_{yy} + 4h_{xy}h_xh_y, \\ &= \|\nabla h\|^2 \Delta h - 2\left[h_x^2h_{yy} + h_y^2h_{xx} - 2h_{xy}h_xh_y\right]. \end{aligned} \tag{3.18}$$

From (3.18) we obtain

$$h_{xx}h_y^2 + h_{yy}h_x^2 - 2h_{xy}h_xh_y = \frac{1}{2}\left[\|\nabla h\|^2 \triangle h - \langle(\nabla h)^2, \omega\rangle\right]. \tag{3.19}$$

Using (2.4), (3.17) and (3.19) we have

$$H = \frac{\|f_1'\|^2}{2\|v\|^3}\left\{\|\nabla h\|^2\left\langle\left(\overline{\frac{f_1''}{f_1'}}\right), \nabla h\right\rangle + \|f_1'\|^2 \triangle h + \frac{1}{2}\left[\|\nabla h\|^2 \triangle h - \langle(\nabla h)^2, \omega\rangle\right]\right\},$$

this equation is equivalent to (3.2). ■

4. Graph-type biharmonic surfaces. In this section we define the graph-type biharmonic surfaces and give some examples.

Definition 4.1. Let X be a biharmonic surface defined in $U \subset \mathbb{C}$ open and simply-connected, given by $X(x,y) = (\langle 1, f_1\rangle, \langle 1, f_2\rangle, \langle 1, f_3\rangle + \langle z, g_3\rangle)$. If $f_2 = -if_1$ then $X(x,y) = (f_1(x,y), \langle 1, f_3\rangle + \langle z, g_3\rangle)$, where $f_i, g_3, i = 1,2,3$ are holomorphic functions, $z \in \mathbb{C}$, is called *graph-type biharmonic surface* .

Theorem 4.1. *Let X be a graph-type biharmonic surface given by $X(x,y) = (f_1(x,y), \langle 1, f_3\rangle + \langle z, g_3\rangle)$. Then the Gaussian curvature K and the mean curvature H of X are given by*

$$K = \frac{2\left\langle\left(\overline{\frac{f_1''}{f_1'}}\right)\xi, \overline{f_3''} + z\overline{g_3''}\right\rangle - \left\|\left(\overline{\frac{f_1''}{f_1'}}\right)\xi\right\|^2 + 4\langle 1, g_3'\rangle^2 - \left\|\overline{f_3''} + z\overline{g_3''}\right\|^2}{\left(\sqrt{\|\xi\|^2 + \|f_1'\|^2}\right)^4}, \tag{4.1}$$

$$H = \frac{\|\xi\|^2\left(\left\langle\left(\overline{\frac{f_1''}{f_1'}}\right), \xi\right\rangle + 2\langle 1, g_3'\rangle\right) + 4\|f_1'\|^2\langle 1, g_3'\rangle - \left\langle \xi^2, \overline{f_3''} + z\overline{g_3''}\right\rangle}{2\|f_1'\|\left(\sqrt{\|\xi\|^2 + \|f_1'\|^2}\right)^3}, \tag{4.2}$$

where $\xi = \overline{f_3'} + g_3 + z\overline{g_3'}$.

Proof: Since the biharmonic surface X is given by $X(x,y) = (f_1(x,y), \langle 1, f_3\rangle + \langle z, g_3\rangle)$, considering $h = \langle 1, f_3\rangle + \langle z, g_3\rangle$, using (2.1) and (2.2) we obtain

$$h_x = \langle 1, f_3'\rangle + \langle 1, g_3\rangle + \langle z, g_3'\rangle\,, \quad h_y = \langle 1, if_3'\rangle + \langle i, g_3\rangle + \langle z, ig_3'\rangle\,. \tag{4.3}$$

Thus, from (2.4) and (2.6)

$$\nabla h = \overline{f_3'} + g_3 + z\overline{g_3'} = \xi. \tag{4.4}$$

From (4.3) it follows

$$\begin{aligned} h_{xx} &= \langle 1, f_3''\rangle + 2\,\langle 1, g_3'\rangle + \langle z, g_3''\rangle\,, \\ h_{xy} &= \langle 1, if_3''\rangle + \langle z, ig_3''\rangle\,, \\ h_{yy} &= -\,\langle 1, f_3''\rangle + 2\,\langle 1, g_3'\rangle - \langle z, g_3''\rangle\,. \end{aligned} \tag{4.5}$$

$$\triangle h = 4\,\langle 1, g_3'\rangle\,. \tag{4.6}$$

Using (2.6), (4.3) and (4.5) in (3.12) we get

$$\omega = 2\left(\overline{f_3''} + z\overline{g_3''}\right) \tag{4.7}$$

Substituting (4.4) and (4.7) in (3.1) and (3.2), we obtain (4.1) and (4.2), respectively. ■

Example 1 Considering $f_1(z) = z^2$, $f_3(z) = -5z + 2 + i$, $g_3(z) = z^2$ in Theorem 4.2, we get the graph-type biharmonic surface (see Fig. 4.1)

$$X(x,y) = (x^2 - y^2, 2xy, 2 - 5x + xy^2 + x^3).$$

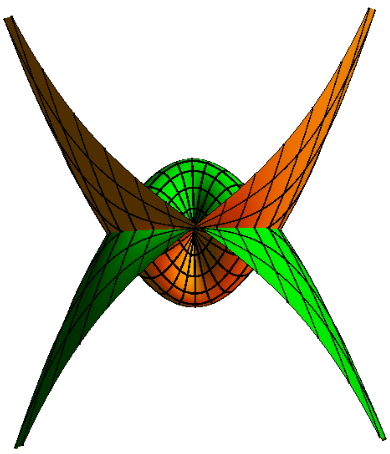

Figure 4.1: Graph-type biharmonic surface. Source: self elaboration

Example 2 Considering $f_1(z) = z^2$, $f_3(z) = z$, $g_3(z) = e^z$ in Theorem 4.2, we get the graph-type biharmonic surface (see Fig. 4.2)

$$X(x, y) = (x^2 - y^2, 2xy, x + xe^x \cos y + ye^x \sin y).$$

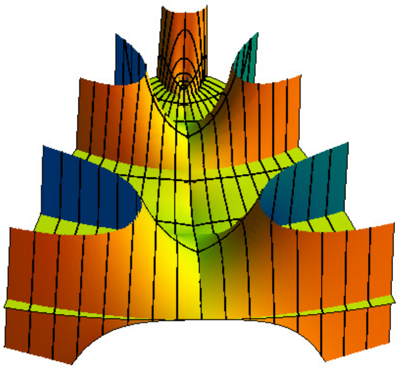

Figure 4.2: Graph-type biharmonic surface. Source: self elaboration

4.1. Special cases of graph-type biharmonic surfaces. In this subsection, we study special classes of graph-type biharmonic surfaces.

Proposition 4.1. *Let X be a graph-type biharmonic surface defined in $U \subset \mathbb{C}$ open and simply-connected, given by $X(x, y) = (z, \langle 1, f_3\rangle + \langle z, g_3\rangle)$, where f_3 and g_3 are holomorphic functions. Then Gaussian curvature K and the mean curvature H of X are given by*

$$K = \frac{4\left\langle 1, g_3'\right\rangle^2 - \left\|\overline{f_3''} + z\overline{g_3''}\right\|^2}{\left(\left\|\overline{f_3'} + g_3 + z\overline{g_3'}\right\|^2 + 1\right)^2}, \tag{4.8}$$

$$H = \frac{2\left\langle 1, g_3'\right\rangle \left(\left\|\overline{f_3'} + g_3 + z\overline{g_3'}\right\|^2 + 2\right) - \left\langle \left(\overline{f_3'} + g_3 + z\overline{g_3'}\right)^2, \overline{f_3''} + z\overline{g_3''}\right\rangle}{2\left(\sqrt{\left\|\overline{f_3'} + g_3 + z\overline{g_3'}\right\|^2 + 1}\right)^3}. \tag{4.9}$$

Proof: The proof it follows from Theorem 4.2, considering $f_1(z) = z$. ■

Proposition 4.2. *Let X be a graph-type biharmonic surface defined in $U \subset \mathbb{C}$ open and simply-connected, given by $X(x, y) = (z, \langle z, g_3\rangle)$ where g_3 is a holomorphic function. Then Gaussian curvature K and the mean curvature H of X are given by*

$$K = \frac{4\left\langle 1, g_3'\right\rangle^2 - \left\|z\overline{g_3''}\right\|^2}{\left(\left\|g_3 + z\overline{g_3'}\right\|^2 + 1\right)^2}, \tag{4.10}$$

$$H = \frac{2\left\langle 1, g_3'\right\rangle \left(\left\|g_3 + z\overline{g_3'}\right\| + 2\right) - \left\langle \left(g_3 + z\overline{g_3'}\right)^2, z\overline{g_3''}\right\rangle}{2\left(\sqrt{\left\|g_3 + z\overline{g_3'}\right\|^2 + 1}\right)^3}. \tag{4.11}$$

Proof: The proof it follows from Theorem 4.2, considering $f_1(z) = z$ and $f_3(z) = 0$. ■

5. Surfaces associated to two harmonic functions. In this section, we introduce the surfaces associated to two harmonic functions (FH2A-surfaces) and give some examples.

Definition 5.1. Consider a plane $\sigma \subset \mathbb{R}^3$, we define the *projection application* $\pi_\sigma : \mathbb{R}^3 \longrightarrow \sigma$ as being the orthogonal projection of $p \in \mathbb{R}^3$ into σ.

Definition 5.2. A regular parameterized surface X is called *surface associated to two harmonic functions (FH2A-surface)*, if the following relation is satisfied

$$\frac{K}{(d_\sigma)^4} = \mu^2 - e^\lambda. \tag{5.1}$$

where K is the Gaussian curvature of X, $d_\sigma = \pi_\sigma \circ N$ and the functions λ, μ are harmonic. The following result shows that a graph-type biharmonic surface with $f_1 = z$ and $f_3 = 0$ is a FH2A-surface.

Theorem 5.1. *Let X be a graph-type biharmonic surface given by $X(x, y) = (z, \langle z, g\rangle)$. Then X is a FH2A-surface.*

Proof: Since, $X(x, y) = (z, \langle z, g\rangle)$, from (3.10) and (4.4) the normal vector to X is given by

$$N = \frac{\left(-(g + z\overline{g'}), 1\right)}{\sqrt{\left\|g + z\overline{g'}\right\|^2 + 1}}. \tag{5.2}$$

Note that

$$d_\sigma = \frac{1}{\sqrt{\left\|g + z\overline{g'}\right\|^2 + 1}}. \tag{5.3}$$

Now, define $\mu = 2\langle 1, g'\rangle$ and $\lambda = \ln \|zg''\|^2$ and replace μ, λ, (5.4) in (4.10) to get (5.1).
To complete the proof, it remains to show that μ and λ are harmonics. From (2.1) and (2.2) we have

$$\Delta\mu = \Delta\left(2\langle 1, g'\rangle\right) = \langle 1, g'''\rangle + \langle 1, -g'''\rangle = 0,$$

thus, μ is harmonic.
Also, differentiating and using (2.1) and (2.2) we get

$$\lambda_x = \frac{2\langle g'' + zg''', zg''\rangle}{\langle zg'', zg''\rangle}, \ \lambda_y = \frac{2\langle i(g'' + zg'''), zg''\rangle}{\langle zg'', zg''\rangle},$$

$$\lambda_{xx} = \frac{\left(2\langle 2g''' + zg'''', zg''\rangle + 2\|g'' + zg'''\|^2\right)\|zg''\|^2 - 4\langle g'' + zg''', zg''\rangle^2}{\langle zg'', zg''\rangle^2}, \tag{5.4}$$

$$\lambda_{yy} = \frac{\left(-2\langle 2g''' + zg'''', zg''\rangle + 2\|g'' + zg'''\|^2\right)\|zg''\|^2 - 4\langle i(g'' + zg''), zg''\rangle^2}{\langle zg'', zg''\rangle^2}. \tag{5.5}$$

From (5.4) and (5.5) we obtain

$$\Delta\lambda = \frac{4\left(\|g'' + zg'''\|^2\|zg''\|^2 - \left[\langle g'' + zg''', zg''\rangle^2 + \langle i(g'' + zg'''), zg''\rangle^2\right]\right)}{\langle zg'', zg''\rangle^2}. \tag{5.6}$$

We observe that

$$\begin{aligned}\langle g'' + zg''', zg''\rangle^2 + \langle i(g'' + zg'''), zg''\rangle^2 &= \left\langle 1, \overline{g'' + zg'''}zg''\right\rangle^2 + \left\langle i, \overline{g'' + zg''''}zg''\right\rangle^2 \\ &= \left\|\overline{g'' + zg'''}zg''\right\|^2 \\ &= \|g'' + zg'''\|^2\|zg''\|^2.\end{aligned}$$

Substituting this expression in (5.6), we have that $\Delta\lambda = 0$, therefore, λ is harmonic. The proof is complete. ■

Example 3 Considering $f_1(z) = z$, $g(z) = e^{z^2}$ in Theorem 5.3, we get the FH2A-surface (see Fig. 5.1)

$$X(x, y) = (x, y, e^{x^2 - y^2}(x\cos(2xy) + y\sin(2xy))).$$

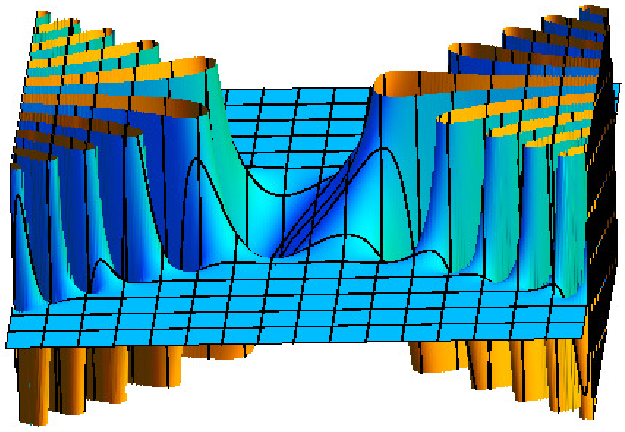

Figure 5.1: FH2A-surface. Source: self elaboration

Example 4 Considering $f_1(z) = z$, $g(z) = \tan z$ in Theorem 5.3, we get the FH2A-surface (see Fig. 5.2)

$$X(x,y) = \left(x, y, \frac{x\sin(2x) + y\sinh(2y)}{\cos(2x) + \cosh(2y)}\right).$$

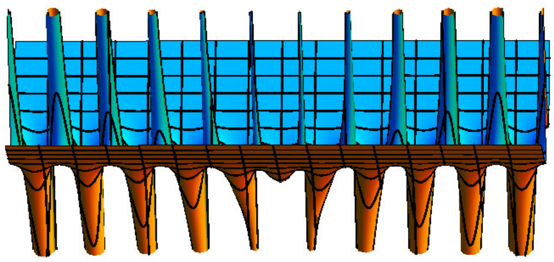

Figure 5.2: FH2A-surface. Source: self elaboration

Example 5 Considering $f_1(z) = z$, $g(z) = z^7$ in Theorem 5.3, we get the FH2A-surface (see Fig. 5.3)

$$X(x,y) = \left(x, y, x^8 - 14x^6y^2 + 14x^2y^6 - y^8\right).$$

Figure 5.3: FH2A-surface. Source: self elaboration

6. FH2A-surfaces of rotation. In the next we will classify the FH2A-surfaces of rotation.

Theorem 6.1. *Let $X(z) = (e^z, h(x))$ be a parametrization of a rotation surface. Then X is a FH2A-surface of rotation if, and only if, h is given by*

$$h(x) = a_2 \pm \int_1^x \sqrt{a_1 e^{2u} - c_2^2 - c_1 c_2(1+2u) - \frac{c_1^2\left(1+2u+2u^2\right)}{2} - \frac{2e^{c_3 u + c_4}}{c_3 - 2}}\, du, \tag{6.1}$$

where $a_1, a_2, c_1, c_2, c_3 \neq 2, c_4 \in \mathbb{R}$,
or

$$h(x) = a_2 \pm \int_1^x \sqrt{a_1 e^{2u} - c_2^2 - c_1 c_2(1+2u) - \frac{c_1^2\left(1+2u+2u^2\right)}{2} - 2ue^{2u+c_4}}\, du \tag{6.2}$$

where $a_1, a_2, c_1, c_2, c_4 \in \mathbb{R}$.

Proof: Using (3.1) we obtain that the Gaussian curvature satisfy

$$\frac{K}{(d\sigma)^4} = h'h'' - (h')^2. \tag{6.3}$$

From (5.1), X is a FH2A-surface if, and only if,

$$h'h'' - (h')^2 = (c_1 x + c_2)^2 - e^{c_3 x + c_4}. \tag{6.4}$$

In fact, since, μ and λ are harmonic functions and h only depend on x we have that

$$\mu^2 - e^{\lambda} = (c_1 x + c_2)^2 - e^{c_3 x + c_4}.$$

The solutions of (6.4) for $c_3 \neq 2$ and for $c_3 = 2$ are given by (6.1) and (6.2), respectively. ■

Example 6 Considering $a_1 = 2, a_2 = 1, c_1 = c_3 = 0, c_2 = \pi, c_4 = \frac{1}{4}$ in Theorem 6.1, we obtain that $h(x) = 1 + \sqrt{e^{1/4} + e^{2(2+x)} - \pi^2} - \sqrt{\pi^2 - e^{1/4}}\arctan\left[\frac{\sqrt{e^{1/4}+e^{2(2+x)}-\pi^2}}{\sqrt{\pi^2 - e^{1/4}}}\right]$, thus, we get the FH2A-surface of rotation (see Fig. 6.1) given by

$$X(x,y) = \left(e^x\cos y, e^x \sin y, 1 + \sqrt{e^{1/4} + e^{2(2+x)} - \pi^2} - \sqrt{\pi^2 - e^{1/4}}\arctan\left[\frac{\sqrt{e^{1/4}+e^{2(2+x)}-\pi^2}}{\sqrt{\pi^2 - e^{1/4}}}\right]\right).$$

Figure 6.1: FH2A-surface of rotation. Source: self elaboration

Example 7 Considering $a_1 = 1, a_2 = 0, c_1 = c_2 = 0, c_3 = 2, c_4 = -1$ in Theorem 6.1, we obtain that $h(x) = \dfrac{e^{-x}\sqrt{e^{2x}\left(1-\frac{2x}{e}\right)}\left(e^{e/2}\sqrt{\pi} + e^x\sqrt{2e-4x} - e^{e/2}\sqrt{\pi}\text{Erf}\left[\sqrt{\frac{e}{2}-x}\right]\right)}{\sqrt{2e-4x}}$, thus, we get the FH2A-surface of rotation (see Fig. 6.2) given by

$$X(x,y) = \left(e^x\cos y, e^x\sin y, \frac{e^{-x}\sqrt{e^{2x}\left(1-\frac{2x}{e}\right)}\left(e^{e/2}\sqrt{\pi} + e^x\sqrt{2e-4x} - e^{e/2}\sqrt{\pi}\text{Erf}\left[\sqrt{\frac{e}{2}-x}\right]\right)}{\sqrt{2e-4x}}\right),$$

where, Erf is the error function.

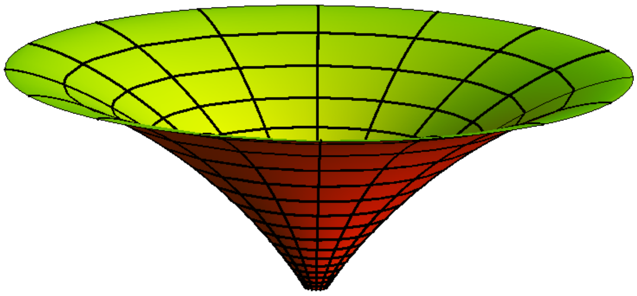

Figure 6.2: FH2A-surface of rotation. Source: self elaboration

7. Conclusions. In this paper, using holomorphic functions we characterize a class of biharmonic surfaces called graph-type biharmonic surfaces. We introduce a special class of surfaces that are associated to two harmonic functions (FH2A-surfaces) and as a first step we classify the FH2A-surfaces of rotation. In this sense, it would be important to study the FH2A-surfaces with certain properties and obtain their classification.

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Carlos M. C. Riveros https://orcid.org/0000-0002-1206-7072
Armando M. V. Corro https://orcid.org/0000-0002-6864-3876
Raquel P. de Araújo https://orcid.org/0000-0002-3944-9863

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