Dynamics of Close Earth Satellites by Picard Iterations

Martin Lara*[1]

[1]SCoTIC, University of La Rioja, Madre de Dios 53, 26006 Logroño, La Rioja, Spain

The proliferation of crowded constellations of small satellites in low Earth orbit (LEO), ranging from the current 150 units of Planet Labs[1] to the eventual 42000 programmed for SpaceX's Starlink satellite megaconstellation,[2] is producing a revival in the use of simple analytical solutions of the artificial satellite problem, which fit quite well for their design and operation [1]. This fact motivates me to revisit the main effects on the LEO dynamics of the gravitational disturbances of Keplerian motion. As is well known, they are due to the dominant effect of the Earth's zonal harmonic of the second degree, whose nondimensional coefficient is customarily denoted J_2 [2].

Beyond the integrable equatorial case [3] the J_2 problem lacks of the needed integrals that would guarantee the existence of a closed form solution [4]. Nevertheless, the non-integrability can be ignored in practice for the small value of the Earth's $J_2 = \mathcal{O}(10^{-3}) > 0$, which makes the size of the regions in which chaos may emerge negligible [5]. Indeed, machine-precision accuracy can be preserved for long times with high order perturbation solutions of the J_2 problem [6]. However, the length of the series involved in this kind of solution, together with the inadequacy of the J_2 model for simulating the real dynamics of circumterrestrial orbits, makes that highly accurate solutions of the J_2 problem are of limited interest in practice.

Conversely, the bulk of the J_2 dynamics is captured by much simpler *intermediary orbits*, which share the mean dynamics of the satellite problem at least up to $\mathcal{O}(J_2)$ effects. To wit, on average, the intermediary orbit must undergo a small linear variation of the right ascension of the ascending node, and a small but steady motion of the argument of the perigee in the orbital plane. These general properties of the oblateness perturbation are usually derived from an average representation of the disturbing function $\mathcal{D}$ of the J_2 problem [7]. That is,

$$\langle\mathcal{D}\rangle_M = \frac{1}{2\pi}\int_0^{2\pi}\mathcal{D}\,\mathrm{d}M = \frac{1}{4}J_2\frac{\mu}{p}\frac{R_\oplus^2}{p^2}\eta^3\left(1-3c^2\right), \quad (1)$$

where μ is the gravitational parameter, $R_\oplus$ is the Earth's equatorial radius, $p = a\eta^2$, a is the orbit semimajor axis, $\eta = (1-e^2)^{1/2}$, e is orbital eccentricity, c is the cosine of the orbit inclination I, and M is the mean anomaly [8].

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[1]www.planet.com/our-constellations/ (as May 3, 2022).

[2]www.space.com/spacex-starlink-satellites.html (as May 3, 2022).

Replacing (1) in the variation of parameters equations, we readily obtain the well-known mean variations of the right ascension of the ascending node Ω and the argument of the perigee ω. Namely,

$$\overline{\frac{\mathrm{d}\Omega}{\mathrm{d}t}} = -nJ_2\frac{R_\oplus^2}{p^2}\frac{3}{2}c, \quad \overline{\frac{\mathrm{d}\omega}{\mathrm{d}t}} = nJ_2\frac{R_\oplus^2}{p^2}\frac{3}{4}(5c^2-1), \quad (2)$$

where $n = (\mu/a^3)^{1/2}$ is the mean motion. While (2) reasonably agrees with the average dynamics, the average rate of variation of the mean anomaly obtained with this procedure [9, 10, 11]

$$\overline{\frac{\mathrm{d}M}{\mathrm{d}t}} = n + nJ_2\frac{R_\oplus^2}{p^2}\frac{3}{4}\eta(3c^2-1), \quad (3)$$

soon yields large in-track errors with respect to which would be expected from a $\mathcal{O}(J_2)$ average solution. Rather than starting from an average disturbing function, I will show that the mean rate in (3) is amended when the average dynamics is computed by neglecting the periodic terms from the true solution. The latter is obtained using Picard's constructive proof for the existence and unicity of solutions to ordinary differential equations [12].

Thus, let

$$\frac{\mathrm{d}\xi_i}{\mathrm{d}\tau} = \chi_i(\xi_j,\tau), \quad \xi_i(\tau_0) = \xi_{i,0}, \quad i,j = 1,\ldots m, \quad (4)$$

be a first order differential system in which τ is the independent variable, and ξ_i are m dependent variables. Assuming that the functions χ_i are analytic, they are replaced by corresponding Taylor series expansions in powers of $\Delta\tau = \tau - \tau_0$. When constraining to such interval $\Delta\tau$ that the differences $\xi_j - \xi_{j,0}$ remain small enough, these differences can be neglected, and hence an analytical approximation to the solution of (4) is computed from the convergent sequence

$$\xi_{i,k} = \xi_{i,0} + \int_{\tau_0}^{\tau}\chi_i[\xi_{j,k-1}(\tau,\xi_{j,0}),\tau]\,\mathrm{d}\tau, \quad (5)$$

which starts from $\xi_{i,1} = \xi_{i,0} + \int_{\tau_0}^{\tau}\chi_i(\xi_{j,0},\tau)\,\mathrm{d}\tau$.

I apply this procedure to the variations of the traditional Keplerian variables a, e, I, Ω, ω, and M, whose detailed expressions can be consulted elsewhere [13]. However, to deal with strict elements the variation of M is replaced by [7]

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = \frac{\mathrm{d}M}{\mathrm{d}t} - \sqrt{\frac{\mu}{a^3}}. \quad (6)$$

Still, M is present in the variation equations through its implicit dependence on the true anomaly f. Then, in order for the variation equations to take a form amenable to solution by Picard iterations, the integration is carried out in a fictitious time τ, given by the differential relation

$$\mathrm{d}t = (r^2/G)\,\mathrm{d}\tau, \tag{7}$$

where $G = \sqrt{\mu p}$ is the specific angular momentum.

Comparison of (7) with Kepler's law of areas shows that τ evolves at the same rate as the argument of the latitude. Alternatively, for an $\mathcal{O}(J_2)$ solution the fictitious time can be replaced by f as the independent variable [14, 15]. Then, the variations in the physical time are replaced by

$$\frac{\mathrm{d}\xi_j}{\mathrm{d}f} = \frac{r^2}{G}\frac{\mathrm{d}\xi_j}{\mathrm{d}t}, \qquad j = 1, \dots 6, \tag{8}$$

which are integrated in closed form of the eccentricity by Picard iterations starting from the initial values a_0, e_0, I_0, Ω_0, ω_0, and $\beta_0 = 0$, for f_0. The first iteration results in

$$\begin{aligned}
a_1 &= a_0 + a_0\varepsilon[a_{1,\mathrm{P}}(f) - a_{1,\mathrm{P}}(f_0)]\\
e_1 &= e_0 + \varepsilon[e_{1,\mathrm{P}}(f) - e_{1,\mathrm{P}}(f_0)]\\
I_1 &= I_0 + \varepsilon c[I_{1,\mathrm{P}}(f) - I_{1,\mathrm{P}}(f_0)]\\
\Omega_1 &= \Omega_0 - 6\varepsilon c\Delta M + \varepsilon c[\Omega_{1,\mathrm{P}}(f) - \Omega_{1,\mathrm{P}}(f_0)]\\
\omega_1 &= \omega_0 + 3\varepsilon(5c^2-1)\Delta M + \varepsilon[\omega_{1,\mathrm{P}}(f) - \omega_{1,\mathrm{P}}(f_0)]\\
\beta_1 &= 3\varepsilon\eta(3c^2-1)\Delta M + \varepsilon[\beta_{1,\mathrm{P}}(f) - \beta_{1,\mathrm{P}}(f_0)]
\end{aligned} \tag{9}$$

where $\varepsilon = \frac{1}{4}J_2R_\oplus^2/p^2$, $\Delta M = M - M_0$, and $\xi_{1,\mathrm{P}}$ are such trigonometric polynomials that $\langle\xi_{1,\mathrm{P}}(f)\rangle_M = 0$. Refer to [13] for detailed expressions.

Next, $M_1 = M_0 + \beta_1(f) + \sqrt{\mu}\int_{t_0}^{t} a_1(t)^{-3/2}\,\mathrm{d}t$, from (6), where the integrand $a_1(t)^{-3/2}$ is replaced by an $\mathcal{O}(J_2)$ approximation to obtain a solution in closed form of e. Thus,

$$M_1 = M_0 + n^*(t - t_0) + \varepsilon\left[M_\mathrm{P}(f) - M_\mathrm{P}(f_0)\right], \tag{10}$$

in which

$$n^* = n\left[1 + 3\varepsilon\eta(3c^2-1) + \tfrac{3}{2}\varepsilon a_{1,\mathrm{P}}(f_0)\right], \tag{11}$$

and the detailed form of M_P can be consulted in [13].

Finally, from (7),

$$t = t_0 + \int_{f_0}^{f}\frac{\left[\left(1-e_1^2\right)a_1\right]^{3/2}}{(1+e_1\cos f)^2\mu^{1/2}}\,\mathrm{d}f. \tag{12}$$

Note that, in a typical ephemeris evaluation, the errors introduced in the physical time determination may be as important as those of the elements [16].

Removing the purely periodic terms $\xi_{1,\mathrm{P}}(f)$ from (9) it is readily obtained that, in the approximation provided by the first Picard iteration, the semi-major axis, eccentricity, and inclination, remain constant on average. On the other hand, (2) is recovered by differentiation of the secular terms of Ω_1 and ω_1 with respect to the physical time. Analogously, the removal of purely short-period terms from (10) shows that, at the precision of the first Picard iteration, M advances, on average, at the rate n^* given in (11), thus amending (3) with the additional term $\frac{3}{8}nJ_2(R_\oplus/p)^2a_{1,\mathrm{P}}(f_0)$.

The accuracy of the first Picard iteration with respect to the true, numerically integrated solution is illustrated in the left column of Fig. 1 for an example eccentric orbit with $a_0 = 9500$ km, $e_0 = 0.2$, $I_0 = 20°$, $\Omega_0 = 6°$, $\omega_0 = 274°$, and $M_0 = 0$ ($\mu = 398600.4415\,\mathrm{km}^3/\mathrm{s}^2$, $R_\oplus = 6378.1363$ km, $J_2 = 0.001082634$). The mild behavior of the errors of M is due to the new secular term in (11). It can be checked that when n^* is replaced in (10) by (3), the error of M grows by about two orders of magnitude at the end of the one-day interval shown in the current example, reaching an amplitude close to 1° — or about 200 km along-track as opposed to the km level reached when using (11).

The first iteration of Picard's method misses the long-period effects of the true solution, which are clearly apparent in Fig. 1 coupled with the short-period errors. A refinement of the analytical solution that captures non-resonant long-period effects of the dynamics is obtained by an additional iteration of (5). To $\mathcal{O}(J_2)$, the whole procedure is equivalent to substituting M by n^*t in (9), and replacing the appearances of the constant ω_0 throughout $\xi_{1,\mathrm{P}}$ by the low frequency $\omega(f) \equiv \omega_0 + 3(5c^2-1)(\varepsilon f)$ [13].

The improvements produced by the second Picard iteration are illustrated in the right column of Fig. 1. While the errors start with the same amplitudes as before, the influence of the long-period terms becomes now evident, and the amplitude of the errors remains mostly constant along the propagation, improving the errors with respect to the first Picard iteration by about one order of magnitude at the end of the one-day propagation interval. Remaining secular and long-period components are a consequence of the $\mathcal{O}(J_2)$ truncation of the Picard iterations solution.

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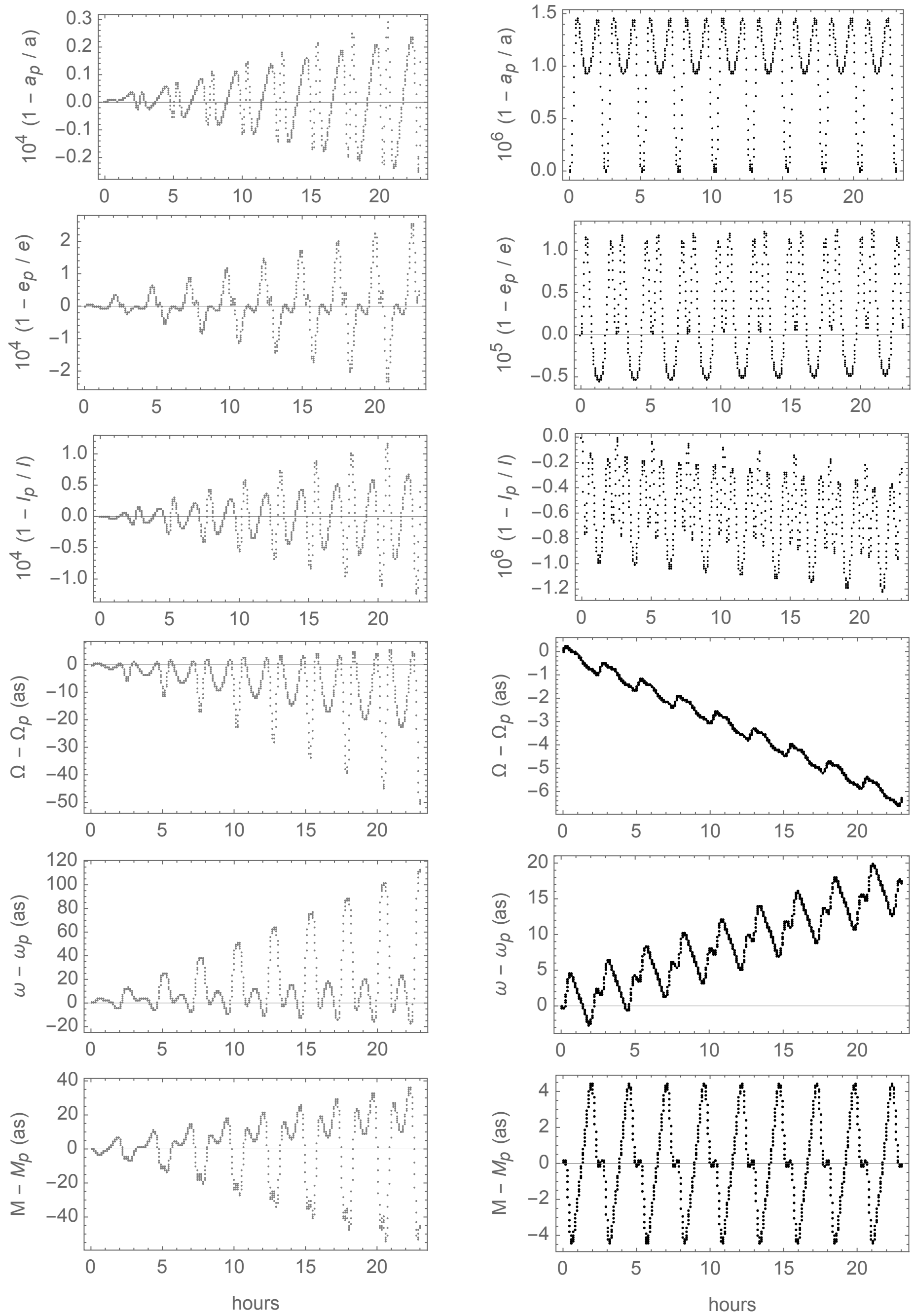

Figure 1: Errors of the first (left column) and second Picard iteration (right colum) of the test orbit [13].

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