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Some integral inequalities involving the k-Beta function using h-convex functions

Algunas desigualdades integrales con la función k-Beta usando funciones h-convexas

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Abstract. The present work deals with the study of the integral of the type

$$\int_a^b (x-a)^{p/k}(b-x)^{q/k} f(x)dx$$

for $p, q, k > 0$, considering some inequalities for h-convex functions. From these results some others integral inequalities for other class of generalized convex functions are obtained.

***Key words*:** Integral inequalities, h-convex functions, k-Beta function.

Resumen. El presente trabajo trata acerca del estudio de la integral del tipo

$$\int_a^b (x-a)^{p/k}(b-x)^{q/k} f(x)dx$$

para $p, q, k > 0$, considerando algunas desigualdades para funciones h-convexas. De estos resultados se derivan algunas otras desigualdades integrales para otras clases de funciones convexas generalizadas.

***Palabras clave*:** Desigualdades integrales, funciones h-convexas, Función k-Beta.

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1 Introduction

Convexity is a basic notion in geometry, but it is also widely used in other areas of mathematics. The use of techniques of convexity appears in many branches of mathematics and sciences, such as Theory of Optimization and Theory of Inequalities, Functional Analysis, Mathematical Programming and Game Theory, Theory of Numbers, Variational Calculus and its interrelation with these branches shows itself day by day deeper and fruitful [7, 9, 14].

Definition 1. A function $f : I \to R$ is said to be convex if for all $x, y \in I$ and $t \in [0, 1]$ the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds.

Over time, several problems and applications have arisen, and these have given rise to generalizations of the concept of convex function, and also numerous works of investigation have been realized extending results on inequalities for this kind of convexity: quasi-convexity [17], s-convexity in the first and second sense [2], logarithmic convexity [1], m-convexity [13], h-convexity [18, 19] and others [20].

The study of special functions has had special attention with the works of R. Diaz and E. Pariguan [4] and A. Rehman et al. [15], in which the subject of the generalized k-Gamma function and generalized k-Beta is treated, in turn, these appear in the study of fractional integrals. Also the research field corresponding to inequalities, generalized convexity and fractional integrals has been extensively studied [8, 21].

Motivated by the works of W. Liu [11] and the corresponding to M. E. Özdemir et al. [12], the purpose of this paper is to study the integral

$$\int_a^b (x-a)^{p/k}(b-x)^{q/k} f(x)dx$$

for $p, q, k > 0$ for h-convex functions and establish some bounds for the aforementioned integral.

2 Preliminaries

Concerning the generalized convexity the following definitions will be necessary.

S. S. Dragomir et al. in [5] introduced the following definition.

Definition 2. Let $I \subset \mathbb{R}$ an interval. A function $f : I \to \mathbb{R}$ is said to belong to the class $P(I)$ or to be $P-$convex if it is non negative and for all $x, y \in I$ and $t \in [0, 1]$, satisfies the following inequality

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

It is useful to mention that the class $P(I)$ contain all the non-negative convex and quasi-convex functions.

Also, M. Alomari et al. in [2] used the concept of s-onvexity in the second sense as Hudzik and L. Maligranda in [10] introduced it. Some interesting properties of this class of functions were given by S.S. Dragomir and S. Fitzpatrick in [6], by example: all functions in K_s^2 are locally Hölder continuous of order s on (a, b) and therefore Riemann integrable on $[a, b]$.

Definition 3. Let $s \in (0, 1]$. A real valued function f on an interval $I \subset [0, \infty)$ is s-onvex in the second sense provided

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b)$$

for all $a, b \in I$ and $t \in [0, 1]$. This is denoted by $f \in K_s^2$.

In 2007, S. Varošanec in [18] introduced a new concept of generalized convexity.

Definition 4. Let J an interval of $\mathbb{R}$ such that $(0, 1) \subset J$ and $h : J \to \mathbb{R}$ a non negative real valued function such that $h \not\equiv 0$. A function $f : I \to \mathbb{R}$ is said to be h-convex or that f belong to the class $SX(h, I)$ if f is non negative and for all $x, y \in I$ and all $t \in (0, 1)$ the following inequality holds

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y).$$

Obviously if $h(t) = t$ all the non-negative convex functions belong to the class $SX(h, I)$; if $h(t) = 1$ then $P(I) \subset SX(h, I)$, and if $h(t) = t^s$ then $K_s^2 \subset SX(h, I)$.

Several works have been published using h-convex functions applied to the Hermite-Hadamard and Hermite-Hadamard-Féjer inequalities [3, 16].

Also, in the development of this work we use the k-Beta function and it is necessary to recall some notes about it. The following definitions and properties are taken from the works by R. Diaz and E. Pariguan [4] and N. Rehman et al. [15]:

Definition 5. Let $x \in \mathbb{C}, k \in \mathbb{R}$ and $n \in \mathbb{N}^+$ For $k > 0$, the Pochhammer k-symbol is given by

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k).$$

Definition 6. For $k > 0$, the k-Gamma function Γ_k is given by

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \qquad x \in \mathbb{C} \setminus k\mathbb{Z}^-.$$

Definition 7. The k-Beta function $B_k(x, y)$ is given by

$$B_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} \qquad Re(x) > 0, Re(y) > 0.$$

As mentioned before

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt,$$

and

$$B_k(x+k,y) = \frac{x}{x+y}B_k(x,y) \text{ and } B_k(x,y+k) = \frac{y}{x+y}B_k(x,y). \tag{1}$$

Some others properties of the k-Beta functions, and also for k-Beta function with several variables, can be found in [15].

3 Main Results

Lemma 1. *Consider I an interval and let $f : I \to \mathbb{R}$ an integrable function on the interval I. Then the equality*

$$\int_a^b (u-a)^{p/k}(b-u)^{q/k} f(u)du = (b-a)^{\frac{p+q}{k}+1}\int_0^1 (1-t)^{p/k} t^{q/k} f(ta+(1-t)b)dt$$

holds for some fixed $p, q, k > 0$, and $a, b \in I$.

Proof. Let $u = ta + (1-t)b$. Then $t = (b-u)/(b-a)$, $1-t = (u-a)/(b-a)$ and $dt = -du/(b-a)$, so

$$\int_0^1 (1-t)^{p/k} t^{q/k} f(ta+(1-t)b)dt = \frac{1}{(b-a)^{\frac{p}{k}+\frac{q}{k}+1}}\int_a^b (u-a)^{p/k}(b-u)^{q/k} f(u)du.$$

The proof is complete. □

The following results for functions whose absolute values are h-convex, including $r-$th powers of them, are established.

Theorem 1. *Let $f : I \to \mathbb{R}$ be an integrable function on I, and $p, q, k > 0$. If $|f|$ is h-convex on $[a,b]$, for $a, b \in I$ and $a < b$, then the following inequality holds*

$$\int_a^b (u-a)^{p/k}(b-u)^{q/k} f(u)du \le (b-a)^{\frac{p+q}{k}+1}\left(I_1(h)|f(a)| + I_2(h)|f(b)|\right), \tag{2}$$

where

$$I_1(h) = \int_0^1 (1-t)^{p/k} t^{q/k} h(t)dt$$

and

$$I_2(h) = \int_0^1 (1-t)^{p/k} t^{q/k} h(1-t)dt.$$

Proof. Using Lemma 1 and the h-convexity of $|f|$, we have

$$\begin{aligned}
&\int_a^b (u-a)^{p/k}(b-u)^{q/k} f(u)du \\
&\leq (b-a)^{\frac{p+q}{k}+1}\int_0^1 (1-t)^{p/k} t^{q/k} |f(ta+(1-t)b)|\, dt \\
&\leq (b-a)^{\frac{p+q}{k}+1}\int_0^1 (1-t)^{p/k} t^{q/k} \left(h(t)\,|f(a,\cdot)| + h(1-t)\,|f(b,\cdot)|\right) dt \\
&= (b-a)^{\frac{p+q}{k}+1}\Bigg(|f(a)|\int_0^1 (1-t)^{p/k} t^{q/k} h(t)dt \\
&\qquad\qquad + |f(b)|\int_0^1 (1-t)^{p/k} t^{q/k} h(1-t)dt\Bigg).
\end{aligned}$$

Using the fact that

$$I_1(h) = \int_0^1 (1-t)^{p/k} t^{q/k} h(t)dt$$

and

$$I_2(h) = \int_0^1 (1-t)^{p/k} t^{q/k} h(1-t)dt$$

we have the desired result. □

Using a suitable choice of the function h it is possible to find other important results.

Corollary 1. Let $f : I \to \mathbb{R}$ be an integrable function on I, and $p, q, k > 0$. If $|f|$ is convex on $[a, b]$, where $a, b \in I$ and $a < b$, then the following inequality holds

$$\begin{aligned}
&\int_a^b (u-a)^{p/k}(b-u)^{q/k} f(u)du \\
&\qquad\leq (b-a)^{\frac{p+q}{k}+1}\frac{kB_k(p,q)}{(p+q)_{3,k}}\left((q)_{2,k}\, p\,|f(a)| + (p)_{2,k}\, q\,|f(b)|\right).
\end{aligned}$$

Proof. Setting $h(t) = t$ for all $t \in [0, 1]$, using Definition 7 and properties given in (1), we have the following identities

$$\begin{aligned}
I_1(h) = \int_0^1 (1-t)^{p/k} t^{q/k} h(t)dt &= \int_0^1 (1-t)^{p/k} t^{q/k+1} dt \\
&= kB_k(p+k, q+2k) \\
&= kB_k(p,q)\frac{(q)_{2,k}\, p}{(p+q)_{3,k}}
\end{aligned}$$

and

$$\begin{aligned} I_2(h) = \int_0^1 (1-t)^{p/k} t^{q/k} h(1-t)dt &= \int_0^1 (1-t)^{p/k+1} t^{q/k} dt \\ &= kB_k(p+2k, q+k) \\ &= kB_k(p,q)\frac{(p)_{2,k}\, q}{(p+q)_{3,k}}. \end{aligned}$$

By replacement of these values in equation (2) we obtain the desired result. □

Corollary 2. Let $f : I \to \mathbb{R}$ be an integrable function on I, and $p, q, k > 0$. If $|f|$ is $P-$convex on $[a, b]$ where $a, b \in I$ and $a < b$, then the following inequality holds

$$\begin{aligned} &\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du \\ &\qquad \le (b-a)^{\frac{p+q}{k}+1} \frac{kpq}{(p+q)_{2,k}} B_k(p,q) \left(|f(a)| + |f(b)|\right). \end{aligned}$$

Proof. Letting $h(t) = 1$ for all $t \in [0, 1]$, using Definition 7 and property (1) we have

$$\begin{aligned} I_1(h) = I_2(h) &= \int_0^1 (1-t)^{p/k} t^{q/k} dt \\ &= \frac{pq}{(p+q)_{2,k}} B_k(p,q). \end{aligned} \tag{3}$$

By replacement of these values in Theorem 1 it is attained the desired result. □

Remark 1. If we take $k = 1$ in Corollary 2 then Theorem 4 from [11] is obtained.

Corollary 3. Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. If $|f|$ is s-convex in the second sense on $[a, b]$ for some $s \in (0, 1]$, where $a, b \in I$ and $a < b$, then the following inequality holds

$$\begin{aligned} &\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du \\ &\quad \le k(b-a)^{\frac{p+q}{k}+1} \left(\frac{p(q+ks)}{(p+q+ks)_{2,k}} B_k(p, q+ks))\, |f(a)| \right. \\ &\qquad\qquad \left. + \frac{(p+ks)q}{(p+q+ks)_{2,k}} B_k(p+ks, q)\, |f(b,\cdot)| \right). \end{aligned}$$

Proof. Letting $h(t) = t^s$ for all $t \in [0,1]$ and some $s \in (0,1]$, using Definition 7 and property (1) we have

$$\begin{aligned} I_1(h) = \int_0^1 (1-t)^{p/k}\, t^{q/k} h(t)dt &= \int_0^1 (1-t)^{p/k}\, t^{q/k+s} dt \\ &= B_k\left(p+k, q+k(s+1)\right) \\ &= \frac{p(q+ks)}{(p+q+ks)_{2,k}} B_k\left(p, q+ks\right) \end{aligned}$$

and

$$\begin{aligned} I_2(h) = \int_0^1 (1-t)^{p/k}\, t^{q/k} h(1-t)dt &= \int_0^1 (1-t)^{p/k+s}\, t^{q/k} dt \\ &= B_k\left(p+k(s+1), q+k\right) \\ &= \frac{(p+ks)q}{(p+q+ks)_{2,k}} B_k\left(p+ks, q\right). \end{aligned}$$

By replacement of these values in Theorem 1 the desired result is obtained. ☐

Theorem 2. *Let $f : I \to \mathbb{R}$ be an integrable function on the interval I and $p, q, k > 0$. If $|f|^r$ is h-convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\begin{aligned} &\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du \qquad (4) \\ &\leq (b-a)^{\frac{p+q}{k}+1} \left(\frac{kpq}{(lp+lq)_{2,k}} B_k(lp, lq)\right)^{1/l} \left(I_1(h)\left|f(a)\right|^r + I_2(h)\left|f(b)\right|^r\right)^{1/r}, \end{aligned}$$

where

$$I_1(h) = \int_0^1 h(t)dt \ , \ I_2(h) = \int_0^1 h(1-t)dt$$

and $(1/l) + (1/r) = 1$.

Proof. From Lemma 1 and using the Hölder inequality we have

$$\begin{aligned} &\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du \qquad (5) \\ &\leq (b-a)^{\frac{p+q}{k}+1} \int_0^1 (1-t)^{p/k}\, t^{q/k} \left|f\left(ta+(1-t)b, \cdot\right)\right| dt \\ &\leq (b-a)^{p+q+1} \left(\int_0^1 (1-t)^{lp/k}\, t^{lq/k} dt\right)^{1/l} \left(\int_0^1 \left|f\left(ta+(1-t)b, \cdot\right)\right|^r dt\right)^{1/r}. \end{aligned}$$

Since $|f|^r$ is a h-convex function then

$$\int_0^1 \left|f\left(ta+(1-t)b, \cdot\right)\right|^r dt \leq \left|f(a)\right|^r \int_0^1 h(t)dt + \left|f(b)\right|^r \int_0^1 h(1-t)dt \qquad (6)$$

and using the definition of the k-Beta function and the property (1), we get

$$\begin{aligned}\int_0^1 (1-t)^{lp/k}\, t^{lq/k} dt &= kB_k(lp+k, lq+k) \\ &= k\frac{pq}{(lp+lq)_{2,k}} B_k(lp,lq) \end{aligned} \tag{7}$$

Besides,

$$I_1(h) = \int_0^1 h(t)dt \text{ and } I_2(h) = \int_0^1 h(1-t)dt$$

So, replacing (6) and (7) in (5) it is attained the required inequality (4).

The proof is complete. □

Corollary 4. Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. If $|f|^r$ is convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds

$$\begin{aligned}&\int_a^b (u-a)^{p/k} (b-u)^{q/k} X(u,\cdot)du \\ &\leq 2^{-1/r} (b-a)^{\frac{p+q}{k}+1} \left(\frac{kpq}{(lp+lq)_{2,k}} B_k(lp,lq)\right)^{1/l} \left(|X(a,\cdot)|^r + |X(b,\cdot)|^r\right)^{1/r},\end{aligned}$$

where $(1/l) + (1/r) = 1$.

Proof. Letting $h(t) = t$ for all $t \in [0, 1]$ we have

$$I_1(h) = I_2(h) = \int_0^1 tdt = \frac{1}{2}.$$

So, by replacement in Theorem 2 it is obtained the desired result.

□

Corollary 5. Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. If $|f|^r$ is P-convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds

$$\begin{aligned}&\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du \\ &\leq (b-a)^{\frac{p+q}{k}+1} \left(\frac{kpq}{(lp+lq)_{2,k}} B_k(lp,lq)\right)^{1/l} \left(|f(a)|^r + |f(b)|^r\right)^{1/r},\end{aligned} \tag{8}$$

where $(1/l) + (1/r) = 1$.

Proof. Letting $h(t) = 1$ for all $t \in [0, 1]$ we have $I_1(h) = I_2(h) = 1$. Then, by replacement in Theorem 2 it is attained the desired result. □

Corollary 6. Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. If $|f|^r$ is s-onvex in the second sense on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du$$
$$\leq \frac{(b-a)^{\frac{p+q}{k}+1}}{(s+1)^{1/r}} \left(\frac{kpq}{(lp+lq)_{2,k}} B_k(lp, lq) \right)^{1/l} \left(|f(a)|^r + |f(b)|^r \right)^{1/r},$$

where $(1/l) + (1/r) = 1$.

Proof. Letting $h(t) = t^s$ for all $t \in [0, 1]$ and some $s \in (0, 1]$, we have

$$I_1(h) = I_2(h) = \int_0^1 t^s dt = \frac{1}{s+1}.$$

By replacement of these values in Theorem 2 it is obtained the result. □

Theorem 3. *Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. If $|X|^r$ is h-convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du \tag{9}$$
$$\leq k (b-a)^{\frac{p+q}{k}+1} \left[\frac{pq}{(p+q)_{2,k}} B_k(p, q) \right]^{1-1/r} \left(I_1(h) |f(a)|^r + I_2(h) |f(b)|^r \right)^{1/r},$$

where

$$I_1(h) = \int_0^1 (1-t)^{p/k} t^{q/k} h(t)dt \ \text{ and } \ I_2(h) = \int_0^1 (1-t)^{p/k} t^{q/k} h(1-t)dt.$$

Proof. From Lemma 1 and using the power mean inequality for $r \geq 1$ we have

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du \tag{10}$$
$$\leq (b-a)^{\frac{p+q}{k}+1} \int_0^1 (1-t)^{p/k} t^{q/k} |f(ta + (1-t)b, \cdot)| dt$$
$$\leq (b-a)^{\frac{p+q}{k}+1} \left(\int_0^1 (1-t)^{p/k} t^{q/k} dt \right)^{1-1/r} \times$$
$$\left(\int_0^1 (1-t)^{p/k} t^{q/k} |f(ta + (1-t)b, \cdot)|^r dt \right)^{1/r}.$$

Making use of the h-convexity of $|f|^r$ and the definition the k-Beta function, we get

$$\begin{aligned}\int_0^1 & (1-t)^{p/k} t^{q/k} \left|f\left(ta+(1-t)b\right)\right|^r dt \qquad (11)\\ &\leq \int_0^1 (1-t)^{p/k} t^{q/k} \left(h(t)\left|f(a)\right|^r + h(1-t)\left|f(b)\right|^r\right) dt \\ &= I_1(h)\left|f(a)\right|^r + I_2(h)\left|f(b)\right|^r,\end{aligned}$$

where

$$I_1(h) = \int_0^1 (1-t)^{p/k} t^{q/k} h(t)dt \text{ and } I_2(h) = \int_0^1 (1-t)^{p/k} t^{q/k} h(1-t)dt.$$

So, replacing (11) and (3) in the inequality (10) it is attained the desired inequality (9).

The proof is complete. □

Corollary 7. Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. If $|X|^r$ is convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds

$$\begin{aligned}\int_a^b & (u-a)^{p/k} (b-u)^{q/k} f(u)du \\ &\leq k^{1+1/r} (b-a)^{\frac{p+q}{k}+1} \left[\frac{pq}{(p+q)_{2,k}}\right]^{1-1/r} B_k(p,q) \times \\ &\qquad \left(\frac{(q)_{2,k}\, p}{(p+q)_{3,k}}\left|f(a)\right|^r + \frac{(p)_{2,k}\, q}{(p+q)_{3,k}}\left|f(b)\right|^r\right)^{1/r}\end{aligned}$$

Proof. Letting $h(t) = t$ for all $t \in [0, 1]$, the values of $I_1(h)$ and $I_2(h)$ are as in the proof of Corollary 1. By replacement of these values in Theorem 3 it is attained the desired result. □

Corollary 8. Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. If $|X|^r$ is $P-$convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds

$$\begin{aligned}\int_a^b & (u-a)^{p/k} (b-u)^{q/k} f(u)du \\ &\leq k(b-a)^{\frac{p+q}{k}+1} \left[\frac{pq}{(p+q)_{2,k}} B_k(p,q)\right] \left(\left|f(a)\right|^r + \left|f(b)\right|^r\right)^{1/r},\end{aligned}$$

Proof. Letting $h(t) = 1$ for all $t \in [0, 1]$ we have

$$I_1(h) = I_2(h) = \int_0^1 (1-t)^{p/k} t^{q/k} dt = \frac{pq}{(p+q)_{2,k}} B_k(p,q).$$

By replacement of these values in Theorem 3 we have the desired result. □

Corollary 9. Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. If $|X|^r$ is s-onvex in the second sense on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du$$
$$\leq k(b-a)^{\frac{p+q}{k}+1} \left[\frac{pq}{(p+q)_{2,k}} B_k(p,q) \right]^{1-1/r} \times$$
$$\left(\frac{p(q+ks)B_k(p,q+ks)}{(p+q+ks)_{2,k}} |f(a)|^r + \frac{(p+ks)qB_k(p+ks,q)}{(p+q+ks)_{2,k}} |f(b)|^r \right)^{1/r}$$

Proof. Letting $h(t) = t^s$ for all $t \in [0, 1]$ and some $s \in (0, 1]$, the values of $I_1(h)$ and $I_2(h)$ are as in the proof of Corollary 3. By replacement of these values we find the result. □

4 Conclusions

In the present work some results concerning the integral

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du$$

are obtained. Here, $p, q, k > 0$ and an integrable function f on the interval $[a, b]$. It was used the concept of h-convexity of a function f in order to obtain several inequalities that bound in the upper case the aforementioned integral as it is shown in Theorem 1, Theorem 2 and Theorem 3, and from these results some others inequalities that involve the classical concept of convexity and the generalized concept of P-convexity and s-onvexity in the second sense were derived as it is shown in the presented Corollaries. Also some special functions were used.

For other kind of generalized convexity, for example: MT-convexity, Godunova-Levin convexity, it is possible to establish similar results with a particular choice of the function h. By example, the MT-convexity is related to the function $h(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$ for all $t \in (0, 1)$ and the Godunova-Levin convexity is related to the function $h(t) = 1/t$ for all $t \in (0, 1)$.

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References

[1] M. Alomari and M. Darus, *On The Hadamard's Inequality for Log-Convex Functions on the Coordinates*, J. Ineq. Appl., **2009** (2009), Article ID 283147, 13 pp,

[2] M. Alomari, M. Darus, S.S. Dragomir and P. Cerone, *Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense*. Appl.Math. Lett., **23** (2010), 1071-1076

[3] M. Bombardelli and S. Varošanec. *Properties of h-convex functions related to the Hermite-Hadamard-Féjer inequalities*. Computers and Mathematics with Applications. **58**(9) (2009), 1869-1877

[4] R. Diaz and E. Pariguan. *On hypergeometric functions and Pochhammer k-symbol*. Divulgaciones Matemáticas, **15**(2) (2007), 179-192

[5] S. S. Dragomir, J. Pečarić and L. E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math., **21**(3) (1995), 335-341.

[6] S.S. Dragomir and S. Fitzpatrick, *The Hadamard Inequalities for s-convex functions in the second sense* , Demostratio Mathematica, **4** (1999), 687-696.

[7] A. Hassibi, J. P. How and S. Boyd, *Low-authority controller design via convex optimization*, AIAA Journal of Guidance, Control, and Dynamics, **22**(6) (1999), 862-872.

[8] J.E. Hernández H, *On a Hardy's inequality for a fractional integral operator*, Annals of the University of Craiova, Mathematics and Computer Science Series, **45**(2) (2018), 232-242

[9] J. -B. Hiriart-Urruty, *Global optimality conditions in maximizing a convex quadratic function under convex quadratic constraints*, J. Global Optim., **21** (2001), 445-455.

[10] H. Hudzik and L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math. **48** (1994), 100-111

[11] W. Liu, *New Integral Inequalities involving Beta function via P-convexity*, Miskolc Mathematical Notes, **15**(2) (2014), 585-591

[12] M. E.Özdemir, E. Set and M. Alomari, *Integral inequalities via several kinds of convexity*, Creat. Math. Inform., **20**(1) (2011), 62-73

[13] Z. Pavić and M. Avci Ardic, *The most important inequalities for m-convex functions*, Turk J. Math., **41** (2017), 625-635.

[14] B. T. Polyak, *Convexity of quadratic transformations and its use in control and optimization*, J. Optim. Theory Appl., **99** (1998), 553-583.

[15] A. Rehman, S. Mubeen, R. Safdar and N. Sadiq. *Properties of k-Beta function with several variables*, Open Math., **13** (2015) , 308-320

[16] M. Z. Sarikaya, E. Set and M. E. Özdemir, *On some new inequalities of Hadamard type involving h-convex functions*, Acta Math. Univ. Comenian., **79**(1) (2010), 265-272.

[17] E. Set, A. Akdemir and N. Uygun, *On new simpson type inequalities for generalized quasi-convex mappings*, Xth International Statistics Days Conference, 2016, Giresun, Turkey

[18] S. Varošanec, *On h-convexity*, J. Math. Anal. Appl. **326**(1) (2007), 303-311.

[19] M. J. Vivas and J. E. Hernández H, *On some new generalized Hermite-Hadamard-Féjer inequalities for product of two operator h-convex functions*, Appl. Math. Inf. Sci. **11**(4) (2017), 983-992

[20] M. J. Vivas-Cortez and Y. C. Rangel Oliveros, *Ostrowski type inequalities for functions whose second derivatives are convex generalized*, App. Math. Inf. Sci., **12**(6) (2018), 1117-1126

[21] M. J. Vivas, C. García and J. E. Hernández H, *Ostrowski-type inequalities for functions whose derivative modulus is relatively (m, h_1, h_2)-convex*, Appl. Math. Inf. Sci., **13**(3) (2019), 369-378

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