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Fourth and Fifth Grade Students' Structuring of 2D Rectangular Arrays

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Abstract

In this article we examine 4th and 5th grade Portuguese students' performance on problems of structuring 2D rectangular arrays of squares, which was analyzed and categorized into the five levels of sophistication described by Battista, Clements, Arnoff, Battista, and Borrow (1998). In general, the results suggest that students in the higher-grade exhibit higher levels of sophistication. Nevertheless, several students from both grades could not perceive the structure in rows and columns of such rectangular arrays; their spatial structuring of such arrays was still inadequate. The result for their learning may be even worse if we think that for students in these grades, many textbooks display rectangular arrays as the representation to teach area measurement.

Keywords: Area measurement, spatial structuring, rectangular arrays, levels of sophistication

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Estudiantes de Cuarto y Quinto Grado Estructurando Arreglos Rectangulares en 2D

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Resumen

En este artículo examinamos el rendimiento de los estudiantes portugueses de 4to y 5to grado en los problemas de estructuración de arreglos rectangulares de cuadrados en 2D, que fue analizado y categorizado en los cinco niveles de sofisticación descritos por Battista, Clements, Arnoff, Battista y Borrow (1998). En general, los resultados sugieren que los estudiantes en el grado superior exhiben niveles más altos de sofisticación. Sin embargo, varios estudiantes de ambos grados no pudieron percibir la estructura en filas y columnas de dichos arreglos rectangulares; su estructuración espacial de tales arreglos todavía era inadecuada. El resultado de su aprendizaje puede ser aún peor si pensamos que para los estudiantes de estos grados, muchos libros de texto muestran arreglos rectangulares como la representación para enseñar la medición del área.

Palabras clave: Mediación de área, estructuración espacial, arreglos rectangulares, niveles de sofisticación

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According to Battista *et al.* (1998), students' spatial structuring of 2D rectangular arrays of squares is essential for the development of the notion of area and it is intimately related to multiplication. The authors define spatial structuring as the mental operation of constructing an organization for an object or a set of objects that determines its configuration through the identification of its spatial components, the combination of these components into composite units, and the establishment of relationships between the components and the composite units. They questioned whether students who are unable to visualize the structure of the rectangular arrays in terms of rows and columns can understand the need to use multiplication for the counting of the squares in these arrays, and, furthermore, the use of the formula to calculate its area.

In accordance with Outhred and Mitchelmore (2000), if students do not understand the basis for the area of a rectangle, they have difficulty in generalizing the procedures they have learned. As stated by the authors, the experiential origin of the rectangle area formula is the action of covering a rectangle with square units. However, whereas this action is one-dimensional and suggests an additive process, the formula is two-dimensional and multiplicative. Therefore, it is essential that students move from an intuitive approach, which emphasizes covering the surface, to a more formal one, which relies on relating the area to the linear measurements of the rectangular figure. Nevertheless, the transition from repeated addition to multiplication as a method of enumerating the square units is not easily accomplished by students.

Clements and Sarama (2009) argue that students need to structure arrays to understand area as two-dimensional, which means that "they need to understand how a surface can be tiled with squares that line up in rows and columns" (p. 175).

Wickstrom (2014) explains that, as detailed in CCSSM (Common Core State Standards for Mathematics) document, to build conceptions about area, students must experience the division of a rectangle in rows and columns composed by same-sized square units and determine the total result.

In spite of the recognized importance of structuring rectangular arrays to develop the concept of area, various studies (Outhred & Mitchelmore, 1992; Batista *et al.*, 1998; Batista, 1999; Outhred & Mitchelmore, 2000; Battista, 2003, 2004) have revealed that many students are unable to "see" the structure in rows and columns of a 2D rectangular array of squares.

Outhred and Mitchelmore (1992) individually interviewed students from grades 1 to 5 while they worked through a sequence of counting, drawing and measuring tasks involving covering rectangular figures. They found that, to enumerate rectangular arrays of squares, 50% of the students in the sample counted by ones; 38% counted by groups of either rows or columns; and only 12% calculate by array multiplication. Although most of the students in the sample had not shown difficulties covering a rectangular shape with tiles and then counting the tiles they had used, 30% could not correctly draw the array they had made and still in front of them. The authors correlated the students' performance in drawing rectangular arrays to the strategies they used in the enumeration of these arrays. In general, unless students were able to structure the arrays in terms of rows and columns in their drawing, they could not determine the number of elements by group counting (of rows or columns) or by multiplication. Outhred and Mitchelmore (1992) concluded that if students do not intuitively interpret the structure of 2D rectangular arrays of squares in terms of rows and columns, their learning about area using diagrams can be hindered. According to the authors, linking counting by groups to the structure of rectangular arrays can be a powerful procedure to develop area concepts.

Batista *et al.* (1998) tried to extend their analysis of spatial structuring, by examining in detail students' structuring of 2D rectangular arrays of squares. They individually interviewed primary-grade students, by carrying out a set of problems of structuring 2D rectangular arrays of squares, in which students should: first, make a prediction about how many squares it would take to completely cover one rectangle (the *original prediction*); second, draw where they thought the squares would be located on the rectangle and, then, make a new prediction how many squares it would take to completely cover the rectangle (the *drawing prediction*); third, cover the rectangle with square tiles and, after this, determine again the number of squares needed. The researchers systematized five levels of sophistication in students' structuring of 2D rectangular arrays of squares. At level 1, *Complete lack of row or column structuring*, students do not use a row or a column of squares as a composite unit. At level 2, *Partial row or column structuring*, students make some use of a row or a column of squares as a composite unit, but they do not use this composite to cover the entire rectangle. At level 3A, *Structuring an array as a set of row or column composites*, students conceptualize the rectangle as being completely covered by copies of row or column composites, but they do not coordinate these composites with the orthogonal dimension. At level 3B, *Visual row or column iteration*, students iterate a row of squares as a composite, by distributing it over the squares of a column. When the drawn squares are available, students use them to guide the iteration. Otherwise, students determine the iterations by visually estimating how the rows fit in the rectangle. Finally, at level 3C, *Row by column structuring: iterative process interiorized*, students iterate a row or a column of squares, using the number of squares in a column or in a row, respectively, to determine the iterations. The original perceptual material, for instance the drawn squares, is not used during the iteration. According to Batista *et al.* (1998), the level of sophistication exhibited by some student on a particular problem should not be interpreted as the level of sophistication of the student, in some general developmental scheme. Instead, these levels describe students' performance on structuring 2D rectangular arrays. Moreover, because a student can exhibit slightly different levels of sophistication on distinct problems, more than one problem should be applied. As stated by Battista (1999), "to construct a proper spatial structuring of 2D arrays of squares, students need numerous opportunities to structure such arrays and to reflect on the appropriateness of their structurings." (p. 174).

Outhred and Mitchelmore (2000) analyze strategies that students from grades 1 to 4 use to solve rectangular covering tasks, before they were taught the area measurement. Children's solution strategies were classified into five developmental levels. At level 0, *Incomplete covering*, the units do not completely cover the rectangle without gaps or overlaps. At level 1, *Primitive covering*, the units completely cover the rectangle without overlaps, but their organization is unsystematic: units considerably vary in size and shape or are incorrectly aligned. At level 2, *Array covering, constructed from unit*, drawings exhibit a correct array structure, with an equal number of rectangular units in rows and in columns; however, the size of each unit is visually determined by students from the given unit, and not from considering the dimensions of the rectangle. At level 3, *Array covering, constructed by measurement*, the rectangles' dimensions are used by students to iterate rows: one dimension is used to find the number of units in each row and the other dimension is used to find the number of rows. Finally, at level 4, *Array implied, solution by calculation*, students do not need to draw units; instead, the number of units in rows and in columns is used to calculate the total number of squares, usually by multiplication, but occasionally by repeated addition. In accordance with Outhred and Mitchelmore (2000), these five levels indicate a developmental nature of the sequence, in the sense that each level is more sophisticated than the previous ones, which does not mean that students necessarily progress through those levels, one at a time. The researchers point out some important considerations about the learning of area measurement: first, an important intermediate aim is that children should perceive the spatial structure of an array; second, a relational understanding of linear measurement is crucial; third, children need to link area measurement to both linear measurement and multiplicative concepts before the rectangle area formula could be significantly learned.

More recently, Battista (2003, 2004) developed an integrated and general model for area and volume, associating the independent models developed for each attribute. According to Battista (2004), locating the position of students in the predefined cognitive trajectory of learning those mathematical ideas (area and volume) makes it possible to know what cognitive processes and conceptualizations students must acquire to progress, and so it provides a fundamental knowledge for teachers to successfully guide students in their construction of these concepts. According to Battista (1999), teaching geometry so that students can give meaning to learn requires understand how students construct their knowledge of various geometric topics and use this understanding to choose appropriate instructional tasks and to assess, as well as support, students' learning.

We focus here on area, particularly on 4th and 5th grade students' structuring of 2D rectangular arrays of squares. To study rectangular arrays structuring, we use Batista *et al.*'s (1998) investigation, which presents a useful way to analyze and categorize students' performance on a set of problems they illustrated. These problems are listed roughly in order of difficulty, giving more or less graphical information about the location of squares. Students' success on these problems depends on an operational understanding of each rectangular array structure. It is important to clarify that this study has occurred prior to teaching rectangle area formula, in the 4th grade class, and prior this formula was covered again, in the 5th grade.

Methodology

This study involved 23 students in a 4th grade class, aged 9-11, and 23 students in a 5th grade class, aged 10-12.

In all problems of structuring 2D rectangular arrays of squares, which are part of a set listed by Battista *et al.* (1998), after being shown how a plastic centimeter square fits exactly into one of the squares drawn on the rectangular arrays of the problems, students were asked to do three things: first, to make a prediction about how many squares it would take to completely cover the array; second, to draw where they thought the squares would be located on the array and, then, to predict again how many squares it would take to completely cover the array; third, to cover the array with plastic squares and, after this, to determine again the number of squares needed.

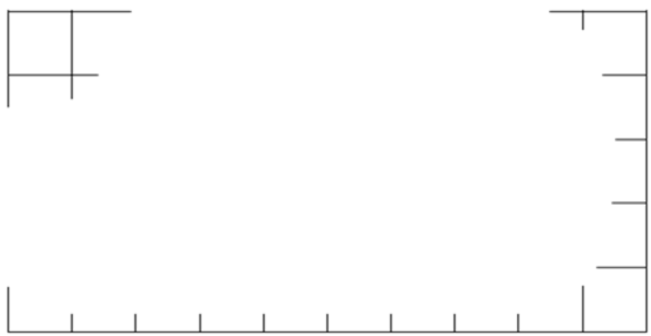

Figure 1. Rectangular array of the first problem.

Students' performance on these problems was analyzed and categorized into the levels of sophistication already described.

The application of the problems was similar for the two schooling years. Each student was individually interviewed at two different times. First, we applied the same problem to all students (Figure 1). Then, depending on the performance shown by students in this first problem, we either applied a less difficult problem than the first (Figure 2) - to students who exhibited level 1 - or a more difficult problem than the first (Figure 3) - to students who exhibited one of the higher levels.

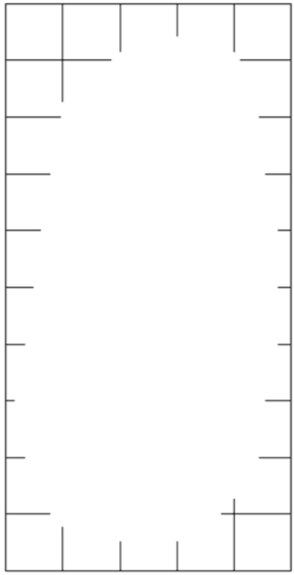

Figure 2. Rectangular array of the second problem less difficult than the first.

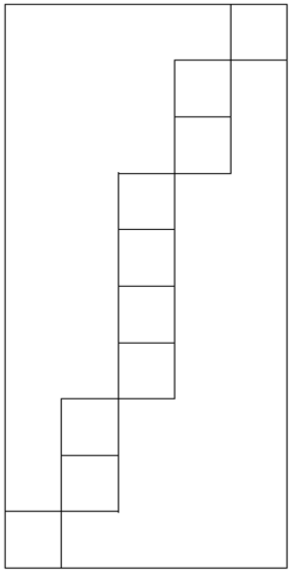

Figure 3. Rectangular array of the second problem more difficult than the first.

The interviews were videotaped and some of them were then transcribed. Students' drawings were all collected. Besides that, the interviewer (first author) took some annotations during students' work on the problems.

Results

Types of Performance Revealed by Students

It was possible to distinguish five types of performance revealed by 4th grade students in the two problems: levels 1 - 1; levels 3A - 1; levels 3A - 3A; levels 3B - 3B; levels 3C - 3C; and also three types of performance revealed by 5th grade students, all of which correspond to some of the previous ones: levels 1 - 1; levels 3B - 3B; levels 3C - 3C.

Illustrative examples of these types of performance are presented below.

Example A. Student DA (first problem: level 1; second problem less difficult: level 1)

First problem

To make a prediction about how many centimeter squares it would take to completely cover the rectangular array shown in figure 1, this student pointed to and counted 35 squares (Figure 4). He showed many difficulties visualizing and enumerating the squares that were not totally drawn or only partially drawn in the array. For that reason, DA first enumerated individual squares that the perceptual material helped to identify (perimeter squares). Then, in the interior of the array (still empty), he tried to guess the number of squares fitting, hesitating repeatedly. He was unable to adequately structure the squares when he tried to visualize them. His prediction for the interior part of the array was not correct. Afterwards, DA drew where he thought the squares would be located within the array (Figure 5). He drew each individual square and once again he started with the squares that the perceptual material available in the array helped to identify. So he drew all perimeter squares first, using the graphic marks presented in the array as guides. Only then DA drew the missing squares one-by-one in the middle of the array, changing direction several times (Figure 6). He drew units almost correctly around the sides of the array, but he could not continue the structure in the middle. He lost the horizontal alignment, as the squares tended to decrease in size from left to right. Then he felt the need to add one last sequence to cover the whole figure. As a consequence, he drew an array with more squares than it should, with little regard to the size of the unit. DA several times explicitly tried to connect the sides of the square he was drawing to other sides. Regardless of that, he easily accepted when these segments did not correctly connect. Although DA's overall drawing seemed to have some organization, his structuring was clearly local, not global. Finally, DA enumerated the squares that he had previously drawn. He first enumerated each individual perimeter square as he pointed to it. Then he started to vertically enumerate each individual interior square, but at a certain time he got confused and stopped (Figure 7), asking if he could restart. He enumerated again the squares in the array as before, only slightly marking the squares with a pencil while he counted them, getting 74. This time he enumerated each square once and only once, avoiding double-counting some squares. At last, DA covered the array with plastic centimeter squares and correctly arrived at a total of 50 squares, by separating them one-by-one as he counted. He had no difficulty covering the rectangular array with tiles and counting them by ones. Apparently, he was unable to count the number of squares by groups of rows or columns.

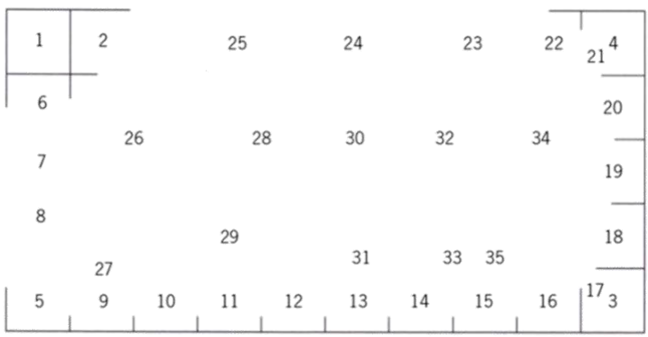

Figure 4. Squares pointed to and counted by DA in the first problem.

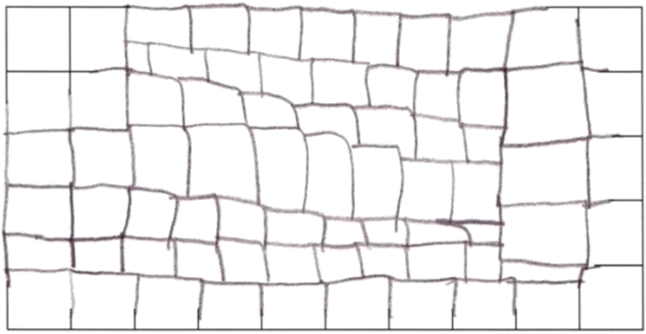

Figure 5. Squares drawn by DA in the first problem.

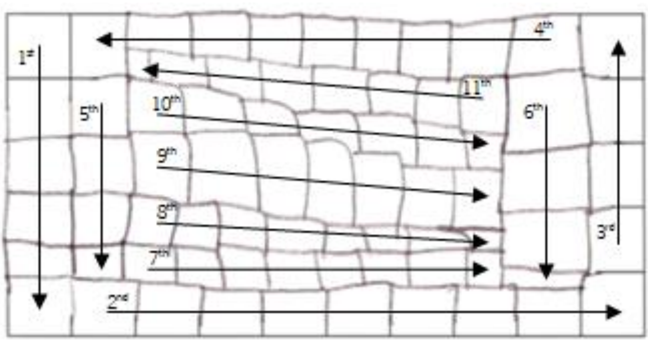

Figure 6. DA's order of drawing in the first problem.

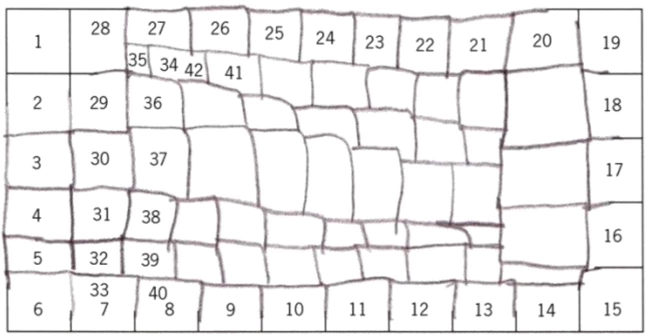

Figure 7. DA's enumeration of drawn squares in the first problem.

Second problem (less difficult)

To predict how many centimeter squares, it would take to completely cover the rectangular array shown in figure 2, the same student pointed to and counted 65 squares (Figure 8). While he was counting, he made the form of each square with his finger. As in the first problem, he showed many difficulties visualizing and enumerating the squares that were not drawn or only partially drawn in the array. Therefore, he started by individually enumerating the squares that graphical cues helped to identify (perimeter squares). Inside the empty array, he tried again to guess the number of squares fitting, now following an anticlockwise spiral of squares. DA did not fail completely to structure the squares. However, his structuring was once again inadequate for the task. Afterwards, he drew where he thought the squares would be located within the array (Figure 9). Like in the first problem, he drew each square individually and started with the most easily identifiable squares (all perimeter squares). He used the hash marks presented in the array as guides and completed them, forming squares along the sides of the array. Only then DA individually drew the missing squares in the middle of the array, predominantly in the vertical direction (Figure 10). His attempts to connect the sides of the square he was drawing to other sides became even more explicit. However, there still was lack of global coordination. In fact, when he drew the last sequence of individual squares, he did not explicitly try to match sides to both right and left squares already drawn. As a consequence, he drew one more square than he should. In addition, he ignored the inconsistency of the last two squares linked to an only square (Figure 11). Therefore, even with a less difficult problem, his structuring clearly remained local, not global. Finally, DA correctly counted one-by-one the 51 squares drawn. To keep track of his counting, DA marked the squares with a pencil while he counted them. He first enumerated each individual perimeter square. Then he did the same to each individual interior square, repeating his anticlockwise spiral method (Figure 12). At last, DA covered the array with plastic centimeter squares and correctly arrived to a total of 50 squares, separating them one-by-one, as before.

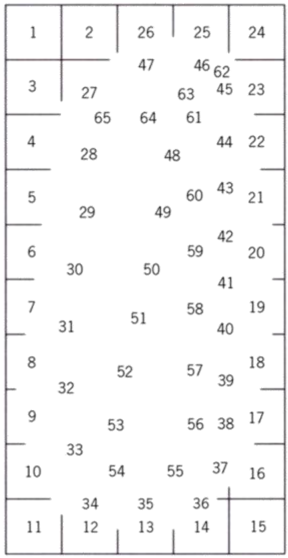

Figure 8. Squares pointed to and counted by DA in the second problem (less difficult).

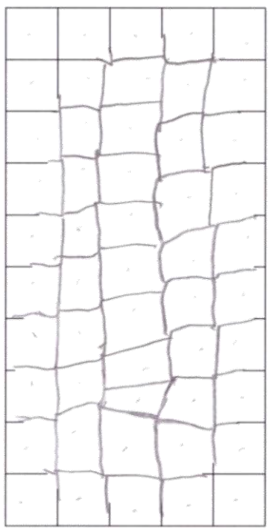

Figure 9. Squares drawn by DA in the second problem (less difficult).

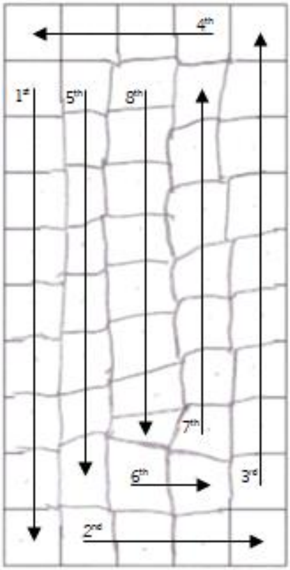

Figure 10. DA’s order of drawing in the second problem (less difficult).

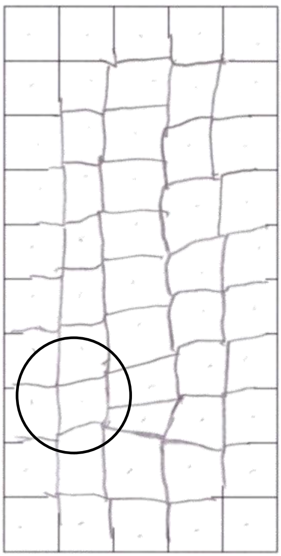

Figure 2. Inconsistency revealed by DA in the second problem (less difficult).

Figure 12. DA's enumeration of drawn squares in the second problem (less difficult).

Example B. Student JR (first problem: level 3A; second problem more difficult: level 1)

First problem

JR was asked to predict how many centimeter squares would cover the rectangular array of the first problem. He almost immediately started enumerating the squares in the top row of the array. But due to the lack of pictorial cues, he got confused and stopped. Then he watched carefully the whole array and said:

> JR: I know. Instead of counting here [pointing to the top row of the array], I'll count there [pointing to the bottom row of the array]. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 [counting from left to right each square partially drawn in the bottom row of the array, while pointing to each one]. 10 [pointing to the bottom row of the array]. Plus 10, plus 10

> (pause), plus 10, plus 10, plus 10 [sweeping his finger across each row from the bottom to the top of the array while counting by tens in each movement]. 60. (Figure 13).

JR seemed to know that he could compose the whole array as rows of 10 squares, which reveals he perceived the bottom row of the array as a composite unit. So, he was aware that rows were congruent, and he counted the total number of squares accordingly. However, he was unsure how to find the number of rows. Actually, he ignored the perceptual markings in the right column of the array (5 squares) indicating the number of rows. He repeated each row of 10 squares using his fingers to roughly estimate how far it extended upward. The simultaneous accuracy in horizontal placement of squares and inaccuracy in vertical indicates that his structuring of the rectangular array (rows of 10 squares) did not apply to squares in a column. Afterwards, he drew where he thought the squares would be located within the array (Figure 14). He explicitly made use of the hash marks given in the array, drawing both horizontal and vertical segments. In particular, he drew horizontal lines, then individual units along the rows. This pattern was systematically continued in the whole figure (Figure 15).

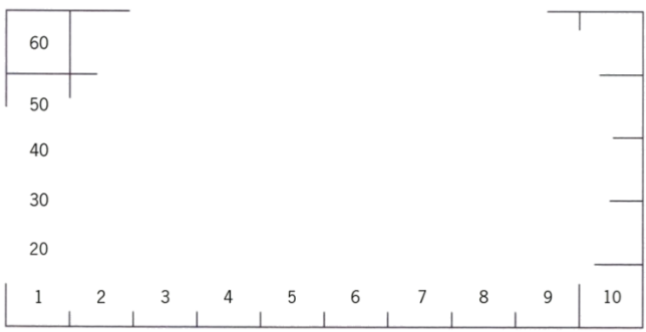

Figure 13. Squares pointed to and counted by JR in the first problem.

Apparently, he perceived that a row could be represented by horizontally extending a line and he was aware of the horizontal alignment of the squares. However, as he marked the squares in each row individually, maybe he still was unaware of the congruence of the rows. Finally, JR counted ten-by-ten the 50 squares that he had previously drawn. His own counting provided further evidence he was structuring the rectangular array into composite units (rows). Furthermore, through this counting he realized he had failed the total number of squares in his initial prediction. He was so surprised with this number that he repeated twice his own counting by tens and then he also counted them by ones. At last, after JR correctly covered the array with plastic centimeter squares, he counted 50 squares by grouping in rows again, confirming the number of squares that he got earlier.

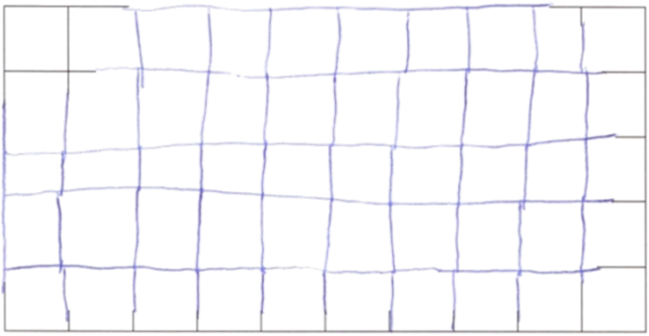

Figure 3. Squares drawn by JR in the first problem

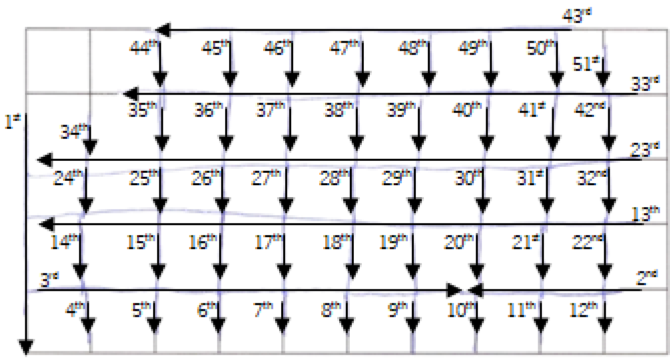

Figure 4 - JR’s order of drawing in the first problem.

Second problem (more difficult)

To make a prediction about how many centimeter squares would cover the rectangular array of the second problem more difficult than the first, the same student pointed to and counted as shown in Figure 16, getting 50 squares. In disagreement with what he had done in the first problem, he did not conceptualize the rectangular array as being completely covered by composite units, as duplicates of rows of squares. Indeed, JR horizontally pointed to and counted one-by-one the 50 squares. He struggled to visualize and count the squares that were not drawn in the array. He used his fingers to measure the space taken by a square and then he tried to replicate it on the empty part of the array, while he enumerated each square. Regardless of JR having tried to use the drawn squares in the array to support his horizontal localization of individual squares, his structuring was obviously local, not global. Afterwards, he drew where he thought the squares would be located within the array (Figure 17).

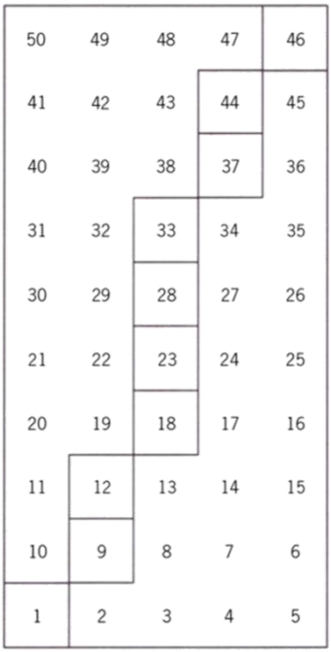

Figure 5. Squares pointed to and counted by JR in the second problem (more difficult).

In his drawing, he independently made squares in each one of the two empty parts visible in the array. As shown in Figure 18, he first drew three vertical segments and eight horizontal segments, making roughly half array, and after that he drew eight horizontal segments and three vertical segments, completing the whole array. Apparently, he drew the array as perpendicular sets of parallel lines. His drawing strategy was inconsistent with the one in the first problem and seemed to be affected by the different pictorial guidance presented in the two arrays. Finally, JR counted one-by-one the squares that he had previously drawn. He stated the correct number of squares, marking each square with a dot while progressed in counting. Despite the fact he was able to represent the array structure properly, he could not count the number of squares that he drew using grouping by rows (or columns) to systematize his counting, which reveals his local structuring of the array. At last, JR covered the array with 50 plastic centimeter squares and correctly found the total number of units into the array. Once again, he counted the squares one-by-one, providing additional evidence of his lack of global organization of the array.

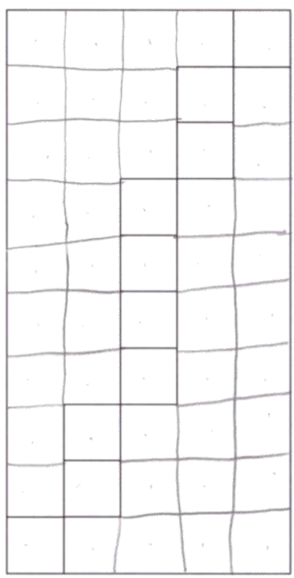

Figure 6. Squares drawn by JR in the second problem (more difficult).

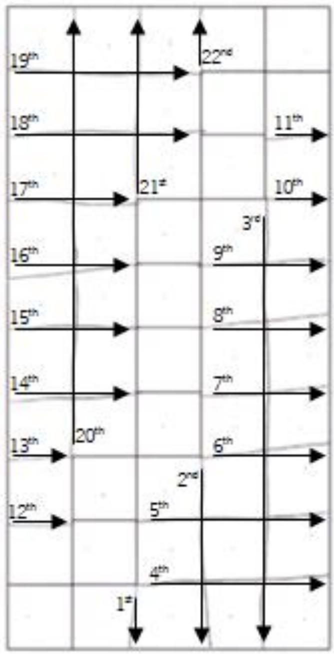

Figure 7. JR's order of drawing in the second problem (more difficult).

Example C. Student IM (first problem: level 3A; second problem more difficult: level 3A)

First problem

To make a prediction about how many centimeter squares would cover the rectangular array of the first problem, this student estimated 40 squares (Figure 19). She began to correctly enumerate the 10 squares partially drawn in the bottom row of the array. Then she tried to cover the whole array with rows of 10 squares, sliding upward her fingers along the left edge of the rectangle, while she counted 10. Accordingly, IM was able to see that she could compose the whole array as rows of 10 squares, meaning that she perceived the bottom row as a composite unit. However, in her iteration of the 10-squares-rows, she did not take into account the given dimension of the right column of the array (5 squares). Instead she used two fingers to simulate the width of a row and then she tried to replicate it in each movement upward. She focused on part of the array structure (iterating rows), but her mental model of organizing the entire array out of a row of 10 did not include duplicating this composite using the squares in a column. Afterwards, she drew where she thought the squares would be located within the array (Figure 20). As shown in Figure 21, she started drawing two individual squares in the top row of the array and then she drew two horizontal segments, matching both sides of the last drawn square to the marks in the right edge of the array. After this, she divided this row in squares, by drawing vertical segments. At some point, she counted by ones the squares she had drawn in this row, getting 9, so she divided in half one of these squares, drawing one more vertical segment.

> Interviewer: Why did you do that? (pause) Why did you divide that square in half?
> IM: Because I only had 9 squares over here [pointing to the top row of the array] and I divided it to get 10.
> Interviewer: And why did you need 10 squares?
> IM: Because I counted the squares down here [pointing to the bottom row of the array with 10 partially drawn squares] and there are 10. And there are always tens by here [sweeping her finger from left to right across the interior of the array, illustrating rows].

The fact that after counting 10 squares in the bottom row, she was able to infer that not only the top row but also the interior rows of the array had 10 squares, suggests again that she treated the bottom line as a composite unit. Thus, she was employing a viable structure for the whole array and not just for the top and bottom rows. Hereafter, as shown in Figure 22, she deliberately linked the sides of the squares that she had drawn in the top row to the hash marks given in the bottom of the array, drawing 10 vertical segments, and then she drew three horizontal segments, using the hash marks given in the right edge of the array. At this time, she was able to make use of parallel lines to draw the remaining part of the array quickly. Finally, IM counted by tens the squares already drawn, making a correct prediction about how many squares it would take to completely cover the array, 50 squares. Her own counting evidenced again that she was structuring the rectangular array into composite units (rows) and also allow her to realize that she had failed the number of squares in her initial prediction. At last, IM covered the array with plastic centimeter squares and counted them by tens, confirming the number of squares that she had pointed out earlier.

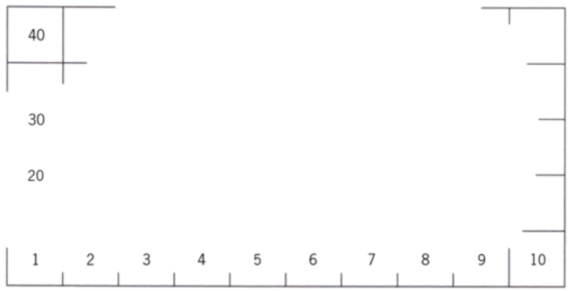

Figure 8. Squares pointed to and counted by IM in the first problem.

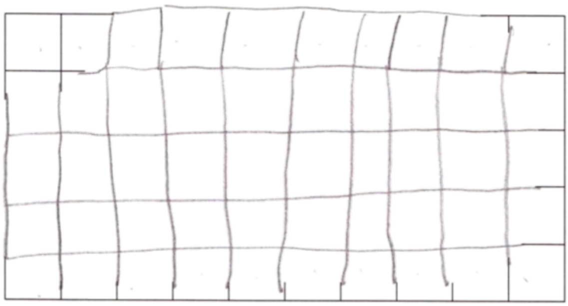

Figure 9. Squares drawn by IM in the first problem.

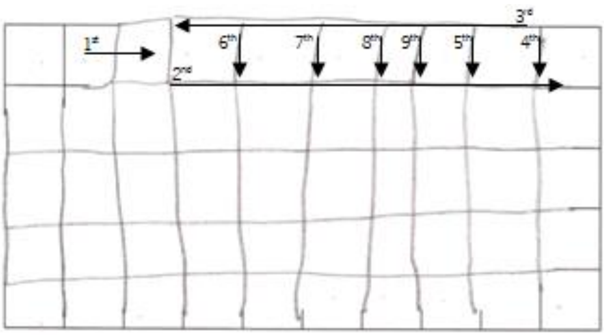

Figure 10. IM's order of drawing in the first problem (part I).

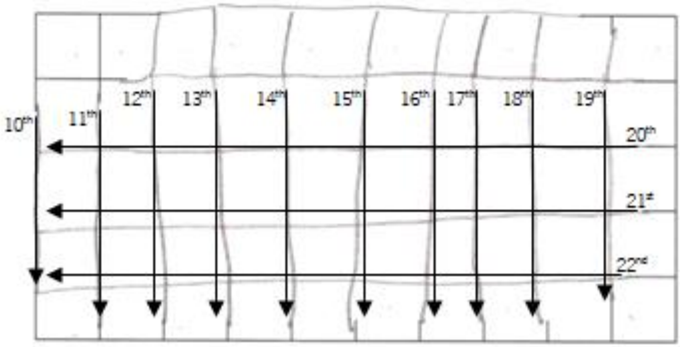

Figure 11. IM's order of drawing in the first problem (part II).

Second problem (more difficult)

To make a prediction about how many centimeter squares it would take to completely cover the rectangular array of the second problem more difficult than the first, the same student estimated 45 squares (Figure 23). During this process, IM began to determine the number of squares in the bottom row of the array. Since the array provided the perceptual material (drawn squares) along one of its diagonals, determining the number of squares in a row was more difficult than in the first problem. Therefore, she used her finger to approximately measure the space occupied by the drawn square in the lower left corner of the array and then she tried to replicate it from the left edge to the right edge. After she counted and pointed to 5 squares in the bottom row, IM tried to cover the whole array with rows of 5 squares, sliding upward her fingers along the left edge of the rectangle, while she counted 5, as she had done in the first problem. Once more IM intentionally maintained a row of 5 squares to compose the whole array, which reveals that she perceived the bottom row as a composite unit. She also ignored the dimension of a column to guide the repetition of the 5-squares-rows that she formed. Like in the first problem, she used two fingers to approximately measure the width of a row and then she tried to replicate it in each movement upward. As before, it was not evident she had related the number of rows in the array to the number of squares in a column. Afterwards, she drew where she thought the squares would be located within the array (Figure 24). As illustrated in Figure 25, she drew six vertical segments by extending lines from the squares drawn along the diagonal of the rectangle and then she drew sixteen horizontal segments in the same way. The consistent matching of both sides of the squares created vertical and horizontal alignment, so a proper representation of the array structure. Apparently, IM perceived that she could construct the array by drawing both vertical and horizontal lines. Finally, IM counted by fives the squares already drawn, making a correct prediction about how many squares it would take to completely cover the array, 50 squares. Again, IM's own counting provided further evidence that she was structuring the rectangular array into composite units (rows), although she had failed the number of rows in her initial prediction. At last, IM covered the array with plastic centimeter squares and counted them by fives, confirming the number of squares that she had pointed out earlier.

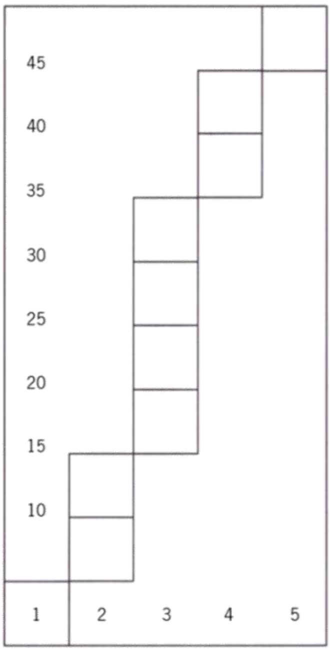

Figure 12. Squares pointed to and counted by IM in the second problem (more difficult).

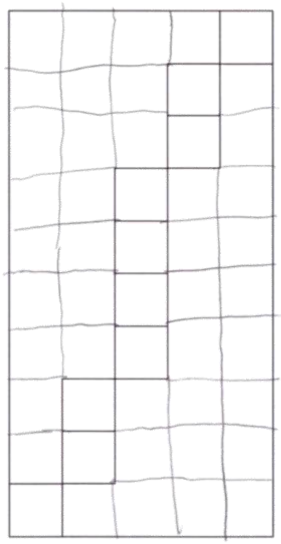

Figure 13. Squares drawn by IM in the second problem (more difficult).

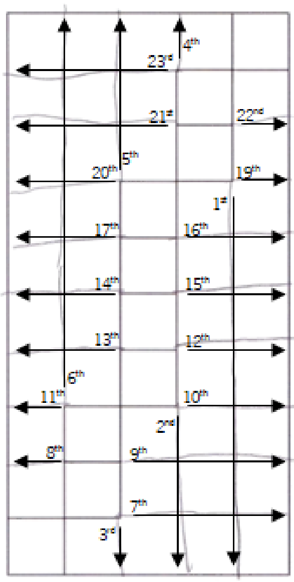

Figure 14. IM's order of drawing in the second problem (more difficult).

Levels of Sophistication Exhibited by Students

The five types of performance already presented had different occurrences. The levels of sophistication exhibited by each of the 4th and 5th grade students in the problems are presented next.

4th grade students

First problem

When applying the first problem to the 23 students in 4th grade: 16 students exhibited level 1; 2 students exhibited level 3A; 1 student exhibited level 3B; and 4 students exhibited level 3C (Figure 26).

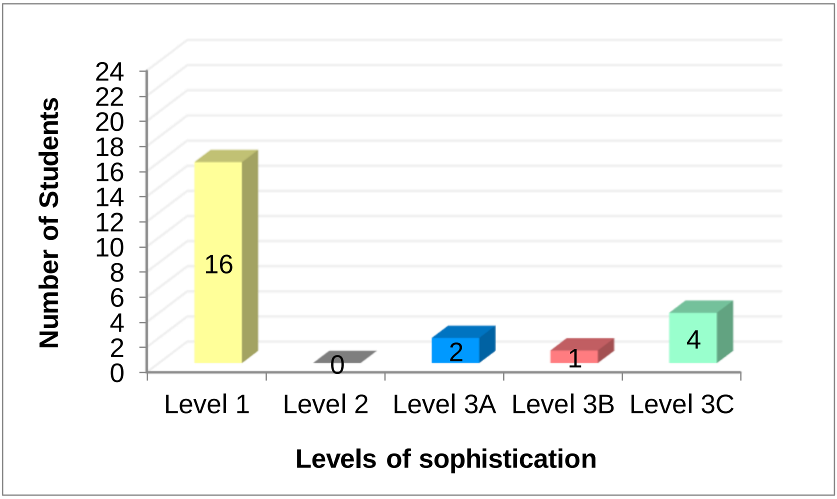

Figure 26. Levels exhibited by the 4th grade students in the first problem.

Second problem

When applying the second problem less difficult than the first to the 16 students who had exhibited level 1 in the first problem, all of these students showed an identical performance, exhibiting level 1 again (Figure 27). So, even with a problem less difficult, in which the rectangular array provided more perceptual guidance, students did not reveal changes in their performance.

When applying the second problem more difficult than the first to the 7 students who had exhibited any level except level 1 in the first problem, one of the 2 students who had exhibited level 3A showed an analogous performance, exhibiting the same level, while interestingly the other student expressed level 1; and the only student who had exhibited level 3B, as well as the 4 students who had exhibited level 3C, remained at the respective levels (Figure 28). In this sense, even with a more difficult problem, in which the rectangular array provided the perceptual material along one of its diagonals, all students remained solid at the levels that they had exhibited, except one student, who revealed a performance two levels below than he had shown in the first problem.

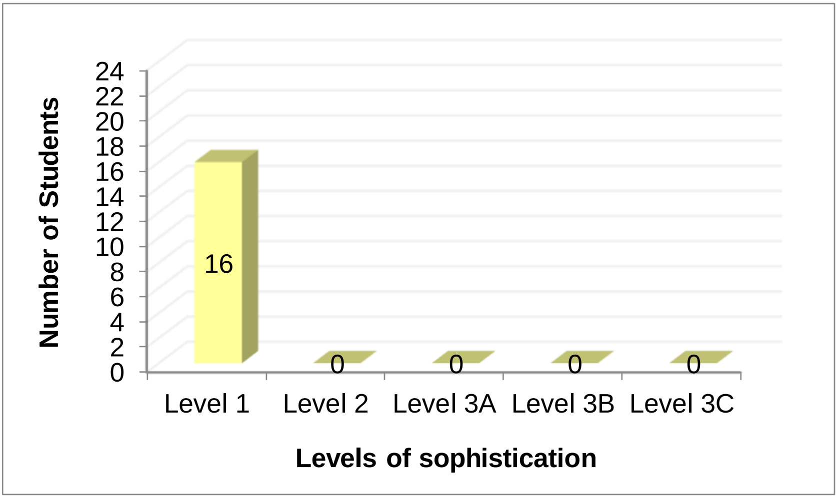

Figure 27. Levels exhibited by the 4th grade students in the second problem (less difficult).

In summary, when applying the structuring of 2D rectangular arrays problems to 4th grade students, it was found that the majority of students exhibited level 1, while the remaining students exhibited level 3A, 3B, or 3C. All 4th grade students exhibited the same level in both problems, except one student who exhibited, respectively, levels 3A-1. No student exhibited level 2 in any problem.

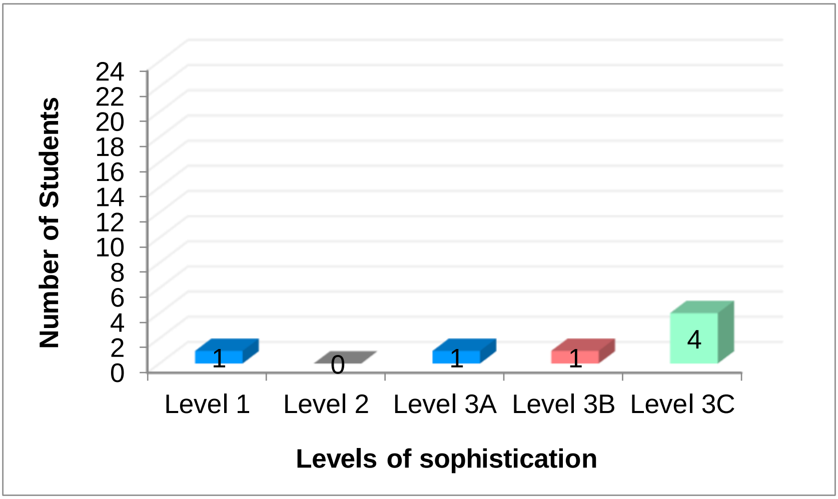

Figure 28. Levels exhibited by the 4th grade students in the second problem (more difficult).

5th grade students

First problem

When applying the first problem to the 23 students in 5th grade: 4 students exhibited level 1; 4 students exhibited level 3B; and 15 students exhibited level 3C (Figure 29).

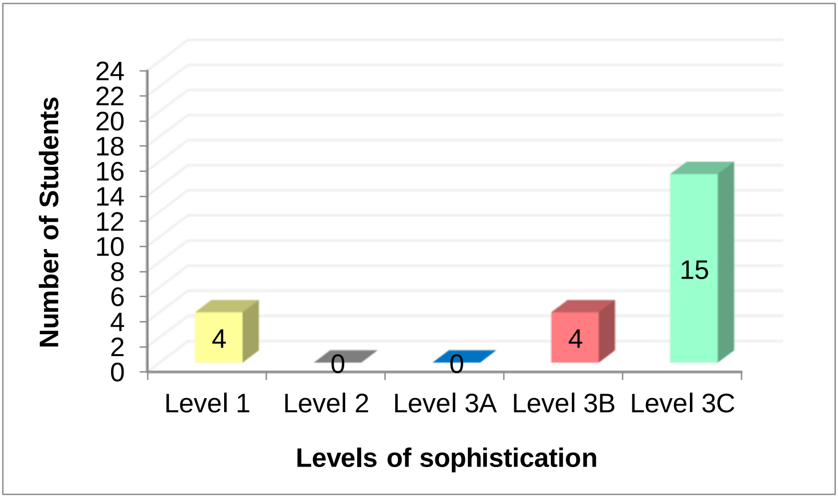

Figure 29. Levels exhibited by the 5th grade students in the first problem.

Second problem

When applying the second problem less difficult than the first to the 4 students who had exhibited level 1 in the first problem, all of these students exhibited level 1 again (Figure 30). Therefore, and similar to what was found in 4th grade, even with a problem less difficult, in which the rectangular array provided more perceptual guidance, students did not reveal changes in their performance.

When applying the second problem more difficult than the first to the 19 students who had exhibited any level except level 1 in the first problem, the performance of all students was consistent with the levels they had exhibited before (Figure 31). Thus, even with a more difficult problem, in which the rectangular array provided the perceptual material along one of its diagonals, students kept their past performance.

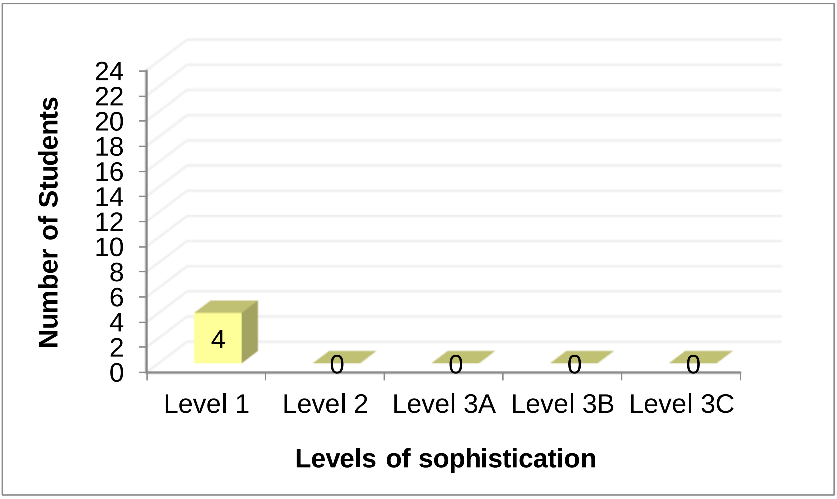

Figure 30. Levels exhibited by the 5th grade students in the second problem (less difficult).

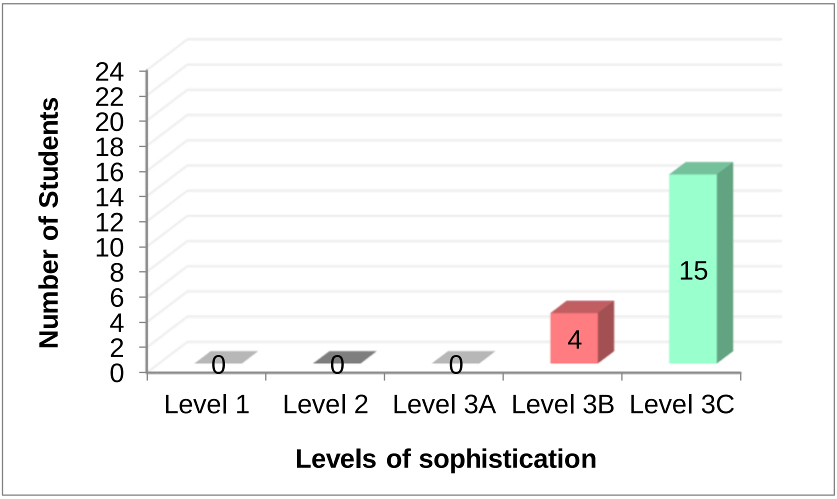

Figure 31. Levels exhibited by the 5th grade students in the second problem (more difficult).

In summary, when applying the structuring of 2D rectangular arrays problems to 5th grade students, it was found that the majority of students exhibited level 3C, while the remaining students equally exhibited level 1 or 3B. All 5th grade students exhibited the same level in both problems. No student exhibited level 2 or level 3A in any problem.

Conclusions

In this article we examined 4th and 5th grade Portuguese students' performance on problems of structuring 2D rectangular arrays of squares, which was analyzed and categorized into the five levels of sophistication described by Battista *et al.* (1998).

As students worked on the problems, they exhibited a wide range of sophistication in spatial structuring of 2D arrays. In the 4th grade class, the majority of students exhibited level 1, and the remaining students exhibited level 3A, 3B, or 3C. In the 5th grade class, the majority of students exhibited level 3C and the remaining students equally exhibited level 1 or 3B. Accordingly, in the 4th grade class, level 1 was the most frequent and, in the 5th grade class, it was level 3C. These results suggest that students in the higher-grade exhibit higher levels of sophistication in structuring 2D rectangular arrays. Moreover, the difference found in the two groups was substantial.

One probable reason for that difference is maturation, intimately connected to age. In fact, when 2D rectangular arrays problems were applied, the 23 students in 5th grade were in average 15 months older than the 23 students in 4th grade, and generally students' development of reasoning about various topics as a result of getting older. Another possible reason is instruction. As is already clear, this study has occurred prior to teaching rectangle area formula, in the 4th grade class, and prior this formula was covered again, in the 5th grade. In view of that, 5th graders had been taught already a procedure for array multiplication (formula for the area of a rectangle - length times width). Therefore, the increase in students' success on these problems with grade level could be the result of students' ability to use more efficient strategies, due to teaching and learning process. In addition, because 4th and 5th graders are not the same group of children, their individual characteristics, for instance experience, education background and mental processing, are inherent and likely contributed too for the difference found between the two groups.

The current results can be compared to those of Battista (1999). According to the author, when he gave a 2D rectangular array problem similar to those we used to students in grades 2-5, “only 19 percent of second graders, 31 percent of third graders, 54 percent of fourth graders, and 78 percent of fifth graders made correct predictions. (…) the other students made errors on this problem because of their inadequate spatial structuring.” (pp. 170-171). As we observe from these results, students in higher grades exhibit more sophisticated levels of structuring.

Most importantly in our view is that numerous students in 4th grade, and even in 5th grade, were unable to perceive the structure in rows and columns of such rectangular arrays, exhibiting level 1. This level corresponds to “the informal, preinstructional reasoning typically possessed by students” (Battista, 2004, p. 186). In this initial level, “students structure arrays as one-dimensional paths. They follow these paths as if they are traveling along a road and have no awareness of their surroundings, as if in a tunnel” (Battista *et al.*, 1998, p. 528). We found no evidence that any of these students were visualizing the row-by-column structure of the rectangular arrays provided in the two problems; we concluded that their spatial structuring of such arrays was still inadequate. Therefore, the results of this study, resembling those of Battista *et al.* (1998), reveal that the structure in rows and columns of 2D rectangular arrays of squares is not obvious in these arrays, but it must be personally constructed by each student. Clements and Sarama (2009) also affirm that, although the row-by-column organization of rectangular arrays “is taken as “obvious” by most adults, most primary grade students had not yet built up this understanding.” (p. 175).

Rectangle area formula implicitly relies on structuring rectangular arrays. Thus, it is expected that students who exhibit lower levels of sophistication in structuring 2D rectangular arrays of squares to have difficulty to meaningfully learn this formula. Some of the students in 4th and 5th grades were unable to perceive the structure in rows and columns of rectangular arrays. These students will afterwards learn rectangle area formula, or it will be revisited. They will need to understand the row-by-column structure of rectangular arrays and the use of multiplication to count the squares in these arrays, but they probably cannot make it yet. The result for their learning may be even worse if we think that for students in these grades, many textbooks display rectangular arrays as the representation to teach area measurement. In this way, the diagnosis of students' performance on problems of structuring 2D rectangular arrays of squares provide an excellent framework, not only to clarify their conceptualizations and reasoning, but also to provide them appropriate teaching strategies.

The most troubling finding is that no student exhibited level 2 in any problem. When in this level, "students decompose the paths into components, dissembled these components from the sequential organization of a 1D path and attempt to situate the components in 2D space." (Battista *et al.*, 1998, p. 529). According to Battista (2004), a set of levels of sophistication describe the major landmarks that students pass through in learning trajectories for the topic (in this case, area). However, because these levels are compilations of empirical observations of many students' thinking, "a particular student might not pass through every level for a topic; he or she might skip some levels or pass through them so quickly that the passage is difficult to detect." (Battista, 2004, p. 187). Therefore, one possible reason for the absence of level 2 could be the variability over described. But even with this variability, the complete absence of level 2 among 92 interviews accomplished on this study would be improbable. This fact impels us to consider that maybe level 2 just occurs between early year students, who possess the informal preinstructional reasoning already mentioned. In this line of thought, instruction would help students progressing from level 1 directly to level 3A or higher levels.

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