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ON THE STANDARD PART OF SOME KINDS OF TWO PARAMETER INTERNAL MARTINGALES

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Resumen. Se definen algunas filtraciones y martingalas no-estándar y se establecen algunas relaciones entre ellas y sus partes estándar con parámetros. Se presentan también algunos resultados acerca de las partes estándar de algunas clases de martingalas internas con dos parámetros.

Abstract. Some nonstandard filtrations and martingales are defined and some relations between them and the standard part of filtrations and martingales on a nonstandard, standard parameter set are established. Some results are also given about standard parts of some kinds of two parameter internal martingales.

Keywords. Nonstardard analysis, martingales, parameters.

0. Preliminaries

We will use the standard two dimensional interval $[0,1]^2$ with the partial order "$\leq$" given by

$$(s,t) \leq (s',t') \Leftrightarrow s \leq s' \text{and } \ t \leq t'.$$

$(s,t) < (s',t')$ means $(s,t) \leq (s',t')$ and $s < s'$ or $t < t'$ and $(s,t) << (s',t')$ means $s < s'$ and $t < t'$. We will write $(s,t)\underline{\triangle}(s',t')$ if $s \leq s'$ and $t \geq t'$. If $z = (s,t) \leq (s',t') = z'$, we denote with $(z,z']$ the set $\{x \in [0,1]^2 : z < x \leq z'\}$, and call it a rectangle.

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In what follows we will use the terminology and notations from nonstandard analysis as presented for example in [1]. In particular, we assume saturation is granted, as is the case when discussing stochastic processes in nonstandard analysis. We review some definitions in [9], and we also restate some results from the same source.

1. Definition. Given sub-σ-algebras $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ in a probability space $(\Omega, \mathfrak{F}, P)$, we say that $\mathfrak{A}$ and $\mathfrak{B}$ are conditionally independent given $\mathfrak{C}$, if for $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$

$$P(A \cap B|\mathfrak{C}) = P(A|\mathfrak{C})P(B|\mathfrak{C}).$$

2. Definition. An adapted two parameter probability space is a structure $\underline{\Omega} = (\Omega, \mathfrak{F}, (\mathfrak{F}_{(s,t)})_{(s,t)\in[0,1]^2}, P)$ such that $(\mathfrak{F}_{(s,t)})_{(s,t)\in[0,1]^2})$ is a family of sub-σ-algebras of $\mathfrak{F}$. We call it a two parameter filtration if the σ-algebras satisfy:

F1 : Given $(s,t) \leq (s',t')$, then $\mathfrak{F}_{(s,t)} \subseteq \mathfrak{F}_{(s',t')}$.
F2 : $\mathfrak{F}_{(0,0)}$ is P-complete.
F3 : For each (s,t), $\mathfrak{F}_{(s,t)} = \bigcap_{(s',t')>(s,t)} \mathfrak{F}_{(s',t')}$.

Additionaly we say that the filtration satisfies F4 if for $(s,t)\underline{\triangle}(s',t')$, $\mathfrak{F}_{(s,t)}$ and $\mathfrak{F}_{(s',t')}$ are conditionally independent given $\mathfrak{F}_{(s,t')}$.

Condition F4 is equivalent to each one of the following:

(a) If $(s,t)\underline{\triangle}(s',t')$ and X is a random variable, then

$$E(E(X|\mathfrak{F}_{(s,t)})|\mathfrak{F}_{(s',t')}) = E(E(X|\mathfrak{F}_{(s',t')})|\mathfrak{F}_{(s,t)}) = E(X|\mathfrak{F}_{(s,t')}).$$

(b) If $(s,t)\underline{\triangle}(s',t')$ and X is an $\mathfrak{F}_{(s',t')}$- measurable random variable, then

$$E(X|\mathfrak{F}_{(s,t)}) = E(X|\mathfrak{F}_{(s,t')}).$$

Given an internal probability space $(\Omega, \mathfrak{B}, \overline{P})$, $(\Omega, L(\mathfrak{B}), P)$ denotes the corresponding Loeb space; that is, $L(\mathfrak{B})$ is the external complete σ-algebra generated by $\mathfrak{B}$ and P is the unique σ-additive extension of $st(\overline{P})$ to $L(\mathfrak{B})$.

1. Filtrations

3. Definition.

(i) Let $L \in {}^*\mathbb{N} - \mathbb{N}$, $N = L!$, $\delta t = 1/N$. The hyperfinite line is $T = \{0, \delta t, 2\delta t, ..., (N-1)\delta t, 1\}$.

(ii) Let $\Omega = \{-1,1\}^{T^2} = \{w : T^2 \longrightarrow \{-1,1\}|\ w \text{ is internal }\}$. The internal hyperfinite cardinality of Ω is $2^{(N+1)^2}$.

(iii) Given $(\underline{s},\underline{t}) \in T^2$, we define on Ω the equivalence relation:

$$w \approx_{(\underline{s},\underline{t})} w' \Leftrightarrow w(\underline{s}',\underline{t}') = w'(\underline{s}',\underline{t}')$$

for all $(\underline{s}',\underline{t}') \leq (\underline{s},\underline{t})$, $(\underline{s}',\underline{t}') \in T^2$, where $w, w' \in \Omega$.

(iv) Using the last equivalence relation we define for $(\underline{s},\underline{t}) \in T^2$,

$$\mathfrak{B}_{(\underline{s},\underline{t})} = \{A \subseteq \Omega | A \text{ is internal and closed under } \approx_{(\underline{s},\underline{t})}\}.$$

This is an internal $^*\sigma$-algebra.

(v) An internal two parameter filtration is an internal family $(\mathfrak{B}_{(\underline{s},\underline{t})})$ with $(\underline{s},\underline{t}) \in T^2$ of internal *sub-σ-algebras of $\mathfrak{B}$ that satisfy property $\overline{F1}$ (that is, the corresponding property F1 in the nonstandard sense). The filtration is $\overline{P}$-complete if $\mathfrak{B}_{(0,0)}$ is complete.

Let $\overline{P}$ be the internal counting probability measure defined by

$$\overline{P}(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{2^{(N+1)^2}},$$

where $|A|$ denotes the internal cardinality of A.

In this paper we will always work with the internal hyperfinite probability space $\underline{\Omega} = (\Omega, (\mathfrak{B}_{(\underline{s},\underline{t})})_{(\underline{s},\underline{t}) \in T^2}, \overline{P})$ with Ω as in (ii), $(\mathfrak{B}_{(\underline{s},\underline{t})})_{(\underline{s},\underline{t}) \in T^2}$ defined as in (iv), and $\overline{P}$, the internal counting probability measure.

4. Definition. The standard part of $\{\mathfrak{B}_{(\underline{s},\underline{t})}\}$ is the filtration $\{\mathfrak{F}_{(s,t)} : (s,t) \in [0,1]^2\}$ defined by

$$\mathfrak{F}_{(s,t)} = \left(\bigcap_{^\circ(\underline{s},\underline{t}) > (s,t)} \sigma(\mathfrak{B}_{(\underline{s},\underline{t})}) \right) \bigvee \mathfrak{N}$$

for $(\underline{s},\underline{t}) \in T^2$, where $\mathfrak{N}$ is the class of P-null sets of $\mathfrak{F}$.

The standard filtration $\{\mathfrak{F}_{(s,t)}\}_{(s,t) \in [0,1]^2}$ satisfies properties F1 to F4 (see [9]).

In a two parameter stochastic analysis we use different kinds of filtrations. We want to associate to each one of them the corresponding nonstandard internal filtrations as follows:

(a) $\mathfrak{B}^1_{(\underline{s},\underline{t})} = \mathfrak{B}_{(\underline{s},1)}$ and $\mathfrak{B}^2_{(\underline{s},\underline{t})} = \mathfrak{B}_{(1,\underline{t})}$.

(b) $\mathfrak{B}^*_{(\underline{s},\underline{t})} = \mathfrak{B}_{(\underline{s},1)} \vee \mathfrak{B}_{(1,\underline{t})}$ is the smallest $^*\sigma$-algebra containing the $^*\sigma$-algebras $\mathfrak{B}^1_{(\underline{s},\underline{t})}$ and $\mathfrak{B}^2_{(\underline{s},\underline{t})}$. $\mathfrak{B}^*_{(\underline{s},\underline{t})}$ is atomic and his atoms are $[w]_{(\underline{s},\underline{t})^*} = [w]_{(\underline{s},1)} \cap [w]_{(1,\underline{t})}$.

5. Lemma. $\mathfrak{B}^*_{(\underline{s},\underline{t})}$ *does not satisfy* $\overline{F4}$.

Proof. Let $\underline{s} = (\underline{s}_1, \underline{s}_2)$, $\underline{t} = (\underline{t}_1, \underline{t}_2)$, $s \triangle t$ and $\underline{u} = (\underline{s}_1, \underline{t}_2)$, we will see that if $[w']_{\underline{s}^*} \in \mathfrak{B}^*_{\underline{s}}$, $[w'']_{\underline{t}^*} \in \mathfrak{B}^*_{\underline{t}}$ and $[w]_{\underline{u}^*} \in \mathfrak{B}^*_{\underline{u}}$, then

$$\overline{P}([w']_{\underline{s}^*} \cap [w'']_{\underline{t}^*} | [w]_{\underline{u}^*}) \neq \overline{P}([w']_{\underline{s}^*} | [w]_{\underline{u}^*}) \overline{P}([w'']_{\underline{t}^*} | [w]_{\underline{u}^*}).$$

It is enough to show that

$$\big|[w']_{\underline{s}^*} \cap [w'']_{\underline{t}^*} \cap [w]_{\underline{u}^*}\big| \neq \big|[w']_{\underline{s}^*} \cap [w]_{\underline{u}^*}\big| \cdot \big|[w'']_{\underline{t}^*} \cap [w]_{\underline{u}^*}\big| / \big|[w]_{\underline{u}^*}\big|.$$

In order to have both sides different from zero, it must be $w \approx_{(\underline{s}_1, \underline{t}_2)} w' \approx_{(\underline{s}_1, \underline{t}_2)} w''$

$$\begin{aligned}
&\big|[w']_{\underline{s}^*} \cap [w'']_{\underline{t}^*} \cap [w]_{\underline{u}^*}\big| \\
&= \Big|[w']_{(\underline{s}_1,1)} \cap [w']_{(1,\underline{s}_2)} \cap [w'']_{(\underline{t}_1,1)} \cap [w'']_{(1,\underline{t}_2)} \cap [w]_{(\underline{s}_1,1)} \cap [w]_{(1,\underline{t}_2)}\Big| \\
&= 2^{(N+1)^2(1-\underline{t}_1)(1-\underline{s}_2)} = 2^{(N+1)^2(1-\underline{s}_2-\underline{t}_1+\underline{s}_2\underline{t}_1)}.
\end{aligned}$$

In the same way, the right side is equal to

$$\begin{aligned}
&\big|[w']_{(\underline{s}_1,1)} \cap [w']_{(1,\underline{s}_2)} \cap [w]_{(\underline{s}_1,1)} \cap [w]_{(1,\underline{t}_2)}\big| \cdot \\
&\cdot \big|[w'']_{(\underline{t}_1,1)} \cap [w'']_{(1,\underline{t}_2)} \cap [w]_{(\underline{s}_1,1)} \cap [w]_{(1,\underline{t}_2)}\big| / \big|[w]_{(\underline{s}_1,1)} \cap [w]_{(1,\underline{t}_2)}\big| \\
&= 2^{(N+1)^2[(1-\underline{s}_1)(1-\underline{s}_2)+(1-\underline{t}_1)(1-\underline{t}_2)-(1-\underline{s}_1)(1-\underline{t}_2)]} \\
&= 2^{(N+1)^2(1-\underline{s}_2-\underline{t}_1+\underline{s}_1\underline{s}_2+\underline{t}_1\underline{t}_2-\underline{s}_1\underline{t}_2)},
\end{aligned}$$

and we see that the term on the right side is different from that on the left.

6. Proposition. *Let* $\{\mathfrak{F}_{(s,t)}\}$ *be the standard part of the filtration* $\{\mathfrak{B}_{(\underline{s},\underline{t})}\}$. *Then*

(a) $\mathfrak{F}^1_{(s,t)} = \mathfrak{F}_{(s,1)} = \cap_{^\circ\underline{s}>s} \sigma(\mathfrak{B}_{(\underline{s},1)}) \vee \mathfrak{N}$
$\mathfrak{F}^2_{(s,t)} = \mathfrak{F}_{(1,t)} = \bigcap_{^\circ\underline{t}>t} \sigma(\mathfrak{B}_{(1,\underline{t})}) \vee \mathfrak{N}$

(b) $\mathfrak{F}^*_{(s,t)} = [(\bigcap_{^\circ\underline{s}>s} \sigma(\mathfrak{B}_{(\underline{s},1)}) \vee \mathfrak{N}] \vee [(\cap_{^\circ\underline{t}>t} \sigma(\mathfrak{B}_{(1,\underline{t})}) \vee \mathfrak{N}]$
$= [\bigcap_{^*(\underline{s},\underline{t})>(s,t)} \sigma(\mathfrak{B}^*_{(\underline{s},\underline{t})})] \vee \mathfrak{N}$

Proof.

(a) is obvious from the definition.

(b) Let $A \in \mathfrak{F}^*_{(s,t)} = \mathfrak{F}^1_{(s,t)} \vee \mathfrak{F}^2_{(s,t)}$, i.e.,

$$A \in \left[\bigcap_{^\circ\underline{s}>s} \sigma(\mathfrak{B}_{(\underline{s},1)}) \vee \mathfrak{N}\right] \vee \left[\bigcap_{^\circ\underline{t}>t} \sigma(\mathfrak{B}_{(1,\underline{t})}) \vee \mathfrak{N}\right]$$

Then $A = (B_1 \cap B_2) \cap (C_1 \cap C_2)$ with

$$B_1 \in \bigcap_{^o\underline{s}>s} \sigma(\mathfrak{B}_{(\underline{s},1)}),\ C_1 \in \bigcap_{^o\underline{t}>t} \sigma(\mathfrak{B}_{(1,\underline{t})})$$

and $B_2,\ C_2 \in \mathfrak{N}$. This implies that $B_1 \in \sigma(\mathfrak{B}_{(\underline{s},1)})$ for all $^o\underline{s} > s$ and $C_1 \in \sigma(\mathfrak{B}_{(1,\underline{t})})$ for all $^o\underline{t} > t$, so that $B_1 \cap C_1 \in \sigma(\mathfrak{B}_{(\underline{s},1)}) \vee \sigma(\mathfrak{B}_{(1,\underline{t})})$ for all $^o(\underline{s},\underline{t}) > (s,t)$. Therefore,

$$\begin{aligned} B_1 \cap C_1 \in & \bigcap_{^o(\underline{s},\underline{t})>>(s,t)} \sigma(\mathfrak{B}_{(\underline{s},1)}) \vee \sigma(\mathfrak{B}_{(1,\underline{t})}) \\ = & \bigcap_{^o(\underline{s},\underline{t})>>(s,t)} \sigma(\mathfrak{B}_{(\underline{s},1)} \vee \mathfrak{B}_{(1,\underline{t})}). \end{aligned}$$

As $\sigma(\mathfrak{B}_{(\underline{s},1)}) \vee \sigma(\mathfrak{B}_{(1,\underline{t})}) = \sigma(\mathfrak{B}_{(\underline{s},1)} \vee \mathfrak{B}_{(1,\underline{t})})$, we also have that

$$\begin{aligned} \bigcap_{^o(\underline{s},\underline{t})>>(s,t)} \sigma(\mathfrak{B}^*_{(\underline{s},\underline{t})}) &= \bigcap_{^o\underline{s}>s} \left[\bigcap_{^o\underline{t}>t} \sigma(\mathfrak{B}_{(\underline{s},1)}) \vee \sigma(\mathfrak{B}_{(1,\underline{t})}) \right] \\ &= \bigcap_{^o\underline{t}>t} \left[\bigcap_{^o\underline{s}>s} \sigma(\mathfrak{B}_{(\underline{s},1)}) \vee \sigma(\mathfrak{B}_{(1,\underline{t})}) \right] \end{aligned}$$

7. Definition. For $\underline{t} \in T$, fixed, we define the semistandard filtration

$$\underline{\mathfrak{F}}_{(s,\underline{t})} = \left[\bigcap_{^o\underline{s}>s} \sigma(\mathfrak{B}_{(\underline{s},\underline{t})}) \right] \vee \mathfrak{N}.$$

For $\underline{s} \in T$, fixed, we similarly define

$$\underline{\mathfrak{F}}_{(\underline{s},t)} = \left[\bigcap_{^o\underline{t}>t} \sigma(\mathfrak{B}_{(\underline{s},\underline{t})}) \right] \vee \mathfrak{N}.$$

We can consider these filtrations as one parameter filtrations when we fix one of the parameters. By fixing the nonstandard parameter, it follows from results in one parameter analysis that the filtrations satisfy the usual properties, that is, they satisfy F1 to F3. But we can also consider both filtrations as two parameter filtrations. Along this line we have the following result:

8. Proposition. *Both σ-algebras introduced in Definition 8 are complete and satisfy properties F1 to F4 for the parameter sets $[0,1] \times T$ and $T \times [0,1]$, respectively.*

Proof. It is enough to show, for example, that $\{\underline{\mathfrak{F}}_{(s,\underline{t})}\}$ satisfy F1 to F4. For the other σ-algebra, the proof is similar. It is obvious that $\underline{\mathfrak{F}}_{(0,\underline{t})}$ is complete for each $\underline{t} \in T$, so we have F2.

Since $\mathfrak{B}_{(\underline{s},\underline{t})} \subseteq \mathfrak{B}_{(\underline{s}',\underline{t}')}$, for $(s,\underline{t}) \leq (s',\underline{t}')$, we have that $\underline{\mathfrak{F}}_{(s,\underline{t})} \subseteq \underline{\mathfrak{F}}_{(s',\underline{t}')}$, and F1 holds.

From the preceding argument we also have that $\underline{\mathfrak{F}}_{(s,\underline{t})} \subseteq \bigcap_{(s',\underline{t}')>>(s,\underline{t})} \underline{\mathfrak{F}}_{(s',\underline{t}')}$. Now, if $A \in \bigcap_{(s',\underline{t}')>>(s,\underline{t})} \underline{\mathfrak{F}}_{(s',\underline{t}')}$, $A \in \underline{\mathfrak{F}}_{(s',\underline{t}')}$ for all $(s',\underline{t}') >> (s,\underline{t})$, so that $A = B \cap C$ with $B \in \bigcap_{^o\underline{s}'>s'} \sigma(\mathfrak{B}_{(\underline{s}',\underline{t}')})$ for all $(s',\underline{t}') >> (s,\underline{t})$ and $C \in \mathfrak{N}$. Then $B \in \sigma(\mathfrak{B}_{(\underline{s}',\underline{t}')})$ for all $(^o\underline{s}',\underline{t}') >> (s',\underline{t}')$ whenever $(s',\underline{t}') >> (s,\underline{t})$ so that $B \in \sigma(\mathfrak{B}_{(\underline{s}',\underline{t}')})$ for all $(^o\underline{s}',\underline{t}') >> (s,\underline{t})$. Hence $A \in \underline{\mathfrak{F}}_{(s,\underline{t})}$, and F3 holds.

Now we prove F4. In fact, given $\overline{s} = (s_1,\underline{s}_2)\triangle(t_1,\underline{t}_2) = \overline{t}$, $\overline{u} = (s_1,\underline{t}_2)$, if $A \in \underline{\mathfrak{F}}_{\overline{s}}$, $B \in \underline{\mathfrak{F}}_{\overline{t}}$, I_A is $\underline{\mathfrak{F}}_{\overline{t}}$ adapted and I_B is $\underline{\mathfrak{F}}_{\overline{s}}$ adapted, there are $\underline{s} = (\underline{s}_1,\underline{s}_2) \approx \overline{s}$ and $\underline{t} = (\underline{t}_1,\underline{t}_2) \approx \overline{t}$ such that $\overline{E}(I_{\overline{A}\cap\overline{B}}|\mathfrak{B}_{(\underline{s}_1,\underline{t}_2)})$ is a lifting of $E(I_{A\cap B}|\underline{\mathfrak{F}}_{\overline{u}})$, where A and B are internal and $P(A\Delta\overline{A}) = 0$ and $P(B\Delta\overline{B}) = 0$. Then

$$\begin{aligned} E\left(I_{A\cap B}|\underline{\mathfrak{F}}_{\overline{u}}\right) &= P(A\cap B|\underline{\mathfrak{F}}_{\overline{u}}) \\ &= st\left(\overline{P}\left(\overline{A}\cap\overline{B}|\mathfrak{B}_{(\underline{s}_1,\underline{t}_2)}\right)\right) \\ &= st\left(\overline{P}(\overline{A}|\mathfrak{B}_{(\underline{s}_1,\underline{t}_2)})\cdot\overline{P}(\overline{B}|\mathfrak{B}_{(\underline{s}_1,\underline{t}_2)})\right) \\ &= st\left(\overline{P}(\overline{A}|\mathfrak{B}_{(\underline{s}_1,\underline{t}_2)})\right)\cdot st\left(\overline{P}(\overline{B}|\mathfrak{B}_{(\underline{s}_1,\underline{t}_2)})\right) \\ &= P(A|\underline{\mathfrak{F}}_{\overline{u}})\cdot P(B|\underline{\mathfrak{F}}_{\overline{u}}), \end{aligned}$$

and F4 follows.

Remark. From the definitions it follows that $\mathfrak{F}_{(s,t)} = \bigcap_{^o\underline{t}>t} \underline{\mathfrak{F}}_{(s,\underline{t})}$.

2. Standard Part of Some Kinds of Internal Martingales

First we recall some definitions and results in [9].

9. Definition. A function $x : [0,1]^2 \longrightarrow \mathbb{R}$ is a larc in $[0,1]^2$ if, for each $(s_o,t_o) \in [0,1]^2$, the quadrantal limits exist and satisfy:

$$\lim_{\substack{s\to s_o^+ \\ t\to t_o^+}} x(s,t) = x(s_o,t_o) \qquad \lim_{\substack{s\to s_o^+ \\ t\to t_o^-}} x(s,t) = x(s_o,t_o^-)$$

$$\lim_{\substack{s\to s_o^- \\ t\to t_o^+}} x(s,t) = x(s_o^-,t_o^+) \qquad \lim_{\substack{s\to s_o^- \\ t\to t_o^-}} x(s,t) = x(s_o^-,t_o^-).$$

We denote with D^2 the set of all larcs in $[0,1]^2$. In this set we can define a metric k_o such that the space (D^2, k_o) is a complete and separable metric space:

$$k_o(x,y) = \inf\left\{\epsilon \in \mathbb{R}^+ : (\exists \rho \in \Lambda[0,1]^2)(\sup_{r\in[0,1]^2} |x(r) - y(\rho(r))| < \epsilon) \wedge d(\rho) < \epsilon\right\}$$

$x, y \in D^2$, $\rho \in \Lambda[0,1]^2$, where $\Lambda[0,1]^2$ is the set of deformations of $[0,1]^2$. We denote with $\mathcal{J}_2$ the topology induced by this metric.

For each point $(\underline{s},\underline{t}) \in {}^*[0,1]^2$ let us consider the following sets

$$Q^1_{(\underline{s},\underline{t})} = \{(\underline{u},\underline{v}) \in {}^*[0,1]^2 : \underline{u} \geq \underline{s} \text{ and } \underline{v} \geq \underline{t}\}$$
$$Q^2_{(\underline{s},\underline{t})} = \{(\underline{u},\underline{v}) \in {}^*[0,1]^2 : \underline{u} < \underline{s} \text{ and } \underline{v} \geq \underline{t}\}$$
$$Q^3_{(\underline{s},\underline{t})} = \{(\underline{u},\underline{v}) \in {}^*[0,1]^2 : \underline{u} < \underline{s} \text{ and } \underline{v} < \underline{t}\}$$
$$Q^4_{(\underline{s},\underline{t})} = \{(\underline{u},\underline{v}) \in {}^*[0,1]^2 : \underline{u} \geq \underline{s} \text{ and } \underline{v} < \underline{t}\}.$$

10. Definition. Let $F \in {}^*D^2$ be such that $F(\underline{s},\underline{t}) \in ns({}^*\mathbb{R})$ for $(\underline{s},\underline{t}) \in {}^*[0,1]^2$. We say that

(a) F is of class SD^2, if for each $(s,t) \in [0,1]^2$ there are points $(\underline{s}_1,\underline{t}_1) \approx (\underline{s}_2,\underline{t}_2) \approx (\underline{s}_3,\underline{t}_3) \approx (\underline{s}_4,\underline{t}_4) \approx (s,t)$ such that:
  i) If $(\underline{u}_1,\underline{v}_1) \approx (s,t), (\underline{u}_1,\underline{v}_1) \in Q^1_{(\underline{s}_1,\underline{t}_1)}$, then $F(\underline{u}_1,\underline{v}_1) \approx F(\underline{s}_1,\underline{t}_1)$
  ii) If $(\underline{u}_2,\underline{v}_2) \approx (s,t), (\underline{u}_2,\underline{v}_2) \in Q^2_{(\underline{s}_2,\underline{t}_2)}$, then $F(\underline{u}_2,\underline{v}_2) \approx F(\underline{s}_2^-,\underline{t}_2)$
  iii) If $(\underline{u}_3,\underline{v}_3) \approx (s,t), (\underline{u}_3,\underline{v}_3) \in Q^3_{(\underline{s}_3,\underline{t}_3)}$, then $F(\underline{u}_3,\underline{v}_3) \approx F(\underline{s}_3^-,\underline{t}_3^-)$
  iv) If $(\underline{u}_4,\underline{v}_4) \approx (s,t), (\underline{u}_4,\underline{v}_4) \in Q^4_{(\underline{s}_4,\underline{t}_4)}$, then $F(\underline{u}_4,\underline{v}_4) \approx F(\underline{s}_4,\underline{t}_4^-)$.

(b) F is of class SD^2J, or a larc lift, if (a) holds with $(\underline{s}_1,\underline{t}_1) = (\underline{s}_2,\underline{t}_2) = (\underline{s}_3,\underline{t}_3) = (\underline{s}_4,\underline{t}_4)$, and $F(\underline{s},\underline{t}) \approx F(0,0) \quad \forall(\underline{s},\underline{t}) \approx (0,0)$ in ${}^*[0,1]^2$.

A function $F : T^2 \longrightarrow {}^*\mathbb{R}$ is of class SD^2 (SD^2J) in T^2 if it is the restriction to T^2 of an SD^2 (SD^2J) function F on ${}^*[0,1]^2$

11. Definition. The standard part of an SD^2 function F on T^2 is the function $st(F)$ defined by

$$st(F)(s,t) = \lim_{{}^o(\underline{s},\underline{t})\downarrow(s,t)} {}^oF(\underline{s},\underline{t}), \qquad (\underline{s},\underline{t}) \in T^2.$$

Remark. The class of functions in ${}^*D^2$ which are nearstandard in the $\mathcal{J}_2$ topology is SD^2J, and $st|_{SD^2J}$ is the standard part map for the $\mathcal{J}_2$ topology.

12. Proposition. *Suppose that $F : T^2 \longrightarrow {}^*\mathbb{R}$ is the restriction of a function in ${}^*D^2$ to T^2 and that $F(\underline{s},\underline{t}) \in ns({}^*\mathbb{R})$ for all $(\underline{s},\underline{t}) \in T^2$. Then F is SD^2 if and only if $st(F)$ exists and belongs to D^2.*

13. Theorem. *If $X : T^2 \times \Omega \longrightarrow {}^*\mathbb{R}$ is an internal map of class SD^2, then there is a positive infinitesimal $\Delta' t \in T$ such that if $T' = \{k\Delta' t : k \in {}^*\mathbb{N},\ k\Delta' \leq 1\} \cup \{1\}$ then $X|_{(T')^2\times\Omega}$ is of class SD^2J.*

14. Definition. An internal stochastic process X is of class SD^2 (SD^2J) if, for almost all w, the mapping $X((\cdot,\cdot),w) : T^2 \longrightarrow {}^*\mathbb{R}$ is of class SD^2 (SD^2J).

If X is SD^2, a process $st(X)$ with sample paths in D^2 is defined by fixing $x_o \in \mathbb{R}$ and requiring that

$$st(X)(s,t) = \begin{cases} st(X)(\cdot,\cdot),w)(s,t), & \text{if } X((\cdot,\cdot),w) \in SD^2 \\ x_o, & \text{otherwise} \end{cases}$$

15. Definition. Let $\{\mathfrak{B}_{(\underline{s},\underline{t})} : (\underline{s},\underline{t}) \in T^2\}$ be an internal filtration satisfying $\overline{F1}$ to $\overline{F4}$. Then

(i) An internal stochastic process $X : T^2 \times \Omega \longrightarrow {}^*\mathbb{R}$ is a $\mathfrak{B}_{(\underline{s},\underline{t})}$- martingale if $\{(X(\underline{s},\underline{t}), \mathfrak{B}_{(\underline{s},\underline{t})}) : (\underline{s},\underline{t}) \in T^2\}$ is an internal martingale, i.e., if $X(\underline{s},\underline{t})$ is $\mathfrak{B}_{(\underline{s},\underline{t})}$ adapted and

$$\overline{E}\left(X(\underline{s}_2,\underline{t}_2)|\mathfrak{B}_{(\underline{s}_1,\underline{t}_1)}\right) = X(\underline{s}_1,\underline{t}_1) \quad \overline{P}\text{-a.s.}$$

whenever $(\underline{s}_1,\underline{t}_1) \leq (\underline{s}_2,\underline{t}_2)$

(ii) X is an S- martingale with respect to $\{\mathfrak{B}_{(\underline{s},\underline{t})}\}$ if X is a $\mathfrak{B}_{(\underline{s},\underline{t})}$- martingale and $|X(\underline{s},\underline{t})|^p$ is S- integrable for all $(\underline{s},\underline{t}) \in T^2$, $p \geq 0$.

(iii) X is a $*$- martingale after Δt for $\Delta t \approx 0$, $\Delta t \in T$, if

$$\overline{E}\left(X(\underline{s}_2,\underline{t}_2)|\mathfrak{B}_{(\underline{s}_1,\underline{t}_1)}\right) = X(\underline{s}_1,\underline{t}_1) \quad \overline{P}\text{-a.s.}$$

whenever $(\underline{s}_1,\underline{t}_1) \leq (\underline{s}_2,\underline{t}_2)$, $(\underline{s}_1,\underline{t}_1),(\underline{s}_2,\underline{t}_2) \in (T')^2$, where $T' = \{k\Delta t : k \in {}^*\mathbb{N}, k\Delta t < 1\} \cup \{1\}$.

(iv) X is a Δt- martingale for some $\Delta t \in T$, $\Delta t \approx 0$, if X is SD^2J, S-integrable for all $(\underline{s},\underline{t}) \in (T')^2$ and a $*$- martingale after Δt.

Remark. From theorem 14 it can inferred that if X is an S-martingale and X is SD^2, there exists an infinitesimal $\Delta t \in T$ such that X is a Δt- martingale.

16. Definition. Let $\{\mathfrak{F}_{(s,t)} : (s,t) \in [0,1]^2\}$ be the standard part of $\{\mathfrak{B}_{(\underline{s},\underline{t})} : (\underline{s},\underline{t}) \in T^2\}$. A stochastic process $x : [0,1]^2 \times \Omega \longrightarrow \mathbb{R}$ is an $\{\mathfrak{F}_{(s,t)}\}$- larcmartingale if it is $\mathfrak{F}_{(s,t)}$- adapted, p-uniformly integrable for some $p \geq 1$, $x((\cdot,\cdot),w) \in D^2$ a.s., and for $(s,t) \leq (u,v)$,

$$E\left(x(u,v)|\mathfrak{F}_{(s,t)}\right) = x(s,t) \quad P\text{-a.s}$$

17. Theorem. *If X is a Δt- martingale, then $st(X) = x$ is a larcmartingale.*

18. Definition. $X(\underline{s},\underline{t})$ is an internal 1-martingale if $X(\underline{s},0)$ is a one parameter internal martingale with respect to $\mathfrak{B}^1_{(\underline{s},0)}$, $X(\underline{s},\underline{t})$ is $\mathfrak{B}^1_{(\underline{s},\underline{t})}$- adapted and $\overline{E}(X(R)|\mathfrak{B}^1_{(\underline{s},\underline{t})}) = 0$, for each rectangle $R = ((\underline{s},\underline{t}),(\underline{s}',\underline{t}')]$, where $X(R)$ denotes the increment $X(\underline{s}',\underline{t}') - X(\underline{s}',\underline{t}) - X(\underline{s},\underline{t}') + X(\underline{s},\underline{t})$.

We have a corresponding definition for an internal 2-martingale.

19. Proposition. *$X(\underline{s},\underline{t})$ is an internal 1-martingale if and only if for each fixed $\underline{t}$, $X(\underline{s},\underline{t})$ is a one parameter $\mathfrak{B}_{(\underline{s},\underline{t})}$ internal martingale. (A similar statement holds for an internal 2-martingale).*

Proof. Let $X(\underline{s},\underline{t})$ be an internal 1-martingale. Since $\mathfrak{B}_{(\underline{s},\underline{t})}$ satisfies $\overline{F4}$, given the rectangle $A = ((\underline{s},0),(\underline{s}+\underline{h},t)]$ we have $\mathfrak{B}_{(\underline{s},\underline{t})} \subseteq \mathfrak{B}_{(\underline{s},1)} = \mathfrak{B}^1_{(\underline{s},0)}$ and $\overline{E}(X(A)|\mathfrak{B}_{(\underline{s},o)}) = 0$, thus

$$\begin{aligned}
&\overline{E}(X(A)|\mathfrak{B}_{(\underline{s},\underline{t})}) = \\
&= \overline{E}\left(\overline{E}\left([X(\underline{s}+\underline{h},\underline{t}) - X(\underline{s}+\underline{h},0) - X(\underline{s},\underline{t}) + X(\underline{s},0)]|\mathfrak{B}_{(\underline{s},1)}\right)|\mathfrak{B}_{(\underline{s},\underline{t})}\right) = 0 \\
&= \overline{E}([X(\underline{s}+\underline{h},\underline{t})) - X(\underline{s},\underline{t})]|\mathfrak{B}_{(\underline{s},\underline{t})}) - \overline{E}\left(\overline{E}([X(\underline{s}+\underline{h},0) - X(\underline{s},0)]|\mathfrak{B}_{(\underline{s},1)})|\mathfrak{B}_{(\underline{s},\underline{t})}\right) \\
&= \overline{E}\left([X(\underline{s}+\underline{h},\underline{t})) - X(\underline{s},\underline{t})]|\mathfrak{B}_{(\underline{s},\underline{t})}\right).
\end{aligned}$$

The last equality holds because $X(\underline{s},0)$ is a one parameter internal martingale with respect to $\mathfrak{B}^1_{(\underline{s},0)} = \mathfrak{B}_{(\underline{s},1)}$.

On the other hand, if for each fixed $\underline{t}$, $X(\underline{s},\underline{t})$ is a one parameter $\mathfrak{B}_{(\underline{s},\underline{t})}$ internal martingale, we have first of all that $X(\underline{s},\underline{t})$ is $\mathfrak{B}_{(\underline{s},\underline{t})}$ adapted. Since $\overline{F4}$ holds, given $A = ((\underline{s},\underline{t}),(\underline{s}',\underline{t}')]$ we then have

$$\begin{aligned}
&\overline{E}\left(X(A)|\mathfrak{B}^1_{(\underline{s},\underline{t})}\right) = \overline{E}\left([X(\underline{s}',\underline{t}') - X(\underline{s}',\underline{t}) - X(\underline{s},\underline{t}') + X(\underline{s},\underline{t})]|\mathfrak{B}_{(\underline{s},1)}\right) \\
&= \overline{E}\left([X(\underline{s}',\underline{t}') - X(\underline{s},\underline{t}')]|\mathfrak{B}_{(\underline{s},1)}\right) - \overline{E}\left([X(\underline{s}',\underline{t}) - X(\underline{s},\underline{t})]|\mathfrak{B}_{(\underline{s},1)}\right) \\
&= \overline{E}\left([X(\underline{s}',\underline{t}') - X(\underline{s},\underline{t}')]|\mathfrak{B}_{(\underline{s},\underline{t}')}\right) - \overline{E}\left([X(\underline{s}',\underline{t}) - X(\underline{s},\underline{t})]|\mathfrak{B}_{(\underline{s},\underline{t})}\right) = 0.
\end{aligned}$$

For fixed $\underline{t}$ it follows from Proposition 20 that we can see an internal 1-martingale as a one parameter internal martingale for $\underline{s}$, thus $X(\underline{s},\underline{t})$ is SD with respect to $\underline{s}$, and therefore there exists $\Delta t \in T$ such that $X|_{T' \times T}$ is an SDJ martingale on $\underline{s}$ with $T' = \{k\Delta t : k\Delta t \in T\} \cup \{1\}$. Hence, we can define the following stochastic process:

$$\underline{st}(X)(s,\underline{t}) = \lim_{{}^o\underline{s}\downarrow s} {}^oX(\underline{s},\underline{t}).$$

We also immediately have that if $X(\underline{s},\underline{t})$ is an S-integrable 1- martingale, $\underline{st}(X)(s,\underline{t})$ is a one parameter cad-lag martingale with respect to $\{\mathfrak{F}_{(s,\underline{t})}\}$ (cad-lag means that is continuous from the right and has limits from the left). See [6]. The same holds for 2-martingales.

20. Theorem. *If $\{X(\underline{s},\underline{t})\}$ is an internal $\mathfrak{B}_{(\underline{s},\underline{t})}$-martingale and $\overline{E}(|X(1,1)|) < +\infty$, then X is SD^2.*

The proof is in [9], Theorem 2.2.27.

21. Theorem. *If $X(\underline{s},\underline{t})$ is an internal martingale, for each fixed s, $\underline{st}(X)(s,\underline{t})$ is an $\underline{\mathfrak{F}}_{(s,\underline{t})}$ martingale, and $\lim_{{}^o\underline{t}\downarrow t} \underline{st}(X)(s,\underline{t})$ exists. Also, for each fixed t, $\underline{st}(X)(\underline{s},t)$ is an $\underline{\mathfrak{F}}_{(\underline{s},t)}$ martingale, $\lim_{{}^o\underline{s}\downarrow s} \underline{st}(X)(\underline{s},t)$ exists and*

$$st(X)(s,t) = \lim_{{}^o\underline{t}\downarrow t}[\lim_{{}^o\underline{s}\downarrow s} {}^oX(\underline{s},\underline{t})] = \lim_{{}^o\underline{s}\downarrow s}[\lim_{{}^o\underline{t}\downarrow t} {}^oX(\underline{s},\underline{t})].$$

Proof. First we prove that, for each fixed s, $\underline{st}(X)(s,\underline{t})$ is an $\underline{\mathfrak{F}}_{(s,\underline{t})}$ martingale. We must show that

$$E\left([\underline{st}(X)(s,\underline{t}+\underline{h}) - \underline{st}(X)(s,\underline{t})]\,|\underline{\mathfrak{F}}_{(s,\underline{t})}\right) = 0.$$

Given s there exist $\underline{s_1}$, $\underline{s_2}$, $\underline{s_1} \approx \underline{s_2} \approx s$ such that for $\underline{u} \approx s$ and $\underline{u} \geq \underline{s_1}$, $X(\underline{u},\underline{t}+\underline{h}) \approx X(\underline{s_1},\underline{t}+\underline{h})$, and for $\underline{u} \approx s$ and $\underline{u} \geq \underline{s_2}$, $X(\underline{u},\underline{t}) \approx X(\underline{s_2},\underline{t})$. Let us take $\underline{s}' = \max\{\underline{s_1},\underline{s_2}\}$, so that for $\underline{u} \approx s$ and $\underline{u} \geq \underline{s}'$, $X(\underline{u},\underline{t}+\underline{h}) \approx X(\underline{s}',\underline{t}+\underline{h})$, $X(\underline{u},\underline{t}) \approx X(\underline{s}',\underline{t})$, $\underline{st}(X)(s,\underline{t}+\underline{h}) = {}^oX(\underline{s}',\underline{t}+\underline{h})$ and $\underline{st}(X)(s,\underline{t}) = {}^oX(\underline{s}',\underline{t})$. Thus,

$$\begin{aligned}
&E\left([\underline{st}(X)(s,\underline{t}+\underline{h}) - \underline{st}(X)(s,\underline{t})]\,|\underline{\mathfrak{F}}_{(s,\underline{t})}\right)\\
&= E\left({}^oX(\underline{s}',\underline{t}+\underline{h}) - {}^oX(\underline{s}',\underline{t})|\underline{\mathfrak{F}}_{(s,\underline{t})}\right)\\
&= E\left(E\left({}^oX(\underline{s}',\underline{t}+\underline{h}) - {}^oX(\underline{s}',\underline{t})|\sigma(\mathfrak{B}_{(\underline{s}'',\underline{t})})\right)|\underline{\mathfrak{F}}_{(s,\underline{t})}\right)\\
&= E\left({}^o\left(\overline{E}(X(\underline{s}',\underline{t}+\underline{h}) - X((\underline{s}',\underline{t})|\mathfrak{B}_{(\underline{s}'',\underline{t})})\right)|\underline{\mathfrak{F}}_{(s,\underline{t})}\right)
\end{aligned}$$

for all $\underline{s}''$ with ${}^o\underline{s}'' > s$. Now, $X(\underline{s}',\underline{t})$ is $\mathfrak{B}_{(\underline{s}',\underline{t})}$ adapted; then it is $\mathfrak{B}_{(\underline{s}'',\underline{t})}$ adapted, and we also have from $F4$ that

$$\overline{E}(X(\underline{s}',\underline{t}+\underline{h})|\mathfrak{B}_{(\underline{s}'',\underline{t})}) = \overline{E}(X(\underline{s}',\underline{t}+\underline{h})|\mathfrak{B}_{(\underline{s}',\underline{t})}),$$

and finally that

$$\begin{aligned}
&\overline{E}\left(X(\underline{s}',\underline{t}+\underline{h}) - X(\underline{s}',\underline{t})|\mathfrak{B}_{(\underline{s}'',\underline{t})}\right)\\
&= \overline{E}\left(X(\underline{s}',\underline{t}+\underline{h})|\mathfrak{B}_{(\underline{s}'',\underline{t})}\right) - \overline{E}\left(X(\underline{s}',\underline{t})|\mathfrak{B}_{(\underline{s}',\underline{t})}\right)\\
&= \overline{E}\left(X(\underline{s}',\underline{t}+\underline{h})|\mathfrak{B}_{(\underline{s}'',\underline{t})}\right) - X(\underline{s}',\underline{t})\\
&= \overline{E}\left(X(\underline{s}',\underline{t}+\underline{h}) - X(\underline{s}',\underline{t})|\mathfrak{B}_{(\underline{s}',\underline{t})}\right) = 0.
\end{aligned}$$

Hence

$$E({}^{o}X(\underline{s}',\underline{t}+\underline{h}) - {}^{o}X(\underline{s}',\underline{t})|\underline{\mathfrak{F}}_{(s,\underline{t})}) = 0,$$

and then

$$E\left([\underline{st}(X)(s,\underline{t}+\underline{h}) - \underline{st}(X)(s,\underline{t})]\,|\underline{\mathfrak{F}}_{(s,\underline{t})}\right) = 0.$$

Now let $\epsilon > 0$. There exists $\delta' > 0$ such that if $(\underline{s},\underline{t}) \in T^2$ and $(s,t) << {}^{o}(\underline{s},\underline{t}) << (s+\delta', t+\delta')$ then

$$|{}^{o}X(\underline{s},\underline{t}) - st(X)(s,t)| < \epsilon/2.$$

For each $\underline{t}$ also exists $\delta'' > 0$ such that for $\underline{s} \in T$ and $s < {}^{o}\underline{s} < s+\delta''$,

$$|{}^{o}X(\underline{s},\underline{t}) - \underline{st}(X)(s,\underline{t})| < \epsilon/2.$$

Let $\delta = \min\{\delta', \delta''\}$. If $t < {}^{o}\underline{t} < t+\delta$, we choose for this $\underline{t}$ an $\underline{s}$, such that $s < {}^{o}\underline{s} < s+\delta$, and so

$$|\underline{st}(X)(s,\underline{t}) - st(X)(s,t)| \leq |\underline{st}(X)(s,\underline{t}) - {}^{o}X(\underline{s},\underline{t})| + |{}^{o}X(\underline{s},\underline{t}) - st(X)(s,t)| < \epsilon.$$

Then we have that

$$st(X)(s,t) = \lim_{{}^{o}\underline{t}\downarrow t}\left[\lim_{{}^{o}\underline{s}\downarrow s} {}^{o}X(\underline{s},\underline{t})\right]$$

and the same holds for the other parameter.

22. Theorem. *If $X(\underline{s},\underline{t})$ is an internal S-integrable SD^2 i-martingale (SD^2J i-local martingale), then $st(X)(s,t)$ is a i-larcmartingale (i-local larcmartingale), $i = 1,2$.*

Proof. The proof for the local martingales follows from the proof for martingales and the properties of the one parameter local martingales. Suppose that $X(\underline{s},\underline{t})$ is an internal S-integrable SD^2 1-martingale. If $X(\underline{s},\underline{t})$ is S-integrable, we have that $st(X)(s,t)$ is uniformly integrable, and $X(\underline{s},\underline{t})$ being SD^2, we have:

(1) There exists a $\Delta t \in T$ such that $X(\underline{s},\underline{t})$ is SD^2J for $(\underline{s},\underline{t}) \in (T')^2$, then $st(X)(s,t)$ is a larc.
(2) For $(s,t) \in [0,1]^2$, $\lim_{{}^{o}(\underline{s},\underline{t})\downarrow(s,t)} {}^{o}X(\underline{s},\underline{t})$ exists.
(3) For each $\underline{t} \in T$, $\lim_{{}^{o}\underline{s}\downarrow s} {}^{o}X(\underline{s},\underline{t})$ exists.
(4) Similarly as in the proof of Theorem 20 we can show that

$$st(X)(s,t) = \lim_{{}^{o}(\underline{s},\underline{t})\downarrow(s,t)} {}^{o}X(\underline{s},\underline{t}) = \lim_{{}^{o}\underline{t}\downarrow t}\left[\lim_{{}^{o}\underline{s}\downarrow s} {}^{o}X(\underline{s},\underline{t})\right]$$

and furthermore we have

$$\begin{aligned}\lim_{{}^o\underline{t}\downarrow t}\left[\lim_{{}^o\underline{s}\downarrow s} {}^oX(\underline{s},\underline{t})\right] &= \lim_{{}^o\underline{t}\downarrow t}\left[\lim_{{}^o\underline{s}\downarrow s} {}^o\overline{E}\left(X(1,\underline{t})|\mathfrak{B}_{(\underline{s},\underline{t})}\right)\right]\\ &= \lim_{{}^o\underline{t}\downarrow t}\left[\lim_{{}^o\underline{s}\downarrow s} {}^o\overline{E}\left(X(1,\underline{t})|\mathfrak{B}_{(\underline{s},1)}\right)\right]\\ &= \lim_{{}^o\underline{t}\downarrow t}\left[\lim_{{}^o\underline{s}\downarrow s} E\left({}^oX(1,\underline{t})|\sigma(\mathfrak{B}_{(\underline{s},1)})\right)\right]\\ &= \lim_{{}^o\underline{t}\downarrow t} E\left({}^oX(1,\underline{t})|\mathfrak{F}_{(s,1)}\right).\end{aligned}$$

That the last equality holds, follows from the reverse martingale Theorem. Now let $\{t_n\}$ be a decreasing sequence in $[0,1]$ that converges to t. Then

$$\begin{aligned}E\left(st(X)(1,t)|\mathfrak{F}_{(s,1)}\right) &= E\left(\liminf_{t_n\downarrow t} {}^oX(1,\underline{t})|\mathfrak{F}_{(s,1)}\right)\\ &\leq \liminf_{t_n\downarrow t} E\left({}^oX(1,\underline{t})|\mathfrak{F}_{(s,1)}\right)\\ &\leq \limsup_{t_n\downarrow t} E\left({}^oX(1,\underline{t})|\mathfrak{F}_{(s,1)}\right)\\ &\leq E\left(\limsup_{t_n\downarrow t} {}^oX(1,\underline{t})|\mathfrak{F}_{(s,1)}\right)\\ &= E\left(st(X)(1,t)|\mathfrak{F}_{(s,1)}\right).\end{aligned}$$

Finally we have that $st(X)(s,t) = E(st(X)(1,t)|\mathfrak{F}_{(s,1)}) = E(st(X)(1,t)|\mathfrak{F}_{(s,t)})$, and therefore $st(X)$ is a 1-martingale.

23. Definition. $X(\underline{s},\underline{t})$ is an internal weak martingale if $X(\underline{s},\underline{t})$ is $\mathfrak{B}_{(\underline{s},\underline{t})}$ adapted and for a rectangle $R = ((\underline{s},\underline{t}),(\underline{s}',\underline{t}')]$ with $(\underline{s},\underline{t}) < (\underline{s}',\underline{t}')$ we have

$$\overline{E}(X(R)|\mathfrak{B}_{(\underline{s},\underline{t})}) = 0.$$

Remark. It holds (Wong- Zakai [11]) that $X(\underline{s},\underline{t})$ is an internal weak martingale if and only if $X(\underline{s},\underline{t}) = M^1(\underline{s},\underline{t}) + M^2(\underline{s},\underline{t})$, where $M^1(\underline{s},\underline{t})$ is an internal 1-martingale and $M^2(\underline{s},\underline{t})$ is an internal 2-martingale.

24. Theorem. *If $X(\underline{s},\underline{t})$ is an internal S- integrable and SD^2 weak martingale then $st(X)(s,t)$ is a weak larc martingale.*

Proof. It follows at once from the remark above, the previous theorem and the fact that every internal i-martingale is an internal weak martingale.

25. Definition. $X(\underline{s},\underline{t})$ is an internal strong martingale if it is $\mathfrak{B}_{(\underline{s},\underline{t})}$ adapted, $X(0,\underline{t}) = 0 = X(\underline{s},0)$, and for each rectangle $\overline{R} = ((\underline{s},\underline{t}),(\underline{s}',\underline{t}')]$ with $(\underline{s},\underline{t}) < (\underline{s}',\underline{t}')$,

$$\overline{E}(X(\overline{R})|\mathfrak{B}^*_{(\underline{s},\underline{t})}) = 0.$$

26. Theorem. *If $X(\underline{s},\underline{t})$ is an internal strong martingale that is S-integrable, then $st(X)(s,t)$ is a strong larc martingale.*

Proof. An internal S-integrable strong martingale is an internal S-integrable martingale (because it is an internal 1- and 2-martingale). Then $X(\underline{s},\underline{t})$ is SD^2 by Theorem 2.2.27, and therefore is SD^2J restricted to $(T')^2$ for some $\Delta t \approx 0$ in T'. So, we have that $st(X)(s,t)$ exists a.s. for each (s,t) and is a larc. Denote with $x(s,t)$ the process $st(X)(s,t)$.

Given $R = ((s,t),(s',t')]$, from property SD^2J we can find $(\underline{u}_1,\underline{v}_1)$ and $(\underline{u}_2,\underline{v}_2)$ in $(T')^2$, $(\underline{u}_1,\underline{v}_1) \approx (s,t)$ and $(\underline{u}_2,\underline{v}_2) \approx (s',t')$, such that if $\overline{R} = ((\underline{s},\underline{t}),(\underline{s}',\underline{t}')]$ with $(\underline{s},\underline{t}) \geq (\underline{u}_1,\underline{v}_1)$, $(\underline{s},\underline{t}) \approx (s,t)$, $(\underline{s}',\underline{t}') \geq (\underline{u}_2,\underline{v}_2)$, and $(\underline{s}',\underline{t}') \approx (s,t)$, then ${}^oX(\overline{R}) = x(R)$.

Let $U \in \mathfrak{F}^*_{(s,t)}$. Given a sequence $\{(\underline{s}_n,\underline{t}_n)\}$ in $(T')^2$ such that $(\underline{u}_1,\underline{v}_1) << (\underline{s}_n,\underline{t}_n)$, and $0 < {}^o(\underline{s}_n,\underline{t}_n) - (s,t) < (1/n,1/n)$, for each $n \in \mathbb{N}$, , there exists, as $U \in \sigma(\mathfrak{B}^*_{(\underline{s}_n,\underline{t}_n)})$, an internal set $\overline{U}_n \in \mathfrak{B}^*_{(\underline{s}_n,\underline{t}_n)}$ such that ${}^oI_{\overline{U}_n} = I_U$ a.s. ($I_A(s,t) = 1$ if $(s,t) \in A$ and $I_A(s,t) = 0$ if $(s,t) \notin A$). By saturation there also exists $\nu \in {}^*\mathbb{N} - \mathbb{N}$ and a $\overline{U}_\nu \in \mathfrak{B}_{(\underline{s}_\nu,\underline{t}_\nu)}$ such that $(\underline{u}_1,\underline{v}_1) \leq (\underline{s}_\nu,\underline{t}_\nu)$, $(\underline{s}_\nu,\underline{t}_\nu) \approx (s,t)$ and ${}^oI_{\overline{U}_\nu} = I_U$ a.s, and then we have $P(U\Delta\overline{U}_\nu) = 0$. Take $\overline{R} = ((\underline{s}_\nu,\underline{t}_\nu),(\underline{u}_2,\underline{v}_2)]$. Then ${}^oX(\overline{R}) = x(R)$ and

$$\int_U E(x(R)|\mathfrak{F}^*_{(s,t)})dP = \int_U x(R)dP = \int_U {}^oX(\overline{R})dP$$
$$= {}^o\left(\int_{\overline{U}} X(\overline{R})d\overline{P}\right) = {}^o\left(\int_{\overline{U}} \overline{E}\left(X(\overline{R})|\mathfrak{B}_{(\underline{s}_\nu,\underline{t}_\nu)}\right)d\overline{P}\right) = 0$$

so that $E(x(R)|\mathfrak{F}^*_{(s,t)}) = 0$ a.s.. Then, $st(X)(s,t)$ is a strong larc martingale.

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