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INCONSISTENCY, PARACONSISTENCY AND ω-INCONSISTENCY

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Abstract. In this paper I'll explore the relation between ω-inconsistency and plain inconsistency, in the context of theories that intend to capture semantic concepts. In particular, I'll focus on two very well known inconsistent but non-trivial theories of truth: **LP** and **STTT**. Both have the interesting feature of being able to handle semantic and arithmetic concepts, maintaining the standard model. However, it can be easily shown that both theories are ω-inconsistent. Although usually a theory of truth is generally expected to be ω-consistent, all conceptual concerns don't apply to inconsistent theories. Finally, I'll explore if it's possible to have an inconsistent, but ω-consistent theory of truth, restricting my analysis to substructural theories.

Keywords: Substructural theories of truth • paraconsistency • ω-inconsistency

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1. Introduction

In this paper I will explore the relation between ω-inconsistency and plain inconsistency, in the context of theories that intend to capture semantic concepts, such as *truth*.[1] As it's very well known, expressing the concept of truth inside a classical formal theory (such as an extension of arithmetic), satisfying very weak principles, leads to paradoxes. The most famous paradox of this kind is the so-called 'liar paradox'. In a classical context, if we were able to express sentences such as the following one:

(λ) The sentence λ is not true

(the liar sentence) we would be in trouble, since we cannot assign to it any stable semantic value. It's easy to check that, in a classical framework, from assuming that λ is true, it follows that is not true, and from assuming that it's not true it follows that is true.

However, when we reason in natural language, we use the concept of truth. So, in order to capture this notion in a formal setting, we would like to build theories capable to express semantical concepts avoiding triviality. In other words, any formal theory of truth should be capable to express this kind of sentences, and to be non-trivial. Given the limitations of classical logic, many theorists have tried to capture the concept of truth appealing to non-classical approaches (for a general view of these approaches, see e.g. Field (2008)).

In particular, I'll consider two inconsistent theories of truth: **LP** and **STTT**. The former is a paraconsistent theory, developed by Priest (see e.g. 1979; 2002; 2006) and semantic paradoxes are avoided allowing true contradictions. On the other hand, **STTT**, even though it's an inconsistent theory of truth, is not paraconsistent. It was developed in Ripley (2012), Cobreros, Egré, Ripley and van Rooij (2013) and the derivation of the usual paradoxes is avoided rejecting the metarule of cut (in fact, the resulting logic is not transitive).

One of the desirable features of a theory of truth based on an arithmetical theory is to keep the standard model. Let's recall that a theory $\mathscr{T}$ is ω-inconsistent if and only if there is a formula ϕ such that $\vdash_{\mathscr{T}} \phi(\overline{n})$, for each $n \in \omega$, but it's also the case that $\vdash_{\mathscr{T}} \neg\forall x \phi(x)$.[2] Informally, if a theory is ω-inconsistent, but consistent, it's easy to see that the models of the theory cannot be the standard one. Intuitively, that's because, if for each natural number n, $\phi(\overline{n})$ it's true, there must be an object in the domain of the model that is not a natural number, given that $\neg\forall x \phi(x)$ is also true.

So far, to rule out the standard model of arithmetic can be seen as a technical limitation of a given arithmetical theory. However, there are many conceptual problems, if the resulting theory is ω-inconsistent, but consistent. For instance, Barrio (2010), p.381 claims that:

> A theory of truth should be ω-consistent because, if it is not, this theory may not be interpreted as speaking (only) about the intended ontology of the theory to which it applies.

Therefore, the preservation of the intended ontology of the base theory is an indispensable condition for being considered an acceptable theory of truth. Also, it's worth noting that if we cannot have the standard model of the base theory, then we are not able anymore to interpret our truth predicate as legitimately expressing the concept of truth. This is because in the domains of the models we don't have only standard numbers anymore and so, the truth predicate is satisfied by constants that are not codifying sentences. In other words, given that ω-inconsistent theories of truth interfere in the ontology of the model of the base theory (ruling out the intended interpretation of the language of arithmetic), the resulting theory cannot be interpreted as intended. The problem is not the existence of non-standard models but that these type of theories only have non-standard models as a consequence of introducing axioms and rules that try to capture the concept of truth. Then, ω-inconsistent, but consistent, theories of truth cannot preserve the standard semantic values for at least some interpretation.

Usually, ω-inconsistency (Field 2008, Leitgeb 2001, Barrio 2010) is taken equivalent to non-standardness (and it is natural since, usually, the context is a consistent one). However, in the context of inconsistent theories of truth, it's possible to avoid the problems that carries out the ω-inconsistency: ω-inconsistency doesn't imply to rule out the standard model. Nonetheless, as far as I know, there has not been in the literature any attempt to analyze this phenomenon in the context of inconsistent theories (except for Fjellstad (2016)). In section 2, I will explore the relation between ω-inconsistency and inconsistency, considering two theories: one inconsistent and paraconsistent theory, **LP**, and the other inconsistent, but not paraconsistent, **STTT**. In section 3, I will try to explore the possibility of building an inconsistent but ω-consistent substructural theory of truth. Lastly, section 4 will be devoted to final remarks.

2. Paraconsistent and Inconsistent theories of truth: the case of LP and STTT

In this section, I'll consider ω-inconsistent, and inconsistent, theories of truth. Instead of doing it in general, we will focus on two very known inconsistent theories of truth: **LP** and **STTT**. The former is a paraconsistent theory, developed by Priest (see e.g. 1979; 2006). The latter, even though it's an inconsistent theory of truth, it's not paraconsistent, was developed in Ripley (2012), Cobreros, Egré, Ripley and van Rooij (2013). Before doing that, for the sake of clarity, we should begin with two definitions:

Definition 2.1. Let's say that a theory $\mathscr{T}$ is inconsistent[3] if and only if there is a sentence A such that $\vdash_{\mathscr{T}} A$ and $\vdash_{\mathscr{T}} \neg A$.

Definition 2.2. Let's say that a theory $\mathscr{T}$ is paraconsistent if and only if there are two sentences A and B, such that $A, \neg A \nvdash_{\mathscr{T}} B$.[4]

2.1. The case of LP

Firstly, we will present the logic *LP*. This logic was introduced by Priest (1979)[5] and is a paraconsistent logic, since it invalidates the rule of explosion. The logic *LP* is defined in a standard first order language, where $\{\neg, \vee, \exists\}$ are taken as the primitive logical vocabulary,[6] $\mathscr{V} = \{0, 1/2, 1\}$ is the set of semantic values, $\mathscr{J} = \{1/2, 1\}$ the set of designated one. Let $\mathbf{M} = \langle \mathscr{D}, \mathscr{I} \rangle$, a *LP*-model, where $D \neq \emptyset$, and $\mathscr{I}$ is a function such that assign each constant an object of the domain $\mathscr{D}$ and for each n-ary predicate, a total function from $\mathscr{D}^n$ to $\mathscr{V}$. Considering also in the usual way variable assignment, we have that for each valuation v from the object variables into the domain D and every two formulae A and B of the language we define $||A||_{\mathbf{M},v}$ to be the semantic value of A in $\mathbf{M}, v$:

- $||A(x,\dots t,\dots)||_{\mathbf{M},v} = I(A)((v(x),\dots,I(t),\dots))$
- negation: $||\neg A||_{\mathbf{M},v} = 1 - ||A||_{\mathbf{M},v}$
- disjunction: $||A \vee B||_{\mathbf{M},v} = max\{||A||_{\mathbf{M},v}; ||B||_{\mathbf{M},v}\}$
- existential: $||\exists x A(x)||_{\mathbf{M},v} = supr\{||A(x)||_{M,v'}|| : v(y) = v'(y), \textit{for all variables except possibly } x\}$

We define validity in the usual way, as preservation of designated value: $\Gamma \vDash_{LP} B$ if and only if for every model, $\mathbf{M}$, if $||\gamma||_{\mathbf{M},v} \in \{1/2, 1\}$, for each $\gamma \in \Gamma$, then $||B||_{\mathbf{M},v} \in \{1/2, 1\}$.

So, we will work in a first order language. However, since it has an heuristic value, let's introduce the truth-functions of the connectives as in the propositional case, defined by the following matrices:

$f^{\mathbf{LP}}_{\rightarrow}$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	1/2
0	1	1	1

$f^{\mathbf{LP}}_{\vee}$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$f^{\mathbf{LP}}_{\neg}$	
1	0
1/2	1/2
0	1

Thus, these truth tables are very indicative of the behavior of the logical connectives and the corresponding interaction between the different semantic values. Usually, in the literature about semantic paradoxes, the intermediate semantic value is interpreted as a glut of truth values. In other words, if a sentence is evaluated as 1/2, the sentence is interpreted as true and false. This way of interpreting the intermediate value is compatible with the philosophical posture adopted and defended by Priest, among others, that is called *dialetheism*. The dialetheists think that there are true contradictions in the world, and to use this logic is a way of making sense to that belief (see v.g. Priest (1979; 1989), among others). If we adopt this way of interpreting the intermediate value, we can say that validity in *LP* is transmission of truth, and, as in the classical case, we will always go from true premises to a true conclusion.[7]

As mentioned before, a very well known feature of this logic is the failure of the explosion rule, which is the main feature of paraconsistency[8]. It's easy to check, inspecting the valuations, that $A \wedge \neg A \nvDash_{LP} B$, where A and B are sentences of the language. It's enough that A be true and false, but B just false.

Going back to our main goal, Priest has shown how to add a transparent truth predicate to an arithmetical theory built over this logic, avoiding triviality. Transparency means that for every model and every sentence A, A and $T\ulcorner A \urcorner$ receive the same semantic value. In particular, in Priest (2002), pp.351–354 he shows how to build a transparent theory of truth based on first order Peano arithmetic with *LP* as the base logic. Obviously, since the truth predicate is transparent the theory will validate all the expected naive properties, such as T-schema[9]: $\vDash T\ulcorner A \urcorner \longleftrightarrow A$, for every sentence A of the language. Also, it satisfies the stronger rules:

$$\text{NEC-CONEC}\ \frac{\vDash A}{\vDash T\ulcorner A\urcorner} \qquad\qquad \neg\text{NEC-}\neg\text{CONEC}\ \frac{\vDash \neg A}{\vDash \neg T\ulcorner A\urcorner}$$

Actually, in (2006), p.236, Priest says:

> Of course, since the extension of the language with a truth predicate is conservative, if we start with a consistent arithmetic, the purely arithmetic fragment of the theory with the truth predicate will also be consistent. So the inconsistency generated by the truth predicate gives no reason, as such, to suppose that the purely arithmetic fragment is inconsistent.[10]

In other words, even though paraconsistent theorists may favour inconsistent arithmetic, they can build inconsistent theories of truth that are consistent with regard to the arithmetic language.[11] So, it's possible to have the standard model of arithmetic, although the theory extended is inconsistent.

Nevertheless, here arithmetic is only used to provide enough resources in order to talk about the sentences of the theory. Let's call **LP** the non-trivial theory consisting in all of the theorems of arithmetic and a transparent truth predicate, built over the logic *LP*. As we mentioned before, the main idea of **LP** solution of semantic paradoxes is to treat paradoxical sentences like being true and false, i.e. dialetheias. Recall the liar sentence, λ, the sentence that says about itself that is not true. Reasoning in a classical frame, if we assume that it's true, then conclude that it's false, and vice versa. In a dialetheist framework, this sentence is considered true and false. So, in every **LP**-model this sentence will receive the intermediate value ($||\lambda||_{\mathbf{M},v}$=1/2). Therefore, looking at the notion of validity, this sentence will have designated value in every model. However, looking at the behaviour of the negation in **LP**, the negation of the liar sentence will receive the intermediate value in every model. In other words, both the liar sentence and his negation are true in every model: $\mathbf{M} \vDash_{\mathbf{LP}} \lambda$ and $\mathbf{M} \vDash_{\mathbf{LP}} \neg\lambda$, for every **M**, **LP**-model. The paradoxical sentences and their negations are in both the extension and the anti-extension of the truth predicate. In this way, paraconsistent theorists avoid the trivialization.[12]

It's worth noting that ω-inconsistency has to do with a syntactical problem. However, we will analyze the case of **LP** from a semantical point of view.[13]

Remark 2.3. **LP** is ω-inconsistent.[14]

Proof. Here we adapt the strategy used in Mcgee (1985). Using arithmetic, we can represent the function f, defined in the following way:

- $f(0,x) = x$
- $f(\overline{1},x) = Tx$
- $f(\overline{n}+1,x) = Tf(\overline{n},x)$.[15]

This is the famous McGee function used in (1985). So, given that f is representable in arithmetic, we can obtain the following instance of the diagonalization lemma:[16]

$$\mu \leftrightarrow \exists x \neg Tf(x, \ulcorner \mu \urcorner)$$

where μ is the so called McGee's sentence (due to Mcgee (1985)), and of course, $\mu \leftrightarrow \exists x \neg Tf(x, \ulcorner \mu \urcorner)$ is true in every **LP**-model.

It is easy to see that, for each model, the McGee sentence and the sentence that is materially equivalent to it must both be designated. As an anonymous reviewer noted, it's worth pointing out that the reason for this is not modus ponens. In fact, trivially, in **LP**, the liar sentence, λ is materially equivalent in every model to the sentence 2+2=5 (in the sense that $\lambda \leftrightarrow 2+2=5$ is the case, since the material conditional is very weak). And, of course, this last sentence (2+2=5) is not true in **LP**. In the particular case we are dealing with, both sentences (μ and $\exists x \neg Tf(x, \ulcorner \mu \urcorner)$) are designated in every model because of the way f was built and the fact that the truth predicate is transparent. More specifically, by transparency and the definition of f, it's easy to show that for every **LP**-model, $||\mu||_{\mathbf{M},v}$ nor $||\exists x \neg Tf(x, \ulcorner \mu \urcorner)||_{\mathbf{M},v}$ can be non-designated. □

Nonetheless, there is a simpler way of proving this. We can derive ω-inconsistency directly from inconsistency. Let's take the liar sentence λ. This sentence is the fixed point of $\neg T(x)$, and in every model, since the truth predicate is transparent, has the intermediate value. In other words, for every model **LP**-model M, $||\lambda||_{\mathbf{M},v} = ||\neg T(\ulcorner \lambda \urcorner)||_{\mathbf{M},v} = 1/2$. But, as $||\neg T(\ulcorner \lambda \urcorner)||_{\mathbf{M},v} = 1/2$, then $||\neg \forall x Tf(x, \ulcorner \lambda \urcorner)||_{\mathbf{M},v} > 0$, and then $\vDash_{\mathbf{LP}} \neg \forall x Tf(x, \ulcorner \lambda \urcorner)$, but $\vDash_{\mathbf{LP}} Tf(\overline{n}, \lambda)$ for each n $\in \omega$, because $||\neg T(\ulcorner \lambda \urcorner)||_{\mathbf{M},v} = ||T(\ulcorner \lambda \urcorner)||_{\mathbf{M},v} = 1/2$. So, ω-inconsistency in the **LP** context can be seen as a consequence of inconsistency, i.e. it doesn't have any substantial impact over the models or the ontology of the theory.[17]

2.2. The case of STTT

In this subsection, we'll focus on a substructural logic for transparent truth: **STTT**. Substructural logics are logics that reject some of the structural rules,[18] i.e. the rules that govern the very notion of logical consequence. So, we must work with a presentation in sequent calculus, because it makes explicit the structural rules and the operational rules (those governing the logical connectives). The sequents are notationally: $\Gamma \Rightarrow \Delta$, where Γ is the multiset of premises and Δ is the multiset of conclusions. The difference between multisets and sets is that in the multisets the different occurrences of a given member count. In this way $\{A,A\}$ and $\{A\}$ are the same set but different multisets. Let's start introducing the logic **ST**.

Definition 2.4. (**ST**) Let Γ, Δ, Π, and Σ be (finite) multisets of formulas, let ϕ and ψ be formulas. The system ST is given by the following rules:

Structural rules

$$\text{Reflexivity}\ \frac{}{\mathrm{A} \Rightarrow \mathrm{A}}$$

$$\text{L C}\ \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \text{R C}\ \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$\text{L W}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \text{R W}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

Operational rules

$$\text{L}\wedge\ \frac{\Gamma, (A), (B) \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\text{L}\vee\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad \text{R}\vee\ \frac{\Gamma, (A), (B) \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}$$

$$\text{L}\rightarrow\ \frac{\Gamma, \Rightarrow A, \Delta \qquad \Sigma, B \Rightarrow \Pi}{\Gamma, \Sigma, A \rightarrow B \Rightarrow \Pi, \Delta} \qquad \text{R}\rightarrow\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta}$$

$$\text{L}\forall\ \frac{\Gamma, A \Rightarrow \Delta}{\forall x A[x/t], \Gamma \Rightarrow \Delta} \qquad \text{R}\forall\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \forall x A[x/a], \Delta} \qquad \text{L}\neg\ \frac{\Gamma, \Rightarrow A, \Delta}{\neg A, \Gamma \Rightarrow \Delta} \qquad \text{R}\neg\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Rightarrow \neg A, \Delta}$$

Where the operational rules are the rules that regiment the logical vocabulary, i.e. the usual connectives and quantifiers ¬, ∧, ∨, → and ∀. On the other hand, structural rules are related with the very notion of logical consequence, and don't involve any particular logical vocabulary (connective or quantifier). In this context, the structural rules are contraction (C), weakening (W) and reflexivity. In the above presentation of **ST**, we didn't include the structural rule of cut:

$$\text{Cut}\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Sigma, A \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Pi, \Delta}$$

It's not difficult to check that, in fact, the above presentation of **ST** is the usual presentation of classical logic (actually, **ST** and classical logic are the same logic). So, not including cut is not a problem at all, since in classical logic cut is admissible. In other words, in classical logic, including cut doesn't modify the set of derivable sequents (for details, see e.g. Paoli (2013)). The main distinction between **ST** and classical logic has to do with the theory of truth **STTT**.

Definition 2.5. (**STTT**) The theory of truth **STTT** consists in adding to the logic **ST** the following rules for the truth predicate T:

$$\text{L T}\ \frac{\Gamma, A \Rightarrow \Delta}{\mathrm{T}\ulcorner \mathrm{A}\urcorner, \Gamma \Rightarrow \Delta} \qquad\qquad \text{R T}\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma, \Rightarrow \Delta, \mathrm{T}\ulcorner \mathrm{A}\urcorner}$$

Where $\ulcorner \mathrm{A}\urcorner$ is the name of the sentence A.[19] We'll call this theory **STTT**.[20] The most attractive feature of **STTT** is that it's just classical logic plus a transparent truth predicate, but surprisingly it's not trivial (Ripley, 2012)! Given that the rule of cut is eliminable in classical logic, **STTT** restricted to logical vocabulary coincides in its inferences with the classical inferences.[21] Because of that, we cannot say that **STTT** is paraconsistent.

Remark 2.6. **STTT** is not paraconsistent.

Proof. : It's easy to derive explosion, given that the rules for negation are classical.

$$\text{R W}\ \dfrac{\text{L}\,\neg\ \dfrac{\mathrm{A} \Rightarrow \mathrm{A}}{\mathrm{A}, \neg \mathrm{A} \Rightarrow}}{\mathrm{A}, \neg \mathrm{A} \Rightarrow \mathrm{B}}$$

□

In Ripley (2012), the author gives a complete semantics for this calculus. Intuitively, given a sequent $\Gamma \Rightarrow \Delta$, a counterexample consists in treating the premises as strictly true and the conclusion as strictly false. More formally, if we take the set of values as $\mathcal{V} = \{0, 1/2, 1\}$ and the interpretation of these values as: strictly true (false) for the value 1 (0) and tolerantly true/false for the value 1/2, a counterexample would be such that $v(\Gamma) = 1$ and $v(\Delta) = 0$. For instance, regarding the negation, checking the rules, if A is strictly true (false) ¬ A is strictly false (true). On the other hand, if A is tolerant then ¬ A is tolerant. In other words, ¬ can be understood as an operator that represents what is called "classical negation" in paraconsistent context. So, the idea is that $\mathrm{A} \vDash \mathrm{B}$ is valid if and only if it cannot be the case that A be strictly true and B strictly false. As mentioned, this semantic (formally developed) is complete with respect to the sequent calculus above presented (in Ripley (2012) it's shown in detail). From this point of view, it's easy to see why explosion is valid: A, ¬ A ⊨ B is valid because it's impossible that the premises be strictly true.

However, although **STTT** is not paraconsistent, it is inconsistent. In this frame, every paradoxical sentence is taken as tolerant, and so, both $\Rightarrow \lambda$ and $\Rightarrow \neg\lambda$ are valid (derivable), because in each case satisfy the definition. In a more formal fashion:

$$\dfrac{\dfrac{\dfrac{\dfrac{T(\ulcorner\lambda\urcorner) \Rightarrow T(\ulcorner\lambda\urcorner)}{\lambda \Rightarrow T(\ulcorner\lambda\urcorner)}\,\mathrm{L}T}{\lambda, \neg T(\ulcorner\lambda\urcorner) \Rightarrow}\,\mathrm{L}\neg}{\lambda, \lambda \Rightarrow}\,\text{Subs. id.}}{\lambda \Rightarrow}\,\mathrm{L\,C} \qquad \dfrac{\dfrac{\dfrac{\dfrac{T(\ulcorner\lambda\urcorner) \Rightarrow T(\ulcorner\lambda\urcorner)}{T(\ulcorner\lambda\urcorner) \Rightarrow \lambda}\,\mathrm{R}T}{\Rightarrow \lambda, \neg T(\ulcorner\lambda\urcorner)}\,\mathrm{R}\neg}{\Rightarrow \lambda, \lambda}\,\text{Subs. id.}}{\Rightarrow \lambda}\,\mathrm{R\,C}$$

but we cannot cut $\lambda \Rightarrow$ and $\Rightarrow \lambda$. Therefore, this is a counterexample to transitivity.

So far, we didn't mention the relation between **STTT** and ω-inconsistency. In Cobreros, Egré, Ripley and van Rooij (2013) the authors show that if we add arithmetic to **STTT** we have a theory that has the standard model. However, later in Cobreros, Egré, Ripley and van Rooij (2013), p.860, comparing **STTT** plus arithmetic with the theory FS,[22] Cobreros et al. say:

> Further, FS is ω-inconsistent, and so can have no standard models. **STTT**, on the other hand, is shown to have standard models by the Kripke construction. In this regard, it is worth noting that $\mathbf{STTT_{PA}}$, which contains the compositional principles, **PA**, and a transparent truth predicate, seems to more than satisfy the conditions for the 'negative result' in McGee 1985, showing that any system meeting weaker conditions than these must be ω-inconsistent. (It is this result that shows FS to be ω-inconsistent.) Nonetheless, the result does not apply to $\mathbf{STTT_{PA}}$, as McGee's argument depends on assuming transitivity.

However, recently, Fjellstad in (2016) has shown that **STTT** plus Robinson arithmetic is ω-inconsistent (even though as it was mentioned it's proven that it doesn't lack the standard model).[23] The idea behind the proof of ω-inconsistency relies on defining the same function used in the last section. Taking the same terminology used above, let's say f is the function and we obtain the following instance of the diagonalization lemma $\Rightarrow \mu \leftrightarrow \exists x \neg T(x, \ulcorner\mu\urcorner)$. Then, Fjellstad shows without cut that $\Rightarrow \exists x \neg T(x, \ulcorner\mu\urcorner)$ and for each n, $\Rightarrow Tf(\overline{n}, \ulcorner\mu\urcorner)$.[24]

From a semantical view, once transitivity is rejected, it's easy to see how ω-inconsistency is compatible with standardness. Let's take γ. As $\Rightarrow \gamma$, then the sentence can be tolerantly true. And, also we have $Tf(n, \ulcorner\gamma\urcorner) \Rightarrow$ for each n in ω, and this doesn't imply that each instance is strictly false. So, once we don't have transitivity and we have transparency of the truth predicate, we have $\Rightarrow \gamma$ and $\gamma \Rightarrow$, that doesn't imply triviality, given that γ is just tolerantly true/false.

3. A generalization

In this section I investigate the relation between inconsistent and ω-inconsistent theories in a more abstract fashion. I would like to answer the following question: can an inconsistent theory be ω-consistent? This is not so easy to answer. Firstly, the very question can be put in doubt by many authors. In the literature the notion of ω-consistency have been confused many times. Perhaps, one of the most important reasons for that is that there is not consensus at all about what *inconsistency* means (and therefore *consistency*). For instance in a recent paper, Walter Carnielli (2011) shows in details these multiple definitions and conceptions about inconsistency.[25] However, when he looks at the notion of ω-inconsistent theories, in Carnielli (2011), p.95 claims:

> Gödel's results (later refined by Rosser) originally presupposed the notion of ω-consistency [...] An ω-consistent theory is not only (syntactically) non-contradictory (that is, does not prove any single contradiction), but also avoids proving certain infinite collections of sentences that are intuitively contradictory.

If this is right, then, it's impossible to build any ω-consistent but inconsistent theory of truth. Let's recall that a theory is ω-consistent if and only if it is not ω-inconsistent. So, if ω-consistency implied consistency, then inconsistency would imply ω-inconsistency. In a classical frame, where negation is explosive is easy to check that it is the case. But, what is the situation in an inconsistent context?

Since the notion of ω-inconsistency is syntactic, let's express the following in a syntactic way. Let's assume an inconsistent theory, such that exists a sentence A, with $\Rightarrow A$ and $A \Rightarrow$. Then, Fjellstad (2016) shows the following derivation of ω-inconsistency from inconsistency, that I'll call $\mathscr{D}_{\mathscr{A}}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{A}\Rightarrow}{\mathrm{A},\, t=t \Rightarrow}\ \mathrm{L\,W}}{\mathrm{A} \wedge t=t \Rightarrow}\ \mathrm{L} \wedge}{\forall x\ (\mathrm{A} \wedge \mathrm{x{=}x}) \Rightarrow}\ \mathrm{L} \forall}{\Rightarrow \neg \forall x\ (\mathrm{A} \wedge \mathrm{x{=}x})}\ \mathrm{R} \neg$$

But, also we have that:

$$\mathrm{R} \wedge\ \dfrac{\Rightarrow \mathrm{A} \qquad \Rightarrow \overline{n} = \overline{n}}{\Rightarrow \mathrm{A} \wedge \overline{n} = \overline{n}}$$

for each n $\in \omega$.

In other words, if the logic accepts the usual operational and structural rules and it's inconsistent, then it's also ω-inconsistent. It's easy to check that it's the case of the theory built over **STTT**.[26] This theory rejects some classical meta rules, but none of the involved in the derivation above mentioned.

So, looking at the derivation $\mathscr{D}_{\mathscr{A}}$ one may wonder if a theory of truth could be inconsistent but ω-consistent. This is not an easy question to answer, because there is not any standard syntactic method to show the ω-consistency of a theory. Usually, as we treat with consistent theories, it's enough to show that the theory has the standard model. Nevertheless, as we've shown in the last section, ω-inconsistency doesn't entail non-standardness in the context of inconsistent theories.

Before doing that, one may wonder why it would be desirable to build an inconsistent but ω-consistent theory of truth. It there would be many reasons for doing that, among others, one could argue as follows. Both of the two inconsistent theories above presented treat McGee's sentence like a paradoxical sentence. However, it's possible to think that the liar sentence and McGee's sentence are different types of sentences. The first one is genuinely paradoxical, i.e. it leads to trivialization of classical theories. However, the second one is perfectly compatible with the usual classical theories. In fact, in a consistent theory of truth, this kind of sentence doesn't lead to paradoxes. So, why would we consider both as being of the same kind? Someone whose aim is to distinguish different kind of semantic oddness could be interested in treating differently to these two sentences. In what follows, we will not insist on this point, since our aim is just exploratory.

3.1. The substructural case

Given that the question whether it is possible to find a ω-consistent and inconsistent theory of truth is very broad, let us simply focus on substructural theories of truth.

As mentioned before (see the Section 2.2), we will consider four structural rules: contraction, weakening, reflexivity and cut. So, in this context we will take a logic to be substructural, if some of these rules is/are not admissible.

Now, turning our attention to the derivation $\mathscr{D}_{\mathscr{A}}$, it seems that the only structural rule playing a role there is weakening. In other words, one might be inclined to think that rejecting weakening is enough to break the relation between ω-inconsistency and inconsistency. In this section, first, we are going to show that it's not the case, i.e. it's not enough to reject weakening in order to get ω-consistency. The reason for this is that the derivation of ω-inconsistency still obtains if we use a different kind of vocabulary, i.e. additive vocabulary. Next, we will show that even if we stick to multiplicative vocabulary and reject weakening, although the derivation $\mathscr{D}_{\mathscr{A}}$ is blocked, the very theory is trivial, due to Curry's paradox (and so the theory is trivially ω-inconsistent). Finally, in order to avoid triviality and block the derivation $\mathscr{D}_{\mathscr{A}}$, two options still remains: either to abandon weakening and contraction in a multiplicative frame (adopting some subsystem of Linear Logic) or to reject weakening and cut, also in a multiplicative frame. The first option is not available for us, since the resulting theory is consistent (see, e.g. Mares and Paoli (2014)). The second one is what we consider the only possible route.

In order to enter into the details referred in the last paragraph, we need to begin with the distinction between additive and multiplicative vocabulary. Once we reject contraction and/or weakening the language can be divided into two categories: the additive and the multiplicative. That is, the rules for the connectives can be presented in two main different ways.[27] For instance, let's take the conjunction. One could present the rules for conjunction in these ways:[28]

$$\text{L}\sqcap\ \frac{(A),\ (B),\ \Gamma \Rightarrow \Delta}{A \sqcap B, \Gamma \Rightarrow \Delta} \qquad\qquad \text{R}\sqcap\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \sqcap B}$$

or:

$$\text{L}\otimes\ \frac{A,\ B,\ \Gamma \Rightarrow \Delta}{A \otimes B, \Gamma \Rightarrow \Delta} \qquad\qquad \text{R}\otimes\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Sigma \Rightarrow \Pi, B}{\Gamma \Rightarrow \Delta, A \otimes B}$$

where the first two rules are the so-called additive rules for conjunction and the second two, the so-called multiplicative ones.[29][30] It's easy to show that $A \otimes B \Leftrightarrow A \sqcap B$ holds in classical logic. In fact, all we need in order to prove this is contraction and weakening. In particular, weakening in one direction:

$$\text{L }\otimes\ \dfrac{\text{R }\sqcap\ \dfrac{\text{L W}\ \dfrac{A \Rightarrow A}{A, B \Rightarrow A} \qquad \text{L W}\ \dfrac{B \Rightarrow B}{A, B \Rightarrow B}}{A, B \Rightarrow A \sqcap B}}{A \otimes B \Rightarrow A \sqcap B}$$

and contraction in the other one:

$$\text{L C}\ \dfrac{\text{L }\sqcap\ \dfrac{\text{L }\sqcap\ \dfrac{\text{R }\otimes\ \dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A, B \Rightarrow A \otimes B}}{A \sqcap B, B \Rightarrow A \otimes B}}{A \sqcap B, A \sqcap B \Rightarrow A \otimes B}}{A \sqcap B \Rightarrow A \otimes B}$$

So, with this structural rules in place, both connectives are the same. However, this equivalence fails once we abandon at least one of these rules. In this way, going back to the derivation of ω-inconsistency from plain inconsistency ($\mathscr{D}_{\mathscr{A}}$), even though we can reject weakening, we must reject also $\sqcap$, because we can build the following derivation ($\mathscr{D}_{\mathscr{B}}$), even without weakening:

$$
\begin{prooftree}
\AxiomC{$A \Rightarrow$}
\LeftLabel{L $\sqcap$}
\UnaryInfC{$A \sqcap t = t \Rightarrow$}
\LeftLabel{L $\forall$}
\UnaryInfC{$\forall x(A \sqcap x = x) \Rightarrow$}
\LeftLabel{R $\neg$}
\UnaryInfC{$\Rightarrow \neg\forall x(A \sqcap x = x)$}
\end{prooftree}
$$

But, also we have that:

$$
\begin{prooftree}
\AxiomC{$\Rightarrow A$}
\AxiomC{$\Rightarrow \overline{n} = \overline{n}$}
\LeftLabel{R $\sqcap$}
\BinaryInfC{$\Rightarrow A \sqcap \overline{n} = \overline{n}$}
\end{prooftree}
$$

for each n $\in \omega$ where, clearly, we didn't use any structural rule explicitly. In other words, what this derivation shows is that it's possible, in principle, to build an inconsistent theory of truth that blocks the usual proof of ω-inconsistency. Here we have shown that it's not enough to reject weakening. We must also reject the additive rules for conjunction.

However, as far as I know, there have not been proposed in the literature any inconsistent theory of truth based on a sequent calculus that accepts the multiplicative operational rules but rejects weakening.[31] The reason for this is that Curry's paradox and related paradoxes still obtain in this kind of framework. Let's recall the rules for the conditional presented in Section 2.2:[32]

$$
\begin{prooftree}
\AxiomC{$\Gamma \Rightarrow \Delta, A$}
\AxiomC{$\Pi, B \Rightarrow \Sigma$}
\LeftLabel{L $\rightarrow$}
\BinaryInfC{$\Gamma, \Pi, A \rightarrow B \Rightarrow \Delta, \Sigma$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$\Gamma, A \Rightarrow B, \Delta$}
\LeftLabel{R $\rightarrow$}
\UnaryInfC{$\Gamma \Rightarrow A \rightarrow B, \Delta$}
\end{prooftree}
$$

Let's define a sentence κ (the Curry's sentence) to be equivalent to $T(\ulcorner\kappa\urcorner) \rightarrow A$, where A is some arbitrary sentence.

So, we can derive A, and therefore trivialize the theory.

$$
\begin{prooftree}
\AxiomC{$T(\ulcorner\kappa\urcorner) \Rightarrow Tr(\ulcorner\kappa\urcorner)$}
\AxiomC{$A \Rightarrow A$}
\LeftLabel{L$\rightarrow$}
\BinaryInfC{$T(\ulcorner\kappa\urcorner), T(\ulcorner\kappa\urcorner) \rightarrow A \Rightarrow A$}
\LeftLabel{Sub., LT}
\UnaryInfC{$T(\ulcorner\kappa\urcorner), T(\ulcorner\kappa\urcorner) \Rightarrow A$}
\LeftLabel{L C}
\UnaryInfC{$T(\ulcorner\kappa\urcorner) \Rightarrow A$}
\LeftLabel{R$\rightarrow$}
\UnaryInfC{$\Rightarrow T(\ulcorner\kappa\urcorner) \rightarrow A$}
\LeftLabel{Sub., LT}
\UnaryInfC{$\Rightarrow T(\ulcorner\kappa\urcorner)$}
\AxiomC{$T(\ulcorner\kappa\urcorner) \Rightarrow T(\ulcorner\kappa\urcorner)$}
\AxiomC{$A \Rightarrow A$}
\LeftLabel{L$\rightarrow$}
\BinaryInfC{$T(\ulcorner\kappa\urcorner), T(\ulcorner\kappa\urcorner) \rightarrow A \Rightarrow A$}
\LeftLabel{Sub., LT}
\UnaryInfC{$T(\ulcorner\kappa\urcorner), T(\ulcorner\kappa\urcorner) \Rightarrow A$}
\LeftLabel{L C}
\UnaryInfC{$T(\ulcorner\kappa\urcorner) \Rightarrow A$}
\LeftLabel{Cut}
\BinaryInfC{$\Rightarrow A$}
\end{prooftree}
$$

Therefore, to reject weakening and reject additive operational rules is not sufficient in order to avoid this kind of derivation. In this vein, it seems that in order to block the derivation above presented and the usual proofs of ω-inconsistency, could be possible to adopt a theory that not only rejects weakening but also rejects cut. This also has to do with the fact that the proof of ω-inconsistency in Fjellstad (2016) that avoids using cut, requires the use of weakening several times.[33] So, the logic that admits only multiplicative vocabulary and rejects cut and weakening would be an interesting candidate for being considered as a base theory of a theory of truth inconsistent but ω-consistent. However, so far, there is not anyone in the literature who defended this logic.

Regarding the other structural rules, we could also consider a theory of truth based on the multiplicative fragment of Linear Logic (a logic without contraction and without weakening). This theory is non-trivial, since is a subtheory of the theory of truth based on Linear Logic. However, this theory is consistent. And, surprisingly, it's paraconsistent.[34][35]

Another interesting point to explore is how is the relation between ω-inconsistency and inconsistency from the point of view of the models. In a more general respect, intuitively, if a theory is ω-inconsistent and has the standard model, can be consistent? Let's see. Assume we have a theory $\mathscr{T}$, such that there is a formula ϕ, $\mathscr{T} \vDash \phi(\overline{n})$, for each $n \in \omega$, and $\mathscr{T} \vDash \neg\forall x \phi(x)$. From the point of view of the models, if $\mathscr{T} \vDash \neg\forall x \phi(x)$ and the domain of the model is composed just for numbers, it must be the case that there must be a constant c that refers to an object of the domain (a number) such that $\mathscr{T} \vDash \neg\phi(c)$, which contradicts $\mathscr{T} \vDash \phi(\overline{n})$, for each $n \in \omega$, or c do not refer to a number and so the theory lacks the standard model. So, in this way, it seems impossible for a theory to be ω-inconsistent, to have the standard model, but to be consistent. In the other way around, if the models of our theory accept a sentence and its negation (suppose A and $\neg$ A), it seems possible to be ω-inconsistent, since it has to do with the interpretation and the kind of quantifiers. However, the question is so general and further research needs to be done, in order to answer it.

4. Conclusion and further research

In this paper I tried to explore the relation between ω-inconsistency and inconsistency. In this vein, I have analyzed the fact that the two most popular inconsistent theories of truth, **STTT** and **LP**, are also ω-inconsistent. However, both of them maintains the standard model, avoiding the usual concerns that affect to ω-inconsistent theories of truth.

On the other hand, I analyzed the relation between inconsistent and ω-inconsistent theories of truth in a more abstract fashion. In this way, the main question of section 3 was: is it possible to build inconsistent but ω-consistent theories of truth? In order to answer it, I focused on substructural theories. In particular, I showed that some derivations of ω-inconsistency are blocked if we consider a theory without weakening, without cut and formulated on the multiplicative vocabulary. One possible motivation to consider this kind of theory is that this allows us to distinguish paradoxical sentences, such as the liar sentence, from pathological but not strictly speaking paradoxical sentences, such as the McGee's sentence. Of course, these kind of theories would be extremely weak, as they should invalidate a lot of very plausible principles. Also, I didn't provide a proof of ω-consistency, because the usual strategy fails in this case, but just showed that the standard derivations are blocked. Also, it would be interesting to analyze if the uses of weakening in other proofs of ω-inconsistency, for instance the one given in Fjellstad (2016), can be replaced. So, I analyzed the phenomenon from a general perspective and I think that further research still needs to be done, in order to evaluate if it's possible to get ω-consistent but inconsistent theories of truth.

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Notes

[1] I will focus on theories of truth. So, many things I will say here are not applicable to other types of theories.

[2] Here I'm assuming familiarity with Gödel codes and overline notation. For details, see e.g. Halbach (2011).

[3] As an anonymous reviewer suggested, it's worth pointing out that here we are considering to be inconsistent as "negation inconsistent" (to imply some formula and its negation). Negation consistency should not be confused with Post-consistency, i.e. non-triviality. For the sake of clarity, in this paper, when we talk about inconsistent theories, we refer to negation inconsistent theories, and not to Post-inconsistent ones. For the latter, we will use the label "trivial theories".

[4] Although we are using $\vdash$ for defining these notions, throughout this paper, we will also use these concepts in a semantical way.

[5] Many authors have pointed out that its propositional part was introduced in Asenjo (1966). However, this discussion is not relevant for the present purposes.

[6] As usual the other connectives and quantifiers as as $\wedge$, $\rightarrow$ and $\forall$ are definable.

[7] Of course, since this is a dialetheist interpretation of the logic, it's possible to go from true premises to a false conclusion (obviously, also true).

[8] In this context, it's not relevant to make the distinction between $A \wedge \neg A \nvDash B$ and $A, \neg A \nvDash B$. Here the comma on the left side is equivalent to conjunction

[9] He also shows how to add to this theory a conditional that satisfies *modus ponens*. Recall that this inference ($A, A \rightarrow B \vDash B$) is invalid in *LP* (take a model M, such that $||A||_{\mathbf{M},v}$=1/2 and $||B||_{\mathbf{M},v}$=0).

[10] Even though, he adds that the truth predicate should be part of the pure theories, arguing for inconsistent arithmetic.

[11] Interesting as it is, however, here we won't discuss if the arithmetic is consistent or not. For consulting details of inconsistent arithmetic, such as finite models, see for instance Priest (1997; 2000).

[12] It's not the aim of this paper to focus on the formal features that allow to this theory to manage semantic vocabulary. So, this brief introduction should be taken just as an intuitive idea of the technical aspects. The details can be consulted in Priest (2002; 2006).
[13] Here, we follow the usual way of introducing these kind of theories, i.e. semantically. However, it's worth mentioning that it's possible to give syntactic presentations, for instance tableaux or sequent-calculi (see Priest (2008), Avron (1991), Ripley (2012), among many others).
[14] Here we generalize the syntactic concept to a semantic one in the obvious way.
[15] Roughly speaking, for a numeral $\overline{n}$, the function $f(\overline{n}, x)$ returns the representation in the theory (a numeral) of applying n-times the T predicate. Usually, this representation is presented using dot notation. However, for readability reasons I will omit under dots. For details about representability of functions and for McGee's function in particular, see Halbach (2011) and Mcgee (1985), respectively.
[16] As an anonymous reviewer suggested, it's not obvious at all how arithmetical theories behave in paraconsistent contexts. For instance, diagonalization holds in this context. In order to see the details, Priest (2006).
[17] As an anonymous reviewer suggested, it's worth pointing out that although we present a derivation of ω-inconsistency not appealing to McGee's sentence, the reason for presenting McGee's argument will become more evident later, when we discuss the possibility of distinguishing paradoxical sentences such as λ and problematic, but not paradoxical sentences, such as McGee's one.
[18] Here I will not discuss if *LP* admits a substructural presentation as it's argued in Shapiro (2016). Instead, I use the usual categorizations (see for instance Beall and Ripley (2014)).
[19] For the time being, it's enough to assume we have an stock of distinguished names for the sentences, such that there is a bijection between each distinguished name and each sentence. For details, see Ripley (2012). In this way, we can talk about the sentences of the language and generate self reference. Later we will build self-referential sentences with arithmetic.
[20] This theory was first introduced semantically. However as we will mention later the theory is sound and complete with respect to this sequent calculus. In this sense, even though we are not following the original order of the presentation, the relevant technical issues are the same.
[21] Obviously, although cut is admissible in classical logic, once we add a transparent truth predicate, cut is not admissible anymore. Also, of course, **STTT** doesn't coincide with classical logic in its metainferences.
[22] FS is an axiomatic theory of truth formulated in a Hilbert-style calculus, with a truth predicate satisfying NEC and CONEC. It was introduced by Friedman and Sheard (1987), and studied by Halbach (1994; 2011). One of the distinctive features of this theory is to be ω-inconsistency. The usual way of proving this is using the McGee's argument above presented (see e.g. Halbach (1994)).
[23] He formulates Robinson Arithmetic with sequents, extending **STTT**, and shows that the McGee sentence can be defined in this framework and leads to ω-inconsistency.
[24] This is just an scheme of the proof. For seeing the details of the derivation, we would have to introduce Robinson Arithmetic and show that each step can be proved. That's done in Fjellstad (2016).

[25] Let's recall that along this paper, and for what follows, we use the definition of inconsistency given above (see Definition 2.1).
[26] From a semantical point of view, the same happens with the theory built over *LP*.
[27] It's not the aim of this paper to take a deep look into this division. To see more details, can be consulted e.g. Paoli (2013).
[28] The parentheses in the additive rules indicate that in order to apply the rule only one of the formulae is required. Thus, this is an abbreviation of two rules.
[29] There are alternative terminologies for these two kinds of connectives. Sometimes, instead of using 'additive/multiplicative', can be found the use of 'extensional/intensional' or 'lattice-/group-theoretic'.
[30] In the case of single-premise rules, to apply the additive rule requires the occurrence in the premise of exactly one of the formulae to be connected. In the case of multi-premise rules, to apply the additive rule requires in both premises the same context.
[31] There are proposals of theories of truth based on Linear Logic (that also rejects contraction), for instance Mares and Paoli (2014), but these theories are consistent.
[32] Note that these rules are multiplicative.
[33] It would be interesting to explore if these uses of weakening could be replaced in some way. If it were the case, then we could claim that every inconsistent theory of truth is ω-inconsistent. Interesting as it is, I let this for further research.
[34] It's easy to check this point, in the following way:

$$\text{L}\neg\,\dfrac{\text{Reflexivity}\,\dfrac{}{A \Rightarrow A}}{A, \neg A \Rightarrow}$$

but since in Linear Logic you don't have weakening, $A, \neg A \not\Rightarrow B$. So the theory is paraconsistent, but it's consistent.
[35] Another possible option is rejecting reflexivity. A non-reflexive theory of truth has been developed in French (2016). However, since the only initial sequents are instances of reflexivity, non reflexive theories doesn't prove any sequent (although they still validate many meta-inferences). Therefore, we can say that the logical relation in the inferential level is the empty relation (because there are not valid sequents), and therefore non-reflexive logics are trivially consistent (since there is not any sentence such that the theory proves it and proves its negation). So this possibility is trivially discarded.

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