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# AXIOM $(\mathbf{cc})_0$ AND VERIFIABILITY IN TWO EXTRACANONICAL LOGICS OF FORMAL INCONSISTENCY

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**Abstract.** In the field of logics of formal inconsistency (LFIs), the notion of "consistency" is frequently too broad to draw decisive conclusions with respect to the validity of many theses involving the consistency connective. In this paper, we consider the matter of the axiom $(\mathbf{cc})_0$—*i.e.*, the schema $\circ\circ\varphi$—by considering its interpretation in contexts in which "consistency" is understood as a type of verifiability. This paper suggests that such an interpretation is implicit in two *extracanonical* LFIs—Sören Halldén's nonsense-logic C and Graham Priest's cointuitionistic logic daC—drawing some interesting conclusions concerning the status of $(\mathbf{cc})_0$. Initially, we discuss Halldén's skepticism of this axiom and provide a plausible counterexample to its validity. We then discuss the interpretation of the operator in Priest's daC and show the equivalence of $(\mathbf{cc})_0$ to the intuitionistic principle of testability. These observations suggest that it may be fruitful for members of the LFI community to look outside the canon for evidence concerning the adoption of principles like $(\mathbf{cc})_0$.

**Keywords:** Logics of formal inconsistency • nonsense logic • Priest–da Costa logic • consistency operators • verifiability.

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## 1. Introduction

In the realm of logics of formal inconsistency (LFIs), in which the unary connective "$\circ$" is read as a "consistency" or "classicality" connective, the scheme $\circ\circ\varphi$—referred to as "$(\mathbf{cc})_0$" in Carnielli, Coniglio and Marcos (2007)—is particularly interesting. $(\mathbf{cc})_0$ is standardly read as the statement that for any formula $\varphi$, it is consistent to assert that $\varphi$ is consistent. Although $(\mathbf{cc})_0$ is a theorem of many LFIs—indeed it is one of the characteristic elements of João Marcos' axiomatization of mCi in (2008)—there appears to be little examination of the philosophical and semantical merits of this formula in the canonical literature on LFIs. In this paper, we analyze $(\mathbf{cc})_0$ modulo the interpretation of consistency as *verifiability* implicit in two semantical frameworks outside the traditional canon on LFIs.

As recognized by Walter Carnielli and Abílio Rodrigues in (2016) and by Hitoshi Omori in (2016), Sören Halldén's logic of nonsense C is an LFI in which Halldén's unary "meaningfulness" connective "∘" acts as a consistency connective.[1] This observation lends new salience to Halldén's discussion of the theorem $\circ\circ\varphi$ in (1949), in which Halldén expresses reservations about its validity but stops short of providing possible counterexamples. After suggesting that Halldén's semantical analysis of meaningfulness comports with the canonical reading of "∘" in the LFI literature, we will find that the positivist analysis of meaningfulness as verifiability favored by Halldén leads to cases in which $(\mathbf{cc})_0$ seems to fail. The upshot is twofold: To the reader of Halldén, this example serves to confirm Halldén's suspicions and suggest that his own intuitions are not adequately captured by C. To the logician working in the LFI tradition, the example provides a novel perspective concerning the interpretation of "∘" and, more generally, the interpretation of the notion of a logic of formal inconsistency.

Secondly, we consider the status of the axiom scheme in the cointuitionistic logic daC described by Graham Priest in (2009), in which the common intuitionist negation of intuitionistic logic is exchanged for a paraconsistent counterpart "$\neg$." According to the Grzegorczyk–Wolter analysis of the dual intuitionistic connectives developed between Grzegorczyk (1964) and Wolter (1998), the correctness of asserting a conegated formula $\neg\varphi$ corresponds to states in which $\varphi$ is not counted among the "initial information" on which a scientific investigation is based. Now, daC is shown to be an LFI in which "∘" is definable in Ferguson (2014), and this definition induces a corresponding Grzegorczyk–Wolter-style interpretation of the consistency connective as asserting a type of verifiability. An examination of the formal features of "∘" in daC, such as the description of the weakest normal extension of daC in which $(\mathbf{cc})_0$ holds, leads to some curious facts, such as the equivalence of $(\mathbf{cc})_0$ to commonly cited theses of intuitionistic logic and LFIs over daC.

## 1.1. Paraconsistency and Inferential Situations

*Paraconsistency* is the property ascribed to deductive systems that do not validate the *principle of explosion*, *i.e.*, systems in which it is not legitimate to infer arbitrary conclusions from an inconsistent set of premises or, equivalently, systems for which there exist inconsistent but nontrivial theories. During philosophical discussions of the notion of paraconsistency, one often encounters the suggestion that even if certain inferences are considered to be invalid *universally*, there may remain particular cases in which their application is admissible. A particularly articulate example is implicit in a long-running dispute initiated by John Burgess in (1981) concerning the invalidity of disjunctive syllogism in relevant—and, by extension, paraconsistent—logics. Burgess argues against relevant logicians that disjunctive syllogism ought to be regarded as valid by describing particular occasions—such as participating in card games—in which a paraconsistent logician's willful abstention from the use of disjunctive syllogism would be foolhardy from a strategic perspective.

Chris Mortensen's reply appeals to a distinction between inferential situations, with Mortensen arguing that the mere fact that there exist *some* situations within which disjunctive syllogism is reasonable does not entail that it is valid in *all* situations. Given the intimate connection between disjunctive syllogism and the principle of explosion,[2] Mortensen's position works equally well for either inference:

> The position I propose is that although [the inference] is not universally valid, it is an acceptable mode of reasoning under certain circumstances... Many relevance people feel suspicious of [the inference] because it seems to break down in what might be called "abnormal" deductive situations, particularly inconsistent situations... On the other hand, [the inference] does seem to be a natural mode of inference in "normal" deductive situations, the kind encountered every day. (Mortensen 1983, p.37)

This act of partitioning particular types of situation reflects a philosophical assumption that is common—if frequently underdeveloped—among paraconsistent logicians. This is the assumption that there are certain circumstancess—*consistent* cases—in which the principle of explosion is "locally valid," a sentiment captured in Walter Carnielli, Marcelo Coniglio, and João Marcos' handbook article (2007) by the slogan:

CONTRADICTIONS + CONSISTENCY = TRIVIALITY

A deductive system that is paraconsistent in general but for which the principle of explosion can be deployed in targeted cases is frequently known as "gently explosive." A difficulty for the adequate formalization of this thesis in, *e.g.*, most species of relevant logic is that the languages they employ are insufficiently expressive to differentiate between the "consistent" situations Burgess describes and the "abnormal" situations to which Mortensen appeals.

Generally speaking, the field of *logics of formal inconsistency* attempts to codify this slogan by studying paraconsistent deductive systems in which the notion of a statement's being *consistent* can be represented in the language itself. Overwhelmingly, the languages employed by LFIs are enriched with a unary "consistency connective" $\circ$—either primitive or defined—that indicates that a formula "behaves consistently" or is "normal" in some sense. da Costa's intended interpretation—an interpretation that essentially lays the groundwork for the field of logics of formal inconsistency—is that the formula $\circ\varphi$

> can be interpreted as expressing the proposition that [$\varphi$] is not paradoxical, or "behaves classically."(da Costa 1974, p.585)

The canonical interpretation of a formula $\circ\varphi$, then, is the assertion "$\varphi$ is consistent," and the operator $\circ$ serves as a formal mark indicating that the formula to which it applies is one about which we may reason classically. Hence, for a unary negation connective $\sim$, paraconsistency can be retained insofar the set $\{\varphi, \sim\varphi\}$ can be satisfied in some situations (*i.e.*, "abnormal" situations). In contrast, the additional expressive power allows us to assume a premise $\circ\varphi$, indicating that the situation in which we work is "normal" or "classical" (at least with respect to $\varphi$), formally represented by the unsatisfiability of the set of formulae $\{\varphi, \sim\varphi, \circ\varphi\}$.

## 1.2. Logics of Formal Inconsistency

In the more precise sense described in Carnielli, Coniglio and Marcos (2007), a deductive system with a consequence relation $\vdash$ is identified as a logic of formal inconsistency when it meets the following criteria:

**Definition 1.** *A deductive system is an LFI with respect to a negation $\sim$ if:*

*a* *there are $\Gamma$, $\varphi$, $\psi$ such that $\Gamma, \varphi, \sim\varphi \nvdash \psi$, and*

*b* *there is a set of formulae $\bigcirc(p)$ depending only on $p$ such that:*

- *there are $\varphi$, $\psi$ such that $\bigcirc(\varphi), \varphi \nvdash \psi$ and $\bigcirc(\varphi), \sim\varphi \nvdash \psi$*
- *for all $\Gamma$, $\varphi$, $\psi$, we have: $\Gamma, \bigcirc(\varphi), \varphi, \sim\varphi \vdash \psi$*

Long before the term was coined, logics of formal inconsistency had been described in the literature. Earliest of these was Stanisław Jaśkowski's *discussive logic* D2 of (1948). Possibly most readily identifiable within the tradition itself is a family of systems described by da Costa in (1974). In each of da Costa's systems $\{\mathsf{C}_n \mid 1 \leq n\}$, the consistency connective is a defined notion corresponding to something like a "depth" of consistency. In, *e.g.*, $\mathsf{C}_1$, the formula $\circ\varphi$ acts as a natural shorthand for the formula $\sim(\varphi \wedge \sim\varphi)$, *i.e.*, a rejection of the claim that $\varphi$ is a contradiction.

The central notion that explosion is a legitimate inference in consistent or classical circumstances receives a very natural representation in $\mathsf{C}_1$—as well as other LFIs—as the axiom scheme:

**(bc1)** $\circ\varphi \to (\varphi \to (\sim\varphi \to \psi))$

Moreover, each member of the hierarchy $\{\mathsf{C}_n \mid 1 \leq n\}$ includes so-called *propagation axioms* that assert that the consistency of two subformulae is necessarily inherited by a complex formula counting them as subformulae:

**(ca1)** $(\circ\varphi \wedge \circ\psi) \to \circ(\varphi \wedge \psi)$
**(ca2)** $(\circ\varphi \wedge \circ\psi) \to \circ(\varphi \vee \psi)$
**(ca3)** $(\circ\varphi \wedge \circ\psi) \to \circ(\varphi \to \psi)$

*N.b.* that when working with deductive systems in which $\circ$ is defined, the explicit statements of these schema are tailored to the particular definition of $\circ$ and may differ from system to system.

In the literature on LFIs, (**bc1**) and the propagation axioms (**ca1–3**) are joined by a wide range of possible axiom schema, each serving as a formal representation of some further property of consistency. Most important for the present investigation are the schema $(\mathbf{cc})_\mathbf{n}$—first identified by Marcos in his axiomatization of the LFI mCi found in (2008). The investigation in the present paper centers around a particular instance of the schema in which $n = 0$, *i.e.*, the axiom scheme $(\mathbf{cc})_0$:

$(\mathbf{cc})_0 \quad \circ\circ\varphi$

According to the canonical reading of the consistency connective, then, $(\mathbf{cc})_0$ is read as the statement that for any formula $\varphi$, the assertion that $\varphi$ is consistent is *itself* consistent.

Now, the axiom scheme (**bc1**) is an apparently perfect codification of the fundamental thesis motivating the development of LFIs and is therefore assumed nearly universally in this setting. The adoption of $(\mathbf{cc})_0$, on the other hand, is far less ubiquitous. While it is a key ingredient in mCi, the axiom scheme fails to hold many of the central logics of formal inconsistency. $(\mathbf{cc})_0$—with "$\circ$" suitably defined—fails to be admissible in any of da Costa's hierarchy of systems $\{\mathsf{C}_n \mid 1 \leq n\}$, for example.

While there is little discussion of the axiom $(\mathbf{cc})_0$ in the canonical literature on LFIs, we now proceed to make an investigation into the intuitive merits of the axiom by looking to some *extracanonical* logics of formal inconsistency.

## 2. Halldén's Logic of Nonsense

In general, *logics of nonsense* are deductive systems which aim to reconcile a theory of deduction with the thesis that some statements are meaningless or nonsense. Where the notion of *meaninglessness* is understood as the state in which a well-formed statement has no corresponding *truthbearer*, such statements cannot be said to be true or false. According to such theories, then, there exist inferential situations in which the classical, bivalent logic championed by Gottlob Frege and Bertrand Russell is inadequate.

The specter of grammatical yet meaningless statements that are neither true nor false is frequently encountered in philosophical contexts. For example, one type of a purportedly meaningless statement is a so-called *category mistake*, *e.g.*, a statement such as "the square root of Socrates is irrational" in which a predicate ("the square root of $x$ is irrational") is applied to an object (Socrates) in an apparently nonsensical fashion. The statement is apparently grammatical; whether it is *meaningful* is less clear. It is arguably plausible to suggest that such statements are indeed nonsense—grammatical yet non-significant—and thus to demand that a correct theory of deduction be flexible enough to give accounts of meaningless statements. Logics of nonsense profess to give such a correct theory. For the interested reader, Krystyna Piróg-Rzepecka's (1977) is a very good sourcebook cataloguing the formal details and discussing the merits of the canonical systems of nonsense logic.

## 2.1. Halldén's Logic of Nonsense

The first work on deduction explicitly aiming to provide a semantical analysis of meaninglessness was Dmitri Bochvar's (1938), in which Bochvar defined the propositional three-valued logic $\Sigma$ and its subsystem $\Sigma_0$ in first-order formulations. While the topic of apparently meaningless statements was taken up by Whitehead and Russell in (1963), the solution posed was *syntactical* in nature, that is, meaningless statements are uniformly dismissed as *ungrammatical*. In contrast, logics of nonsense posit a *semantical* solution to the treatment of *prima facie* meaningless statements.

In his (1949), Sören Halldén independently introduces his own logic of nonsense $\mathsf{C}$ that shares the truth-functional matrices with $\Sigma$. Halldén's system differs from $\Sigma$ primarily with respect to the connectives considered to be primitive and in Halldén's (not uncontroversial) decision to include the semantical state corresponding to meaninglessness among the designated values. Halldén, like Bochvar, considers the standard truth-functional connectives in two modes: The "internal" connectives—identified as those used in the *Principia Mathematica*—and the "external" connectives that implicitly assert their meaningfulness.

As presented by Halldén, the language of $\mathsf{C}$ takes as primitive connectives the meaningfulness operator and the "internal" connectives of negation and conjunction. Insofar as the "internal" versions of disjunction and the material conditional are definable in the standard way, we include these connectives in our presentation. Given a countable set of propositional atoms **At**, the language may be represented in Backus-Naur form with $p \in \mathbf{At}$:

$$\varphi ::= p \mid {\sim}\varphi \mid \circ\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi$$

Semantically, $\mathsf{C}$ employs a set of truth values $\mathscr{V}_\mathsf{C} = \{\mathfrak{t}, \mathfrak{n}, \mathfrak{f}\}$ (interpreted as corresponding to "truth," "nonsense," and "falsity," respectively) and designated values $\mathscr{D}_\mathsf{C} = \{\mathfrak{t}, \mathfrak{n}\}$. Halldén, like Bochvar, takes internal negation and conjunction as primitive, but includes only the "meaningfulness" operator $\circ$ as a primitive external operator.

The many-valued semantics for $\mathsf{C}$ is represented by the following logical matrix:

**Definition 2.** *The matrix* $\mathfrak{M}_\mathsf{C}$ *is a tuple* $\langle \mathscr{V}_\mathsf{C}, \mathscr{D}_\mathsf{C}, f_\mathsf{C}^{\sim}, f_\mathsf{C}^{\circ}, f_\mathsf{C}^{\wedge} \rangle$ *such that:*

- $\mathscr{V}_\mathsf{C} = \{\mathfrak{t}, \mathfrak{n}, \mathfrak{f}\}$ *is a set of truth values*
- $\mathscr{D}_\mathsf{C} = \{\mathfrak{t}, \mathfrak{n}\}$ *is a set of designated values*
- *the truth functions* $f_\mathsf{C}^{\sim}$, $f_\mathsf{C}^{\circ}$, $f_\mathsf{C}^{\wedge}$ *are as follows:*

| $f_{\mathsf{C}}^{\sim}$ | |
|---|---|
| $\mathfrak{t}$ | $\mathfrak{f}$ |
| $\mathfrak{f}$ | $\mathfrak{t}$ |
| $\mathfrak{n}$ | $\mathfrak{n}$ |

| $f_{\mathsf{C}}^{\circ}$ | |
|---|---|
| $\mathfrak{t}$ | $\mathfrak{t}$ |
| $\mathfrak{f}$ | $\mathfrak{t}$ |
| $\mathfrak{n}$ | $\mathfrak{f}$ |

| $f_{\mathsf{C}}^{\wedge}$ | $\mathfrak{t}$ | $\mathfrak{f}$ | $\mathfrak{n}$ |
|---|---|---|---|
| $\mathfrak{t}$ | $\mathfrak{t}$ | $\mathfrak{f}$ | $\mathfrak{n}$ |
| $\mathfrak{f}$ | $\mathfrak{f}$ | $\mathfrak{f}$ | $\mathfrak{n}$ |
| $\mathfrak{n}$ | $\mathfrak{n}$ | $\mathfrak{n}$ | $\mathfrak{n}$ |

*Truth functions for the connectives* $\vee$ *and* $\rightarrow$ *may be defined from these matrices by the usual definitions.*

Semantical validity in C is understood as the preservation of designated values (*i.e.*, the preservation of non-falsity) from premisses to conclusion with respect to all valuations.

The foregoing many-valued semantics for C reflect an assumption that there are certain inferential situations in which the inference of *detachment* fails without qualification. Suppose, for example, that a valuation $v$ assigns the parameter $p$ a value of $\mathfrak{n}$ and the parameter $q$ the value of $\mathfrak{f}$. Then although the formulae $p$ and $p \rightarrow q$ will both take a designated value, $q$ will not be designated, witnessing that in general a formula $\psi$ cannot be inferred from the set of premises $\{\varphi, \varphi \rightarrow \psi\}$. In particular, the inference fails precisely in those inferential situations in which a conditional includes a meaningless antecedent and a false consequent; in all other cases—*i.e.*, those that behave "classically"—detachment succeeds in preserving designated values from premisses to conclusion. The existence of special cases in which the rule preserves designated values corresponds to the validity of a *restricted* version of detachment in C.

The cases in which the rule may be applied are distinguished by Halldén's introduction of the notion of a formula's being *covered*.

**Definition 3.** *An instance of a formula* $\varphi$ *is said to be* covered in *a formula* $\psi$ *if a)* $\varphi$ *is a subformula appearing in* $\psi$ *and b) every instance of* $\varphi$ *appearing in* $\psi$ *appears within the scope of the meaningfulness connective "*$\circ$*."*

This semantical motivation informs the proof theory for C. From an axiomatic perspective, the syntactical consequence relation $\vdash_{\mathsf{C}}$ can be given a standard, Hilbert-style account. Let "$\varphi \equiv \psi$" be shorthand for the formula $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. Then consequence in C can be defined as follows:

**Definition 4.** *The syntactic consequence relation* $\vdash_{\mathsf{C}}$ *is defined so that* $\Gamma \vdash_{\mathsf{C}} \varphi$ *holds when there exists a proof of* $\varphi$ *from premises in* $\Gamma$ *by appealing to substitution instances of the following axiom schema:*

- **CL** *Standard axioms for classical propositional logic*
- **C1** $\circ p \equiv \circ{\sim}p$
- **C2** $\circ(p \wedge q) \equiv (\circ p \wedge \circ q)$
- **C3** $p \rightarrow \circ p$

*and by applying the following rules:*

**CR1** *If $\varphi$ and $\varphi \rightarrow \psi$ are theorems and every variable covered in $\psi$ is also covered in $\varphi$, then $\psi$ is a theorem*

**CR2** *If $\varphi$ is a theorem and $p \in$ **At**, then for all formulae $\psi$, $\varphi[p := \psi]$ is a theorem*

*where "$\varphi[p := \psi]$" is the formula in which instances of p appearing in $\varphi$ are uniformly replaced by the formula $\psi$.*

During the first few decades of sustained study of logics of formal inconsistency, the fact that Halldén's C is in fact an LFI went unacknowledged; no mention of Halldén is made, for example, in the historical remarks found in Carnielli, Coniglio, and Marcos' excellent handbook article (2007). It is only very recently that this fact has been acknowledged; in (2016), Carnielli and Rodrigues mention *en passant* that C is an LFI and Omori makes a more thorough formal investigation into this feature in (2016). Insofar as we are interested in Halldén's system *qua* LFI, the features of C that lead it to be counted in this family are important to identify. In order to bring these features to the fore, it will behoove us to explicitly—if succinctly—demonstrate this fact.

We may easily provide an explicit demonstration that C meets the criteria outlined in Definition 1:

**Observation 1** (Carnielli, Rodrigues, Omori)**.** C *is an LFI with respect to* $\sim$.

*Proof.* It may be easily confirmed that clause *a* is satisfied, as $p, \sim p \nvdash q$ in C. To verify clause *b*, we let the set $\{\circ p\}$ serve as our $\bigcirc(p)$ and note that for any $\psi$, if $\circ\psi$ takes a designated value, then either $\psi$ or $\sim\psi$ must be assigned a value of $\mathfrak{f}$. □

From the short proof of Observation 1 we observe that, when C is viewed as an LFI, Halldén's unary meaningfulness connective "∘" plays the role of the familiar consistency connective.

Despite this, it also must be conceded that the progenitors of logics of nonsense and LFIs afforded distinct interpretations to the consistency connective. This divide is perhaps exacerbated by the fact that proponents of logics of nonsense and proponents of LFIs both allow extraordinarily diverse ranges of applications for their respective frameworks. The interpretations given to "∘" in the setting of logics of nonsense include, for example, interpretations according to which $\circ\varphi$ asserts the syntactical well-formedness of the string of symbols $\varphi$ (in Lennart Åqvist's (1962)) or in which $\circ\varphi$ indicates that there is a primitive recursive function calculating $\varphi$ (an implicit consequence of Stephen Kleene's discussion of Halldén's matrices in (1952).

The interpretations of "∘" in the LFI tradition are similarly diverse; in Carnielli, de Amo, and Marcos' (2000), "$\circ\varphi$" corresponds to cases in which a database does not count both $\varphi$ and "$\sim\varphi$ among its entries, does not include both "$\varphi$" and "$\sim\varphi$." Despite the apparent heterogeneity of the proposed applications for the two frameworks, there remains a great deal of convergence. The grounds for the identification of Halldén's interpretation consistency connective with the canonical interpretation following da Costa is, consequently, not limited to the mutual satisfaction of Definition 1 by their constituent connectives. Rather, we can offer semantical and philosophical evidence for their identification.

Despite the *prima facie* asymmetry between the terms "meaningfulness" and "consistency," on further inspection, the semantical behavior Halldén ascribes to "$\circ$" accords with the canonical reading of $\circ$ as a mark of consistency. By freely identifying the meaningful with the consistent, we are able to recognize the restriction Halldén imposes on detachment to be a tacit recognition that inference in consistent situations differs significantly from inference in inconsistent situations. In other words, in Halldén's framework, the connective "$\circ$" acts as a signpost reflecting when classical inferences are admissible, mirroring the motivations expressed by da Costa. Indeed, the infectiousness of nonsense and the propagation of consistency can be understood as two sides of the same medal.

As Halldén observes, many of the theses concerning nonsense are made concrete by theorems of $\mathsf{C}$. In particular, each of the $\mathsf{C}_1$ axioms $(\mathbf{ca1})$–$(\mathbf{ca3})$ are easily derived from the axiom **C2**. Moreover, in virtue of Halldén's treating the nonsense value $\mathfrak{n}$ as a designated value, the formula $\circ\varphi$ is true in a model precisely when $\varphi$ is "consistent" in the sense that $\varphi$ and $\sim\varphi$ do not both hold. Semantically, then, the operator "$\circ$" is, quite literally, a consistency connective.

## 2.2. Meaningfulness and $(\mathbf{cc})_0$

In contrast to the lack of a consensus with respect to the validity of $(\mathbf{cc})_0$ the LFI tradition, proponents of logics of nonsense have more or less taken its status for granted. Preceding Halldén, an analogue of the thesis can be confirmed to hold of Bochvar's $\Sigma$ (as "it is not meaningless that it is not meaningless that *p*"). In Halldén's wake, the same can be confirmed of the further logics of nonsense produced by, *e.g.*, Lennart Åqvist in (1962), Krister Segerberg in (1965), and Peter Woodruff in (1973). In Åqvist's $\mathsf{A}$, Segerberg's $\mathsf{D}$ (and his $\mathsf{E}$ and $\mathsf{F}$), and Woodruff's intuitionistic $\mathsf{DI}$, notational variants of $(\mathbf{cc})_0$—whether the notion of "meaningfulness" is primitive or defined—can be shown to hold.

Now, the validity the axiom $(\mathbf{cc})_0$ is one thing about which Halldén expressed strong reservations and that the thesis should enjoy such constancy throughout the many critical examinations of Halldén's 1949 is curious. In contrast to his successors who picked up his project of developing logics of nonsense, Halldén is explicitly suspicious of the validity of $(\mathbf{cc})_0$:

> We will now take into consideration the possibility that the calculus C... is not wholly correct. According to it all propositions of the type *p is meaningful* should be meaningful. This is a rather far-reaching assertion which should be met with some doubt. (Halldén 1949, p.56)

While considering the theoremhood of the formula $\circ\circ\varphi$, Halldén complements C by defining a weaker calculus K. Although the full details are outlined in (1949), K is defined by imposing a stronger restriction that detachment holds for a sentence $\varphi \to \psi$ if every variable appearing in $\psi$ within the scope of $n$ many "∘"s appears within the scope of *at least* $n$ many instances of the symbol "∘" in $\varphi$. Notably, as Halldén shows, $(\mathbf{cc})_0$ is not a theorem of K. This calculus is defined proof-theoretically and no semantics are offered for it.

Despite his reservations, Halldén concedes that he "know[s] no conclusive reason for the falsity of '$\circ\circ p$'."(1949, p.56) Stopping short of providing a concrete counterexample or philosophical picture in which $(\mathbf{cc})_0$ should fail, Halldén merely speculates that counterexamples could emerge if "∘" is taken to be "systematically ambiguous," so that "we may interpret the two occurrences of '∘' in the discussed formula as expressing two different concepts of meaningfulness." (1949, p.57) Despite Halldén's omission of counterexamples, the *context* of Halldén's analysis indeed appears fertile enough to bring the validity of $(\mathbf{cc})_0$ into question.

The range of philosophical matters Halldén intends to treat is quite diverse, including the analysis of vague predicates and the semantics of the ethical theory of emotivism. Chief among these proposed applications, however, is Halldén's suggestion that C adequately models the species of meaninglessness yielded by a central tenet in the school of logical positivism: The *empiricist criterion of meaning*.

Each among the cluster of theories of meaning that fall under this label stipulates that a necessary condition for the meaningfulness of a statement is that it can be *verified* by either mathematical or empirical means. One of the archetypal characterizations of the empiricist criterion of meaning is found in Carl Hempel's assertion that a statement is meaningful

> only if it is either (1) analytic or self-contradictory or (2) capable, at least in principle, of experiential test. (Hempel 1950, p.108)

Famously, a number of claims—*e.g.*, theological and metaphysical theses—fail to meet these criteria and were judged by the positivists to be meaningless. Rudolph Carnap—an important figurehead in the development of this school—provided the case of *category mistakes* as a salient example of a statement (or a *pseudo-statement*) meaningless in virtue of failing to meet this criterion. For example, the sentence "Caesar is a prime number," Carnap asserts, is meaningless in virtue of the fact that the predicate "$x$ is a prime number" can "be neither affirmed nor denied of a person."

(Carnap 1931, p.68) While, perhaps, one can provide a meaningful test of the statement "Caesar is a general"—by, say, consulting historical records—no meaningful test of the *primeness* of Caesar can be articulated, much less put into practice.

On this interpretation of C, "$\circ p$" is read as "there is an experiential means of verifying (or falsifying) the statement $p$," and the validity of $(\mathbf{cc})_0$ is equivalent to a statement along the following lines:

For any sentence $p$, there exists a method to verify whether or not there exists a method to verify $p$.

So—on the Halldén-style reading of LFIs—the existence of statements whose verification conditions cannot in principle be verified is equivalent to the failure of $(\mathbf{cc})_0$. In this setting, I would like to consider an apparent counterexample to $(\mathbf{cc})_0$.

## 2.3. A Plausible, Carnapian Counterexample

To provide a case in which $(\mathbf{cc})_0$ appears to fail, we consider another of Carnap's illustrations. Initially, Carnap carefully distinguishes between several species of meaninglessness. In particular, Carnap asserts that the type of meaninglessness triggered upon asserting a category mistake—the semantical category to which "Caesar is even" belongs—is distinct from the sort of meaninglessness that emerges when one attempts to employ *a priori* nonsensical predicates, an illustration of which Carnap provides by the example of a predicate "$x$ is *teavy*." According to the Carnapian line, the *artificiality* of the term "teavy" is so great that in principle we are unable to envision—much less engineer—the sort of criterion that could be employed in an empirical test of a sentence in which "teaviness" appears. Consequently, the positivist argument suggests that with the possibility of an experiential test eliminated, a statement like "the dog is teavy" must be rejected as nonsense.

There remains an important distinction between the cases of "Caesar is prime" and "Caesar is teavy," as Carnap concedes. The term "teaviness" is unlike the term "primeness" to the extent that "teaviness" is utterly inscrutible. It is the *alien* nature of the predicate that leads to Carnap to reject the possibility of any method to verify whether the sentence "the dog is teavy" holds. The case of "teaviness" is especially salient, as its artificiality appears to yield not only the meaninglessness of statements such as "the dog is teavy," but also seems to induce the very type of second-order unverifiability that calls $(\mathbf{cc})_0$ into question.

To bring this into the light, let us consider the procedure by which we conclude that the assertion of the meaningfulness of the statement "Caesar is a prime number" is itself a meaningful statement. In the positivist's language, we may decode this sentence as the statement asserting that there is some experiential test capable of determining whether or not there exists an experiential test verifying the truth of "Caesar is a prime number." The positivist position, of course, is that there is no such test and, more importantly, that this is *demonstrably* so by appeal to the following *experiential test*: The first step is to determine the *type* of the predicate "...is a prime number" (or, in Halldén's terminology, to determine its *v-range*) and then determine the *type* of the name "Caesar." Having successfully extrapolated the types of these terms, we note that the predicate "...is a prime number" applies only to natural numbers, a class of which the referent of "Caesar" is not a member. Because of the type mismatch between "Caesar" and "...is a prime number," we reason that there can be no experiential test of the statement and we may conclude that the complex sentence is meaningless.

But in making this experiential test explicit, "primeness" and "teaviness" are revealed to be markedly different beasts. Within the positivist framework, the above algorithm presupposes that *one is capable of deducing the specific* type *of the relevant predicate*, that is, of determining the predicate's second-order properties. Because we are so well-acquainted with the notion of *primeness*, evaluating the meaningfulness of "Caesar is a prime number" is on its face a trivial exercise. In contrast, the term "teaviness" *does not correspond to a unique property* and, consequently, the predicate "...is teavy" *cannot be verified to be of any particular type*. In a strong sense, any claim of the form "the predicate '...is teavy' is of such-and-such a type" is a second-order category mistake. Hence, we are in principle unable to surmise the type of this predicate and are unable to effect the procedure that would tell us whether "Caesar is teavy" is or is not a category mistake, *i.e.*, the matter of whether or not Caesar is teavy *is not verifiable*. But this is to say that the matter of whether or not "Caesar is teavy" is verifiable is not itself verifiable.

One may object that the Carnapian line explicitly gives an effective means of verifying that there can be no experiential test of "Caesar is teavy," and thus, that the statement is meaningless. And indeed, this procedure shows that it is *not true* that such an experiential test can be given. In the context of meaningless statements, however, there is an important distinction between a statement's *untruth* from its *falsity*, *e.g.*, that "Caesar is prime" is not true does not entail its falsity. Hence, to successfully apply the empiricist criterion of meaning to draw the conclusion that "Caesar is teavy" is meaningless requires that we have shown the stronger condition that it is *false* that such an experiential test exists.

## 3. $(\mathbf{cc})_0$ in Priest's Cointuitionistic Logic daC

We now turn our attention to a further extracanonical logic of formal inconsistency: The cointuitionistic logic daC described by Graham Priest in (2009). The calculus is a fragment of Cecylia Rauszer's propositional *Heyting–Brouwer logic* (HB), developed in, *e.g.*, (1974) and (1977) (and, hence, much of what follows applies to HB as well as to daC). HB enriches propositional intuitionistic logic (Int) by the addition of *coimplication* and *conegation* connectives, symbolized in this paper by "$\prec$" and "$\neg$." As coimplication and conegation are in a sense dual to implication and negation, the formal aspects of HB have proven to be interesting objects of study. In (2009), Priest isolated the $\{\neg, \wedge, \vee, \rightarrow\}$-fragment of HB and offered it as a sort of intuitionistic logic with a "dualised" negation.

Priest demonstrated that the system is a conservative extension of da Costa's $\mathsf{C}_\omega$—motivating Priest's use of the nomenclature "da Costa logic." Although $\mathsf{C}_\omega$ is not itself an LFI, it is described as a "limit" of such systems; hence, it is a natural task investigate whether or not daC is a logic of formal inconsistency. Priest leaves this matter unaddressed, but the question is taken up in Ferguson (2014), in which daC (and, consequently, HB) is shown to be an LFI in which the consistency connective is definable.

## 3.1. Priest–da Costa Logic as an LFI

As stated, the language of daC exchanges the familiar intuitionistic negation for a paraconsistent negation, which will be represented here as "$\neg$." More precisely, in Backus-Naur form, the language of daC is as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi$$

Unsurprisingly, the Kripke semantics for daC hew to the familiar semantics for Int.

Recall that a *Kripke frame* $\mathfrak{F} = \langle W, R\rangle$ is a set of points $W$ and a *Kripke model* is a frame together with a valuation $v$ mapping atomic formulae $p_0, p_1, \ldots$ to subsets of points such that for any atom $p$, if $w \in v(p)$ and $wRw'$, then $w' \in v(p)$. More formally, we have the following definition:

**Definition 5.** *A* daC model *is a tuple* $\langle W, R, v\rangle$ *where* $W$ *is nonempty and* $R$ *is a reflexive, transitive, and antisymmetric*[3] *binary relation on* $W$. $v$ *is a function from* **At** *to* $\wp(W)$ *obeying the* heredity constraint *that if* $w \in v(p)$ *and* $wRw'$, *then* $w' \in v(p)$.

Notably, this property of *heredity* for truth also entails that whenever $w \notin v(p)$ and $w'Rw$, then $w' \notin v(p)$.

The model determines a forcing relation, which we properly define as a relation holding between a point in a model and a formula, *e.g.*, $\mathfrak{M}, w \Vdash \varphi$, though when the model is clear from context, this will be abbreviated to $w \Vdash \varphi$. This forcing relation is defined recursively as follows:

- $w \Vdash p$ *iff* $w \in v(p)$ for atoms $p$
- $w \Vdash \neg\varphi$ *iff* there is some $w'$ such that $w'Rw$ for which $w' \nVdash \varphi$
- $w \Vdash \varphi \wedge \psi$ *iff* $w \Vdash \varphi$ and $w \Vdash \psi$
- $w \Vdash \varphi \vee \psi$ *iff* $w \Vdash \varphi$ or $w \Vdash \psi$
- $w \Vdash \varphi \rightarrow \psi$ *iff* at all $w'$ with $wRw'$, if $w' \Vdash \varphi$ then $w' \Vdash \psi$

Validity of an inference $\Gamma \vDash_{\mathsf{daC}} \varphi$ is once again just the familiar intuitionistic notion, *i.e.*, that the inference is valid when for all daC models and points $w$, if all formulae in $\Gamma$ are true at $w$, then $\varphi$, too, is true at $w$. In particular, we will say that an inference $\Gamma \vDash \varphi$ is valid with respect to some class of models if this can be said for each point and each model in that class.

Two points about definable connectives in daC should be made. For one, the full system HB of which daC is a fragment includes a coimplication operator "$\prec$" with the following truth condition:

- $w \Vdash \varphi \prec \psi$ *iff* there is a $w'$ such that $w'Rw$, $w' \Vdash \varphi$, and $w'\varphi \nVdash \psi$

In case that we have a truth constant $\top$ that is true at all worlds by fiat, then these truth conditions clearly show that cointuitionistic negation is definable so that $\neg\varphi =_{df} \top \prec \varphi$.

Secondly, as shown in Priest (2009), daC can be thought of as a *conservative extension* of intuitionistic logic to the extent that intuitionistic negation is definable. Recall the truth condition for the intuitionistic negation "$\sim$":

- $w \Vdash \sim\varphi$ *iff* for all $w'$ such that $wRw'$, $w' \nVdash \varphi$

Then it is easy to confirm that a unary connective "$\dot{\sim}$" defined by the scheme $\dot{\sim}\varphi =_{df} \varphi \rightarrow \neg(\varphi \vee \neg\varphi)$ faithfully captures the behavior of intuitionistic negation.

A Hilbert-style proof theory for daC was first described by Castiglioni and Ertola in (2014), which is presented here.[4]

**Definition 6.** *The syntactic consequence relation $\vdash_{\mathsf{daC}}$ is defined so that $\Gamma \vdash_{\mathsf{daC}} \varphi$ holds when there exists a proof of $\varphi$ from premises in $\Gamma$ by appealing to substitution instances of the following axiom schema:*

**Int**$^+$ *Standard set of axioms for positive intuitionistic logic*
**EM** $p \vee \neg p$

*and by the application of the following rules:*

**dR1** *from* $\varnothing \vdash_{\mathsf{daC}} p \vee q$, *infer* $\varnothing \vdash_{\mathsf{daC}} \neg p \rightarrow q$
**dR2** *from* $\varphi$ *and* $\varphi \rightarrow \psi$ *infer* $\psi$

After having established that daC is an LFI, one of the contributions of Ferguson (2014) was to explore the sense in which daC and its extensions ("sdc-logics") exhibit the standard features discussed in the literature of LFIs. In order to succinctly describe the behavior of the definable consistency connective, we first review some perspicuous notation:

**Definition 7.** *For a Kripke frame* $\langle W, R\rangle$ *and a* $w \in W$*, we employ the following definitions:*

- $w^{\uparrow} = \{w' \in W \mid wRw'\}$
- $w^{\downarrow} = \{w' \in W \mid w'Rw\}$
- $w^{\searrow} = \bigcup\{w'^{\downarrow} \mid w' \in w^{\uparrow}\}$

The definition of the consistency connective described in Ferguson (2014) identifies the formula $\circ\varphi$ with $\dot{\sim}(\varphi \wedge \neg\varphi)$. Let the notation "$v(\varphi)$" denote the set of points at which formula $\varphi$ is true. Then, by considering the truth condition of the definiens, the defined consistency operator $\circ$ can be provided with the following forcing conditions:

- $w \Vdash \circ\varphi$ *iff* either $\begin{cases} w^{\searrow} \subseteq v(\varphi), \text{ or} \\ w^{\searrow} \cap v(\varphi) = \varnothing \end{cases}$

Then in Ferguson (2014) the following observation is proven:

**Observation 2.** daC *is an* **LFI** *with respect to* $\neg$

One interesting fact is that the standard propagation axioms break apart. It is shown in Ferguson (2014) that propagation holds over the *extensional* contexts of conjunction and disjunction:

**Observation 3.** $(\mathbf{bc1})$*,* $(\mathbf{ca1})$*, and* $(\mathbf{ca2})$ *are theorems of* daC

In contrast, propagation fails to hold for the *intensional* connective of implication:

**Observation 4.** $(\mathbf{ca3})$ *is not valid in* daC

We can observe here that $(\mathbf{cc})_0$, too, fails to hold in daC:

**Observation 5.** $(\mathbf{cc})_0$ *is not valid in* daC

*Proof.* Consider an arbitrary frame $\langle W, R\rangle$ including points $w, w', w'' \in W$ such that $wRw'$ and $wRw''$ for which $w'^{\uparrow} \cap w''^{\uparrow} = \varnothing$. We construct a valuation on $\langle W, R\rangle$ witnessing that $(\mathbf{cc})_0$ is invalid in such a logic.

Define $v$ in such a way that $v(p) = w'^{\uparrow}$ for some fixed atom $p$. Because $w'^{\uparrow}$ and $w''^{\uparrow}$ are disjoint, $v(p) \cap w''^{\uparrow}$ is empty, entailing that $v(p) \cap w''^{\searrow}$ is also empty. Hence, $w'' \Vdash \circ p$. However, at $w'$, *both* $w' \Vdash p$ and $w' \Vdash \neg p$ hold, so $w'$ witnesses that $\dot{\sim}(p \wedge \neg p)$—*i.e.*, $\circ p$—must fail at $w$. Because $w \nVdash \circ p$ and $wRw''$, we also conclude that $w'' \Vdash \neg\circ p$.

So $wRw''$ and $w'' \Vdash \circ p \wedge \neg\circ p$, whence we infer that $w \nVdash \dot{\sim}(\circ p \wedge \neg\circ p)$, *i.e.*, $w \nVdash \circ\circ p$. □

That both (**ca3**) and $(\mathbf{cc})_0$ fail to hold in daC suggests that there exists some connection between them. As we will see, the fates of the two axiom schema are closely tied in this system. Before examining this relationship, we will first discuss the matter of interpreting "$\circ$" in daC.

### 3.2. Consistency-as-Verifiability in daC

Unlike intuitionistic logic, whose formalization was ancillary to the philosophical assumptions underlying intuitionistic practice, daC has given resistance to a natural interpretation. This is somewhat expected due to the fact that daC is a fragment of HB. Rauszer's (1980) describes the motivation for studying HB only in terms of the elegance of its formal aspects. She notes that from the investigations of Rauzer (1974) and (1977), the system is best motivated by model-theoretic concerns:

> [I]t appeared that an intuitionistic logic with two negations and two implications, dual to itself, would have a more elegant algebraic and model-theoretic theory than an ordinary intuitionistic logic. (Rauzer 1980, p.5)

For his part, Priest omits any discussion of interpreting the paraconsistent negation of daC. Interpreting the dual intuitionistic connectives has historically been more complicated than the intuitionistic connectives. It is on its face difficult to extend, for example, the Brouwer-Heyting-Kolmogorov interpretation to the case of HB—and thus, to daC (although see the interpretation provided by Heinrich Wansing in (2010). In his study of the general case of the dual intuitionistic connectives in (1998), Frank Wolter concedes as much. Recalling that cointuitionistic negation $\neg$ is definable in HB as $\top \prec \varphi$, Wolter writes:

> We do not see however a natural interpretation of $\prec$ in terms of the interpretation of Int as the logic of constructive proofs. (Wolter 1998, p.384)

However, as Wolter points out, the interpretation of Andrzej Grzegorczyk of intuitionistic logic as a logic of *scientific research* in (1964) is far more amenable to the task of interpreting dual intuitionistic operations, and may be extended naturally to the dual intuitionistic negation.

Grzegorczyk works with the following intuitive description of scientific research:

> Scientific research (*e.g.* an experimental investigation) consists of the successive enrichment of the set of data by new established facts obtained by means of our method of inquiry. When making inquiries we question Nature and offer her a set of possible answers. Nature chooses one of them. (Grzegorczyk 1964, p.596)

The formal picture becomes sharper as Grzegorczyk proceeds to suggest that "scientific research may be conceived as a triple $R = \langle J, \mathrm{o}, P\rangle$," (1964, p.596) where $J$ is the set of all potential experimental data, o represents a researcher's initial information, and $P$ represents possible paths along which the research can unfold.

Wolter again suggests that coimplication—and, implicitly, the paraconsistent negation of daC—enjoy an entirely natural interpretation in the context of scientific research.

> [F]or the interpretation of Int as the logic of scientific research in the sense of Grzegorczyk [*cf.* (1964)], the connective $-\!\!<$ has a clear meaning. (Wolter 1998, p.384)

This "clear meaning" works for both coimplication and the paraconsistent negation of daC. Wolter's adaptation of the Grzegorczyk-style interpretation of coimplication is that "truth of $\varphi -\!\!< \psi$ at a point $x$ means that at some moment in the past (of $x$) $\varphi$ was known while $\psi$ was unknown." (Wolter 1998, p.354) It follows, on this picture, that to assert a conegated formula $\neg\varphi$ means simply that $\varphi$ was not included among a researcher's (or among some researchers') initial information, *i.e.*, o. This temporal picture is reinforced by Piotr Łukowski's translation of the connectives of HB into the tense logic $\mathsf{K}_t\mathsf{T4}$ described in (1996), in which "$\neg\varphi$" is read as "$\varphi$ was at some past point false." Let us now consider how this Grzegorczyk–Wolter-style interpretation treats the consistency connective.

In a strong sense, intuitionistic logic plays a the role of an *oracle* in daC to the extent that whenever $\varphi$ is provable in Int, $\neg\varphi$ is unsatisfiable in daC, *i.e.*, $\neg\varphi$ cannot be asserted without triviality. Likewise, for intuitionistic theorems $\sim\varphi$, the assumption of $\varphi$ as a hypothesis triggers triviality in daC. We enjoy paraconsistency with respect to some wide field of propositions—empirical statements of the sciences, for example—but there remain propositions for which inconsistency is catastrophic. If we think along Jaśkowskian lines, a collection of individual researchers embarking on a joint investigation may disagree with respect to, *e.g.*, political, empirical, or moral assertions but there must be unanimity in the case of what is logically—that is, intuitionistically—demonstrable.

In this sense, intuitionistic provability *bounds* reasoning in Priest–da Costa logic. The reasoner in daC enjoys the ability to hold inconsistent premises without triviality *so long* as one respects intuitionistic demonstrations, that is, within the bounds allowed by Int. The consistency connective, analyzed in this way, again allows an interpretation of consistency-as-verifiability. Rehearsing the Grzegorczyk–Wolter interpretation, the assertion of a formula $\circ\varphi$ at some stage in an investigation is tantamount to a type of decidability or verifiability in principle:

$$\circ\,\varphi \text{ holds iff } \begin{cases} \text{there have never---and will never be---any doubts about} \\ \quad \text{the truth of } \varphi\text{, or} \\ \text{there will never be any confirmation of } \varphi. \end{cases}$$

In other words, the truth of a formula $\circ\,\varphi$ is to say that the rules of the investigation stipulate—with the force of logic—that there can be no disagreement concerning $\varphi$.

This interpretation of the consistency connective in daC is underscored by its relationship with the intuitionistic axiom of the *principle of weak excluded middle*. Because of the definability of intuitionistic negation within daC, the standard extraintuitionistic theses—and the superintuitionistic logics that they determine—are reflected in the structure of extensions of daC. Hence, we have a defined version of this axiom:

**WEM**$^{\dot{\sim}}$ $\quad \dot{\sim} p \vee \dot{\sim}\dot{\sim} p$

The principle of weak excluded middle, although first considered axiomatically by V. A. Jankov in (1968), was analyzed in the context of intuitionistic mathematics by L. E. J. Brouwer himself, as a principle of *testability*:

> Another corollary of the *simple* principle of the excluded third is the *simple principle of testability* saying that *every assignment* $\tau$ *of a property to a mathematical entity can be tested, i.e., proved to be either non-contradictory or absurd.* (Brouwer 1983, p.92)

In the broader field of intuitionistic logic, a formula $\varphi$ is said to be *testable* in a theory precisely when $\sim\varphi \vee \sim\sim\varphi$ is provable in that theory. Now, evidence for the interpretation of $\circ$ as a type of verifiability can be found in its *hypertestability*, *i.e.*, that if $\circ\varphi$ is provable in a theory, then $\varphi$ is testable in that theory:

**Observation 6.** *For a* daC *theory* $T$, *if* $T \vdash_{\mathsf{daC}} \circ\,\varphi$ *then* $T \vdash_{\mathsf{daC}} \dot{\sim}\varphi \vee \dot{\sim}\dot{\sim}\varphi$, i.e., $\varphi$ *is* testable *in* $T$.

*Proof.* We prove that $\circ\,\varphi \to (\dot{\sim}\varphi \vee \dot{\sim}\dot{\sim}\varphi)$ is a theorem of daC. Suppose for contradiction that in some model, $w \Vdash \circ\,\varphi$—*i.e.*, $w \Vdash \dot{\sim}(\varphi \wedge \neg\varphi)$—although $w \nVdash \dot{\sim}\varphi \vee \dot{\sim}\dot{\sim}\varphi$. Then $w \nVdash \dot{\sim}\varphi$ and $w \nVdash \dot{\sim}\dot{\sim}\varphi$.

From this, we draw two inferences. In the first case, because $w \nVdash \dot{\sim}\varphi$, there exists a point $w'$ such that $wRw'$ at which $w' \Vdash \varphi$. In the second case, because $w \nVdash \dot{\sim}\dot{\sim}\varphi$, there exists a point $w''$ accessible from $w$ such that $w'' \Vdash \dot{\sim}\varphi$ and, consequently, such that $w'' \nVdash \varphi$.

By the property of heredity, we know that $w' \Vdash \dot{\sim}(\varphi \wedge \neg\varphi)$ as well, whence we infer that $w' \nVdash \varphi \wedge \neg\varphi$ and that either $w' \nVdash \varphi$ or $w' \nVdash \neg\varphi$. Both cases lead to contradiction. The first case directly contradicts the observation that $w' \Vdash \varphi$. In the second case, that $wRw'$ entails that $w \Vdash \varphi$ and by heredity, this means that $w'' \Vdash \varphi$, contradicting the earlier observation that $w'' \nVdash \varphi$. □

However, this notion of consistency-as-verifiability is not *equivalent* to testability.

**Observation 7.** *The statement "*$\circ\,\varphi$*" is strictly stronger than the statement that* $\varphi$ *is testable.*

*Proof.* Consider a simple Kripke frame with two points $w$ and $w'$ such that $wRw'$, and consider a model such that $v(p) = \{w'\}$. Then as $w' \Vdash p$, $w' \Vdash \dot{\sim}\dot{\sim}p$ and, consequently, $w' \Vdash \dot{\sim}p \vee \dot{\sim}\dot{\sim}p$, *i.e.*, $p$ is testable at $w'$. However, because $w \nVdash p$, $w' \Vdash \neg p$, entailing that $w' \Vdash p \wedge \neg p$. This requires that $w' \nVdash \dot{\sim}(p \wedge \neg p)$, *i.e.*, that $w' \nVdash \circ\, p$. □

So, that $\circ\,\varphi$ is true corresponds to $\varphi$'s satisfying some epistemic property that stronger than intuitionistic testability; it is a type of *hypertestability*.

Despite the inequivalence of the consistency of a formula $\varphi$ and its testability, there remains a surprising equivalence between the *consistency of the consistency* of $\varphi$ and its testability, that is, an equivalence between $(\mathbf{cc})_0$ and the principle of weak excluded middle.

## 3.3. The Convergence of $(\mathbf{cc})_0$ and Testability

The aforementioned principle of weak excluded middle $\mathbf{WEM}^{\dot{\sim}}$ plays a particularly important role during the discussion daC as an LFI in Ferguson (2014). Intuitionistically, the axiom scheme corresponds to a superintuitionistic logic known as the "logic of weak excluded middle" or "Jankov's logic." In Ferguson (2014), the name "KC" was employed, so that the corresponding extension of daC was called "$\mathsf{KC}^{\dot{\sim}}$."

Formally, we define this system by using the notation "$\mathsf{L} \oplus \mathbf{A}$" to indicate the deductive system corresponding to a logic L enriched with axiom **A** and closed under the rules of L.

**Definition 8.** $\mathsf{KC}^{\dot{\sim}} = \mathsf{daC} \oplus \mathbf{WEM}^{\dot{\sim}}$

In order to semantically characterize $\mathsf{KC}^{\dot{\sim}}$, we recall a well-known frame condition of *forward convergence*:

**Definition 9.** *A frame* $\langle W, R\rangle$ *is* forward convergent *if for all* $w, u, v \in W$*:*

$$\textit{if } wRu \textit{ and } wRv \textit{ then there is a } y \in W \textit{ such that } uRy \textit{ and } vRy$$

We will say that a model is forward convergent if its underlying frame is forward convergent.

Now, in the case of superintuitionistic logics, KC itself is characterized by forward convergent models for intuitionistic logic.[5] As intuitionistic negation is definable in Priest–da Costa logic, the arguments immediately transfer to the case of $\mathsf{KC}^{\dot{\sim}}$, whence:

**Observation 8.** $\Gamma \vdash_{\mathsf{KC}^{\dot{\sim}}} \varphi$ *iff the inference* $\Gamma \vDash \varphi$ *is valid with respect to the class of forward convergent models.*

We are now able to articulate an interesting convergence between the superintuitionistic principle of weak excluded middle, the propagation axiom (**ca3**), and the axiom $(\mathbf{cc})_{\mathbf{0}}$. Over daC, the latter two theses—central theses in the context of logics of formal inconsistency—are equivalent not only to each other, but also to $\mathbf{WEM}^{\dot{\sim}}$.

To show this, we employ the method of canonical models, recalling the following from Priest(2009) or Ferguson (2014):

**Definition 10.** *For a logic* L *extending* daC*, the* canonical model $\mathfrak{M}_{\mathsf{L}}$ *is a model in which:*

- $W = \{\Gamma \mid \Gamma$ *is a prime and non-trivial* L *theory*$\}$
- *R is the subset relation*
- *for all atoms p,* $v(p) = \{\Gamma \in W \mid p \in \Gamma\}$

As a theory in an extension of daC is *a fortiori* a daC theory, the standard proofs of appropriateness of a canonical model—such as that described in Priest (2009)—apply immediately to any normal extension of daC.

**Observation 9.** *For a logic* L *extending* daC *for which the rules* **dR1** *and* **dR2** *are admissible, we have the following in the canonical model* $\mathfrak{M}_{\mathsf{L}}$*:*

$$\Gamma \Vdash \varphi \textit{ iff } \varphi \in \Gamma$$

Now, let us consider two *prima facie* novel sdc-logics by adding (**ca3**) and $(\mathbf{cc})_{\mathbf{0}}$, respectively, as axiom schema. These will turn out to be merely alternative axiomatizations for $\mathsf{KC}^{\dot{\sim}}$, but this fact must be established.

Hence, we will independently define the following extensions of daC:

**Definition 11.** *The logics* $\mathsf{daC} \oplus (\mathbf{cc})_{\mathbf{0}}$ *and* $\mathsf{daC} \oplus (\mathbf{ca3})$ *are defined as the extensions of* daC *by the axiom schema* $(\mathbf{cc})_{\mathbf{0}}$ *and* (**ca3**)*, respectively.*

To show the equivalence, we first prove soundness of syntactic consequence in $\mathsf{daC} \oplus (\mathbf{cc})_{\mathbf{0}}$ with respect to the class of forward convergent models:

**Observation 10.** *If* $\Gamma \vdash_{\mathsf{daC}\oplus(\mathbf{cc})_{\mathbf{0}}} \varphi$*, then the inference* $\Gamma \vDash \varphi$ *is valid with respect to the class of forward convergent models.*

*Proof.* Clearly, as the rules and axioms of daC hold for all Kripke models, they hold *a fortiori* for forward convergent models. The problem, then, reduces to showing the validity of $(\mathbf{cc})_{\mathbf{0}}$ with respect to such models.

Suppose for contradiction that there exists a point $w$ in a forward convergent model such that $w \nVdash \circ\circ\varphi$. Then—because the formula is shorthand for the formula $\dot{\sim}(\circ\varphi \wedge \neg\circ\varphi)$—there exists a point $w'$ such that $wRw'$, $w' \Vdash \circ\varphi$, and $w' \Vdash \neg\circ\varphi$. From the latter of these observations, we infer the existence of a point $w''$ such that $w''Rw'$ at which $w'' \nVdash \circ\varphi$. That $\circ\varphi$ fails at $w''$ itself entails the existence of a point $w'''$ such that $w''Rw'''$ and both $w''' \Vdash \varphi$ and $w''' \Vdash \neg\varphi$.

That the frame is by hypothesis forward convergent yields a contradiction. We have $w'$—at which $\circ\varphi$ holds—and $w'''$—at which both $\varphi$ and $\neg\varphi$ hold—such that both $w''Rw'$ and $w''Rw'''$. By forward convergence, we have a point $u \in (w'^{\uparrow} \cap w'''^{\uparrow})$. Because $w' \Vdash \circ\varphi$, neither $\varphi$ nor $\neg\varphi$ may hold at $u$. But by the heredity condition, that $w'''Ru$ entails that $u \Vdash \varphi$ and $u \Vdash \neg\varphi$. □

By employing the method of canonical models, completeness may be established by showing the canonical frame for $\mathsf{daC} \oplus (\mathbf{cc})_0$ to be forward convergent:

**Observation 11.** *If the inference* $\Gamma \vDash \varphi$ *is valid with respect to the class of forward convergent models, then* $\Gamma \vdash_{\mathsf{daC}\oplus(\mathbf{cc})_0} \varphi$.

*Proof.* We begin by showing that the canonical model $\mathfrak{M}_{\mathsf{daC}\oplus(\mathbf{cc})_0}$ is forward convergent. Suppose otherwise for contradiction. Then there exists a non-trivial, prime theory $\Delta$ with non-trivial, prime extensions $\Gamma$ and $\Theta$ such that the deductive closure of $\Gamma \cup \Theta$ is trivial. By compactness, there exist formulae $\varphi_\Gamma \in \Gamma \smallsetminus \Delta$ and $\varphi_\Theta \in \Theta \smallsetminus \Delta$ such that $\Delta, \varphi_\Gamma, \varphi_\Theta \vdash \bot$ in $\mathsf{daC} \oplus (\mathbf{cc})_0$.

Now, for any non-trivial extension $\Gamma' \supseteq \Gamma$, also $\Gamma', \varphi_\Theta \vdash \bot$, whence we easily infer that $\Gamma, \varphi_\Theta, \neg\varphi_\Theta \vdash \bot$ and, consequently, that $\Gamma \vdash (\varphi_\Theta \wedge \neg\varphi_\Theta) \rightarrow \bot$, *i.e.*, that $\Gamma \vdash \circ\varphi_\Theta$.

On the other hand, as $\Delta \subseteq \Gamma$, also $\varphi_\Theta \notin \Delta$ and, because $\varphi_\Theta \vee \neg\varphi_\Theta \in \Delta$, by primeness, we infer that $\neg\varphi_\Theta \in \Delta$. Because $\Theta \supseteq \Delta$, also $\neg\varphi_\Theta \in \Theta$, whence $\varphi_\Theta \wedge \neg\varphi_\Theta \in \Theta$. By the non-triviality of $\Theta$, $\circ\varphi_\Theta$ cannot thereby be included in $\Theta$, *i.e.*, $\circ\varphi_\Theta \notin \Theta$. Because $\Delta \subseteq \Theta$, also $\circ\varphi_\Theta \notin \Delta$ and, because $\circ\varphi_\Theta \vee \neg\circ\varphi_\Theta \in \Delta$ and primeness of $\Delta$, we infer that $\neg\circ\varphi_\Theta \in \Delta$. As $\Gamma$ is an extension of $\Delta$, this entails that $\neg\circ\varphi_\Theta \in \Gamma$ as well.

Because $\circ\varphi_\Theta \in \Gamma$ and $\neg\circ\varphi_\Theta \in \Gamma$, also $\circ\varphi_\Theta \wedge \neg\circ\varphi_\Theta \in \Gamma$. But because $\circ\circ\varphi_\Theta$ is by hypothesis a member of all theories in the model, this entails that $\bot \in \Gamma$, contradicting the assumed nontriviality of $\Gamma$.

Hence, whenever $\Gamma \nvdash_{\mathsf{daC}\oplus(\mathbf{cc})_0} \varphi$, we can extend $\Gamma$ to a prime theory $\Gamma^\star$ such that $\varphi \notin \Gamma^\star$. Then the canonical model $\mathfrak{M}_{\mathsf{daC}\oplus(\mathbf{cc})_0}$ provides us an instance of a forward convergent model witnessing that the inference $\Gamma \vDash \varphi$ fails. □

As a corollary, we infer that:

**Observation 12.** $\mathsf{daC} \oplus (\mathbf{cc})_0$ *is characterized by the class of models based on forward convergent frames.*

Now, we repeat the same steps to characterize consequence in the system $\mathsf{daC}\oplus(\mathbf{ca3})$. In Ferguson (2014), the validity of $(\mathbf{ca3})$ in $\mathsf{KC}^{\dot{\sim}}$ was proven. Given the characterization of $\mathsf{KC}^{\dot{\sim}}$, this gives us the soundness of $\mathsf{daC}\oplus(\mathbf{ca3})$ with respect to forward convergent models:

**Observation 13.** *If* $\Gamma \vdash_{\mathsf{daC}\oplus(\mathbf{ca3})} \varphi$*, then the inference* $\Gamma \vDash \varphi$ *is valid with respect to the class of forward convergent models.*

Finally, completeness for $\mathsf{daC}\oplus(\mathbf{ca3})$ follows from a canonical model argument:

**Observation 14.** *If the inference* $\Gamma \vDash \varphi$ *is valid with respect to the class of forward convergent models, then* $\Gamma \vdash_{\mathsf{daC}\oplus(\mathbf{ca3})} \varphi$.

*Proof.* Suppose for contradiction that the canonical model $\mathfrak{M}_{\mathsf{daC}\oplus(\mathbf{ca3})}$ is not forward convergent. As before, this hypothesis commits us to the existence of a non-trivial, prime theory $\Delta$ with extensions $\Gamma$ and $\Theta$ such that any prime theory extending $\Gamma\cup\Theta$ is trivial.

As before, we have a $\varphi_\Theta \in \Theta$ such that $\Gamma, \varphi_\Theta \vdash \bot$. From this, we may infer that $\circ\varphi_\Theta \in \Gamma$. Additionally, by selection of $\bot$, we are able to infer that $\circ\bot \in \Gamma$ and, therefore, that $\circ\varphi_\Theta \wedge \circ\bot \in \Gamma$. As all instances of $(\mathbf{ca3})$ are included in $\Gamma$, by deductive closure, we infer that $\circ(\varphi_\Theta \to \bot) \in \Gamma$. By the assumption of the non-triviality of $\Gamma$, this requires that $(\varphi_\Theta \to \bot) \wedge \neg(\varphi_\Theta \to \bot) \notin \Gamma$.

However, our hypothesis that $\Gamma, \varphi_\Theta \vdash \bot$ holds also entails that $\Gamma \vdash \varphi_\Theta \to \bot$. However, this cannot be said for $\Delta$ itself. Were $\varphi_\Theta \to \bot \in \Delta$, because $\Theta \supseteq \Delta$, it would follow that $\varphi_\Theta \to \bot \in \Theta$. But because $\varphi_\Theta \in \Theta$, deductive closure would entail that $\bot \in \Theta$, contradicting the assumption of $\Theta$'s non-triviality. So $\varphi_\Theta \to \bot \notin \Delta$. By hypothesis, though, $(\varphi_\Theta \to \bot) \vee \neg(\varphi_\Theta \to \bot) \in \Delta$, and by primeness, we infer that $\neg(\varphi_\Theta \to \bot) \in \Delta$. Because $\Gamma \supseteq \Delta$, also $\neg(\varphi_\Theta \to \bot) \in \Gamma$, whence $(\varphi_\Theta \to \bot) \wedge \neg(\varphi_\Theta \to \bot)$ is a member of the theory $\Gamma$, contradicting our earlier inference.

Again, we finish by noting that whenever $\Gamma \nvdash_{\mathsf{daC}\oplus(\mathbf{ca3})} \varphi$, $\mathfrak{M}_{\mathsf{daC}\oplus(\mathbf{ca3})}$ includes a point extending $\Gamma$ at which $\varphi$ is not true, witnessing that the inference $\Gamma \vDash \varphi$ does not hold with respect to the class of forward convergent models. □

Insofar as the systems $\mathsf{KC}^{\dot{\sim}}$, $\mathsf{daC}\oplus(\mathbf{cc})_0$, and $\mathsf{daC}\oplus(\mathbf{ca3})$ are characterized by precisely the same class of models, they share precisely the same counterexamples. Hence, we infer the following corollaries:

**Corollary 1.** *The logics* $\mathsf{KC}^{\dot{\sim}}$*,* $\mathsf{daC}\oplus(\mathbf{cc})_0$*, and* $\mathsf{daC}\oplus(\mathbf{ca3})$ *are equivalent.*

**Corollary 2.** $\mathbf{WEM}^{\dot{\sim}}$*,* $(\mathbf{cc})_0$*, and* $(\mathbf{ca3})$ *are equivalent over* $\mathsf{daC}$.

The coincidence of the axiom schema $\mathbf{WEM}^{\dot{\sim}}$, $(\mathbf{ca3})$, and $(\mathbf{cc})_0$ in $\mathsf{daC}$ is extraordinarily suggestive. These principles, on their faces, appear to encode very distinct notions:

- In its primitive form, $\mathbf{WEM}^{\dot{\sim}}$ was considered as a principle of *testability* by Brouwer, so that weak excluded middle holds of a sentence when it is either demonstrably absurd or demonstrably unfalsifiable. Importantly, this statement does not appeal to the paraconsistent negation of daC and, therefore, is independent of the matter of interpreting "$\neg$."

- The logic daC allows propagation of consistency through *extensional* contexts only, that is, through conjunction and disjunction. The axiom (**ca3**) corresponds to the propagation of consistency through *intensional* contexts as well and its inclusion indicates a sort of *total propagation*.

- Given the reading of "$\circ$" as a type of *verifiability* (or *hypertestability*), $(\mathbf{cc})_0$ is the assertion that the verifiability of any formula is itself verifiable. Alternately —according to the Grzegorczyk–Wolter-style analysis—the adoption of $(\mathbf{cc})_0$ corresponds to scientific investigations in which there is no disagreement concerning what may or may not be disagreed upon. As $(\mathbf{cc})_0$ also includes $\circ\circ\circ\varphi$, $\circ\circ\circ\circ\varphi$, and so forth, this also may be understood as a type of common knowledge among researchers concerning the rules—or scope—of a Grzegorczyk-type investigation.

The equivalence to weak excluded middle also indicates that the operator "$\circ$" may admit a natural interpretation in standard intuitionistic logic; whether or not there the consistency operator may be fruitfully imported to Int itself is intriguing, but set aside for now.

## 4. Concluding Remarks: The Upshot for the LFI Community

As described by da Costa in (1974), the notions of "classicality" or "consistency" are too general to conclusively decide many facts concerning the semantic behavior of the operator "$\circ$," and hence, to decide the validity or invalidity of many axiom schema in which it appears. In particular, the inchoate notion of "classicality" provides little evidence for or against the status of axiom $(\mathbf{cc})_0$. Even interpretations that precisify the notion of "consistency"—such as the interpretation in "evolutionary databases" in Carnielli, Marcos and de Amos (2000)—are inconclusive. *E.g.*, the status of $(\mathbf{cc})_0$ in this setting stands or falls with whether one should allow databases in which both $\circ\varphi$ and $\sim\circ\varphi$ may appear, but there appears to be little evidence to settle this normative question.

It may be worthwhile to ask whether the model theories for the more canonical LFIs can be reevaluated in this light. The most commonly encountered semantical treatments—*e.g.*, bivaluations, possible translation semantics, and Nmatrices—share a strongly non-deterministic character, in which the evaluation of a complex formula is informed by—but not wholly determined by—the evaluation of its subformulae. Although the semantics for C and daC are wholly compositional, connections between consistency-as-verifiability and non-deterministic treatments of the consistency connective might well exist. Exporting the verifiability readings of these extracanonical systems to the core logics of formal inconsistency is worth considering in the future.

In any case, by broadening the search and looking to contexts in which the corresponding notion of "consistency" is more well-defined, we are able to search for evidence that weighs not only on the adoption of $(\mathbf{cc})_0$, but other axiom schema of interest. This paper has focused on the interpretation of consistency-as-verifiability and, I hope, has shown that viewing the consistency operator under a different light provides new ways of looking at logics of formal inconsistency in general.

## References

Åqvist, L. 1962. Refections on the logic of nonsense. *Theoria* **28**(2): 138–157.

Ayer, A. J. ed. 1959. *Logical Positivism*. New York: The Free Press.

Bochvar, D. A. 1938. On a three-valued logical calculus and its application to the analysis of contradictions. *Matematicheskii Sbornik* **4**(2): 287–308. Translated in Bochvar 1981.

Bochvar, D. A. 1981. On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus. *History and Philosophy of Logic* **2**(1-2): 87–112. Translated by M. Bergmann.

Brouwer, L. E. J. 1983. Consciousness, philosophy, and mathematics. In: P. Benacerraf; H. Putnam (eds.) *Philosophy of Mathematics: Selected Readings*, pp.90–96. New York: Cambridge University Press.

Burgess, J. P. 1981. Relevance: A fallacy? *Notre Dame Journal of Formal Logic* **22**(2): 97–104.

Carnap, R. 1931. Überwindung der Metaphysik durch logische Analyse der Sprache. *Erkenntnis* **2**(1): 219–241. Reprinted in Ayer 1959, pp.60–81.

Carnielli, W.; Marcos, J.; de Amo, S. 2000. Formal inconsistency and evolutionary databases. *Logic and Logical Philosophy* **8**: 115–152.

Carnielli, W.; Coniglio, M. E.; and Marcos, J. 2007. Logics of formal inconsistency. In: D. Gabbay; F. Guenthner (eds.) *Handbook of Philosophical Logic* **14**, pp.15–107. Netherlands: Springer.

Carnielli, W.; Rodrigues, A. 2016. On the philosophy and mathematics of the logics of formal inconsistency. In: J.-Y. Beziau; M. Chakraborty; S. Dutta (eds.) *New Directions in Paraconsistent Logic*, pp.57–88. New Delhi: Springer.

Castiglioni, J.; Ertola, R. C. 2014. Strict paraconsistency of truth-degree preserving intuitionistic logic with dual negation. *Logic Journal of the IGPL* **22**(2): 268–273.

da Costa, N. 1974. On the theory of inconsistent formal systems. *Notre Dame Journal of Formal Logic* **15**(4): 497–510.

Ferguson, T. M. 2014. Extensions of Priest–da Costa logic. *Studia Logica* **102**(1): 145–174.

Grzegorczyk, A. 1964. A philosophically plausible formal interpretation of intuitionistic logic. *Indagationes Mathematicae (Proceedings)* **67**: 596–601.

Halldén, S. 1949. *The Logic of Nonsense*. Lund: Lundequista Bokhandeln.

Hempel, C. 1950. Problems and changes in the empiricist criterion of meaning. *Revue Internationale de Philosophie* **41**(11): 41–63. Reprinted in Ayer 1959, pp.108–129.

Jankov, V. A. 1968. The calculus of the weak law of excluded middle. *Mathematics of the USSR Izvestiya* **2**(5): 997–1004.

Jaśkowski, S. 1948. Rachunek zdańdla systemów dedukcyjnych sprzecznych. *Studia Societatis Scientiarum Trunensis*, Sectio A I(5): 57–77. Translated in Jaśkowski 1999.

Jaśkowski, S. 1999. A propositional calculus for inconsistent deductive systems. *Logic and Logical Philosophy* **7**: 35–56. Translated by O. Wojtasiewicz, trans. with corrections and notes by J. Perzanowski.

Kleene, S. C. 1952. *Introduction to Metamathematics*. Amsterdam: North-Holland Publishing Company.

Łukowski, P. 1996. Modal interpretation of Heyting-Brouwer logic. *Bulletin of the Section of Logic* **25**(2): 80–83.

Marcos, J. 2008. Possible-translations semantics for some weak classically-based paraconsistent logics. *Journal of Applied Non-Classical Logics* **18**(1): 7–24.

Mortensen, C. 1983. The validity of disjunctive syllogism is not so easily proved. *Notre Dame Journal of Formal Logic* **24**(1): 35–40.

Omori, H. 2016. Halldén's logic of nonsense and its expansions in view of logics of formal inconsistency. In: *Proc. 27th International Conference on Database and Expert Systems Applications (DEXA 2016)*, pp. 129–133. Los Alamitos, CA: IEEE Computer Society.

Osorio, M.; Borja, V.; Arrazola, J. 2016. Revisiting da Costa logic. *Journal of Applied Logic* **16**: 111–127.

Piróg-Rzepecka, K. 1977. *Systemy Nonsense-Logics*. Warsaw: PWN.

Priest, G. 2009. Dualising intuitionistic negation. *Principia* **13**(2): 165–184.

Rauszer, C. 1974. Semi-Boolean algebras and their application to intuitionistic logic with dual operations. *Fundamenta Mathematicae* **83**(1): 219–249.

———. 1977. Applications of Kripke models to Heyting-Brouwer logic. *Studia Logica* **36**(1-2): 61–71.

———. 1980. An algebraic and Kripke-style approach to a certain extension of intuitionistic logic. *Dissertationes Mathematicae* **167**: 1–62.

Wolter, F. 1998. On logics with coimplication. *Journal of Philosophical Logic* **27**(4): 353–387.

Segerberg, K. 1965. A contribution to nonsense-logics. *Theoria* **31**(3): 199–217.

Wansing, H. 2010. Proofs, disproofs, and their duals. In: L. D. Beklemishev; V. Goranko; V. Shehtman (eds.) *Advances in Modal Logic 8*, pp.483–505. London: College Publications.

Whitehead, A.; Russell, B. 1963. *Principia Mathematica*, vol. 1. Second ed. Cambridge: Cambridge University Press.

Woodruff, P. 1973. On constructive nonsense logic. In: *Modality, Morality, and Other Problems of Sense and Nonsense*, pp.192–205. Lund: GWK Gleerup Bokforlag.

## Notes

[1] Although Halldén employs the symbol "+" as the unary "meaningfulness" connective, we appeal to the presentation in Omori (2016)—in which the more familiar "∘" symbol is em-

ployed—in recognition of C's status as an LFI. Hence Halldén's "+" operator is assumed to be a *notational variant* of the consistency connective, no more unusual than Newton da Costa's (1974) use of the notation "$^\circ\varphi$" rather than the contemporary "$\circ\varphi$."

[2] C. I. Lewis' famous proof of the validity of the principle of explosion in classical logic requires the use of disjunctive syllogism. To those philosophically opposed to the principle of explosion, the rejection of disjunctive syllogism is sometimes described as a *strategy* of resisting the soundness of Lewis' argument.

[3] Priest does not assume antisymmetry in (2009) but daC is shown in Ferguson (2014) to be sound and complete to this class of Kripke frames as well.

[4] Priest offers a natural deduction calculus and a tableaux-style proof theory in Priest (2009). A *restricted* Hilbert calculus was independently described by Osorio, Borja, and Arrazola in (2016).

[5] *N.b.* that this is typically just known as "convergence" in the context of superintuitionistic logic because the dual property of *backward convergence* can be assumed to hold without loss of generality. As backward convergence is nontrivial in extensions of daC, we take care to distinguish the two in this case.

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