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Geometry Students' Arguments About a 1-Point Perspective Drawing

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Abstract

The practice of formulating and justifying claims is a fundamental aspect of doing mathematics, and in geometry, students' use of diagrams is integral to how they establish arguments. We applied Toulmin's model to examine 23 geometry students' arguments about figures included in a 1-point perspective drawing. We asked how students' arguments drew upon their knowledge of 1-point perspective and their use of the diagram provided with the problem. Students warranted their claims based upon their knowledge of perspective, both in an artistic context as well as from experiences in everyday life. Students engaged in multiple apprehensions of the diagram, including using the given features, adding features, or measuring components, to justify claims about the figures. This study illustrates the importance

of students' prior knowledge of a context for formulating arguments, as well as how that prior knowledge is integrated with students' use of a geometric diagram.

Los Argumentos Geométricos de los Estudiantes sobre Dibujos en Perspectiva de 1 Punto

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Resumen

La práctica de formular y justificar las afirmaciones es un aspecto fundamental de la matemática y, en la geometría, el uso que hacen los estudiantes de los diagramas es una parte integrante de cómo se establecen los argumentos. Aplicamos el modelo de Toulmin para examinar 23 argumentos de estudiantes de geometría sobre las figuras incluidas en un dibujo en perspectiva de 1 punto. Preguntamos cómo los argumentos de los estudiantes se basan en su conocimiento de la perspectiva de un punto y del uso del diagrama proporcionado con el problema. Los estudiantes justifican sus afirmaciones basadas en su conocimiento de la perspectiva, tanto en un contexto artístico como de experiencias en la vida cotidiana. Los estudiantes se involucraron en múltiples aprehensiones del diagrama, incluyendo el uso de rasgos dados, añadiendo rasgos, o midiendo componentes, para justificar afirmaciones sobre los dibujos. Este estudio ilustra la importancia del conocimiento previo de los

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estudiantes de un contexto para formular argumentos, así como dicho conocimiento previo es integrado con el uso que los estudiantes hacen del diagrama geométrico.

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S tudents' development of mathematical arguments is a central component of their mathematical learning. We define an argument as a claim based on some available data, which is supported by a justification that may be explicit or left implicit (Toulmin, 1958). In mathematics, the activities of making and justifying claims about mathematical objects and relationships are integral components of problem solving, reasoning, and proving (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 2000). By formulating arguments, students have the opportunity to participate in the practice of doing mathematics (Schoenfeld, 1992). To support students in this endeavor, it is necessary to understand how students engage in the process of argumentation, and in particular how they establish justifications for their claims across different areas of mathematics.

In geometry, students' use of diagrams is integral to the arguments they develop (Sinclair, Pimm, Skelin, & Zbiek, 2012). Geometric proof, which requires an argument in support of a particular claim, is often used for students to describe relationships between objects that can be visually represented. In mathematics classrooms, students typically perceive that whatever diagram is necessary for producing such an argument will be provided by a teacher or by the task (Herbst & Brach, 2006). When students make use of diagrams, they can vary in whether they choose to add elements to the diagram, whether they choose to measure components of a diagram, and in how they interpret the objects represented through a diagram (Herbst, 2004). All of these choices shape students' arguments and help explain the different ways that students can make and justify claims in geometry settings.

We examined students' arguments on a problem about a 1-point perspective drawing. Students needed to determine whether certain pairs of figures represented objects of equal or different heights. The context of the problem allowed students to draw upon their knowledge of art and their experiences with perspective from outside of mathematics. We asked: How did students' arguments draw upon their knowledge of 1-point perspective and their use of the diagram provided with the problem? This study contributes to research on students' arguments in geometry by considering the different types of prior knowledge that students use to justify claims. In addition, we examine how students' use of a diagram interacts with their prior knowledge of mathematics and of the context of a problem.

Students' Use of Diagrams for Solving Problems in Geometry

There are multiple ways that students may interact with diagrams in geometry. Duval (1995) has described four different types of interactions, which he referred to as *apprehensions* of diagram. Perceptual apprehension, which is always present, refers to the interpretation of a diagram as a visual object. In combination with perceptual apprehension, individuals may also employ sequential, discursive, or operative apprehensions of a given diagram. Sequential apprehension refers to the process of constructing a figure or describing a construction in a way that relies upon mathematical properties rather than visual cues. Discursive apprehension of a diagram indicates an individual's use of propositions to justify particular actions. Finally, operative apprehension indicates the modification of a diagram in some way, such as adding features to the original diagram or adjusting size or orientation for a specific purpose. These four different apprehensions help explain how students use a diagram, either relying upon provided features or acting on the diagram in some way, for the purpose of formulating arguments.

There are examples in research of the importance of operative apprehensions, in particular in the case of adding auxiliary features that were not included in a given diagram but are necessary for justifying a particular claim. For example, students may experiment by adding auxiliary components as a strategy for transitioning from empirical evidence to abstract arguments in a geometry class (Marrades & Gutiérrez, 2000). By adding elements such as auxiliary lines, the use of a diagram becomes relevant not only for illustrating a particular argument or claim but also for its use in the process of justifying that claim (Netz, 1999). Geometry textbooks often provide any necessary auxiliary lines in a diagram and use specific notation to distinguish auxiliary lines from the rest of the diagram (Dimmel & Herbst, 2015; Herbst, 2004). This tradition surfaced in the historical development of proving in part to provide students with the resources they would need to complete a proof (Herbst, 2002). Perhaps because of this feature, students tend to have a disposition not to add auxiliary lines to a diagram unless a teacher suggests doing so (Herbst & Brach, 2006). Thus, although adding auxiliary features is an important

aspect of formulating an argument, it is not necessarily the first step that students are likely to take when interacting with a diagram.

In addition to, or separately from, adding new features to a given diagram, students often take measurements of the figures included for the purpose of describing relationships or justifying claims. Measuring can be considered a component of a discursive apprehension of a diagram, in that it allows students to identify properties from which they can build arguments. Students' use of measurements can allow them to achieve multiple purposes, including formulating conjectures or confirming whether the initial conditions of a particular argument are met (González & Herbst, 2009). One particular challenge of measuring is that students may see empirical evidence such as measurements as sufficient for justifying a claim, and because of this they may not perceive the need for more formal deductive proof (Chazan, 1993). By the time that students reach the undergraduate level, it is possible that they become more cautious about using measurements in the justification of geometric arguments (Hollebrands, Conner, & Smith, 2010). At the secondary level, however, measuring seems to be an integral component of the work that students do with diagrams, both in ways that can support or obscure the development of sophisticated arguments.

The different ways that students interact with diagrams in geometry tend to be connected to broader ideas about how diagrams should be interpreted. A diagram has a dual nature in that it serves as both a static drawing as well as the object or figure it represents (Parzysz, 1988). Parzysz has described this distinction between *drawings* (i.e., the picture as it is presented) and *figures* (i.e., the object(s) represented by the drawings) as one that students must manage in order to make productive use of a diagram in formulating a geometric argument. Moreover, there is a common view in mathematics that diagrams may be misleading (Davis, 2006; Inglis & Mejía-Ramos, 2009). With efforts to promote more rigorous use of mathematical relationships and assumptions, teachers may be disinclined to allow students to make assumptions based upon how a diagram looks (Nachlieli & Herbst, 2009). As a result, students may not trust that a drawing gives a faithful representation of the figure it is intended to represent. By investigating a 1-point perspective drawing, students in this study examined a diagram that represented not only a collection of geometric figures and relationships, but also a scenario that could be understood in contexts

beyond mathematics. It was necessary to understand how students formulated arguments through different sources of knowledge related to their use of the diagram.

Using the Toulmin Model to Study Students' Arguments

We assume the perspective that learning is participation in a community of practice, in this case the classroom (Lave & Wenger, 1991). An integral aspect of students' learning in mathematics classrooms is participating in mathematical discussions (Forman, Larreamendy-Joerns, Stein, & Brown, 1998; Forman, McCormick, & Donato, 1997; Moschkovich, 2012; Zahner, 2012). Recent policy documents emphasize that students should have opportunities to formulate arguments and critique the reasoning of their peers (NGAC, 2010; NCTM, 2000). This practice is supported by research suggesting that devising and refining arguments can support mathematical understanding (Jahnke, 2008; Stephan & Akyuz, 2012; Yackel, 2002). Because of the value of mathematical argumentation, and the difficulty of it in a small group setting, there is opportunity to better understand how students work together to build mathematical arguments.

We use Toulmin's (1958) model of an argument to examine how students working together in groups built arguments about a 1-point perspective drawing. There are six components to an argument: the *data* or evidence on which the argument is based; the *claim* of the argument; the *warrant*, or justification for the claim based on the data; a *backing*, or underlying rationale for the warrant; a *rebuttal* which offers a contradiction to the claim; and a *qualifier* that communicates the strength of the claim. Students do not always make the warrants of their claims explicit, in particular when they perceive the warrant may be implied through a shared diagram (Hollebrands et al., 2010). Particularly important for this study, the ways that students warrant their claims or offer rebuttals to others' arguments can indicate the sources of prior knowledge they draw upon to formulate those arguments.

In *collective argumentation*, multiple students provide the components of an argument (Forman et al., 1998; Moore-Russo et al., 2011; Yackel, 2002). Whether or not students perceive one another's justifications as valid can vary according to the person and context (Forman et al., 1998; Krummheuer, 1995). In addition, once students' justifications become

taken as shared (Yackel & Cobb, 1996), students do not always make the warrants of their claims explicit (Stephan & Rasmussen, 2002). Students' prior experiences with a given context may shape their perception of the validity of a particular argument. They may draw upon multiple mathematical knowledge bases (Drake et al., 2015; Turner et al., 2012), specifically their prior knowledge of geometry or of the context of a particular problem, when formulating arguments about a perspective drawing. In everyday settings, arguments based upon empirical evidence are generally perceived as more appropriate than theoretical arguments, although in mathematical settings theoretical arguments are typically more acceptable (Jahnke, 2008). This contrast illuminates a potential challenge for students making arguments about a 1-point perspective drawing, which lies in the intersection of students' knowledge of mathematics and their experiences with art and in everyday settings.

Data Sources and Methods

We conducted an after-school focus group session in the spring of 2014, during which students worked on a problem known as the "perspective drawing" problem. Students studied a diagram drawn in 1-point perspective (Figure 1). Students needed to determine whether the pair of houses represented houses of equal height, and similarly whether the pair of trees represented trees of equal height. They also needed to justify their decisions. Figure 1 was designed so that the houses included in the figure are similar figures, which means that they would represent houses of the same height in the perspective drawing. The trees included in the figure are not similar, and the tree represented as farther back in the picture is taller than the tree towards the front. We designed the perspective drawing problem to emphasize the properties of dilation as determined by a center of dilation and scale factor. In a 1-point perspective drawing, objects aligned along shared perspective lines extending from a single vanishing point are similar figures representing objects of the same size (as in the houses in Figure 1). Objects aligned along different perspective lines extending from a single vanishing point represent objects of different heights (as in the trees in Figure 1).
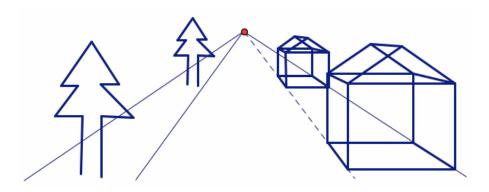
Figure 1. The perspective drawing diagram

Although research on students' use of perspective drawings in upper grades is limited, findings from earlier grades can help to anticipate how secondary students might make geometric arguments about a perspective drawing. Children as early as kindergarten can make connections between 2-dimensional representations and the 3-dimensional objects they represent (van den Heuvel-Panhuizen, Elia, & Robitzsch, 2015), but they may struggle to articulate their rationale for doing so (Clements, Swaminathan, Hannibal, & Sarama, 1999; Hallowell, Okamoto, Romo, & La Joy, 2015). Historically, perspective drawing provided a context in which the development of geometry was informed through art, and it offers an opportunity for students to examine geometric similarity through an artistic context (Bartolini Bussi, 1996). Students' informal experiences can inform their understanding of 2-dimensional representations, although these experiences may not always support formal geometric reasoning.

Data Collection and Analysis

We conducted this study in an after-school focus group session, recruiting 23 students from an urban, public high school serving a population of students who are racially and socioeconomically diverse. We advertised the session in the geometry classes at the school, telling students that we were interested in learning about the different ways that students solve

problems in geometry. All the participants were current geometry students at the time of the session, which took place near the end of the school year.

The first author acted as the instructor and spent approximately 5 minutes introducing the problem at the start of the session. In the introduction, the instructor presented two examples of artwork that had been created through the use of 1-point perspective. The instructor asked whether anyone was familiar with perspective drawing, and multiple students contributed that perspective drawing is a way to indicate which objects are farther away and which objects are nearer to a viewer. When the instructor asked about features of perspective that were included in the two examples, students noted that people and objects towards the back of the picture appeared smaller, and that there was a point towards the center of each picture to which everything collapsed. The instructor introduced this point as the "vanishing point" of a perspective drawing. After the introduction, the instructor told students that they would be working on a problem in which they would use the properties of perspective drawing to make some observations about the picture included in Figure 1.

The 23 students were organized into seven groups of 3-4 students each for their work on the perspective drawing problem. Groups spent between 8-18 minutes discussing the questions, specifically the questions of whether the pair of houses and, respectively, the pair of trees, represented in the diagram would be of equal height in the real world. We provided rulers as well as tracing paper that students could use for their work on the problem. We video and audio recorded each group of students, and we also collected copies of all of the written work that students produced. We produced a complete transcript for each group of students, which provided the basis for our analysis of students' arguments.

We analyzed students' arguments regarding the heights of the houses and trees using Toulmin's (1958) model. We also noted specific linguistic cues, such as "I think" or "because," that would indicate specific components of an argument (González & Herbst, 2013). We allowed for the possibility that multiple students could contribute to a single argument (Forman et al., 1998; Moore-Russo et al., 2011). When students provided warrants, backings, qualifiers, or rebuttals, we noted whether there was evidence of students' use of mathematics or knowledge of the context of the problem. In several groups, students vacillated between different claims. Because of this, students often repeated the same claim multiple times over

the course of the discussion. For the purpose of understanding the flow of students' arguments, we would enumerate these claims as part of two separate arguments. The first author analyzed the transcripts of five different groups of students, and the second author analyzed the transcripts of four groups of students. Thus, we overlapped on two groups, which we used to resolve disagreements in our coding.

Table 1.
An example of our analysis with the Toulmin model

Turn #	Speaker	Turn	Argu-ment #	Compo-nent	Linguis-tic Cues	Prior Know-ledge
136	Marisha	Okay. The houses are the same,	18	Claim		
		because they have the same width and the same like, measurement. You know what I'm talking about?	18	Warrant (1)	Becau-se	Ratio
137	Stella	Yeah.				
138	Marisha	Because they have the same measurements, just, yeah. Like yall get it? Oh my goodness.	18	Warrant (2)		Ratio
139	Stella	Now I'm kind of thinking	19	Qualifi-er		
		they aren't.	19	Claim		

We illustrate our coding through the example in Table 1. The excerpt includes a conversation between Marisha[i] and Stella as they discussed whether or not the houses would be the same height. Turn numbers enumerate turns of speech within the group's discussion. In turns 136 and 138, Marisha argued that the houses were the same height. Marisha's warrant, "they have the same measurements," seemed to express that the front of each house was a square. We identified these warrants as using prior knowledge of ratio. In turn 139 Stella offered a qualified claim that the houses were not the same height.

Results

We identified 100 claims related to the relative sizes of the objects represented in the figure across the seven groups. We provide a summary of the different types of warrants that students provided for these claims, followed by a more in depth examination of how students constructed arguments based upon different sources of prior knowledge related to the problem.

Warrants of Students' Claims

Students' arguments included 61 claims comparing the sizes of the two houses and 39 claims comparing the sizes of the two trees (Table 2). Although the prompts included in the problem specifically addressed the relative heights of the objects, many students addressed both height and width in their discussions of the diagram. Arguments about the widths of the houses or trees were important to consider because students could establish a relationship between height and width in order to determine whether the figures were the same height. For that reason, we include arguments about both the heights as well as the widths of the figures in the diagram, and we refer to these more generally as arguments about size.

Table 2.
A summary of the warrants that students provided for their claims

	#Claims	#Warrants			
		Based Upon the Diagram	Based Upon Measure-ments	Based Upon the Context	No Warrant Provided
The sizes of the houses	61	20 (33%)	9 (15%)	7 (11%)	25 (41%)
The sizes of the trees	39	13 (33%)	4 (10%)	1 (3%)	21 (54%)

We organized the warrants of students' claims into three different categories: warrants based upon information inferred from the diagram; warrants based upon students' measurements; and warrants based upon students' knowledge of the context of the problem. We describe each of these categories more fully in the sections that follow, but from Table 2 it is

evident that students most often drew upon information from the diagram to warrant their claims about the relative sizes of the houses and of the trees. Students warranted their claims based upon their own measurements of the diagram less than half as often as they referenced the visual features of the diagram. Students also used their knowledge of the context, but they did so less frequently overall. That students almost never addressed the context of the problem in their claims about the trees seemed related to the fact that the question about the trees was the second question on the handout. Students engaged with the problem context primarily as an entry point towards comparing the heights of the objects represented in the diagram. By the time students progressed to discussing the trees, they had established justifications based upon their perception of the diagram or based upon some measurements and calculations.

Also illustrated in Table 2 is the finding that a substantial number of students' claims were made without warrants. This should be considered in light of our methods of analysis, in particular that we counted each instance of a particular claim as part of a separate argument. As students generated ideas about the figures in the diagram, they often made claims for which they did not vocalize a warrant, and which they later returned to in order to provide an explicit warrant. In addition, once students' justifications became taken as shared, they often summarized their claims without again making their justifications explicit. Finally, students typically applied similar logic to their arguments about the trees that they had for the houses. Because the question about the trees came second, students were not always as explicit about the justifications of these claims.

Students' Arguments Based Upon the Context of the Problem

Students warranted seven claims about the houses, and one claim about the trees, with information based upon their knowledge of perspective, both from their everyday experiences and from their knowledge of the use of perspective in artistic contexts. Students' knowledge of the context informed their ideas about whether two figures that were clearly different sizes on the given handout could represent objects of the same size. Three of the four students in group 6, for example, immediately argued that the two houses would be the same size, prior to addressing any specific component of the diagram or taking any measurements. Taisha warranted

this claim (Figure 2) by noting that the house towards the front only looked bigger because the drawing was in perspective. Taisha's argument was related to the launch of the problem, in which the class had looked at examples of paintings done in 1-point perspective and noted that objects towards the front appeared larger than objects farther away. The backing of Taisha's warrant, although she did not make it explicit, seemed to be the assumption that, for two objects of the same size, the closer object would look larger in a perspective drawing.
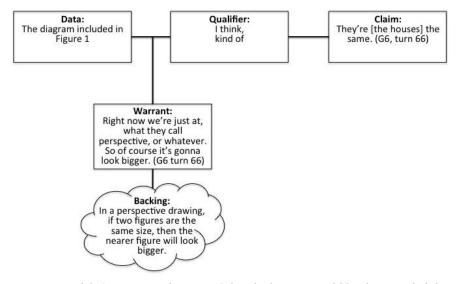

Figure 2. Taisha's argument in group 6 that the houses would be the same height. Following examples from prior research (e.g., Hollebrands et al., 2010), elements of the argument outlined in rectangles indicate explicit evidence from students' work, while the cloud-shaped outline of the backing indicates our inference of Taisha's backing.

Latasha was the only student in group 6 who initially argued that the two houses were different sizes. As is summarized in Figure 3, Latasha appealed to the diagram to warrant her claim, noting that the drawing of the front house was clearly larger than the drawing of the back house. Latasha argued that the two houses as they were drawn on the page, and not necessarily the objects they represented, were different sizes. Latasha did

not provide a warrant for her claim, possibly because the different sizes in the diagram seemed obvious to her. We include Latasha's argument here, because Emiliano's rebuttal to her claim illustrated further use of the context of the problem to reason about the sizes of the houses. Rather than drawing upon 1-point perspective as an artistic technique, Emiliano used the students' experiences viewing objects from a distance. He posited that, from a viewer's perspective, a house nearby would look larger than a house farther off in the distance. Both Taisha and Emiliano drew upon prior knowledge of the context of the problem in slightly different ways, Taisha by building upon the discussion of 1-point perspective drawing that had surfaced in the launch of the problem and Emiliano by posing an example from students' out of school experiences.
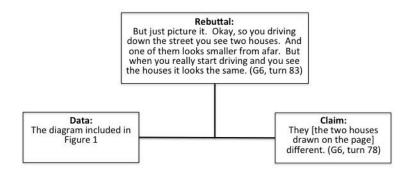

Figure 3. Latasha's claim, and Emiliano's rebuttal, about the relative heights of the houses.

The warrant offered by Taisha, as well as Emiliano's rebuttal to Latasha's claim, relied upon an implicit backing related to the sizes of objects drawn in perspective. Namely, Taisha and Emiliano used the assumption that, for two objects of equal size, an object represented as nearer to a viewer will look larger than an object represented to be farther away in a perspective drawing. This underlying assumption is true, however Taisha's warrant was not fully valid for claiming that the two houses would be of equal size. Namely, Taisha did not account for the possibility that the house towards the back of the picture might have actually been smaller than the house towards the front, in which case it

would also look smaller in the diagram. Formally, the backing that "a house of equal size will look smaller if farther back" does not imply that "a house farther back that looks smaller is necessarily of equal size." In general, students' use of their knowledge of the context of the problem, in isolation from the specific qualities of the given diagram, was insufficient to justify their claims about the relative sizes of the houses or trees. There were ways in which students integrated their understanding of perspective with their use of the diagram or the measurements they performed, to apply their prior knowledge in ways that were directly relevant to the given problem.

Students' Use of the Diagram

Students appealed to the features of the given diagram, such as the perspective lines and vanishing point, to help justify their intuitive claims about the heights of the objects represented. For example, the students in group 2 had worked together to determine that the houses were of equal height, as were the trees. When the instructor stopped by the group to inquire about their justifications for these claims, Stephen summarized the group's argument about the houses (Figure 4). Stephen argued that the houses would be the same height because they were placed on the same pair of perspective lines, backed by his knowledge of the relationship between the location of an object relative to the vanishing point and its size on the diagram. Stephen's use of the diagram indicated an interaction between his knowledge of 1-point perspective drawing and the specific features that we had provided on the diagram. Stephen could explicitly note that the houses aligned between the same two perspective lines, which the diagram included, extending towards the vanishing point. That information, backed by his intuition about how houses of the same size should look on the diagram, justified Stephen's claim about their equal size. With this argument, however, Stephen did not account for the fact that the perspective lines included in the diagram only addressed the widths of the houses, but not their heights.
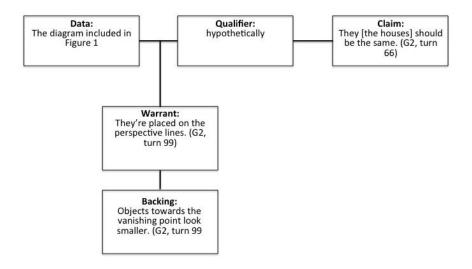
Figure 4. Stephen's argument that the houses would be the same height.

Another way in which students warranted claims about the relative heights of the represented objects was by adding perspective lines to the given diagram. An example of this comes from the students in group 1. When prompted by the instructor about his responses to the two questions, Asher commented, "if you actually were to use a ruler, they actually do measure from the top." Rather than measuring the dimensions of the house, Asher used the ruler as a straightedge to draw a line segment through the roofs of the two houses (Figure 5), and he noted that this segment served as a perspective line extending from the vanishing point. Thus, the line that Asher added to the diagram served as a warrant for his claim that the houses "do measure from the top," which seemed to be a way of claiming the houses were the same height. Asher also argued, "but the trees not, because of the fact that, if we were to measure the top." With his comment, Asher drew a line segment through the tops of the trees and noted that the segment extended above the vanishing point. For the trees to have been the same height, the tops of the trees would need to be collinear with the vanishing point.
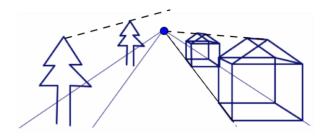
Figure 5. Asher's auxiliary lines added to the diagram (dashed lines), which warranted his claim.

Students' Use of Measurements

Compared to their use of visual features of the diagram, students performed measurements relatively infrequently to justify their claims. Students in the study had previously studied similarity in their geometry classes, and they could have calculated the ratios of width to height of each house to determine whether the drawings in the diagram were similar. Avery articulated this type of argument when the instructor asked the students in group 1 how they had been working on the problem. Avery calculated the ratio of height to width of each house drawn on the diagram, and noted that because the ratios were equivalent the houses were similar. Based upon this information, Avery claimed that the houses represented by the diagram would be of equal height. The underlying backing of this argument, which Avery did not make explicit, was the fact that the houses in the diagram had the same width because they were situated between the same two perspective lines, which meant that they would need to be similar in order to have equal height. It would be possible for a pair of houses to exist with equal height but different widths. In this case, the scale factor between height and width would not be constant. The perspective lines that we provided in the diagram provided the backing for Avery's warrant, although it was not clear whether she recognized the need for this backing for the validity of her argument.

Students' use of measurements at times conflicted with other warrants, as became apparent in group 7. Shanise, in group 7, proposed a pair of arguments about the relative heights of the houses and trees, based upon her prior knowledge of perspective drawing (Figure 6). Similarly to Taisha in Figure 2, Shanise claimed that the two houses would be the same height, because the house "farther out" looked smaller. Shanise also added the argument that, because the trees looked to be the same height on paper, they objects represented by the trees must not be the same height. Although Shanise's warrants did not fully justify the claims, Shanise's claims were true, and there may have been opportunity to build upon these arguments.
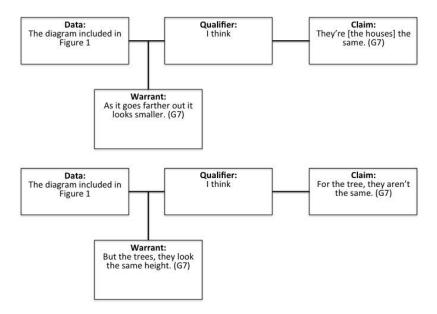

Figure 6. Shanise's arguments about the houses and trees based on her knowledge of perspective.

Teddy, in contrast to Shanise, claimed that the houses would be different heights, warranted by his use of scale factors. After doing several calculations on his own, Teddy summarized his argument about the heights of the houses for his group:

>Teddy: Uh, the heights are not the same, because when the second house's height in perspective was multiplied by the scale factor…the two heights were not equal.

Teddy's argument was based upon his calculation of the scale factor between the widths of the houses. Because of an error in his measuring, Teddy had found two different scale factors for the two different houses. In fact, the houses included on the diagram were similar. Once Teddy concluded that the houses would not be the same height, the other students in the group accepted his argument. It is possible that the students in the group saw Teddy's calculations as a more mathematically valid warrant than their own perceptions of the diagram. Alternatively, it is possible that the students trusted Teddy, who seemed confident and knowledgeable in his work. The students did not address the disconnect between Shanise's claims based upon her use of the diagram and Teddy's claims based upon his measurements.

Discussion

A great deal of research has examined how students use diagrams to formulate arguments in geometry. This study has added to that discussion by considering students' use of a diagram that represented geometric objects as well as a real-world context. Students applied their knowledge—both of perspective as an artistic technique and of their experiences with perspective in the world—to articulate arguments about the objects represented in the diagram. Young children develop the capacity to make connections between 2-dimensional representations and 3-dimensional realities (van den Heuvel-Panhuizen et al., 2015). In this way, students' experiences in the real world strengthen the intuitions they develop about the information represented through a diagram. At the same time, students' knowledge of real-world contexts can have benefits as well as costs to students' mathematical reasoning (Zahner, 2012). We saw multiple examples of students who translated the diagram we provided into a hypothetical experience, namely to imagine themselves looking down a street at a collection of houses and trees. This imagination provided an entry point towards students being able to make claims about how two

houses, which were clearly different sizes on paper, may actually represent houses of the same height.

It is also important to consider the significance of the way a teacher introduces a problem for the different sources of prior knowledge that students will use. Mathematics problems that draw on multiple mathematical knowledge bases, including those from outside of school, have the opportunity to make tasks more accessible to students (Drake et al., 2015). At the same time, not all students in a class have the same prior knowledge, and part of the teacher's work is to establish a shared language through which students can discuss the context of a problem (Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013). In this case, the instructor introduced the perspective drawing problem by sharing examples of artwork created using 1-point perspective. Students made observations during this introduction about the sizes of people and things represented in the pictures and the location of the vanishing point. Students' initial arguments about the diagram provided with the problem included some of these same general observations as warrants. In these cases, the instructor pointed out that such broad observations might not necessarily provide valid warrants to students' specific claims about the given diagram. It was not that students' intuitions about the figures based upon their prior experiences were not valuable; several students established accurate claims based upon their observations about objects nearer to a viewer versus farther in the distance. To sufficiently warrant their claims about the diagram, students needed not only to use multiple mathematical knowledge bases, but also to make connections between those knowledge bases (Drake et al., 2015).

One of the ways in which students supplemented their prior knowledge of the context of perspective drawing was by using features of the diagram. It has been noted that students are often reluctant to add auxiliary features to a diagram in geometry (Herbst & Brach, 2006), which in this case was most relevant to whether students added perspective lines to the drawing or relied upon what was given. This distinction can be viewed in terms of whether students apprehended the diagram discursively, using deduction to make claims about the diagram, or operatively, adding things to the diagram in order to make claims (Duval, 1995). These different apprehensions with respect to the perspective lines had important implications for the validity of students' arguments. Students who only

used the perspective lines that had been included could only make valid claims about the widths of the given figures. Without performing some additional operation (such as measuring and calculating scale factors), students could not warrant claims about the relative heights of the figures. However, perhaps because students expected the necessary information to be included in the diagram, some students attempted to justify their claims based on the lines that were given.

Students' use of measuring and calculating scale factors highlights another way in which students apprehended the diagram discursively. Avery provided a case of a student who warranted a claim about the equal heights of the two houses based upon her finding of a constant ratio between the height and width of each house. Although the participants of the study had studied similarity in their geometry classes, few students used the numerical relationships between similar figures to justify their claims. One possible explanation for this is the way such an argument in the context of the problem relies upon the dual drawing/figure nature of a diagram (Parzysz, 1988). To make an argument about the similarity of the figures, students needed to treat the houses (or, equivalently, trees) as drawings. By taking measurements of those drawings and calculating scale factors, students could conclude that the two houses were similar. Extending this argument to a claim about the houses that the picture was meant to illustrate required students to view the houses also as figures representing objects in real life. The fact that the drawings were similar implied that figures represented by those drawings were equal in size. Translating between the features of an object and its image can be a complex process for students (Bartolini Bussi, 1998). It is possible that students did not see the connection between geometric similarity in the case of the drawing and equal height in the case of the real-world scenario, and therefore did not view similar figures as relevant knowledge for their arguments.

In addition to the conceptual challenges of using geometric similarity to make arguments about the houses or trees, we also saw a case in which an error in a student's measurement led to an invalid argument. Teddy's use of measurements and scale factors directly contradicted Shanise's claims based upon her knowledge of perspective. The students in Teddy's group accepted his arguments, perhaps because students are generally trained not to trust diagrams (Nachlieli & Herbst, 2009; Davis, 2006; Inglis & Mejía-

Ramos, 2009); because Teddy's numerical calculations seemed more mathematically valid; or because of the group dynamics. Students' visual perceptions of geometric similarity, however, can serve as a resource for supporting students to apply proportional reasoning (Cox, 2013). The study of similarity through geometric transformations including dilation is a relatively new feature of the geometry curriculum (NGAC, 2010). This work suggests that there is opportunity to strengthen students' visual reasoning by providing more experiences with dilations, which can contribute to more robust knowledge of similarity overall. The connections between 1-point perspective and geometric dilations can serve as a resource for students to notice when figures are related through a dilation in mathematics.

Conclusion

Students' collective argumentation during group work can support improved understanding of fundamental concepts (Moore-Russo et al., 2011; Stephan & Akyuz, 2012; Yackel, 2002; Zahner, 2012). Moreover, although students' real-life experiences are not always compatible with school mathematics, arguments built upon prior experiences and intuition are a valuable step towards establishing more formal mathematical logic (Inglis, Mejía-Ramos, & Simpson, 2007). This study builds upon research on students' argumentation in geometry by considering students' use of a diagram for formulating arguments about a 1-point perspective drawing. We offer an entry point to considering how students' experiences viewing and representing 3-dimensional space can provide a context for reasoning about the geometric transformation of dilation.

This study also illustrates how students integrate multiple sources of prior knowledge to formulate arguments through the use of a diagram. Students' knowledge of perspective drawing, and experiences viewing 3-dimensional objects in real life, helped them formulate claims about the objects represented through the diagram we provided. In addition to their perceptual apprehensions of the diagram informed by their knowledge of perspective, students needed to engage with discursive or operative apprehensions (Duval, 1995) to warrant their claims about the houses and trees. A teacher can use a problem as an opportunity to facilitate multiple apprehensions of a diagram among students (González, 2013). In this way,

diagrams can become resources for students to think with as they develop geometric arguments.

The study of similarity through geometric dilation is a relatively new component of the geometry curriculum (NGAC, 2010). As such, it is important to explore ways to connect this topic with students' multiple sources of prior knowledge, including their knowledge of school mathematics, their mathematical practices, and their knowledge of contexts. One-point perspective drawing offers one avenue through which students can explore the concept of dilation and its connection to real-world phenomena. Through this process, students have the opportunity to consider the subtle distinctions between a geometry diagram as a drawing and the more abstract figure that the drawing represents. The use of a real-world context may make this distinction more explicit, as the figures represented in a drawing may be connected to tangible objects in the real world to which students can relate.

Acknowledgement

Notes

[1] All names are pseudonyms.

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