DOCUMENTO DE TRABAJO. E2011

New relative indices of inequality base don the lorenz curve: perimeter and longitudinal ratio measures?

ÓSCAR MARCENARO GUTIERREZ
JOSÉ LUIS ALBÁNCHEZ BLANCO

JUNTA DE ANDALUCIA
Centro de Estudios Andaluces
CONSEJERÍA DE LA PRESIDENCIA

Centro de Estudios Andaluces

El Centro de Estudios Andaluces es una entidad de carácter científico y cultural, sin ánimo de lucro, adscrita a la Consejería de la Presidencia de la Junta de Andalucía.
El objetivo esencial de esta institución es fomentar cuantitativa y cualitativamente una línea de estudios e investigaciones científicas que contribuyan a un más preciso y detallado conocimiento de Andalucía, y difundir sus resultados a través de varias líneas estratégicas.

El Centro de Estudios Andaluces desea generar un marco estable de relaciones con la comunidad científica e intelectual y con movimientos culturales en Andalucía desde el que crear verdaderos canales de comunicación para dar cobertura a las inquietudes intelectuales y culturales.

Las opiniones publicadas por los autores en esta colección son de su exclusiva responsabilidad

Centro de Estudios Andaluces
CONSEJERÍA DE LA PRESIDENCIA

E2010/11

NEW RELATIVE INDICES OF INEQUALITY BASED ON THE LORENZ CURVE: PERIMETER AND LONGITUDINAL RATIO MEASURES

José Luis Albanchez Blanco

Oscar David Marcenaro Gutierrez,[*] [+]

[+]Fundación Centro de Estudios Andaluza y Universidad de Málaga

Resumen

En este trabajo proponemos dos medidas alternativas, de medición de la desigualdad, al popular índice de Gini: las medidas exceso de proporción perimetral y longitudinal cuadráticas. Éstas constituyen indicadores de desigualdad que aportan simplicidad y reducen el coste operativo asociado al índice de Gini, presentando propiedades más deseables. Junto al desarrollo de estos nuevos indicadores se ha efectuado una aplicación empírica de estas medidas para estimar la desigualdad del ingreso en España, comparando sus resultados con los obtenidos por otros índices. La información proporcionada en la Memoria de la Administración Tributaria del año 2005 ha sido la fuente estadística empleada.

[*] Corresponding author: odmarcenaro@uma.es. This author acknowledges the financial support granted by the Fundacion Centro de Estudios Andaluces under research contract PRY066/09. Corresponding author:odmarcenaro@uma.es.

Abstract

In this paper we propose two alternatives measures to the most used measure of relative inequality (the Gini index): *excess perimeter* and *longitudinal quadratic ratio*. These inequality indicators bring simplicity and reduce the operating costs associated with the Gini index, also presenting best properties. Along with the development of these new indicators we present an empirical application carried out to estimate income inequality in Spain, comparing its results with those obtained by other more traditional indices. The information provided in the Report of the Tax Administration in 2005 has been the source of statistics used for this empirical approach.

Keywords: inequality, relative indices of inequality.
JEL Classification: H24, C43.

1. Introduction.

The inherent difficulty in establishing a universally accepted concept of inequality is unquestionable. At least that is what is seen in the subjective perceptions of people who have been required to materialize their notion through questionnaires that support research aimed at this purpose[1]. If to the wide ambiguity of the concept we add the selection problem of the variable to which it refers -hereinafter, for simplicity, we will denotes it as income-[2], it is clear the enormous complexity involved in the measurement of inequality.

In this sense, the instruments that have traditionally been used to analyze the inequality, the Lorenz Curve (1905) and the Gini index (1912, 1921) -graphical and analytical measures, respectively-, present several limitations. Among these, the inability to generate complete ordering of inequality when there are intersecting Lorenz curves delimiting a single area of concentration[3]. However both measures satisfy the property of invariance to scale changes and equiproportional changes in subject recipients of different income levels, or simultaneously in both magnitudes.

We propose two measures, "*perimeter and longitudinal ratio measures*", as indicators of inequality that bring simplicity and reduce the operating costs associated with the Gini index (G in what follows), without losing its character as a *relative index of inequality*. To this end, in section 2 we briefly present the framework for the measures we analyze, explaining the concept and properties of relative inequality indices for the full range of families of indicators, which include absolute and intermediate measures. Then, in the third section, we present in detail the proposed

[1] Among others Amiel & Cowell (1992, 1999) and Ballano & Ruiz-Castillo (1992).
[2] See Sen (1995) who ask in his research "why the equality" and "equality of what?".
[3] The Gini index also has the limitation of non-additive decomposability in the usual sense, so it is not possible to obtain the inequality of a population partitioned into strata based on the inequality of the groups and between-groups. However Dagum (1997, 2001) proposes a decomposability which includes, other than the between-groups and within-group, an index of relative economic affluence between distributions

new measures and its reformulation in a disaggregated form (according to the between/within-group inequality), for which we need the derivation of these measures from the relations of advantage. This section ends with the verification of compliance of the properties required to the relative indices. The usefulness of the proposed new indices is illustrated in the fourth paragraph, which makes an empirical approach to the study of inequality in the recent distribution of tax revenue in Spain, considering breakdowns by income group (income from labor and capital). Along with the proposed measures we apply traditional indices in order to make a comparative analysis between the results of the new measures and the ones reported by the group of traditional indicators. The paper concludes with a section summarizing the main conclusions.

2. The relative inequality indices (RII).

The RII are measures of concentration that verify what is called inequality of the Lorenz curve, i.e. their value is diminished when the Lorenz curve moves closer to the line of equal distribution, and vice versa. Among these measures, which are consistent with the criterion of Lorenz partial order, the most prominent are the *generalized entropy index* (Shorrocks, 1980), Atkinson (1970) and the generalized *Gini coefficients*[4]. Importantly, the classification of the above measures as RII has certain limitations. The first, the different theoretical basis, described by Newberry (1970), which hampers the assessment of its capacity as complementary measures of inequality. Moreover, except for G, none of the above indices support a direct expression from the Lorenz curve, which makes it complex to extract information about their characteristics.

[4] The RII has constituted the cornerstone, and sometimes exclusively, of the works of authors such as Cowell (1977), Nigärd and Sandström (1981), Morris and Preston (1986), Jenkins (1989) and Champernowne and Cowell (1993). In Foster (1985) thorough analysis of such measures.

It is essential to bear in mind that when the Lorenz curves, of the distributions which we are comparing in terms of inequality, intersect and the criterion of Lorenz dominance is not resolving, then the rankings obtained by different RII does not have to match, even though they are all Lorenz-consistent.

Like the Lorenz curve, the RII remain unaffected by proportional changes in income, or by the changes experienced by recipients of different income levels, or both simultaneously[5]. In other words, are invariant under scale changes.

If we add scale invariance symmetry[6] and sensitivity to transfers which constitute to Shorrocks (1988, pp. 432) "the most important defining characteristic of an index of inequality," we come to the RII. To precisely define the concept of RII we denote by f(x) and f(y) the density functions of income distribution of two populations X and Y with averages μ_x and μ_y, respectively. In this context if the distribution f(y) dominates to f(x), i.e. if $L_y \geq L_x$, and we make the transformation $z = \frac{\mu_y}{\mu_x} x$ then RII(z)=RII(x), due to scale invariance. After the transformation f(y) dominates to f(z) despite the coincidence between the averages of both distributions. In fact, the change of f(z) to f(y) can be achieved carrying out a sequence of income transfers from higher levels to the lowest. Therefore RII(y)<RII(z), i.e. the dominant distribution will present a less unequal distribution whatever is the RII used.

In contrast, absolute inequality indices would require invariance to the same changes in all incomes, instead of scale invariance, as inferred from the comprehensive study on the subject by Blackorby and Donaldson (1980).

[5] The scale invariance is the second revised principle of Dalton (1920), and independence from changes in proportion of population size room. Also, check RII also reduce transfers are not altering the relative income position of both (first principle) and equal increases in income (third principle).

[6] Also called anonymity; this condition is ethically acceptable provided that individuals or families have comparable requirements.

Even less important than absolute inequality indices are the indices of intermediate inequality (III), defined as those which are invariant under a convex combination of absolute and relative changes so that III(y)=III(Z) if:

$$z = \left[y + c\left(\alpha y + (1-\alpha)e\right)\right] \qquad \forall\ 0 \leq \alpha \leq 1 \qquad [1]$$

where "c" is an scalar and "e" a unitary vector[7]. If $\alpha = 0$ then z=y+c, thus z is the result of applying a change of origin to y. If $\alpha = 1$, z results from applying a change of scale (z=y+cy). For the extreme values of α the intermediate indices match the absolute or relative, respectively.

Dalton (1920) is considered the first investigator to detect the problem of establishing a complete ordering of distributions of income or wealth, because different measures may yield different rankings. His attempt to sort this out focused on fixing a number of principles that bear his name, which should reduce inequality each time that, starting from a default situation, an event that could be classified within one or several of the principles, simultaneously.

But as happens with the measures we propose, all the measures relating to the methods considered involve different weighting schemes. Hence it is extremely difficult to opt for a particular measure as suitable for the quantification of inequality, to the extent that some authors classify them through the application of synthetic indicators resulting from the application of multivariate analysis methods[8].

3. Relative inequality indices based on the length and perimeter of the Lorenz curve.

[7] See Bossert and Pfingsten (1990).

[8] See Garcia *et al*. (2002) for an empirical application of the method of principal components to synthesize a series of indicators of the Spanish Autonomous Communities.

In this section we present two new indices of inequality of undoubted theoretical and practical relevance. These proposed new measures are defined from the Lorenz curve and help to enrich their characterization, and may incorporate judgments about inequality as with the Atkinson indexes.

Adopting the perspective of the distribution of the outcome variables (income, etc ...) is possible to derive the new measures proposed. These, like G, are obtained by the comparison procedure for ratio differences, simply by replacing the magnitude of the surface area of concentration by its perimeter or the length of the Lorenz curve. In fact, instead of considering the proportion represented by the excess of surface concentration between any distribution and the equal-maximum, with respect to the differential area of extreme concentration (1/2) -G defining reason-, we propose using the ratio defined by the following alternative expressions:

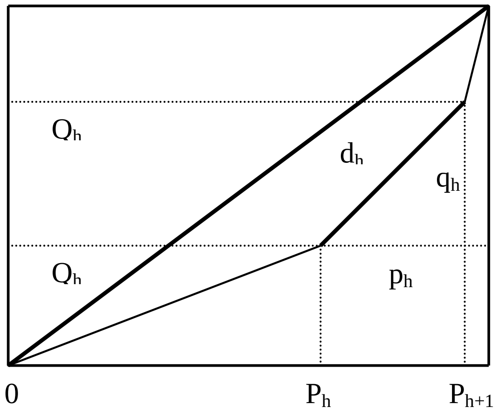
a) The excess perimeter of an area over another, being this excess estimated by the Euclidian distances between points on the Lorenz curve, or by the squared distances.

b) The excess of the length of the Lorenz curve to the minimum-equal curve, whether estimated by Euclidean distances or by these squared.

The estimate using the proposed methods of inequality for a population of N individuals, with incomes belonging to the interval $[x_1, x_N]$, requires the prior determination of the distance between the N+1 consecutive points that make up the Lorenz curve. Based on this idea, assume that $L_h(P_h, Q_h)$ and $L_{h+1}(P_{h+1}, Q_{h+1})$ represent two points on the Lorenz curve (see Figure 1)[9], and d_h is the distance between them initially obtained in the classical Euclidean form.

[9] P_h and Q_h being the difference between cumulative proportions of population and income, respectively, for pairs of consecutive values.

Figure 1. Geometric distance between two points of the Lorenz curve.



Then,

$$d_h = \sqrt{(P_{h+1}-P_h)^2+(Q_{h+1}-Q_h)^2} = \sqrt{p_h^2+q_h^2} = \sqrt{\left(\frac{n_h}{N}\right)^2+\left(\frac{x_h \cdot n_h}{N\mu_x}\right)^2} = \left[\sqrt{\left(\frac{x_h}{\mu_x}\right)^2+1}\right] f_h \quad [2]$$

If we generalize to the N distances between consecutive points of the Lorenz curve we obtain[10]:

$$Lo = \sum_i \left[\sqrt{\left(\frac{x_i}{\mu_x}\right)^2+1}\right] f_i \quad [3]$$

and thus, for the case of the square of distance between points:

$$L2 = \sum_i \left[\left(\frac{x_h}{\mu_x}\right)^2+1\right] f_i^2 \quad [4]$$

Lo represents the length of the Lorenz curve and *L2* the addition of the quadratic distances. Regardless of the method chosen to compute the distance, one can see immediately the invariance of the proposed indices to changes of scale, for what should be included amongst the RII. The proportions of the excess perimeter and length differentials compared to the maximum surplus, that results from comparing the values associated with extreme situations[11], would synthesize in the following algebraic expression:

$$D^{(P)} = \frac{Lo + \cancel{\sqrt{2}} - \cancel{\sqrt{2}}}{2 + \cancel{\sqrt{2}} - \cancel{\sqrt{2}}} = \frac{\sum_i \left[\sqrt{\left(\frac{x_i}{\mu_x}\right)^2+1}\right] \cdot f_i}{2} \quad [5]$$

[10] Applying instead distances from the metric space of Minkowski:

$$Lo^{(M)} = \sum_i \left[\sqrt[p]{\left(\frac{x_i}{\mu_x}\right)^p+1}\right] f_i$$

In the case that $p \to \infty$ we are in the metric space of Chebycheff. We do not go deeper into this line of enquiry as this is not the main focus of our paper. See Egge and Roussean (1990).

[11] These extreme values correspond to situations of equal and maximum inequality, respectively.

Those extreme values correspond to situations of maximum equality and maximum inequality, respectively. Just computing the corresponding Euclidean distances we can easily infer that the extreme values would be:

	Magnitude	
	Length	Perimeter
Maximum-equality	$\sqrt{2}$	$\sqrt{2}$
Maximum-inequality	$2+\sqrt{2}$	2

As we approach the Lorenz curve to the situation of maximum inequality, the N-1 initial values of q_i tend to cancel and the last q_n to be equal to unity. Therefore, the length, in quadratic distances, of the curve representative of the maximum concentration would be that appearing in the denominator of the expression:

$$D^{(P2)} = \frac{\sum_i \left[\left(\frac{x_i}{\mu_x}\right)^2 + 1\right] f_i^2}{\sum_i f_i^2 + 1} \qquad [6]$$

$$D^{(L)} = \frac{Lo-\sqrt{2}}{2-\sqrt{2}} = \frac{D^{(P)} - \frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}\sum_i \left[\sqrt{\left(\frac{x_i}{\mu_x}\right)^2 + 1}\right] \cdot f_i - 1}{\sqrt{2}-1} \qquad [7]$$

$$D^{(L2)} = \frac{\sum_i \left[\left(\frac{x_i}{\mu_x}\right)^2 - 1\right] f_i^2}{1-\sum_i f_i^2} \qquad [8]$$

So far we have assumed that the distribution of the analyzed variable takes a finite set of values. However, in case one knows the functional form of the Lorenz curve[12] the methodology developed can be easily adapted to the case of continuous

[12] Being L=g(F).

distributions. Thus, transforming the equation (4) we come to the following expression:

$$Lo = \int_0^1 \left[\sqrt{\left(\frac{dL}{dF} \right)^2 + 1} \right] dF \qquad [9]$$

with *Lo*=Length of the Lorenz curve.

As an additional benefit of these new indices presented ($D^{(P2)}$ and $D^{(L2)}$) we highlight the possibility of decomposition in an aggregative way in the *within* and *between-groups* components, which will enrich the studies of inequality addressed from these indices. Opposite to these, the indices based on simple Euclidean distances (not squared) only allow to obtain the within-group component. In particular, $D^{(P)}$ has the unique property of allowing aggregative decomposition, so that the contribution of each subpopulation to the inequality would be given by:

$$A^{(i)} = \frac{1}{2} \left[\sqrt{\left(\frac{x_i}{\mu_x} \right)^2 + 1} \right] f_i \qquad [10]$$

and the percentage contribution:

$$C^{(i)} = \frac{A^{(i)}}{D^{(P)}} \qquad [11]$$

That is, with the RII the global inequality is the result of adding only the one in the subpopulations (within-group component) because there is no between-groups-component.

In cases where distributions are presented partitioned in quartiles, the contributions of each subpopulation quartile, i.e. the within-group inequality and its participation in the global inequality, grow with the rank of the quartile. In this way the range of variation would be between the shares and contributions for the first and the *n-th* value of the variable under analysis, so that $A^{(1)} < A^{(i)} < A^{(n)}$ and $P^{(1)} < P^{(i)} < P^{(n)}$.

While the traditional aggregate decomposition of $D^{(P2)}$ can be addressed directly operating from the algebraic expression, is not the case with $D^{(L2)}$. To address the decomposition of the latter we have to resort to the concept of relationship of advantage. Following Alker (1973, p. 40), a basic measure of inequality is the so-called benefit relationship of advantage ($_i$), defined as the percentage excess of the measured value relative to the mean of the distribution. As indices based on quadratic distances, simple distance-based can be expressed in terms of relations without further advantage to operate in the expression $D^{(P)}$[13]:

$$D^{(P)} = \frac{Lo}{2} = \frac{\sum_i \left[\sqrt{\left(\frac{x_i - \mu_x + \mu_x}{\mu_x} \right)^2 + 1} \right] f_i}{2} = \frac{\sum_i \left[\sqrt{\left(\frac{x_i - \mu_x}{\mu_x} + 1 \right)^2 + 1} \right] f_i}{2} = \quad [12]$$

$$= \frac{\sum_i \left[\sqrt{(\varpi_i + 1)^2 + 1} \right] \cdot f_i}{2} = \frac{\sum_i \left[\sqrt{\varpi_i^{\,2} + 2\varpi_i + 2} \right] \cdot f_i}{2}$$

The term $D^{(P2)}$ also allows its aggregative decomposition from relations of advantage, but in this case the decomposition is the traditional, so that within-groups inequality of each subpopulation would be given by:

$$W^{(j)} = \frac{\left(\frac{x_j}{\mu_x} f_j \right)^2}{\sum_j f_j^2 + 1} \quad [13]$$

and the percentage contribution:

$$C^{(j)} = \frac{\left(\frac{x_j}{\mu_x} f_j \right)^2}{\sum_j \left[\left(\frac{x_j}{\mu_x} \right)^2 + 1 \right] f_j^2} \quad [14]$$

[13] Obviously the resulting decomposition would not be traditional, since it would not allow us to discern the within and between groups component.

But with this RII global inequality is the result of adding the internal ones reported by subpopulations – *W(j)* – plus the between-groups inequality -*B(j)*- that may come about in two ways:

1) Depending only on the number of members of the subpopulation and the size of the population analyzed. In this case the sum would be given by:

$$B^{(j)} = \frac{\sum_j f_j^2}{1+\sum_j f_j^2} \qquad [15]$$

2) In terms of relationships of advantage. Then resulting:

$$D^{(P2)} = \underbrace{\frac{\sum_j \varpi_j^2 f_j^2}{\sum_j f_j^2 + 1}}_{W^{(j)}} + \underbrace{\frac{2\left(\sum_j \varpi_j^2 f_j^2 + \sum_j f_j^2\right)}{\sum_j f_j^2 + 1}}_{B^{(j)}} \qquad [16]$$

This second way of expressing inequality between subpopulations is more accurate because it incorporates not only population and subpopulations size but also a measure of the level reached by the variable under analysis in each subpopulation. As outlined above, $D^{(L2)}$ is only susceptible to be decomposed in the traditional manner from the relations of advantage, so that under this premise we would get:

$$D^{(L2)} = \underbrace{\frac{\sum_j \varpi_j^2 f_j^2}{1-\sum_j f_j^2}}_{W^{(j)}} + \underbrace{\frac{2\left(\sum_j \varpi_j^2 f_j^2\right)}{1-\sum_j f_j^2}}_{B^{(j)}} \qquad [17]$$

For quartile-partitioned distributions, inputs and contributions obtained from $D^{(P2)}$ and $D^{(L2)}$ also grow with the quartile range, its values ranging between the first and the nth value, verifying: $A^{(1)} < A^{(i)} < A^{(n)}$ and $P^{(1)} < P^{(i)} < P^{(n)}$.

The estimate of inequality for a subpopulation of k individuals with incomes belonging to the interval $\left[x_h, x_m\right]$ could be estimated following two possible routes:

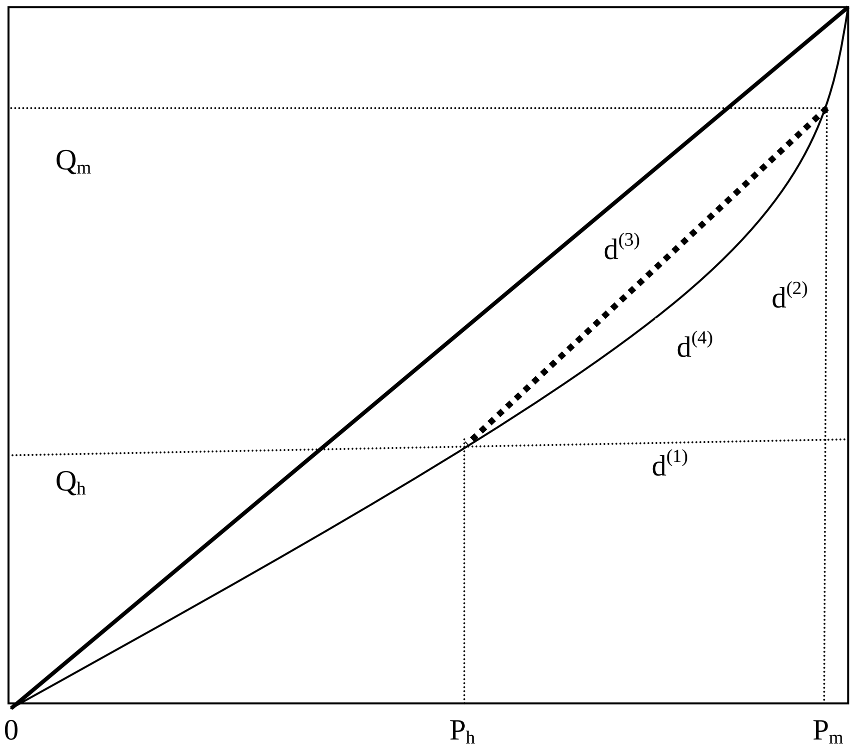
a) Directly applying the above mentioned expressions to the distribution of the subpopulation.

b) Using any of the procedures implemented to obtain the previous indices, but restricting them to the segment of the population Lorenz curve delimited by the points associated with the extremes of the subpopulation distribution range.

The first of these pathways involves simply implementing the indices on the Lorenz curve defined for the subpopulation, while the second channel is based on the population Lorenz curve delimited by the intermediate values associated with the extremes of the subpopulation.

In figure 2 we represent the situation that would arise in estimating the subgroup inequality resulting from a partition. Such a magnitude would be measurable in an analogous manner to the global inequality, after obtaining the distances marked $d^{(1)}$, $d^{(2)}$, $d^{(3)}$ and $d^{(4)}$.

Figure 2. $D^{(P2)}$ and $D^{(L2)}$ applied to quartile partitions.



where:

$$d^{(1)} = P_m - P_h = \frac{N_m}{N} - \frac{N_h}{N} = \frac{\sum_{i=h}^{i=m} n_i}{N} = \sum_{i=h}^{i=m} f_i \qquad [17]$$

$$d^{(2)} = Q_m - Q_h = \frac{\sum_{i=1}^{m} x_i n_i}{N\mu_x} - \frac{\sum_{i=1}^{h} x_i n_i}{N\mu_x} = \frac{\sum_{i=1}^{m} x_i n_i}{N\mu_x} = \sum_{i=h}^{i=m} x_i f_i \qquad [18]$$

$$d^{(3)} = \sqrt{\left(\sum_{i=h}^{i=m} f_i\right)^2 + \left(\sum_{i=h}^{i=m} x_i f_i\right)^2} \qquad [19]$$

$$d^{(4)} = \sum_{i=h}^{m}\left[\sqrt{\left(\frac{x_i}{\mu_x}\right)^2 + 1}\right] f_i \qquad [20]$$

The inequality measured by $D^{(P)}$, $D^{(Lo)}$, $D^{(P2)}$ y $D^{(L2)}$, in the considered subpopulation (J), would be:

$$RI_J^{(P)} = \frac{\sum_{j=h}^{m}\left[\sqrt{\left(\frac{x_j}{\mu_x}\right)^2 + 1}\right] f_j}{\sum_{j=h}^{m} f_j + \sum_{j=h}^{m} \frac{x_j}{\mu_x} f_j} \qquad [21]$$

$$RI_J^{(L)} = \frac{\sum_{j=h}^{m}\left[\sqrt{\left(\frac{x_j}{\mu_x}\right)^2 + 1}\right] f_j - \sqrt{\left(\sum_{j=h}^{m} f_j\right)^2 + \left(\sum_{j=h}^{m} \frac{x_j}{\mu_x} f_j\right)^2}}{\sum_{j=h}^{m} f_j + \sum_{j=h}^{m} \frac{x_j}{\mu_x} f_j - \sqrt{\left(\sum_{j=h}^{m} f_j\right)^2 + \left(\sum_{j=h}^{m} \frac{x_j}{\mu_x} f_j\right)^2}} \qquad [22]$$

$$RI_J^{(P2)} = \frac{\sum_{j=h}^{m}\left[\left(\frac{x_j}{\mu_x}\right)^2 + 1\right] f_j^2}{\sum_{j=h}^{m} f_j^2 + \left(\sum_{j=h}^{m} \frac{x_j}{\mu_x} f_j\right)^2} \qquad [23]$$

$$RI_J^{(L2)} = \frac{\sum_{j=h}^{m}\left[\left(\frac{x_j}{\mu_x}\right)^2 - 1\right] f_j^2}{\left(\sum_{j=h}^{m} \frac{x_j}{\mu_x} f_j\right)^2 - \sum_{j=h}^{m} f_j^2} \qquad [24]$$

The indices $D^{(P)}$, $D^{(Lo)}$, $D^{(P2)}$ and $D^{(L2)}$ are synthesis procedures to obtain a unique value -scalar-of the relations of advantage in a distribution. This way you can quantify the inherent inequality (global), and make comparisons. These indices verify the properties required from an RII, including the first fundamental principle of Dalton (i.e. transfers), without any restriction. They are also continuous measures, differentiable, nonnegative, normalized, symmetric and invariant to changes of scale and replicas of the population, being modified by changes of origin and scale-origin.

Nevertheless, $D^{(P)}$ and $D^{(Lo)}$ have drawbacks that limit their goodness as indices[14]: none of them are decomposable in within-group and between-groups. $D^{(P)}$ only takes the value zero for the case of absence of surface concentration, because of the coincide of the Lorenz curve with the equidistribution line; if the area exists, its share becomes $\frac{\sqrt{2}}{2}$. Thus the index shows little sensitive (higher robustness) to significant increases in inequality -as accounted for other measures-. However, this latter limitation can be resolved to some extent in the index by introducing a parameter of aversion to inequality (α) of the form $\left[D^{(P)}\right]^{\alpha}$. Despite their limitations, have an obvious utility if we think that is the theoretical basis of $D^{(P2)}$ and $D^{(L2)}$.

Inequality indices founded on quadratic distances can overcome the major drawback associated with the ranking criteria of Lorenz and Gini. Specifically, these indices can not reach a complete management of the variable under analysis when Lorenz curves intersect defining a single area of concentration. Indeed, with the indices of inequality based on quadratic distances, different Lorenz curves, surfaces generating identical concentration, yield different values for $D^{(P2)}$ and $D^{(L2)}$, thus obtaining complete arrangements in inequality.

In this sense, the indices $D^{(P)}$, $D^{(L)}$, $D^{(P2)}$ and $D^{(L2)}$ are found to be ordinally equivalent to each other and with the Gini index, with the exception of the idea previously mentioned regarding the coincidence of the concentration areas. In summary, $D^{(P)}$ and $D^{(L)}$ do not allow a complete sorting of a group of distributions, by contrast $D^{(P2)}$ and $D^{(L2)}$ allow to order in full. Sen (1973, 1978) and others have expressed

[14] These limitations are shared with G.

reservations about the inequality measures that provide this type of sorting –the complete-, arguing that inequality does not have any innate property of completeness, but many facets that can point in different directions. However, these measures can help assess the consistency of the procedures for synthesizing multiple indicators and provide a resource to rank in inequality.

Finally we should highlight that all the proposed indexes, both based on simple distances and quadratic distances, are readily applicable to obtain the concentration of a quartile subpopulation.

4. Empirical analysis of inequality in Spain.

Empirical studies on inequality involve preliminary methodological decisions that are not always neutral with respect to the conclusions obtained. The greatest influence on the final results are those targeted to the choice of databases, definition of income and equivalence scales. By way of example, serve the disparity of conclusions about the evolution of inequality in Spain as use the Household Budget Survey or the Fiscal Panel Data reporting income tax (see Ruiz-Huerta *et al*. (1993) and Cantó *et al*. (2000)). While the former shows a reduction in inequality, the results derived from fiscal data point in the opposite direction.

With the application presented in this section we wanted to contrast the proposed measures of inequality, using data from cross-sectional from the statistics on income tax, for the year 2005. Obviously the fact of opting for tax data implies accepting the notion that we restrict the figures to the income tax subject to the legislation. We must also point out that no equivalence scales have been applied to reporting units for each interval of income (taxable income) based on the average family size. Consequently, the results obtained should be associated with the distribution of income tax filers -bound and not bound-tax, assuming homogeneity in personal and family circumstances of them all. Such an assumption implies the distortion of the results of the inequality measures applied; this distortion would be reduced according to the dispersion of the distribution of family size. However, our aim is limited to contrast and compare the proposed indicators for which data on the distribution of tax revenues are, nevertheless, a valuable reference and starting point.

In our empirical approach we compare the results of the implementation of a wide range of indexes of various kinds, under the following families:

a) The *objective indices*, among which include the new proposed in this article, G, Schutz index (S)[15], the Coefficient of Variation (CV) -which is unsuitable for integration into an abbreviated social welfare function-, and Logarithmic Variance (LV) that verifies the principle of the transfers.

b) The *generalized entropy*, complex to interpret and conceptualize, as an index of entropy of order *c* is defined as:

$$I_c = \frac{1}{Nc(c+1)} \sum_i \frac{x_i}{\mu_x} \left[\left(\frac{x_i}{\mu_x} \right)^2 - 1 \right] \qquad [25]$$

Where *c* is a parameter of aversion to inequality.

c) The *normatives*, more common in recent studies based on the concept of equally distributed equivalent income, with values 0.5 and 1 for the Atkinson index, so that:

$$A_{0.5} = 1 - \left[\sum_i \frac{1}{N} \left(\frac{x_i}{\mu_x} \right)^{(1-0.5)} \right]^{(1-0.5)^{-1}} \qquad [26]$$

[15] This index measures the distance between the Lorenz curve and the line of equal distribution in terms of maximum vertical separation, which represents the proportion of total income that would have to be transferred from higher income individuals to recipients with low incomes to achieve perfect equality. This indicator is rarely used because it is based on a point on the Lorenz curve and does not verify the principle of transfers.

$$A_1 = 1 - \prod_i \left(\frac{x_i}{\mu_x} \right)^{\frac{1}{N}} \qquad [27]$$

These indices provide normative content in the empirical application presented in this section.

The empirical evidence derived from this application indicates that income inequality in Spain for 2005 reaches the values presented in table 1:

Table 1. Indices of inequality, by source.

Indices	Total Income	Wages	Capital Income
G ; S	0.42 ; 0.28	0.41 ; 0.28	0.51 ; 0.35
C.V. ; L.V.	0.82 ; 0.12	0.79 ; 0.14	0.99 ; 0.17
$T_{C->-1}$	0.27	0.29	0.36
$T_{C->0}$	0.26	0.25	0.31
$T_{C=0}$	0.34	0.31	0.38
$A_{0,5}$	0.19	0.19	0.23
A_1	0.24	0.25	0.28
$D^{(P)}$	0.74	0.74	0.76
$D^{(Lo)}$	0.13	0.11	0.20
$D^{(P2)}$	0.23	0.22	0.31
$D^{(L2)}$	0.06	0.05	0.16

Source: Authors' own calculations from data provided in the Spanish Report of the Tax Administration (2005).

The results obtained showed how the sensitivity of each indicator to changes in the distribution is very uneven. So if we take as a basis for comparison the distribution of total income before tax the reduction of capital gains barely reduces the concentration, with the exception of LV and $T_{c->-1}$, which show contrasting results as a consequence of their weighting schemes. If the comparison is between total income and capital, we see that inequality is increasing although in very different proportions, $D^{(P)}$ only increases by 2.7%, much lower than that of G which reaches 21.42%. However $D^{(P2)}$, $D^{(L2)}$ and $D^{(L)}$ outweigh that, the latter being the most affected by the change experienced in the Lorenz curve.

Additionally, the estimation of inequality using the battery of indices for the subpopulations representing the stratified social groups (table 2), without continuity[16] -only three quartiles-, shows that all indicators are consistent with G, although the sensitivity to changes in within-groups inequality is very uneven. Again longitudinal indices $D^{(Lo)}$ and $D^{(L2)}$ happen to be the ones with widest variation compared to the rigidity that show $D^{(P)}$ and $D^{(P2)}$.

Table 2. Inequality indices by strato.

Indices	First Stratum (Bottom quartile)	Intermediate Stratum (Third quartile)	Top stratum (Top quartile)
G ; S	0.29 ; 0.14	0.064 ; 0.03	0.25 ; 0.12
C. V.; L.V.	0.58 ; 0.08	0.13 ; 0.017	0.5 ; 0.068
$T_{C->-1}$	0.128	0.019	0.11
$T_{C->0}$	0.116	0.015	0.099
$T_{C=0}$	0.128	0.017	0.11
$A_{0,5}$	0. 14	0.03	0.12
A_1	0. 16	0.036	0.14
$D^{(P)}$	0.73	0.70	0.72
$D^{(Lo)}$	0.07	0.003	0.05
$D^{(P2)}$	0.20	0.18	0.19
$D^{(L2)}$	0.024	0.001	0.022

Source: Authors' own calculations from data provided in the Spanish Report of the Tax Administration (2005).

5. Conclusions.

In this study we sought to present new indicators of inequality grounded in longitudinal differential proportions, either of the Lorenz curve or of the area of concentration, by assessing the length by squared Euclidean distances and quadratic distances. These measures can be derived from an individual perspective from the concept of comparative advantage. In particular, special attention has been paid to the indices estimated based on the concept of quadratic distances, due to a number of reasons: are more sensitive to transfers that do not alter the order of the subject,

[16] To avoid the distortions that arise when an individual of a group is closer to the lower or higher extreme groups than that in which has been integrated.

have the additive decomposition property and are easily expressible in terms of the values observed and their frequencies, allowing us to demonstrate, with relative ease, their properties as RII and interpret in relation to the Lorenz curve.

The sensitivity for redistribution in the Lower tail of the distribution is much higher for indices formulated in terms of excess Longitudinal percentage, showing significantly surpassing the traditional Gini index and the expressed from the percentage differential perimeter. The same applies to redistributions between the upper and lower tail, although in these cases the magnitude of the change in $D^{(L)}$ and $D^{(L2)}$ tends to be equated. We also noted by mean of multiple simulations that, as a result of the incorporated weighting system, these measures have a great aversion to inequality, penalizing the presence of reduced income.

In continuous distributions, the determination of $D^{(P)}$ and $D^{(L)}$ can be very complex because of the need to integrate the square of the function derived from the Lorenz curve, specially bearing in mind the complexity of the latter function in most models of income distribution.

In short, we have presented complementary indices with the same properties as G, but with the advantage of providing comprehensive measures of inequality based on the Lorenz curve and therefore interpretable using this. These new indices can be very useful to assess concordance with other procedures for completeness or to erect for themselves in alternative procedures for complete sorting.

References

Alker, H. R. (1973). Introduction a la Sociologie mathematique. París. Larousse.

Amiel, Y. and Cowell, F. A. (1992). "Measurement of Income Inequality: Experimental Test by Questionnarie ", *Journal of Public Economics*, 47: 233-259.

Amiel, Y. and Cowell, F. A. (1999). Thinking about Inequality, University Press, Cambridge.

Atkinson, A. B. (1970). "On the measurement of inequality", *Journal of Economic Theory*, 2: 244-263.

Ballano, C. and Ruiz-Castillo, J. (1992). "Searching by Questionnaire for the Meaning of Income Inequality", *Revista Española de Economía*, 10: 233-259.

Blackorby, C. and Donaldson, D. (1980). “A theoretical treatment of indices of absolute inequality”, *International Economic Review*, 21-1: 107-136.

Bossert, W. and Pfingsten, A. (1990). “Intermediate inequality: concepts, indices, and welfare implications“, *Mathematical Social Sciences*, 19-2: 117-134.

Cantó, B., Rio, C. del and Gradin, C. (2000). "La situación de los estudios sobre desigualdad y pobreza en España", in I. Bazaga, J. A. Ramos y M. Tamayo (coord.) *Pobreza y desigualdad en España: Enfoques, Fuentes y Acción Pública*, Cuadernos de Gobierno y Administración, 2: 25-94.

Ruiz-Huerta, J., Martínez, R. and Ayala, L. (1993). “La distribución de la en España en los años ochenta:una perspectiva comparada”, in I Simposio sobre Igualdad y distribución de la renta y la riqueza, vol. II, *La distribución de la renta*, Fundación Argentaria, Madrid: 101-137.

Cowell, F. A. (1977). Measurig Inequality, Oxford: Philip Allan.

Champernowne, D. G. and Cowell, F. A. (1993). Inequality and Income Distribution, Cambridge: Cambridge University Press.

Dagum, C. (1997). "A new descomposition of the Gini income inequality ratio", *Empirical Economics*, 22: 515-531.

Dagum, C. (2001). "Desigualdad del rédito y bienestar social, descomposición, distancia direccional y distancia métrica entre distribuciones", *Estudios de Economía Aplicada*, 17: 5-52.

Dalton, H. (1920). “The measurement of the inequality of incomes”, *Economic Journal*, 30: 348-361.

Egghe, L. and Rousseau, R. (1990). *Introduction to informetrics*, Elsevier, Amsterdam.

Foster, J. E. (1985). "Inequality measurement", en Fair Allocation, *Proceedings of Symposia in Applied Mathematics*, 33.

Garcïa, C., Nuñez, J.J., Rivera, L.F. and Zamora, A.J. (2002). "Análisis comparativo de la desigualdad a partir de una bateria de indicadores. El caso de las Comunidades Autónomas españolas en el período 1973-1991", *Estudios de Economía Aplicada*, 20(I): 137-154.

Gini, C. (1912). “Variabilità e mutabilità, Studi Economico-giuridici”, Università di Cagliari, III.

Gini, C. (1921). "Measurement of Inequality of Incomes", *The Economic Journal*, 31: 124-128.

Jenkins, S. (1989). “The measurement of economic inequality“, en Readings on Economic Inequality, (ed. L. Osberg) Armonk, NY:Sharpe.

Lorenz, M. O. (1905). "Methods of Measuring concentration and Wealth", *Journal of the American Statistical Association*, 9: 209-219.

Morris, C. N. and Preston, I. P. (1986). ”Inequality, poverty and the redistribution of income”, *Bulletin of Economic Research*, 38: 277-344.

Newberry, D. M. (1970). The distribution and redistribution of income, Manchester University Press.

Nigärd, F. and Sandström, A. (1981). Measuring Income inequality, Estocolmo: Almqvist & Wiksell.

Sen, A. (1973). *On economic inequality*. Oxford University Press.

Sen, A. (1978). “Ethical measurement of inequality: some difficulties” in. Krelle and A. Shorrocks, eds. Personal Income Distribution, North-Holland.

Sen, A. (1995). Nuevo examen de la desigualdad. Alianza Economía.

Shorrocks, A. F. (1980). “The class of additively descomposable inequality measures", *Econometrica*, 48: 613-625.

Shorrocks, A. F. (1988). “Aggregation issues in inequality measurement“, en *Measurement in Economics: Theory and Applications of Economic Indices*, (ed. W. Eichhorn), Heidelberg: Physica-Verlag.