On the design of group strategy-proof mechanisms: domains, ranges and special conditions*

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Abstract: Group strategy-proofness is a very attractive requirement of incentive compatibility. But in many cases it is hard or impossible to find nontrivial social choice functions satisfying even the weakest condition of individual strategy-proofness. However, there are a number of economically significant domains where interesting rules satisfying individual strategy-proofness can be defined, and for some of them, all these rules turn out to also satisfy the stronger requirement of group strategy-proofness. In other cases, this equivalence does not hold. In a previous paper we provide sufficient conditions defining domains of preferences guaranteeing that individual and group strategy-proofness are equivalent for all rules defined on these domains. In this paper, we show that our results extend in two directions. On the one hand, to intermediate versions of strategy-proofness, defined to exclude manipulations by small group of agents. On the other hand, we provide guidelines on how to restrict the ranges of functions defined on domains that only satisfy our condition partially. Finally, we provide a partial answer regarding the necessity of our condition and give applications to particular settings. Some applications of our results are also provided.

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1 Introduction

Strategy-proofness is a demanding condition that most mechanisms will fail to satisfy, unless they are defined on properly restricted environments. Group strategy-proofness is an even more stringent requirement, but also a much more attractive one. Indeed, what use would it be to guarantee that no single individual could cheat, if a handful of them could jointly manipulate the system? Yet, the literature on mechanism design has concentrated mostly in analyzing the weakest of these two conditions: after all, it is hard enough to meet, and often impossible except for trivial procedures.

We are now well beyond the disquieting negative message of the Gibbard-Satterthwaite theorem (Gibbard, 1973 and Satterthwaite, 1975). After years of research, we know of a considerable number of instances where non-trivial mechanisms can be found to be strategy-proof when defined on some domains of interest. They include the case of voting when preferences are single-peaked or separable, or when the outcomes are lotteries; they also include families of cost sharing, matching and allocation procedures, again for the case where the preferences of agents are conveniently restricted. When looking at the rich literature on strategy-proof rules over restricted domains, one discovers, somewhat surprisingly, that some of the non-trivial strategy-proof mechanisms that arise in these environments are, indeed, also group strategy-proof or non manipulable by groups of size $k > 1$! It is as if, after a hard search for a solution to the challenge of strategy-proofness, the additional blessing of non manipulability by groups of size $k > 1$ would arise automatically, as an extra gift. In Barberà, Berga, and Moreno (2010; say BBM) we exhibit conditions on families of preference profiles such that any strategy-proof social choice function whose domain is one of these families will also be necessarily group strategy-proof.

In this work we define weaker variants of group strategy-proofness and discuss their connection with strategy-proofness. These variants are important, because it would be hard to avoid manipulations if groups of agents could distort the rules to their advantage, even if no single individual was able to. For this same reason, it is also interesting to examine those cases where groups could only manipulate if they were large enough. In these second best worlds, one could still hope that coordination costs and other restrictions might avoid actual manipulation. In this paper we discuss an adaptation of sequential inclusion sequential inclusion, defined in our previous work, guaranteeing that, when strategy-proof rules exist on given domains, they are also immune to manipulations by groups smaller than a given size.

Similarly, we know that on any given domain of preferences, individual and group strategy-proofness can be obtained at the cost of reducing the range of social choices to only include two alternatives. And we also prove in this paper that both conditions become equivalent if we restrict the range when the domain of preferences satisfies sequential inclusion on that range.

There are interesting frameworks where this range restriction is reasonable and our results in this paper can be applied and allow us to state the equivalence between individual and group strategy-proofness while our previous results in BBM did not.

For example, range restrictions have been analyzed in the following problems: in the problem of provision of public bad when agents are single-dipped, it is known that strategy-proof rules defined on single-dipped preferences have at most two alternatives in the range (see Manjunath, 2009, Barberà, Berga and Moreno, 2010). Also, in exchange economies, 2 agents and 2 goods and classical preferences, Lemma 3 in Barberà and Jackson (1995) state that "the range of each strategy-proof social choice function is diagonal". Thus, our result would allow us to affirm in both settings that strategy-proof rules with range B are group strategy-proof (already proved by the corresponding papers).

The paper is structured as follows: in Section 2 we describe the model and define the weak variants of group strategy-proofness. In Section 3 we introduce our new domain conditions based on weakening sequential inclusion in two different directions: to get immunity to manipulations by groups that are not "too large" and controlling for the alternatives in the range. We state the two main results. In Section 4 we mention some applications to emphasize the relevance of our contribution. Section 5 concludes.

2 The model and weak variants of group strategy-proofness

Note that the model we define below is the one defined in Barberà, Berga, and Moreno (2010).[1] Our model encompasses problems related to the provision of public good(s), voting for candidates to join a club, house allocation, exchange economies, among others.

Let A be the set of *alternatives* and $N = \{1, 2, ..., n\}$ be the set of *agents* (with $n \geqslant 2$). Let capital letters $S, T \subseteq N$ denote subsets of agents while lower case letters s, t denote their cardinality.

Let $\mathcal{R}$ be the set of complete, reflexive, and transitive orderings on A and $\mathcal{R}_i \subseteq \mathcal{R}$ be *the set of admissible preferences for agent $i \in N$.* A *preference profile,* denoted by $R_N = (R_1, .., R_n)$, is an element of $\times_{i\in N}\mathcal{R}_i$. As usual, we denote by P_i and I_i the strict and the indifference part of R_i, respectively. We will write the profile $R_N = (R_C, R_{N\backslash C}) \in \times_{i\in N}\mathcal{R}_i$ (or like (R_C, R_{-C})) when we want to stress the role of coalition C. Then the subprofiles $R_C \in \times_{i\in C}\mathcal{R}_i$ and $R_{N\backslash C} \in \times_{i\in N\backslash C}\mathcal{R}_i$ denote the preferences of agents in C and in $N\backslash C$, respectively.

The following concept is crucial in all our analysis. For any $x \in A$ and $R_i \in \mathcal{R}_i$, define *the lower contour set of R_i at x* as $L(R_i, x) = \{y \in A : xR_iy\}$. Similarly, the strict lower contour set at x is $\overline{L}(R_i, x) = \{y \in A : xP_iy\}$.

A *social choice function* (or a rule) is a function $f : \times_{i\in N}\mathcal{R}_i \to A$. Let A_f denote the range of the social choice function f.

We will focus on rules that are nonmanipulable, either by a single agent or by a coalition of agents. We first define what we mean by a manipulation and then we introduce the well known concepts of *strategy-proofness* and *group strategy-proofness.*

Definition 1 *A social choice function f is group manipulable on $\times_{i\in N}\mathcal{R}_i$ at $R_N \in \times_{i\in N}\mathcal{R}_i$ if there exists a coalition $C \subset N$ and $R'_C \in \times_{i\in C}\mathcal{R}_i$ ($R'_i \neq R_i$ for any $i \in C$) such that $f(R'_C, R_{-C})P_if(R_N)$ for all $i \in C$. We say that f is individually manipulable if there exists a possible manipulation where coalition C is a singleton.*

[1] We introduce the basic aspects here. The interested reader can check more details in Barberà, Berga, and Moreno (2010) when required for going through the proofs.

Definition 2 *A social choice function f is group strategy-proof on $\times_{i\in N}\mathcal{R}_i$ if f is not group manipulable for any $R_N \in \times_{i\in N}\mathcal{R}_i$. Similarly, f is strategy-proof if it is not individually manipulable.*

These concepts are well-known in literature. They were the crucial properties in Barberà, Berga, and Moreno (2010) where we motivated their interest.

Let us present an example to motivate the interest of the research in this paper. The example, that extends to larger $k's$, $n's$ and different $q's$, it not only proves that one can have strategy-proof rules that are not group strategy-proof on that domain, but it also suggests that the rules may be extremely fragile, as they can be manipulated by groups composed of two agents alone. So, a natural question would be if this is always the case of there exists the possibility of having group strategy-proofness for low-size group of agents without having full group strategy-proofness. This is a point we try to answer in the paper.

Example 1 *Separable preferences and its subdomains*
The domain of separable preferences, which are described in Barberà, Sonnenschein, and Zhou (1991), is an example of a domain admitting strategy-proof rules that are not group strategy-proof. Yet, we will also define a subdomain of these preferences where the equivalence still holds.
Barberà, Sonnenschein, and Zhou (1991) analyze the problem of selecting subsets from a set K of objects and k its cardinality. They have characterized the family of social choice functions on the domain $\mathcal{S}_k$ of separable preferences on 2^K that are strategy-proof. In separable preferences objects are divided between good and bad ones. Then, a preference relation is separable if for any set $A \subset K$, and any object $x \notin A$, $\{x\} \cup A$ is preferred to A if and only if x is a good object.
An example of these rules, which are in addition neutral and anonymous, is given by quota rules. When society consists of n agents, a quota is a number between 0 an n. Then, given the preferences of the agents, the rule with quota q chooses the objects that are ranked first for at least q agents.
Consider the case $K = \{a, b\}$, $n = 2$, $q = 1$. The following preferences are separable.

P_1	P_2
a	b
$\varnothing$	$\varnothing$
$\{a,b\}$	$\{a,b\}$
b	a

Under the quota 1 rule, the outcome is $\{a, b\}$, and no individual can manipulate. Yet, both agents would prefer $\varnothing$, and they can obtain this result if they both declare $\varnothing$ to be their best alternative. Observe that this preference profile does not satisfy sequential inclusion.

By requiring group strategy-proofness we avoid manipulations by means of coalitions of any size. However, we could be interested in a strategic concept avoiding manipulations by coalitions of particular sizes. Specifically, we could weaken the requirement of group strategy-proofness by just imposing that we want to avoid manipulations of coalitions of size less or equal than k, for a fixed $k < n$, that is, imposing "k-group strategy-proofness".

First, we formally introduce our new property that we call k-group strategy-proofness.[2] From now on, let $k \in \mathbb{Z}$ such that $k < n$.

Definition 3 *A social choice function f is k-group strategy-proof on $\times_{i \in N} \mathcal{R}_i$ if for any $R_N \in \times_{i \in N} \mathcal{R}_i$, there is no coalition $C \subseteq N$ with $\#C \leq k$ that manipulates f on $\times_{i \in N} \mathcal{R}_i$ at R_N.*

Note that if f is k-group strategy-proof then f is also l-group strategy-proof for $l < k$. The converse is not true (see Example 2).

Example 2 *Let $N = \{1, 2, 3\}$ and $A = \{y, z, a_1, a_2, a_3\}$. Suppose that $\mathcal{R}_i = \{R_i^1, R_i^2, R_i^3\}$ for each agent $i \in N$ and is given by:*

R_i^1	R_i^2	R_i^3
a_1	a_2	a_3
y	y	y
a_2	a_3	a_1
z	z	z
a_3	a_1	a_2

Define a social choice function f as follows:

R_3^1				R_3^2				R_3^3			
	R_2^1	R_2^2	R_2^3		R_2^1	R_2^2	R_2^3		R_2^1	R_2^2	R_2^3
R_1^1	a_1	a_1	a_1	R_1^1	a_1	a_2	a_3	R_1^1	a_1	y	a_3
R_1^2	a_1	a_2	z	R_1^2	a_2	a_2	a_3	R_1^2	a_2	a_2	a_3
R_1^3	a_1	a_1	a_1	R_1^3	y	a_2	a_3	R_1^3	a_3	a_3	a_3

We can check that f is 2-group strategy-proof (thus, strategy-proof) but not (3-)group strategy-proof. To see the latter, consider the profile $R_N = (R_1^2, R_2^3, R_3^1)$ and $R'_N = (R_1^3, R_2^1, R_3^2)$. Observe that $f(R_N) = z$ while $f(R'_N) = y$. Since y is strictly preferred to z by all agents, the whole coalition manipulates f at R_N via R'_N.

3 Domain conditions and results

In Barberà, Berga, and Moreno (2010) we introduced a domain condition called sequential inclusion (also its indirect version) that if satisfied, any strategy-proof rule defined on it was also group strategy-proof. After understanding sequential inclusion and observing why it failed in some frameworks, we realized that our domain condition could be weakened in two reasonable directions. Moreover, these weakened versions still allow us to obtain

[2] This generalizes Serizawa's (2006) concept of "effectively pairwise strategy-proofness". His concept requires that not only agents, but also pairs of agents should not be able to manipulate. In our case, we require that no group of size less than k can do it.

equivalence results with the same flavour as the ones in our original paper but with important consequences that will be worth to mention.

In this section we define our two new domain restrictions and state our new equivalence results, emphasizing in each case its peculiarities and importance.

3.1 Immunity to manipulation by groups that are not "too large"

First we propose a condition on preference profiles, that we call *k-size sequential inclusion.* Then, we establish the equivalence between individual and k-group strategy-proofness for social choice functions defined on domains satisfying that condition. Before that, let us introduce some notation that will be important and was already relevant for sequential inclusion.

Let $R_N \in \times_{i\in N}\mathcal{R}_i$, and y, z be a pair of alternatives. Denote by $S(R_N; y, z) \equiv \{i \in N : yP_iz\}$, that is, the set of agents who strictly prefer y to z according to their individual preferences in R_N.

Definition 4 *Given a preference profile $R_N \in \times_{i\in N}\mathcal{R}_i$ and a pair of alternatives $y, z \in A$, we define a binary relation $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ as follows:*[3]

$$i \succsim (R_N; y, z)j \text{ if } L(R_i, z) \subseteq \overline{L}(R_j, y).$$

Note that the binary relation $\succsim$ must be reflexive but not necessarily complete. As usual, we can define the strict and the indifference binary relations associated to $\succsim$. Formally, $i \sim j$ if $L(R_i, z) \subseteq \overline{L}(R_j, y)$ and $L(R_j, z) \subseteq \overline{L}(R_i, y)$. We say that $i \succ j$ if $L(R_i, z) \subseteq \overline{L}(R_j, y)$ and $\neg[L(R_j, z) \subseteq \overline{L}(R_i, y)]$.

Now, we formally define k-size sequential inclusion, a weaker version of sequential inclusion.

Definition 5 *A preference profile $R_N \in \times_{i\in N}\mathcal{R}_i$ satisfies the k-size sequential inclusion condition if for any pair $y, z \in A$, $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is complete and there is no cycle of l agents, for any $l \leq k$. A domain $\times_{i\in N}\mathcal{R}_i$ satisfies k-size sequential inclusion if any preference profile $R_N \in \times_{i\in N}\mathcal{R}_i$ satisfies k-size sequential inclusion.*

Some observations and an important Remark are in order:

First, note that by definition, if a preference profile R_N satisfies k-size sequential inclusion it also satisfies the l-size version, where $l \leq k$. The converse does not hold. In particular, if a domain satisfies sequential inclusion (i.e. n-size sequential inclusion) it also satisfies k-size sequential inclusion for each $k < n$. However, the converse does not hold as we show in Example 3.

Example 3 *Consider the data in Example 2. It is easy to see that $\times_{i\in N}\mathcal{R}_i$ satisfies 2-size sequential inclusion. To see a violation of (3-size) sequential inclusion, consider the following profile $R_N = (R_1^1, R_2^2, R_3^3)$ and the pair of alternatives y, z. Note that $S(R_N; y, z) = \{1, 2, 3\}$.*

[3] In what follows, and when this does not induce to error, we may omit the arguments R_N, y and z and just write $\succsim$.

Since $L(R_1^1, z) \subseteq \overline{L}(R_2^2, y)$ *but* $\neg[L(R_2^2, z) \subseteq \overline{L}(R_1^1, y)]$ *then* $1 \succ 2$. *Since* $L(R_2^2, z) \subseteq \overline{L}(R_3^3, y)$ *but* $\neg[L(R_3^3, z) \subseteq \overline{L}(R_2^2, y)]$ *then* $2 \succ 3$. *Since* $L(R_3^3, z) \subseteq \overline{L}(R_1^1, y)$ *but* $\neg[L(R_1^1, z) \subseteq \overline{L}(R_3^3, y)]$ *then* $3 \succ 1$. *Therefore,* $1 \succ 2 \succ 3$ *and* $3 \succ 1$*: there is a cycle involving three agents and thus* R_N *violates* 3*-size sequential inclusion implying that* $\times_{i \in N}\mathcal{R}_i$ *violates(*3*-size) sequential inclusion.*[4]

Second, note also that if a preference profile satisfies 2-size sequential inclusion, this means that for any pair $y, z \in A$, the relation $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is complete. If a preference profile satisfies k-size sequential inclusion, for $k > 2$, this means that for any pair $y, z \in A$, $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is not only complete but also that there are no cycles of any number of agents lower or equal than k.

In the following result we show that if the k version of sequential inclusion holds then strategy-proofness implies k-group strategy-proofness.

Theorem 1 *Let* $\times_{i \in N}\mathcal{R}_i$ *be a domain satisfying the* k*-size sequential inclusion condition. Then, any strategy-proof social choice function on* $\times_{i \in N}\mathcal{R}_i$ *is* k*-group strategy-proof.*

The proof follows a similar argument to that of Theorem 1 in Barberà, Berga, and Moreno (2010).

Examples 2 and 3 illustrate the interest of the result in Theorem 1. Note that since sequential inclusion is violated in the framework defined the mentioned examples we can not apply the result in Barberà, Berga, and Moreno (2010). However, we know that there exist strategy-proof rules that although violating group strategy-proofness satisfy an intermediate version of it. Theorem 1 tells us that this is not by luck, but it is because the preference domain satisfies the "adequate" condition, 2-size sequential inclusion in the example.

3.2 Controlling for the alternatives in the range

In this section we also provide a result that could be used by a designer to eventually decide whether or not to limit the range of a social choice function, as a method to enhance some of the good properties of a mechanism. In our case, limiting the range could help avoiding manipulation by groups, but the type of analysis we suggest here might be applicable for other purposes as well.

When there are only two social alternatives to choose from, it is possible to design non-dictatorial and (group) strategy-proof social choice functions on the universal domain of all possible preference profiles. The same is also true if society faces more alternatives, but the range of the social choice function is restricted to only contain two of them. Of course, artificially restricting the range, so that it does not contain all conceivable alternatives, may have some negative consequences, especially in terms of efficiency. But it may also have the advantage of limiting the strategic behavior of agents. Therefore, it is useful, in this and

[4] The way to construct this example is the following: there is a pair of alternatives y, z that are the second best and second worst alternatives, respectively in all individual preferences. Moreover, we need additional k alternatives, each one being the best for some individual preference and the worst for some other. The other $k-2$ alternatives are worse than y and better than z.

other contexts, to study the trade-offs that a designer may face when deciding whether or not to allow all alternatives to be in the range of a social choice function.

The literature has seldom mentioned the possibility of imposing limits to the range as part of a deliberate action of design (except, of course, that it has spent much effort to study the case where there are only two possible social choices). Our specific results suggest that we may gain by paying attention to such possibility.

Our next result suggests possible limitations of the range as a tool to strengthen the resilience of functions to be manipulated by large groups. What will matter is no longer the number of alternatives that may be chosen, but their specific names. The idea is that, if our condition of either direct or indirect sequential inclusion holds for a given subset of alternatives, then the range independent functions defined on this subset will satisfy our equivalence result.

Let us now define sequential inclusion on $B \subseteq A$.[5]

Definition 6 *Given a preference profile $R_N \in \times_{i\in N}\mathcal{R}_i$ and a pair of alternatives $y, z \in B$, we define a binary relation $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ as follows:*

$$i \succsim (R_N; y, z) j \text{ if } L(R_i, z) \cap B \subseteq \overline{L}(R_j, y) \cap B.$$

Definition 7 *A preference profile $R_N \in \times_{i\in N}\mathcal{R}_i$ satisfies sequential inclusion on $B \subseteq A$ for $y, z \in B$ if the binary relation $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is complete and acyclic.*

Definition 8 *A preference profile $R_N \in \times_{i\in N}\mathcal{R}_i$ satisfies sequential inclusion on $B \subseteq A$ if for any pair $y, z \in B$ the binary relation $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is complete and acyclic. A domain $\times_{i\in N}\mathcal{R}_i$ satisfies sequential inclusion on B if any preference profile in this domain satisfies it.*

Note that, by definition, sequential inclusion is equivalent to sequential inclusion on $B = A$. Thus, sequential inclusion implies sequential inclusion on B, for any $B \subseteq A$. However, sequential inclusion on B for some $B \subsetneq A$ does not in general imply sequential inclusion: our new condition is weaker as we show in Example 4.

Example 4 *Consider the data in Example 2. Let $B = \{z, a_1, a_2, a_3\} \subsetneq A$. The domain $\times_{i\in N}\mathcal{R}_i$ satisfies sequential inclusion on B. To check it we first need to observe that for each R_N, and each pair $x, t \in B$, $\#S(R_N; x, t) \leq 2$ (this is obvious). Then, we have to prove that the relation $\succsim (R_N; x, t)$ is always complete. To check the latter, consider the case where $x = a_1$, $t = a_2$ (a similar argument - or trivial whenever $\#S = 1$- can be done for any other pair of alternatives in B). Note that $L(R^3, a_2) \cap B = \{a_2\} \subseteq \overline{L}(R^1, a_1) \cap B = \{a_2, z, a_3\}$ but $\neg[L(R^1, a_2) \subseteq \overline{L}(R^3, a_1)]$; thus the agent with type 3 $\succ$agent with type 1 preference. Thus, the relation $\succsim (R_N; a_1, a_2)$ is complete. This finishes the proof.*
As we already checked in Example 3, $\times_{i\in N}\mathcal{R}_i$ violates sequential inclusion.

[5] We could have also defined indirect sequential inclusion on a subset $B \subseteq A$ and obtain similar results imposing our indirect sequential in Barberà, Berga, and Moreno (2010) on B. For sake of simplicity, we stick to the sequential inclusion version. Moreover, we could have complicated even more our search for further equivalence results by combining restrictions on the set of alternatives with those on the subsets of agents analized in the previous subsection. Similar equivalence results would be obtained. We prefer to keep our analysis neat and leave the interested reader go deeper to check the details.

An important consequence of this example is that with the new domain constraint we are able to analyze some frameworks not encompassed in Barberà, Berga, and Moreno (2010).

Before stating the main result in this subsection, two interesting results are in order. First, we observe that sequential inclusion is guaranteed as a direct consequence of a very specific range restriction.

Proposition 1 *Let f be a social choice function on $\times_{i\in N}\mathcal{R}_i$ such that $\#A_f \leq 3$. Then, any profile of preferences $R_N \in \times_{i\in N}\mathcal{R}_i$ satisfies sequential inclusion on A_f.*[6]

Proof of Proposition 1.
If $\#A_f = 2$ or $\#A_f = 1$ sequential inclusion on A_f trivially holds. Therefore, we concentrate on the case where A_f consists of three distinct alternatives x, y, and z and $R_N \in \times_{i\in N}\mathcal{R}_i$. Without loss of generality, choose two of these three alternatives, say y and z. Define the following partition: $S(R_N; y, z) = S_z(R_N; y, z) \cup S_{\{x,z\}}(R_N; y, z)$ where $S_z(R_N; y, z) = \{j \in S(R_N; y, z)$ such that $L(R_j, z) \cap A_f = \{z\}\}$ and $S_{\{x,z\}}(R_N; y, z) = \{k \in S(R_N; y, z)$ such that $L(R_k, z) \cap A_f = \{x, z\}\}$, respectively. Since their lower contour set in A_f at z coincide, for any $j, l \in S_z(R_N; y, z)$, $j \sim (R_N; y, z)l$. Similarly, for any $k, h \in S_{\{x,z\}}(R_N; y, z)$, $k \sim (R_N; y, z)h$. Moreover, for any $j \in S_z(R_N; y, z)$, $\{z\} = L(R_j, z) \cap A_f \subseteq L(R_k, y) \cap A_f$ for any $k \in S_{\{x,z\}}(R_N; y, z)$, thus $j \succsim (R_N; y, z)k$. Therefore, $\succsim (R_N; y, z)$ is complete and acyclic showing that sequential inclusion on A_f holds. ■

Second, notice that there are two distinct ways to violate sequential inclusion on B (similarly for k-size sequential inclusion): by lack of completeness and because of cycles. Both aspects of the definition are essential in what follows, but could be factored out for other purposes, as their implications are different. The following result states the relevance of completeness in the definition of sequential inclusion on B. We say that a preference profile $R_N \in \times_{i\in N}\mathcal{R}_i$ satisfies completeness on $B \subseteq A$ if for any pair $y, z \in B$ the binary relation $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is complete. A domain $\times_{i\in N}\mathcal{R}_i$ satisfies completeness if each preference profile in it satisfies completeness.

Proposition 2 *Let f be a social choice function on $\times_{i\in N}\mathcal{R}_i$ such that the domain satisfies completeness on A_f and $\#A_f \leq 4$. Then, any profile of preferences $R_N \in \times_{i\in N}\mathcal{R}_i$ satisfies sequential inclusion on A_f.*

We state and prove a useful lemma.

Lemma 1 *Let $R_N \in \times_{i\in N}\mathcal{R}_i$ and $y, z \in A_f$ such that $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ has a cycle of size k, $3 \leq k \leq s$ but $\succsim (R_N; y, z)$ does not have any cycle of lower size. Then, there exist at least $k+2$ alternatives in A_f, say y, z, a_1, a_2,...,$a_k \in A_f$. There also exist k individual preference relations in R_N, say R^1, R^2, ..., R^k, such that for any $l = 1, ..., k \in N$, the following holds: $[(1)\ a_l R^l y$, $(2)\ z R^l a_{l-1}$, for any $j = 1, ..., k-2$, $(3)\ y P^l a_{l+j}$, and (4) $a_{l+j} P^l z]$. Moreover, if for any alternative $w \in A_f$, the binary relation $\succsim (R_N; y, w)$ on $S(R_N; y, w)$ is complete, then (5) for any $l = 1, ..., k$, $[a_{l+j} P^l a_{l+1+j}$, for $j = 1, ...k-3]$.*

[6] Its proof is similar to the proof of Proposition 1 in Barberà, Berga, and Moreno (2010). We include it to show the readers how one can proceed to check sequential inclusion on a subset.

Proof of Lemma 1. We first prove that there must exist alternatives and individuals whose preferences satisfy (1) and (2).

Without loss of generality, let the cycle of k agents be $1 \succ 2 \succ 3 \succ ... \succ k$ and $k \succ 1$. Thus, their individual preferences in R_N, say R_1, R_2, ..., R_k, are such that $L(R_l, z) \cap A_f \subseteq \overline{L}(R_{l+1}, y) \cap A_f$ and $\neg \left[L(R_{l+1}, z) \cap A_f \subseteq \overline{L}(R_l, y) \cap A_f\right]$ for $l = 1, ..., k$. From the latter expression, for any $l = 1, ..., k$, there exists $a_l \in A_f \backslash \{z, y\}$ such that $zR_{l+1}a_l$ and $a_l R_l y$. Note that for any $l = 1, ..., k$, $a_l \neq z$ and $a_l \neq y$ since $yP_l z$ and $yP_{l+1}z$, respectively. Thus, there exist k individual preference relations in R_N, in particular: R_1, R_2, ..., R_k, such that the following expressions hold: [(1) $a_l R^l y$ and (2) $zR^l a_{l-1}$, for any $j = 1, ..., k-2$]. We now show that the alternatives a_l so defined are all distinct. Note that we only need to show the following:

For any i, $1 \leq i \leq k-1$, [$a_i \neq a_j$, for any j, $k \geq j > i$]:
Case 1. Let i, j such that $j - i = 1$. Given the above cycle of k agents, $j \succ j+1$, thus $a_{j-1} \in L(R_j, z) \cap A_f \subseteq \overline{L}(R_j y) \cap A_f$. But we also proved that $a_j \notin \overline{L}(R_j, y) \cap A_f$. Thus, $a_j \neq a_{j-1}$.
Case 2. Let i, j such that $j - i = 2$. Given the above cycle of k agents, $j-1 \succ j$, thus $a_{j-2} \in L(R_{j-1}, z) \cap A_f \subseteq \overline{L}(R_j, y) \cap A_f$. But since $j \succ j+1$, by $a_j \notin \overline{L}(R_j, y) \cap A_f$. Thus, $a_j \neq a_{j-1}$.
Case 3. Let i, j such that $j - i = t$, for $3 \leq t \leq k-1$. By assumption, there is no cycle of agents in $S(R_N; y, z)$ of size t. Thus, given the above cycle of k agents, $i+1 \succ i+2 \succ ... \succ i+(t-1) \succ j$ and $i+1 \succsim j$. Thus, $a_i \in L(R_{i+1}, z) \cap A_f \subseteq \overline{L}(R_j, y) \cap A_f$. But we also proved that $a_j \notin \overline{L}(R_j, y) \cap A_f$. Thus, $a_i \neq a_j$.

We now prove that the preferences in question satisfy expression (3).
For any $l = 1, ..., k$, [$a_{l+j} P_l z$ for any $j = 1, ..., k-2$].
Take any j. By assumption, there is no cycle of agents in $S(R_N; y, z)$ of size $j+1$ (since $2 \leq j+1 \leq k-1$). Thus, given the above defined cycle of k agents, $l \succ l+1 \succ ... \succ l+j \succ j$ and $l \succsim l+j$. Thus, $L(R_l, z) \cap A_f \subseteq \overline{L}(R_{l+j}, y) \cap A_f$. If $zR_l a_{l+j}$ then $a_{l+j} \in \overline{L}(R_{l+j}, y) \cap A_f$ which contradicts expression (1) above proved.

We now show that the preferences also satisfy expression (4).
For any $l = 1, ..., k$, [$yP_l a_{l+j}$ for any $j = 1, ..., k-2$].
Take any j. By assumption, there is no cycle of agents in $S(R_N; y, z)$ of size $k-j$ (since $2 \leq k-j \leq k-1$). Thus, given the above defined cycle of k agents $l-(k-j-1) \succ l-(k-j) \succ l-(k-j+1) \succ ... \succ l-3 \succ l-2 \succ l-1 \succ l$ and $l-(k-j-1) \succsim l$. Using our equivalence, $l+j+1 \equiv l+j+1-k \pmod{k}$ and thus $R_{l+j+1} = R_{l+j+1-k}$. Thus, $L(R_{l+j+1}, z) \cap A_f \subseteq \overline{L}(R_l, y) \cap A_f$. By the previous lemma, we know that $a_{l+j} \in L(R_{l+j+1}, z) \cap A_f$, and thus $a_{l+j} \in \overline{L}(R_l, y) \cap A_f$.

For $k = 3$ the Lemma is proved.

For $k \geq 4$, we finally prove that the preferences also satisfy expression (5).
For any $l = 1, ..., k$, [$a_{l+j} P_l a_{l+1+j}$, for $j = 1, ..., k-3$].

We first show that for any $l = 1, ..., k$, $a_{l+1} P_l a_{l+1+j}$ for $j = 1, ..., k-3$. (*)
Fix l and j. By contradiction, suppose that $a_{l+1+j} R_l a_{l+1}$. Consider the pair of alternatives (y, a_{l+1+j}). By hypothesis, the binary relation $\succsim (R_N; y, a_{l+1+j})$ on $S(R_N; y, a_{l+1+j})$ is com-

plete. Since $l, l+1 \in S(R_N; y, a_{l+1+j})$, either $l \succsim l+1$ or $l+1 \succsim l$ should hold. By hypothesis, $a_{l+1} \in L(R_l, a_{l+1+j}) \cap A_f$ and, by conditions (1) to (4) already proved, $a_{l+1} \notin \overline{L}(R_{l+1}, y) \cap A_f$. Then, $\neg \left[L(R_l, a_{l+1+j}) \cap A_f \subseteq \overline{L}(R_{l+1}, y) \cap A_f\right]$, or equivalently, $\neg [l \succsim l+1]$. On the other hand, by conditions (1) to (4), $a_l \notin \overline{L}(R_l, y) \cap A_f$, but also $a_l \in L(R_{l+1}, z) \cap A_f$ and $a_{l+1+j} P_{l+1} z$ holds. The latter two expressions imply that $a_l \in L(R_{l+1}, a_{l+1+j}) \cap A_f$. Then, $\neg \left[L(R_{l+1}, a_{l+1+j}) \cap A_f \subseteq \overline{L}(R_l, y) \cap A_f\right]$, or equivalently, $\neg [l+1 \succsim l]$ which is the desired contradiction.

For $k = 4$ the Lemma is proved.

Now, for $k > 4$, we show that for any $l = 1, ..., k$, $a_{l+t} P_l a_{l+t+1}$ for any t, $t = 2, ..., k-3$.
Fix l and t. By contradiction, suppose that $a_{l+t+1} R_l a_{l+t}$. Consider the pair of alternatives (y, a_{l+t+1}). By hypothesis, the binary relation $\succsim (R_N; y, a_{l+t+1})$ on $S(R_N; y, a_{l+t+1})$ is complete. Since $l, l+t \in S(R_N; y, a_{l+t+1})$, either $l \succsim l+t$ or $l+t \succsim l$ should hold. By hypothesis, $a_{l+t} \in L(R_l, a_{l+t+1}) \cap A_f$ and, by conditions (1) to (4), $a_{l+t} \notin \overline{L}(R_{l+t}, y) \cap A_f$. Then, $\neg \left[L(R_l, a_{l+t+1}) \cap A_f \subseteq \overline{L}(R_{l+t}, y) \cap A_f\right]$, or equivalently, $\neg [l \succsim l+t]$. On the other hand, by conditions (1) to (4), $a_l \notin \overline{L}(R_l, y) \cap A_f$. Using the modulo k equivalence, and by the previous statement (*) applied for $j = k-1-t$ and for $l \equiv l+t$, $a_{l+t+1} P_{l+t} a_l$ (that is, $a_l \in L(R_{l+t}, a_{l+t+1}) \cap A_f$). Then, $\neg \left[L(R_{l+t}, a_{l+t+1}) \cap A_f \subseteq \overline{L}(R_l, y) \cap A_f\right]$, or equivalently, $\neg [l+t \succsim l]$ which is the desired contradiction.

Observe that the preferences of the k agents, $1, ..., k$, satisfy the conditions predicated by the lemma. Again define $R^i = R_i$ for all $i = 1, ..., k$. This ends the proof. ■

Now we prove Proposition 2.

Proof of Proposition 2. Suppose that a profile R_N violates sequential inclusion on A_f: Let $R_N \in \times_{i \in N} R_i$ and $y, z \in A_f$ such that $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ has a cycle of size k, $3 \leq k \leq s$. Without loss of generality, assume that k is the minimum cycle size, that is, $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ has a cycle of size k, $3 \leq k \leq s$ but $\succsim (R_N; y, z)$ does not have any cycle of lower size.
By Lemma 1, there exist at least $k+2$ different alternatives in A_f. Observe that this is the desired contradiction since this would mean that there exist at least 5 alternatives in A_f which is not the case.

■

The following example show the relevance of completeness in Proposition 2.

Example 5 *Let $N = \{1, 2\}$, $A = \{x, y, z, t, w, s\}$, and $\widetilde{\mathcal{R}} = \{R^1, R^2, R^3\}$ where:*

R^1	R^2	R^3
t	s	s
x	w	t
y	y	y
z	z	x
w	x	w
s	t	z

Take y, $z \in A$ and consider $R_N = (R^1, R^2)$. Observe that R^1 and R^2 are not comparable according to $\succsim (R_N; y, z)$. Although, $R^3 \succsim R^1$, $R^3 \succsim R^2$. Define a social choice function f as follows:

f	R^1	R^2	R^3
R^1	x	z	x
R^2	z	w	w
R^3	x	w	y

Thus, $\left(\widetilde{\mathcal{R}}\right)^n$ is not complete in $A_f = \{x, w, y, z\}$. Moreover, observe that f is strategy-proof but violates group strategy-proofness: N manipulates f at $R_N = (R^1, R^2)$ via $R'_N = (R^3, R^3)$ since $f(R_N) = z$ while $f(R'_N) = y$.

To finish this subsection we now state the main equivalence result.

Theorem 2 *Let $\times_{i\in N}\mathcal{R}_i$ be a domain of preferences satisfying sequential inclusion on $B \subseteq A$. Then, any strategy-proof social choice function with range B is also group strategy-proof.*

The proof follows a similar argument to that of Theorem 1 in Barberà, Berga, and Moreno (2010).

We should notice that Theorem 2 allows us to state the equivalence between individual and group strategy-proofness in settings not encompassed in our previous work. For example, the framework defined in Example 2. In purpose, we use an abstract but easy framework to check several properties and results along the paper. However, in Section 4 we present more interesting and well-known frameworks that can also be embedded in our framework.

3.3 Necessity

We finish the section providing a result that establishes the partial necessity of k-size sequential inclusion to guarantee that individual and k-group strategy-proofness become equivalent. In fact, this result generalizes the one in Theorem 4 in Barberà, Berga, and Moreno (2010).

Theorem 3 *Let $2 \leq k \leq n$. Let $\times_{i\in N}\mathcal{R}_i$ be a domain that allows for opposite preferences and such that any strategy-proof social choice function on $\times_{i\in N}\mathcal{D}_i \subset \times_{i\in N}\mathcal{R}_i$ is also k-group strategy-proof. Then, $\times_{i\in N}\mathcal{R}_i$ satisfies k-size sequential inclusion.*

The proof of Theorem 3 follows the same lines as Theorem 4 in Barberà, Berga, and Moreno (2010).[7]

[7] In fact, we do prove the $k = 2$ version of Theorem 3 as part of the proof of Theorem 4 in Barberà, Berga, and Moreno, 2010 (see Lemma 3 there).

4 Applications of range restrictions

An interesting point we want to make in this paper is that we can encompass interesting economic frameworks that could not be analyzed in Barberà, Berga, and Moreno (2010) since sequential inclusion was violated, mainly because of intrinsic characteristics of the setting itself. We present below a house allocation framework where we are able to apply the results in the present paper. Some results in exchange economies mentioned in the introduction can be obtained too.

A second point we make is that given Theorem 2 we can obtain some existing results in the literature as corollaries.

4.1 Single-dipped preferences

We define now the domain of single-dipped preferences. They allow us to analyze cases where distance to a reference "worse" point is preferred, as it is the case when one must allocate a public bad. This is in contrast with the opposite motivation of single-peaked preferences, where being closer to the reference "best" point is preferred. Formally:

Definition 9 *A preference profile R_N is single-dipped iff there exists a linear order $>$ of the set of alternatives such that*
(1) Each of the voters' preferences has a unique minimal element $d_i(A)$, called the dip of i, and
(2) For all $i \in N$, for all $d_i(A)$, and for all $y, z \in A$

$$[z < y \leqslant d_i(A) \text{ or } z > y \geqslant d_i(A)] \rightarrow \quad zP_iy.$$

Two recent works have deeply analyzed different aspects related to the set of single-dipped preferences (see Manjunath, 2010 and Barberà, Berga, and Moreno, 2012). In the latter papers the authors showed that when agents' preferences are single-dipped, any strategy-proof rule has at most two alternatives in the range. That is, a range restriction appears.

Considering the mentiones result and Proposition 1, by applying Theorem 2 we obtain that "any strategy-proof rule defined on single-dipped preference profiles is also group strategy-proof", a result already stated in the literature (it was obtained as a corollary of Theorem 1 in Barberà, Berga, and Moreno, 2010).

4.2 House allocation

First we show how to fit the classical house allocation model in ours. For each agent i, let B_i be the set of individuals objects or houses that agent i can be assigned to. Define the set of alternatives $A \subseteq B_1 \times ... \times B_n$ be the set of possible assignments of houses to agents such that each agent receives a distinct house.

Each agent i has preferences denoted by $R_i \in \mathcal{R}_i$ on A. From preferences on A we can induce preferences on B_i as follows: For any $a, b \in B_i$, $a_iR_ib_i$ if and only if aR_ib. That is, when evaluating different alternatives, agents are selfish and care only about their assignment. Note that, abusing notation we use the same symbol R_i to denote preferences on A and on B_i.

For any $i \in N$, we assume that the set of preferences $\mathcal{R}_i$ are all possible strict orders over objects (for example, for 3 objects there are 6 linear orders). Note, however, that by selfishness preferences on assignments are not strict and have many indifferences: two different alternatives assigning the same object to agent i are indifferent for i.

We use an example to make our point clearer. Consider an example where $N = \{1, 2\}$, $B_1 = \{a_1, a_2, a_3\}$ and $B_2 = \{a_1, a_2\}$; that is, two agents and three objects, but agent 1 can not be assigned to object 3 (suppose that owner of house 3 does not want pets and agent 1 has a pet). Thus, we have an exante range constraint and the set of feasible assignments is $B = \{(a_1, a_2), (a_1, a_3), (a_2, a_3), (a_2, a_1)\} \subsetneq A$. Note that for each agent i, $\#\mathcal{R}_i = 6$, that is, i has six strict preferences over objects. Such preferences induce other 6 preferences for agent i over A:

As an example: let R^1 be such that $a_1 P^1 a_2 P^1 a_3$, then the preference of agent 1 over A is $R_1^1 : \{(a_1, a_2), (a_1, a_3)\}\, P_1^1\, \{(a_2, a_3), (a_2, a_1)\}\, P_1^1\, \{(a_3, a_1), (a_3, a_2)\}$ where assignments (a_1, a_2) and (a_1, a_3) are indifferent for agent 1 since he obtains the same object. Similarly, the preference of agent 2 over A is $R_2^1 : \{(a_2, a_1), (a_3, a_1)\}\, P_1^1\, \{(a_1, a_2), (a_3, a_2)\}\, P_1^1\, \{(a_1, a_3), (a_2, a_3)\}$.

Note that we can not apply our results in Barberà, Berga, and Moreno (2010) about equivalence of individual and group strategy-proofness since the domain $\times_{i \in N} \mathcal{R}_i$ violates sequential inclusion (i.e. sequential inclusion on A). Take profile $R_N = (R_1^1, R_2^3)$ and alternatives $y = (a_1, a_2)$, $z = (a_2, a_1)$ where $y P_1^1 z$ and $y P_2^3 z$; observe that $L(R_1^1, z) \subseteq \overline{L}(R_2^3, y)$ ($(a_3, a_2) \in L(R_1^1, z)$, $(a_3, a_2) \notin \overline{L}(R_2^3, y)$) nor $L(R_2^3, z) \subseteq \overline{L}(R_1^1, y)$ holds ($(a_1, a_3) \in L(R_2^3, z)$, $(a_3, a_2) \notin \overline{L}(R_1^1, y)$), thus, R_N does not satisfy sequential inclusion on A since the binary relation $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ is not complete.

Observe that although R_N above satisfies sequential inclusion on B not any profiles does: (R_1^4, R_2^6) with (a_2, a_3), (a_1, a_2).

However, by Proposition 1 above, if $\#B = 3$ then any profile satisfies sequential inclusion on B. Thus, applying our Theorem 2, any strategy-proof rule with range B is also group strategy-proof.

5 Public domains

Let $\mathcal{R}_i = \widetilde{\mathcal{R}} \subseteq \mathcal{R}$ for any $i \in N$. Let $y, z \in A$ be a pair of alternatives. Denote by $\mathfrak{R}(y, z) \equiv \left\{ R \in \widetilde{\mathcal{R}} : yPz \right\}$, that is, the set of preferences for which y is strictly preferred to z.

Definition 10 *Given a pair of alternatives $y, z \in A$, we define a binary relation $\succsim (y, z)$ on $\mathfrak{R}(y, z)$ as follows:*[8]

$$R \succsim (y, z) R' \text{ if } L(R, z) \subseteq \overline{L}(R', y).$$

Note that the binary relation $\succsim$ must be reflexive but not necessarily complete. As usual, we can define the strict and the indifference binary relations associated to $\succsim$. Formally, $R \sim$

[8] In what follows, and when this does not induce to error, we may omit the arguments y and z and just write $\succsim$.

R' if $L(R,z) \subseteq \overline{L}(R',y)$ and $L(R',z) \subseteq \overline{L}(R,y)$. We say that $R \succ R'$ if $L(R,z) \subseteq \overline{L}(R',y)$ and $\neg[L(R',z) \subseteq \overline{L}(R,y)]$.If $R \succsim R'$, we'll say that R may precede R'. If $R \succ R'$, then we'll say that R must precede R'.

We say that an individual set of preferences $\widetilde{\mathcal{R}}$ is *complete* if for all pair of alternatives $y,z \in A$, $\succsim (y,z)$ on $\mathfrak{R}(y,z)$ is complete. We say that a individual set of preferences $\widetilde{\mathcal{R}}$ is *transitive* if for all pair of alternatives $y,z \in A$, and for any triple R, R', $R'' \in S(y,z)$, if $R \succsim R'$ and $R' \succsim R''$ then $R \succsim R''$.

Observe that the set of separable preferences is neither complete nor transitive.

Example 6 *Consider $\widetilde{\mathcal{R}}$ the set of all separable preferences over two candidates a and b.*

R^1	R^2	R^3	R^4	R^5	R^6	R^7	R^8
$\varnothing$	$\varnothing$	a	a	b	b	$\{a,b\}$	$\{a,b\}$
a	b	$\varnothing$	$\{a,b\}$	$\varnothing$	$\{a,b\}$	a	b
b	a	$\{a,b\}$	$\varnothing$	$\{a,b\}$	$\varnothing$	b	a
$\{a,b\}$	$\{a,b\}$	b	b	a	a	$\varnothing$	$\varnothing$

Take alternatives $y = a$, $z = b$ and consider $\succsim (y,z)$ on $S(y,z) = \{R^1, R^3, R^4, R^7\}$. Note that neither $R^1 \succsim R^7$ nor $R^7 \succsim R^1$, thus $\widetilde{\mathcal{R}}$ is incomplete. Observe also that $R^1 \sim R^4$, $R^4 \sim R^7$ but neither $R^1 \succsim R^7$ nor $R^7 \succsim R^1$, thus $\widetilde{\mathcal{R}}$ is not transitive.

Proposition 3 *Let each individual set of preferences $\widetilde{\mathcal{R}}$ be complete. Any anonymous and strategy-proof social choice function f on $\left(\widetilde{\mathcal{R}}\right)^n$ such that $\#A_f = 5$ is also group strategy-proof.*

Proof. To be finished.

By contradiction, suppose that f is manipulable by some coalition $C \subseteq N$. That is, there exist a coalition C, $R_N \in \times_{i \in N} \widetilde{\mathcal{R}}_i$, and $\widetilde{R}_C \in \times_{i \in C} \mathcal{R}_i$, such that for any agent $i \in C$, $f(\widetilde{R}_C, R_{-C}) P_i f(R_N)$. Let $y = f(\widetilde{R}_C, R_{-C})$ and $z = f(R_N)$. If R_N satisfies sequential inclusion, by applying the same argument as in the proof of Theorem 1 in Barberà, Berga, and Moreno (2010) we get a contradiction to strategy-proofness. Thus, R_N violates sequential inclusion and, by completeness of $\widetilde{\mathcal{R}}$, $\succsim (R_N; y, z)$ on $S(R_N; y, z) = C$ has a cycle. Moreover, by Lemma 1 and the fact that $\#A_f = 5$, the cycle can not be of size strictly higher than 3. That is, $\succsim (R_N; y, z)$ on $S(R_N; y, z)$ has a cycle of size $k = 3$. Lemma 1 then ensures us that 3 individual preference relations in R_N, say without loss of generality by anonimity R_1, R_2, R_3, must be such that for any $l = 1, 2, 3$, the following holds: [(1) $a_l R_l y$, (2) $z R_l a_{l-1}$, for any $j = 1, ..., k-2$, (3) $y P_l a_{l+j}$, and (4) $a_{l+j} P_l z$]. Observe also that the preferences in R_N of agents in $C \backslash \{1, 2, 3\}$ can not be such that a cycle of a larger size is formed.[9]

Step 1: Assume that we have only three agents and that $\widetilde{\mathcal{R}} = \{R_1, R_2, R_3\}$. ■

Example 7 shows the relevance of completeness in the result: in it we present a domain violating completeness for which there exist anonymous and strategy-proof rules violating group strategy-proofness.

[9] One can check considering all possible preferences in $A_f = \{a_1, a_2, a_3, y, z\}$ such that y is strictly preferred to z that no higher cycle will be formed.

Example 7 *Let $N = \{1, 2\}$, $A = \{y, z, a_1, a_2, a_3\}$, and $\widetilde{\mathcal{R}} = \{R^1, R^2, R^3\}$ where:*

R^1	R^2	R^3
y, a_1	a_2	a_3
a_2	y	a_1
z	a_3	y
a_3	z	z
	a_1	a_2

Take y, $z \in A$ and consider $\succsim (y, z)$ on $\mathfrak{R}(y, z) = \widetilde{\mathcal{R}}$. Observe that $R^3 \succsim R^1$, $R^1 \succsim R^2$ but $\neg R^3 \succsim R^2$. Thus, $\widetilde{\mathcal{R}}$ is not transitive. Define a social choice function f as follows:

f	R^1	R^2	R^3
R^1	y	a_2	a_1
R^2	a_2	a_2	z
R^3	a_1	z	z

Note that f is strategy-proof but violates group strategy-proofness: N manipulates f at $R = (R^3, R^2)$ via $R' = (R^1, R^1)$ since $f(R) = z$ while $f(R') = y$.

6 Conclusions

To be written.

An open question that remains is to study the relationship between k-group strategy-proofness and group strategy-proofness. Defining a domain condition guaranteeing the equivalence of these two strategic properties would be one of interesting point in the literature.

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To be completed.

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