teorema
Vol. XXIX/3, 2010, pp. 103-116
ISSN: 0210–1602
[BIBLID 0210-1602 (2010) 29:3; pp. 103-116]

The Non–Involutive Routley Star: Relevant Logics Without Weak Double Negation

Gemma Robles

RESUMEN

En este trabajo, se definen una serie de lógicas incluidas en la lógica de la relevancia *R* sin el axioma de introducción de la doble negación.

PALABRAS CLAVE: *semántica relacional ternaria, lógicas de la relevancia, lógicas no involutivas, operador asterisco de Routley.*

ABSTRACT

In this paper, I define a series of logics included in relevance logic R and without the axiom of introduction of double negation.

KEYWORDS: *Routley-Meyer ternary relational semantics, relevant logics, non-involutive logics, "Routley-star" operator.*

I. INTRODUCTION

As is well-known, negation in standard relevant logics is involutive in the sense that are valid the double negation axioms

dn1. $A \to \neg\neg A$

dn2. $\neg\neg A \to A$

It is also a well-known fact that negation in standard semantics for relevant logics is explained by the "Routley–star", a unary operator on the set of all points in the models [cf., e.g., Routley and Meyer (1973)]. Now, in Sylvan (formerly, Routley) and Plumwood (2003), the authors succinctly define the logic B_M and some of its extensions in two and a half pages. The logic B_M is, when negation is present, the *basic logic* in (Routley and Meyer) ternary relational semantics in the same sense that B_+ [see Routley and Meyer (1972)] is the basic positive (i.e., without negation) logic in the same semantics. In B_M, neither dn1 nor dn2 hold.

In Robles (submitted) a series of extensions of BM with dn1 but without dn2 are studied. Two are the main results of that note:

(a) Let R_+ be the positive fragment of relevance logic R [see, e.g., Anderson and Belnap (1975)]. Then, the logic RMO_{lcNI} is the result of adding to R_+ the mingle, *LC* and weak contraposition axioms, i.e.,

$$A \to (A \to A)$$

$$(A \to B) \vee (B \to A)$$

$$(A \to \neg B) \to (B \to \neg A)$$

respectively, and the principle of *tertium non datur*,

$$A \vee \neg A$$

A Routley-Meyer semantics is provided for RMO_{lcNI}, and it is shown that dn2 is not derivable in RMO_{lcNI}.

(b) The constructive reductio axioms such as

$$(A \to \neg B) \to ((A \to B) \to \neg A)$$

$$(A \to B) \to \neg(A \wedge \neg B)$$

etc. are theorems of RMO_{lcNI}. Then, the non–constructive reductio axioms such as

$$(\neg A \to \neg B) \to ((\neg A \to B) \to A)$$

$$(\neg A \to B) \to ((A \to B) \to B)$$

etc. are added to RMO_{lcNI}. This logic is labelled R_{MNI}. It is shown that dn2 is still not derivable in R_{MNI}. But, on the other hand, it is conjectured that R_{MNI} is not representable in Routley–Meyer semantics.

The aim of this paper is to carry on a similar study on logics including B_M with dn2 and lacking dn1. To be more precise, in *relevant logics* including B_M and lacking dn1. Because, as certainly has been noted, in Robles (submitted) non-relevant logics included in *R–Mingle* (cf. Anderson and Belnap (1975)) are considered. Then, symmetrically (but relevantly) reflecting Robles (submitted), two are the main results of the present paper:

(a) As pointed out above, R_+ is the positive fragment of relevance logic R. Then, the logic R_M is the result of adding to R_+ the minimal negation defined in Sylvan and Plumwood (2003). Next, the logic R_{MNI2} (a second version of R_M with a non-involutive negation: notice that R_{MNI} commented above is defined in Robles (submitted)) is the result of adding to R_M the strong contraposition axiom

$(\neg A \rightarrow B) \rightarrow (\neg B \rightarrow A)$

and the principle of *tertium non datur*

$A \vee \neg A$

A Routley–Meyer semantics is provided for R_{MNI2} and it is shown that dn1 is not a theorem of it.

(b) The constructive, as well as the non–constructive, axioms (cf. supra) are added to R_{MNI2}. It is shown that thesis dn1 is not derivable in the resulting logic, which, unfortunately, seems to be not representable in the present semantical framework. Acquaintance with Routley–Meyer semantics for relevant logics is presupposed.

II THE LOGIC R_M AND ITS SEMANTICS

The positive logic of relevance R_+ can be axiomatized as follows [see, e.g., Anderson and Belnap (1975)]:

Axioms:

A1. $A \rightarrow A$

A2. $(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

A3. $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$

A4. $A \rightarrow ((A \rightarrow B) \rightarrow B)$

A5. $(A \wedge B) \rightarrow A$ and $(A \wedge B) \rightarrow B$

A6. $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$

A7. $A \rightarrow (A \vee B)$ and $B \rightarrow (A \vee B)$

A8. $((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$

A9. $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)$

Rules:

Modus ponens (MP): if $\vdash A \rightarrow B$ and $\vdash A$, then $\vdash B$

Adjunction (Adj): if $\vdash A$ and $\vdash B$, then $\vdash A \wedge B$

Then, the logic R_M is the result of adding to R_+ the axioms:

A10. $\neg(A \wedge B) \rightarrow (\neg A \vee \neg B)$

A11. $(\neg A \wedge \neg B) \rightarrow \neg(A \vee B)$

and the rule

Contraposition (Con): If $\vdash A \rightarrow B$ then $\vdash \neg B \rightarrow \neg A$

Next, we define the semantics. An R_M-*model* is a structure $\langle K, O, R, *, \vDash \rangle$ where O is a non–empty subset of K; R is a ternary relation on K, and $*$ is a unary operation on K subject to the following definitions and postulates for all $a, b, c, d \in K$:

d1. $a \leq b =_{\text{df}} (\exists x \in O)Rxab$

d2. $R^2abcd =_{\text{df}} (\exists x \in K)(Rabx$ and $Rxcd)$

P1. $a \leq a$

P2. If $a \leq b$ and $Rbcd$, then $Racd$

P3. If R^2abcd, then $(\exists x \in K)(Rbcx$ and $Raxd)$

P4. If $Rabc$, then R^2abbc

P5. If $Rabc$, then $Rbac$

P6. If $a \leq b$, then $b^* \leq a^*$

Finally, $\vDash$ is a relation from K to the formulas of the propositional language such that the following conditions are satisfied for all propositional variables p, wff A, B and $a \in K$:

i. If $a \leq b$ and $a \vDash p$, then $b \vDash p$

ii. $a \vDash (A \wedge B)$ iff $a \vDash A$ and $a \vDash B$

iii. $a \vDash (A \vee B)$ iff $a \vDash A$ or $a \vDash B$

iv. $a \vDash (A \rightarrow B)$ iff for all b, $c \in K$, if $Rabc$ and $b \vDash A$, then $c \vDash B$.

v. $a \vDash \neg A$ iff $a^* \nvDash A$.

A is *R_M-valid* ($\vDash_{\mathrm{RM}} A$) iff $a \vDash A$ for all $a \in O$ in all R_M–models.

Next, we sketch a proof of the soundness of R_M. First, two useful lemmas:

LEMMA 1. *For any wff A and a, $b \in K$,* if $a \leq b$ and $a \vDash A$, then $b \vDash A$.

Proof. Induction on the length of A. The conditional case is proved with P2, and the negation case with P6. □

LEMMA 2. *For any wff A, B, $\vDash_{\mathrm{RM}} (A \rightarrow B)$ iff if $a \vDash A$* then *$a \vDash B$, for all $a \in K$ in all R_M–models.*

Proof. By Lemma 1 and P1 (with d1). □

Then, by using Lemma 2, it is proved [cf., e.g., Routley et al., (1982)]:

THEOREM 1 (*Soundness of R_M*). *If $\vdash_{\mathrm{RM}} A$, then $\vDash_{\mathrm{RM}} A$.*

Proof. A1, A5-A11, MP, Adj and Con are immediate. Then, A2, A3 and A4 are proved R_M–valid with P3, P4 and P5, respectively. □

III. COMPLETENESS OF R_M

Regarding completeness, the R_M-canonical model is the structure $\langle K^C, O^C, R^C, *^C, \vDash^C \rangle$ where K^C is the set of all prime theories (K^T is the set of all theories), O^C is the set of all regular prime theories, and R^C, $*^C$ and $\vDash^C$ are defined as follows:

R^T: for any $a,b,c \in K^T$, $R^T abc$ iff if $(A \rightarrow B) \in a$ and $A \in b$, then $B \in c$, for any wff A,B.

Then, R^C is the restriction of R^T to K^C.

$*^C$: for any $a \in K^C$, $a^{*C} = \{A \mid \neg A \notin a\}$.

$\vDash^C$: for any $a \in K^C$, $a \vDash^C A$ iff $A \in a$.

A *theory* is a set of formulas closed under adjunction and R_M–entailment (that is, a is a theory iff (i) if $A\in a$ and $B\in a$, then $(A \wedge B)\in a$ (ii) if $A \rightarrow B$ is a theorem of R_M and $A\in a$, then $B\in a$). A theory a is *prime* if whenever $(A \vee B)\in a$, then $A\in a$ or $B\in a$; and a theory is *regular* iff it contains all theorems of R_M.

Then, the essential lemmas are [cf., e.g., Routley et al. (1982)]:

LEMMA 3. *Let $R^T abc$ for $a,b\in K^T$ and $c\in K^C$, then there are some $x,y\in K^C$ such that (a) $a\subseteq x$ and $R^T xbc$ (b) $b\subseteq y$ and $R^T ayc$.*

Proof. By a "maximizing" argument [see, e.g., Routley and Meyer (1973) or Routley et al. (1982)]. □

LEMMA 4. *Let $a,b\in K^T$. Then the set $x = \{B \mid$ there exists A such that $(A \rightarrow B)\in a$ and $A\in b\}$ is a theory such that $R^T abx$.*

Proof. It is easily shown that x is a theory. Then, that $R^T abx$ holds is obvious. □

LEMMA 5. *For any $a,b\in K^C$, $a \leq^C b$ iff $a\subseteq b$.*

Proof. (a) From left to right, it is immediate. (b) Given that any theory a is closed by R_M–entailment, obviously, $R^T{}_{R_M}aa$. Then, by Lemma 3(a), there is some (regular) member x in K^C such that $R^C xaa$. So, $R^C xab$, and by definition, $(x\in O^C)$, $a \leq^C b$. □

LEMMA 6. *$*^C$ is an operation on K^C.*

Proof. By A10, A11 and Con. □

LEMMA 7. *Postulates P1–P6 hold in the R_M–canonical model.*

Proof. P1 and P2 are trivial by Lemma 5 and P6 is immediate by Lemmas 5 and 6; then, P3, P4 and P5 are proved by, respectively, A2, A3 and A4 with the assistance of Lemmas 3 and 4. □

LEMMA 8. *Let $a\in K^T$ and A be a wff such that $A\notin a$. Then, there is some $x\in K^C$ such that $a\subseteq x$ and $A\notin x$.*

Proof. As in the case of Lemma 3, by a "maximizing" argument. □

LEMMA 9. *Clauses (i)–(v) are satisfied by the R_M–canonical model.*

Proof. Clause (i) is trivial by Lemma 5, and clauses (ii), (iii), (v) and (iv) (from left to right) are immediate. Concerning clause (iv) from right to left, suppose $(A \to B) \notin a$ $(a \in K^C)$. Define the theories $x = \{C \mid \vDash_{\mathrm{RM}} (A \to C)\}$, and $y = \{C \mid$ there exists D such that $(D \to C) \in a$ and $D \in x\}$ such that $R^T axy$ (cf. Lemma 4), $A \in x$, $B \notin y$. By Lemmas 3(b) and 8, x and y are extended to prime theories b and c such that $R^C abc$, $A \in b$ and $B \notin c$, as required. □

A corollary of Lemma 8 is:

LEMMA 10. *If A is not a theorem of R_M, then A fails to belong to some regular, prime theory.*

Now, by Lemmas 7, 9 and 10, the R_M–canonical model is an R_M–model whence, by Lemma 10, we immediately have:

THEOREM 2 (*Completeness of R_M*). *If $\vDash_{\mathrm{RM}} A$, then $\vdash_{\mathrm{RM}} A$.*

IV. THE LOGIC R_{MNI2} AND ITS SEMANTICS

The logic R_{MNI2} is the result of adding to R_+ (cf. §2) the axioms:

A10. $\neg(A \wedge B) \to (\neg A \vee \neg B)$

A11. $(\neg A \wedge \neg B) \to \neg(A \vee B)$

of R_M plus

A12. $(\neg A \to B) \to (\neg B \to A)$

A13. $A \vee \neg A$

Some theorems and rules of inference of R_{MNI2} are (a proof is sketched to the right of each one of them):

T1. $\neg\neg A \to A$ — A12

T2. $(A \to B) \to (\neg B \to \neg A)$ — A12, T1

T3. $(\neg A \vee \neg B) \to \neg(A \wedge B)$ — T2

T4. $\neg(A \vee B) \to (\neg A \wedge \neg B)$ — T2

T5. $\neg(A \wedge \neg A)$ — A13, T3

T6. If $\vdash (A \to B)$ and $\vdash (A \to \neg B)$ then $\vdash \neg A$ — T2, T5

T7. If $\vdash (A \to \neg A)$, then $\vdash \neg A$ T6

T8. If $\vdash (\neg A \to A)$, then $\vdash A$ A13

T9. If $\vdash (\neg A \to \neg B)$ and $\vdash (\neg A \to B)$, then $\vdash A$ A12, T8

Therefore, from an intuitive (syntactical) point of view, R_{MNl2} can be described as having the following theses: (a) the principle of non–contradiction and the principle of "tertium non datur" (T5, A13), (b) the De Morgan laws (A10, A11, T3, T4), (c) the axiom of elimination of double negation (T1), (d) the constructive and non-constructive reductio principles *as rules* (T6, T9), (e) one of the constructive contraposition axioms (T2) and one of the non-constructive contraposition axioms (A12).

Nevertheless, we have:

PROPOSITION 1. *The following are, for example, not derivable in* R_{MNl2}*:*

(a) *The axiom of introduction of double negation* $A \to \neg\neg A$.

(b) *The constructive contraposition axiom* $(A \to \neg B) \to (B \to \neg A)$ *(it is not even derivable as a rule).*

(c) *The non-constructive contraposition axiom* $(\neg A \to \neg B) \to (B \to A)$ *(it is not even derivable as a rule).*

(d) *The constructive reductio axioms*

(d1) $(A \to \neg A) \to \neg A$,

(d2) $(A \to B) \to ((A \to \neg B) \to \neg A)$,

(d3) $(A \to B) \to \neg(A \wedge \neg B)$.

(e) *The non-constructive reductio axioms*

(e1) $(\neg A \to A) \to A$,

(e2) $(\neg A \to B) \to ((A \to B) \to B)$,

(e3) $(\neg A \to \neg B) \to ((\neg A \to B) \to A)$.

Proof. By MaGIC, the matrix generator developed by J. Slaney [see Slaney (1995)]. □

So, R_{MNI2} is really a non-standard logic. It is, of course, a sublogic of relevance logic R, but it is not included in, for example, any of Łukasiewicz's many-valued logics.

Next, we provide a semantics for R_{MNI2}: an R_{MNI2}*-model* is defined, similarly, as an R_M-model except for the addition of the postulates:

P7. If $Rabc$, then Rac^*b^*

P8. $a^{**} \leq a$

P9. If $a \in O$, then $a^* \leq a$

As in the case of R_M, A is R_{MNI2}*–valid* ($\vDash_{\mathrm{RMNI2}} A$) iff $a \vDash A$ for all $a \in O$ in all R_{MNI2}–models.

In order to prove the soundness of R_{MNI2}, we note the following:

LEMMA 11. *For any* $a \in K$ *and wff* A*, if* $a \nvDash A$*, then* $a^* \vDash \neg A$.

Proof. Suppose $a \nvDash A$. By P8 and Lemma 1, $a^{**} \nvDash A$. So, $a^* \vDash \neg A$ by clause (v) (cf. §2). □

Notice, however, and this is important, that the converse of Lemma 11, i.e., if $a^* \vDash \neg A$, then $a \nvDash A$ is not provable, the converse of P8 (i.e., $a \leq a^{**}$) being absent.

Then, we prove:

THEOREM 3 (*Soundness of* R_{MNI2}) *If* $\vdash_{\mathrm{RMNI2}} A$*, then* $\vDash_{\mathrm{RMNI2}} A$.

Proof. Given the soundness of R_M (Theorem 1), it is clear that we just have to prove that A12 and A13 are valid. Now, that A12 is valid can be proved as in the semantics for E or R [use Lemma 11, cf., e.g., Routley et al. (1982)]. So, we prove that A13 is valid. Suppose then, $a \nvDash A$, $a \nvDash \neg A$ for A and $a \in O$ in some model. By Lemma 1 and P9, $a^* \nvDash A$; but, by definitions (clause (v)), $a^* \vDash A$ □.

Regarding completeness, the R_{MNI2}–canonical model is defined in a similar way to which the R_M–canonical model was defined, of course, its items being now referred to R_{MNI2}–theories (i.e., theories closed by R_{MNI2}–entailment).

Now, we set:

DEFINITION 1. *Let* a *be an* R_{MNI2}*–theory. Then,* a *is* w–inconsistent *(inconsistent in a weak sense) iff* $A \in a$*,* $\neg A$ *being some theorem of* R_{MNI2}. (*a is* w–consistent *iff a is not w–inconsistent*).

REMARK 1. *The notion of w-consistency (in fact, those of w1-consistency and of w2-consistency) are introduced in Robles and Méndez (2008) (the reader is referred to the cited paper for details).*

Then, the two following lemmas are useful:

LEMMA 12. *For any wff A and $a \in K^T$, if $A \notin a$*, then $\neg A \in a^*$.

Proof. By T1 and definitions. □

Notice, however, and this is important, that the converse of Lemma 12 does not hold generally if the axiom $A \rightarrow \neg\neg A$ is not present (cf. Lemma 11 and the commentary following it).

LEMMA 13. *Let $a \in O^C$. Then, a^* is w-consistent.*

Proof. Suppose $a \in O^C$ and $A \in a^*$, $\neg A$ being a theorem of R_{MNI2}. By definition, $\neg A \notin a$ contradicting the regularity of a. □

Finally, we prove:

THEOREM 4 (*Completeness of R_{MNI2}*). *If $\vDash_{\mathrm{RMNI2}} A$, then $\vdash_{\mathrm{RMNI2}} A$.*

Proof. Given the completeness of R_M (Theorem 2), it is clear that we just have to prove that P7, P8 and P9 hold canonically. Now, that P7 and P8 hold can be proved as for E or R (use Lemma 12, cf. Routley et al. (1982)). So, we prove that P9 holds. Suppose, then, $a \in O^C$, $A \in a^*$ and, for reductio, $A \notin a$ for some wff A. By Lemma 12, $\neg A \in a^*$. Therefore, $(A \wedge \neg A) \in a^*$. Consequently, a^* is w-inconsistent by T5, which is impossible by Lemma 13. □

V. R_{MNI2} AND THE REDUCTIO AXIOMS

We add to R_{MNI2} the reductio axiom

A14. $(A \rightarrow \neg B) \rightarrow ((A \rightarrow B) \rightarrow \neg A)$

Then, in addition to T1-T9, the following are, for example, provable:

T10. $(\neg A \rightarrow \neg B) \rightarrow ((\neg A \rightarrow B) \rightarrow A)$ A14, T1

T11. $(A \rightarrow \neg A) \rightarrow \neg A$ A14

T12. $(\neg A \rightarrow A) \rightarrow A$ T10

T13. $(\neg A \rightarrow B) \rightarrow ((A \rightarrow B) \rightarrow B)$ A12, T12

T14. $(A \rightarrow B) \rightarrow \neg(A \wedge \neg B)$ A14

T15. $(A \to B) \to ((A \to \neg B) \to \neg A)$ A14, T2

T16. $(A \to \neg B) \to \neg(A \wedge B)$ T15

Nevertheless, we have:

PROPOSITION 2. *Thesis dn1, as well as the contraposition rules*

(a) *If* $\vdash(A \to \neg B)$, *then* $\vdash(B \to \neg A)$

(b) *If* $\vdash(\neg A \to \neg B)$, *then* $\vdash(B \to A)$

are not derivable in R_{MNI2} *plus A14.*

Proof. By MaGIC. □

Regarding semantics, we end this section by noting the following:

REMARK 2. *In Robles and Méndez (2004), corresponding postulates for the reductio axioms such as, e.g., A14, T10 or T12 are provided in the context of Routley and Meyer's positive logic B [cf., e.g., Routley et al. (1982)] plus the contraposition axiom A12. Unfortunately, these postulates are not adequate if the converse of P8, i.e.,* $a \leq a^{**}$, *is not present. Therefore, it seems not possible to provide adequate models for* R_{MNI2} *plus A14 in the present semantical framework.*

VI. CONCLUSION

In this paper, the logics R_M, R_{MNI2} and the one that results from adding the reductio axioms to the latter are considered. They are strong relevant logics lacking dn1. Now, it is clear that any relevant logic included in R_{MNI2} plus the reductio axiom lacks dn1. Moreover, we remark that we have implicitly provided a semantics for a series of logics including Sylvan and Plumwood's B_M and included in R_{MNI2}. Let us briefly discuss the matter.

The logic B_M can be axiomatized as follows [cf. Sylvan and Plumwood (2003), Robles (2008)]: A1, A5-A11, MP, Adj, con, and, in addition, the rules:

Suffixing (Suf): If $\vdash(A \to B)$, then $\vdash(B \to C) \to (A \to C)$

Prefixing (Pref): If $\vdash(B \to C)$, then $\vdash(A \to B) \to (A \to C)$

Then, a B_M–*model* is a structure $\langle K,O,R,*,\vDash\rangle$ where K, O, R,* and $\vDash$ are defined similarly as in a R_M–model except that P3, P4 and P5 are dropped. As in the case of R_M, validity is defined in respect of all points in O.

In Sylvan and Plumwood (2003) or Robles (2008), soundness and completeness of B_M relative to the semantics just sketched are proved.

Then, the logic B_{Mdn2} is axiomatized by adding dn2 to B_M. A B_{Mdn2}-*model* is defined similarly as a B_M–model except for the addition of P8 $a^{**}\leq a$. And validity is defined similarly as in B_M or R_{Mdn2}.

Now, we note:

PROPOSITION 3. *Given the logic B_M and B_M–semantics, P8 is the corresponding postulate (c.p) to T1* $\neg\neg A \rightarrow A$.

That is, given the logic B_M, it is proved that P8 holds canonically; and given B_M–semantics, T1 is valid by P8.

Proof. See, e.g., Routley et al. (1982). (Cf. Theorems 3, 4). □

Consider now the following axioms and semantical postulates:

A2. $(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

A3. $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$

A4. $A \rightarrow ((A \rightarrow B) \rightarrow B)$

A15. $((A \rightarrow A) \rightarrow B)) \rightarrow B$

A16. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$

A12. $(\neg A \rightarrow B) \rightarrow (\neg B \rightarrow A)$

A13. $A \vee \neg A$

P3. If R^2abcd, then there exists $x\in K$ such that $Rbcx$ and $Raxd$

P4. If $Rabc$, then R^2abbc

P5. If $Rabc$, then $Rbac$

PA15. There exists $x\in Z$ such that $Raxa$ (where Zx iff if $Rxyz$ then there exists $u\in O$ such that $Ruyz$)

PA16. If R^2abcd, then there exists $x\in K$ such that $Racx$ and $Rbxd$

P7. If *Rabc*, then *Rac*b**

P9. If $a \in O$, then $a^* \leq a$

It is proved:

PROPOSITION 4. *Given the logic B_{Mdn2} and B_{Mdn2} semantics, P3, P4, P5, PA15, PA16, P7 and P9 are the c.p to A2, A3, A15, A16, A12 and A13, respectively.*

Proof. Regarding P3, P4, P5, PA15, PA16 and P7, consult again Routley et al. (1982). As for P9, see Theorems 3, 4 above. □

Now, let SB_{Mdn2} be any extensión of B_{Mdn2} with any selection of A2, A3, A4, A15, A16, A12 and A13. And let SB_{Mdn2}-models be defined similarly as B_{Mdn2}-models except for the addition of the c.p to the axioms added. We clearly have:

THEOREM 5. *SB_{Mdn2} is sound and complete in respect of SB_{Mdn2}-models.*

Proof. Immediate from the soundness and completeness of B_{Mdn2} and Proposition 4. □

Dpto. de Historia y Filosofía de la Ciencia, la Educación y el Lenguaje
Universidad de La Laguna
Campus de Guajara, 38071, La Laguna, Tenerife, Spain
E-mail: gemmarobles@gmail.com

NOTES

*Work supported by research projects FFI2008-05859/FISO and FFI2008-01205/FISO financed by the Spanish Ministry of Science and Innovation. G. Robles is currently supported by Program Juan de la Cierva, financed by the Spanish Ministry of Science and Innovation.

REFERENCES

ANDERSON, A. R. & BELNAP N. D. JR. (1975), *Entailment. The Logic of Relevance and Necessity*, Vol I., Princeton University Press.

ROBLES, G. (2008), "A Note on the Non-Involutive Routley Star", *Bulletin of the Section of Logic*, 37/1, pp. 19–28.

ROBLES, G. & MÉNDEZ, J.M. (2004), "The Logic B and the *Reductio* Axioms", *Bulletin of the Section of Logic*, 33, pp. 43–54.

— (2008), “The Basic Constructive Logic for a Weak Sense of Consistency”, *Journal of Logic Language and Information*, 17/1, pp. 89–107.

ROUTLEY, R. & MEYER, R. K. (1972), “Semantics of Entailment III”, *Journal of Philosophical Logic*, 1, 192-208.

— (1973), “Semantics of Entailment I”, in LEBLANC, H.(ed.), *Truth, Syntax and Modality*, North Holland, pp. 199–243.

ROUTLEY, R., MEYER, R. K., PLUMWOOD & V., BRADY, R. T. (1982), *Relevant Logics and Their Rivals*, Vol. 1, Atascadero, CA, Ridgeview Publishing Co.

SLANEY, J. (1995), *MaGIC, Matrix Generator for Implication Connectives: Version 2.1, Notes and Guide*, Canberra, Australian National University. URL: *http://users.rsise.anu.edu.au/~jks*.

SYLVAN, R. & PLUMWOOD, V. (2003) “Non-Normal Relevant Logics”, in BRADY, V. (ed.), *Relevant Logics and Their Rivals*, Vol.II, Ashgate.