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From Human Activity to Conceptual Understanding of the Chain Rule

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From Human Activity to Conceptual Understanding of the Chain Rule

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Abstract

This article reports on a study which investigated first year university engineering students' construction of the definition of the concept of the chain rule in differential calculus at a University of Technology in South Africa. An APOS (Action-Process-Objects-Schema) approach was used to explore conceptual understanding displayed by students in learning the chain rule in calculus. Structured worksheets based on instruction designed to induce construction of conceptual understanding of the chain rule were used. A number of students used the straight form technique in differentiating complicated tasks while very few used either the link and Leibniz form techniques. In this manner differentiation of each function within the composite function was accomplished. Students either operated in the Inter- or Trans stages of the Triad. It was found that even students who had inadequate understanding of composition of functions, performed well in the application of the chain rule.

Keywords: calculus, chain rule, APOS, genetic decomposition.

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De la Actividad Humana a la Comprensión Conceptual de la Regla de la Cadena

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Resumen

Este artículo presenta un estudio sobre la construcción de la definición del concepto de regla de la cadena en el cálculo diferencial en el marco de estudiantes de primer año de ingeniería, en la Universidad Tecnológica de Sudáfrica. Se utiliza el enfoque APOS (Acción-Proceso-Objeto-Esquema) para explorar la comprensión conceptual que los estudiantes muestran en el aprendizaje de la regla de la cadena en cálculo. Se utilizaron fichas de trabajo estructuradas basadas en una instrucción diseñada para inducir la construcción de la comprensión conceptual de la regla de la cadena. Una parte de los estudiantes usaron utilizaron la técnica "directa" para diferenciar tareas complicadas, mientras que muy pocos de ellos utilizaron o bien el método de la conexión, o bien el enfoque de Leibniz, como técnicas de resolución. De esta manera se logró diferenciar cada una de las funciones simples en las funciones compuestas presentadas. Los estudiantes operaron tanto en las etapas inter, como intra, de la triada. Se encontró que incluso aquellos estudiantes con una comprensión no adecuada de las funciones compuestas, aplicaron la regla de la cadena correctamente.

Palabras Clave: cálculo, regla de la cadena, APOS, descomposición genética.

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Informal discussions held with other lecturers, revealed that the chain rule is one of the most complicated calculus tools, despite being one of the basic tools for a mathematician. Calculus is one of the topics introduced to matric learners at high school, yet a large number of them receive inadequate mathematics education and join the university mostly under-prepared for the study of differential calculus. Furthermore the chain rule is not part of the South African school syllabus. In our experience many first year university students have difficulty in understanding the chain rule in differentiation. This phenomenon was also observed by Orton (1983) who indicated that students: (1) had problems in the understanding of the meaning of the derivative when it appeared as a fraction or the sum of two parts and application of the chain rule for differentiation, and (2) had little intuitive understanding of solving differentiation problems as well as fundamental misconceptions about the derivative. He further asserts that some students are introduced to differentiation as a rule to be applied without much attempt to reveal the reasons for and justifications of the procedure. When asked about the chain rule, most students will simply provide an example of what it is rather than explain how it works (Clark et al., 1997; Cottrill, 1999). The literature related to studies in calculus provides evidence that students develop more procedural understanding rather than conceptual understanding in differentiation. However, very few studies investigate the characteristics of student's understanding of composition of functions and the chain rule.

Also in our experience some teachers at high school are less comfortable with calculus and its applications. This indicated that there was a need to engage with a study on students' understanding of the concept of the chain rule. The chain rule states that if $g(x)$ is a function differentiable at c and f is a function differentiable at $g(c)$, then, the composite function fog given by $(fog)(x)=f(g(x))$ is differentiable at c and that $(fog)'(c)=f'(g(c))\times g(c)$. This paper reports on the last part of a study conducted with first year engineering students exploring APOS in the conceptual understanding of the chain rule where questionnaires were used to explore the mental constructions formed by students in understanding the chain rule.

Research Questions

The research questions for this study were:

- How do students construct various structures to recognize and apply the chain rule to functions in the context of calculus?
- How should the teaching of the concept of the chain rule in differential calculus be approached?
- What insights would an APOS analysis of students' understanding of the chain rule in differential calculus reveal?

Theoretical Framework

This study was conducted according to a specific framework for research and curriculum development in mathematics education, which guided the systematic enquiry of how students acquire mathematical knowledge and what instructional interventions contribute to student learning. The framework consists of three components: theoretical analysis, instructional treatment, and collection and analysis of data observed when students learn as proposed by Asiala et al (2004). This is also well illustrated in other papers (Maharaj, 2010; Jojo et al 2011).

Theoretical Analysis

The study is based on APOS theory –Actions, Processes, Objects and Schema– (Dubinsky & McDonald, 2001). This approach starts with a statement of an overall perspective of what it means to learn and know something in mathematics as prescribed by Asiala et al:

> An individual's mathematical knowledge is his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing and reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations. (Asiala et al, 2004, p. 7)

They further believe that understanding a mathematical concept begins with manipulating previously constructed mental or physical objects to form *actions*; actions are then interiorised to form *processes* which are then encapsulated to form *objects*. They say that these objects could be de-encapsulated back to the processes from which they are formed, which would be finally organized in *schemas*. For an elaboration of these concepts refer to Maharaj (2010, p. 43).

Construction of knowledge in this study was analysed through reflective abstraction at the heart of which is APOS (Dubinsky, 1991b) which then incorporates Piaget's Triad mechanism. The Triad mechanism occurring in three stages explained other constructions in the mind implicating mental representations and transformations in the analysis of schema formations. These stages are: The *Intra* stage focuses on "a single entity", followed by *Inter*– which is "study of transformations between objects" and *Trans*– noted as "schema development connecting actions, processes and objects."

Reflective abstraction has two components: (a) a projection of existing knowledge onto a higher plane of thought and (b) the reorganization of existing knowledge structures (Dubinsky, 1991a). Reflective abstraction is therefore a process of construction and Dubinsky outlines five kinds of construction in reflective abstraction:

> *Interiorisation*: Actions conceived structurally as objects are interio-rised into a system of operations.
> *Co-ordination*: Two or more processes are co-ordinated in order to form a new process, e.g. the chain rule for differentiation requires the co-ordination of composition of functions with derivatives.
> *Encapsulation*: This is where the construction of mathematical understanding extends from one level to the other, where new forms of the process are built drawing from the previous ones to form an object.
> *Generalisation*: An existing schema is applied to a wide range of contexts. This would happen for example when the student is able to see that after finding the derivatives of the various functions in a composition, they now have to be multiplied to put the chain rule into application.
> *Reversal*: A new process can be constructed by means of reversing the existing one.

In extension of this theory, Dubinsky et al (1991) isolated someessential features of reflective abstractions reorganized and re-constructed them to form a coherent theory of mathematical knowledge and its construction, APOS. Jojo (2011) used the flow diagram (see Figure 2) to explain the activities involved in construction of the chain rule concept and illustrate APOS extended.

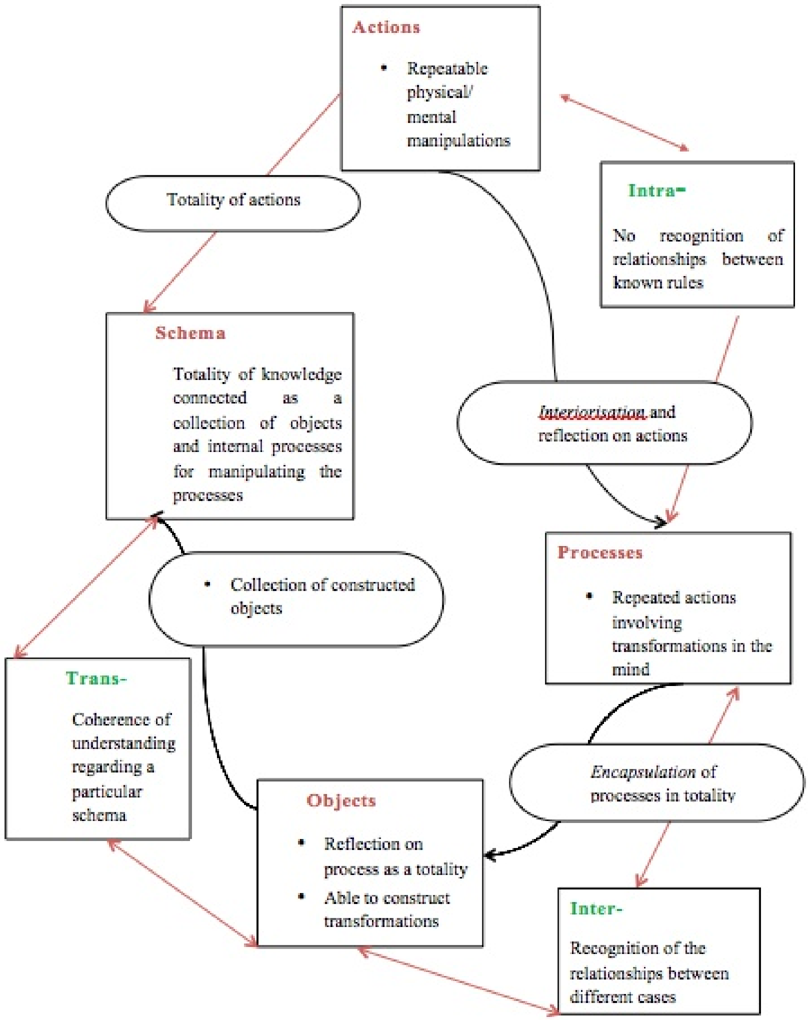

Figure 1. APOS theory extended

A structured set of mental constructs which might describe how the concept can develop in the mind of an individual is called the genetic decomposition of that particular concept. Based on the above discussion, the researchers arrived at the following genetic decomposition:

For a student to have his or her function schema, he or she:

(i) has developed a process or object conception of a function and
(ii) has developed a process or object conception of a composition of functions.

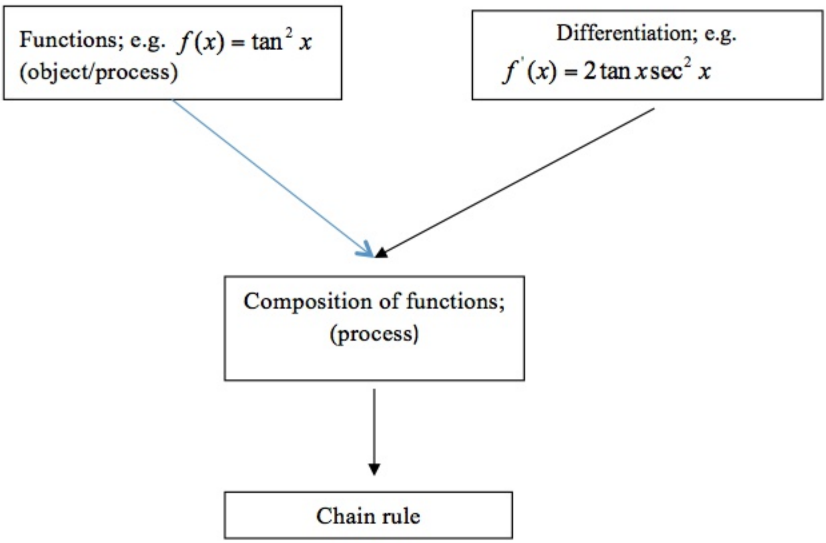

Figure 2. Initial genetic decomposition of the chain rule.

For a derivative schema,

(iii) has developed a process conception of differentiation;
(iv) the student then uses the previously constructed schemas of functions, composition of functions and derivative to

define the chain rule. In this process the student recognizes a given function as the composition of two functions, takes their derivatives separately, and then multiplies them.

(v) The student recognizes and applied the chain rule to specific situations. The initial genetic decomposition is modeled in figure 2.

Literature Review

The chain rule is used to find the derivatives of composite functions. Kaplan (1984) referred to the chain rule as a function of functions. A composite function is a function that is composed of two or more functions. For the two functions f and g, the composite function or the composition of f and g, is defined by $(f \times g)(x)=f(g(x))$. Despite the importance of the chain rule in differential calculus and its difficulty for students, the chain rule has been studied in mathematics educational research (Clark et al, 1997; Gordon, 2005; Uygur & Ozdas, 2007; Webster, 1978). Students' difficulties included the inability to apply the chain rule to functions and also with composing and decomposing functions (Clark et al, 1997; Cottrill, 1999, Hassani, 1998). In our experience the University of Technology students experience most problems in differential calculus.

Burke, Erickson, Lott & Obert (2001) assert that there is growing research support for designing classroom instruction that focuses on developing deep knowledge about mathematics procedures. When instruction is focused only on skillful execution, students develop automated procedural knowledge that is not strongly connected to any conceptual knowledge network (Star, 2000). This instruction resulted in procedures not executed "intelligently" and systematically. Understanding could be achieved, however, if students were given opportunities to develop a framework for understanding appropriate relationships, extended and applied what they knew, reflected on their experiences, and made mathematical knowledge their own (Carpenter & Lehrer, 1999). Further (1) when mathematical knowledge is understood, that knowledge is more easily remembered and more readily applied in a variety of situations (Hiebert & Carpenter, 1992; Kieran, 1992), (2) when a unit of knowledge is part of a well-connected network of mathematical understandings, parts of the network can facilitate recall (and even recreation) of other parts, and (3) when knowledge is understood it becomes easier to incorporate new knowledge into existing networks, so that current understanding facilitates future learning (Hiebert & Carpenter, 1992). It is therefore important to develop teaching methods that help students develop mathematical understanding.

Brijlall & Maharaj (2009) used the APOS theory in a study where they investigated fourth-year undergraduate teacher trainee students' understanding of the two fundamental concepts monotonicity and boundedness of infinite real sequences. They found that: (1) the structured worksheets encouraged group work and fostered an environment conducive to reflective abstraction, (2) the students demonstrated the ability to apply symbols, language, and mental images to construct internal processes as a way of making sense of the concepts of monotonocity and boundedness of sequences, (3) the students could apply actions on objects (sequences) which were interiorized into a system of operations, and (4) the conceptualization of the concept of boundedness of sequences and monotonocity enabled the formulation of new schema which could be applied in various contexts.

It can be agreed (Dubinsky & McDonald, 2001) that mathematical ideas begin with human activity and then proceed to be abstract concepts. It is therefore important for us to understand how the construction of concepts in the mind, lead to abstraction of mathematical knowledge. This interpretation of the relevant knowledge construction processes is essential since it points to the contributions we get from APOS analysis. These include (1) understanding the importance of human thought, and (2) pointing to effective pedagogy for a particular concept. An experimental, constructivist approach, was explored in teaching differentiation in calculus. Classroom activities used included working in teams, individual work, class discussions, sometimes, a mini-lecture summarizing the results of students' work, and providing examples on the use of chain rule in differentiation.

It is evident from the above discussion that, many well-known functions have simple expressions for their derivatives while composite functions require the use of the chain rule for differentiation. Functions having fairly complicated expressions have explicit formulas for derivatives. It was the development of formulas and rules such as the chain rule enabling mathematicians to calculate derivative that motivated the use of the name calculus for this mathematical discipline.

Participants, Instructional Design and Methodology

A qualitative study where worksheets were used to collect data from 12 groups of 76 first year civil engineering students was conducted. There were twelve groups, eight of which had six members and the other four had seven members. Instruction was designed using worksheets with four tasks on the use of the chain rule. There was space provided below each task in the worksheet for students' responses. This was done to reinforce the learning that took place in three sequential lesson components based on the proposed genetic decomposition of the concept of the chain rule. The aim was to provide students with opportunities to make applications of the chain rule they learnt and prepare them for the mathematics in which chain rule would be applied. Discussions would ensue between students working on each of the four problems, after which an agreed upon answer would be documented on the worksheet. Selected students from the groups were then interviewed and responded in explanations regarding their corresponding group presentations and responses.

The instructional design based on APOS theory included Activities, Classroom discussions and Exercises done outside of the classroom. The activities which form the first step of the ACE teaching cycle were designed to foster the students development of mental structures called for by APOS analysis. Students were requested to reflect on chosen activities on the use of the chain rule in differentiating composite trigonometric functions collaboratively. Classroom discussions ensued in each of the 12 groups and they listened to others' explanations and agreed upon a mathematical meaning to be presented in the worksheet. Exercises in the form of homework were then given to re-enforce the knowledge obtained in the activities and classroom discussions.

Whilst working in groups students discussed their results and listened to explanations given by fellow students. The students worked

collaboratively on mathematics tasks designed to help them use the mental structures that they had built during instructional design. In some cases, students worked on a task as a group, whilst in other cases they worked as individuals and then compared notes, and then negotiated a group solution to the problem. They then wrote their agreed upon solution on the spaces provided in the worksheets. During this process, the emphasis was on: (1) discussions, (2) reflection on explanations by the researchers where appropriate, (3) completion of the tasks by the students, and (4) understanding the use and application of the chain rule. The comparisons between three different techniques were made in chain rule differentiation. The first technique was the one using *Leibniz form technique*. The second one was the one where we differentiate from the innermost function and move outwards. We shall henceforth refer to this method of chain rule differentiation as a *link form technique* of the chain rule. The third one involves straight application of the chain rule in differentiation. We shall refer to this method of differentiation as a *straight form technique*. In this technique students used the chain rule mechanically by finding the derivatives of all the functions starting with the function on the outside of the given problem and multiplying out. For example, consider differentiating $y=\ln sinx^3$. We have characterized the three forms of the chain rule:
(1) *Leibniz* form technique gives, we let $y=\ln u$; then $dy/du = 1/u$; where $u=\sin v$; and $v=x^3$ so that $dv/dx=3x^2$; and $du/dv=\cos u$, and $dy/dx=dy/du\times du/dv\times dv/dx=1/u\times\cos u\times 3x^2$. This would give $3x^2\cos x^3/\sin x^3$. (2) *Link* form technique gives, we get $3x^2\times\cos x^3\times 1/\sin x^3$. (3) Using the *Straight* form technique we get, $1/\sin x^3\times\cos x^3\times 3x^2$. Answers using the three techniques were simplified to see if they were the same.

As the researchers moved from group to group, she noticed that some students used a lead pencil to record their responses on the worksheet. They were trying to avoid mistakes and allow correction of an incorrect response without spoiling the worksheet. In some groups, after transcriptions of agreed responses, all the members of the group satisfied themselves that the submitted response was appropriate. They argued from time to time of the positions where brackets should be inserted. Even after submissions of completed worksheets, other students continued convincing and teaching the inquisitive students on how the chain rule works.

Analysis and Discussion

The worksheets were analyzed for meaning which is one of the mechanisms necessary for understsnding a concept. These included detecting (1) the connections made by students to other concepts, (2) calculations made using the chain rule, (3) the chain rule technique used, and (4) mental images on which the chain rule was based. In what follows each of the four group tasks are first presented, and group responses are discussed. Where relevant interview extracts are also included to support the discussions. The task analysis indicating mechanisms used and percentage (correct to one decimal place) for each of the four tasks are illustrated in Tables 1 to 4 below.

Differentiate:

$$y = \tan^2(3x + e^{\sqrt{x^2+1}})$$

Figure 3. Task 1.

Table 1 summarizes the analysis of task 1 using the responses presented by the groups in this task.

Table 1
Analysis of task 1

	Incorrect responses	Partially correct	Completely correct	Chain rule preference	Connection to other concepts
Number of groups	6	4	2	12	7
% groups	50	33,3	16,7	100	58,3

All the groups applied the chain rule to the first task $y=\tan^2(3x + e^{\sqrt{x^2+1}})$ correctly using the straight form technique although only 16,7% of the groups presented a solution with brackets, when they differentiated the composite function inside the brackets in the given task. One of the groups who left out the bracket then went on to detach the derivative 3 of $3x$ from the + sign. This 3 now multiplied the first two functions (see Figure 3).

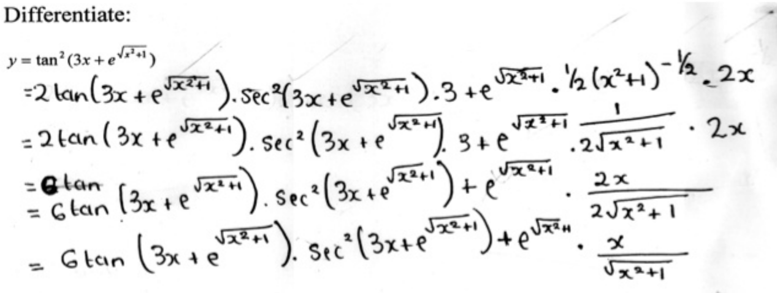

Figure 3. One group's presentation of task 1

This mistake was not detected by any of the other members of the same group. Those students struggled with the connection of previously learnt algebraic skills like use of brackets where appropriate and manipulation of algebraic terms in a function. The calculations presented after differentiating using the chain rule successfully were therefore not correct for 58,3% responses received. When one representative was interviewed and asked to state the chain rule, he

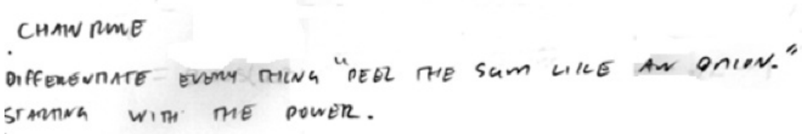

Figure 4. Chain rule in human terms

This student thought of differentiation in human terms. He had a mental picture of an onion being peeled from the outer layer (power in his terms), to the innermost layer. He pictured the straight form technique in human terms.

Also the given function was represented as equal to its derivative. The derivative should have been indicated as *y'*. The mental images constructed by the 58,3% in using the chain rule were incomplete. Although the actions were interiorized into processes, the processes were not encapsulated to objects. This could partly be attributed to previous knowledge of algebraic skills which were just actions and never interiorized. According to the Triad students in the said groups saw the chain rule as a procedure of differentiation which could not be connected or related to other processes applied to functions. Thus most students operated in the Intra- stage regarding task 1. According to APOS, we observed that most students could only go as far as the interiorizing the action to a process stage.

Differentiate:

$$y = (\cos^2 x + e^{\sin x})^2$$

Figure 5. Task 2.

Table 2 summarizes the analysis of task 2 using the responses presented by the groups in this task.

Table 2
Analysis of task 2

	Incorrect responses	Partially correct	Completely correct	Chain rule preference	Connection to other concepts
Number of groups	2	4	6	11	1
% groups	16,7	33,3	50	91,7	8,3

The solution to second differentiation problem y=(cos^2x+e$^{\sin x}$)^2 was presented correctly by 50% of the groups. Only one group avoided the use of the chain rule by squaring the given function and then differentiating. This was a brilliant idea but still required them to apply chain rule on the individual terms, cos^4x, 2cos^2x×e$^{\sin x}$ and e$^{2\sin x}$. They then used *straight* form technique to differentiate (see Figure 6). Those students were connecting the given function to a square of a binomial. Thus a part of understanding the concept of the chain rule is a mental process involving sorting out the given function, dealing with its composition, and connecting the two to find the derivative. They indicated a process construction of mental images since they transformed the given function to a trinomial which was operated on by repeating the actions of differentiation.

$$y = \cos^4 x + 2\cos^2 x \, e^{\sin x} + e^{2\sin x}$$

$$y' = 4\cos^3 x \cdot \sin x - 4\cos x \, e^{\sin x} \cdot \sin x + 2\cos^2 x \, e^{\sin x} \cdot \cos x + \cos x \, e^{2\sin x}$$

Figure 6. Chain rule application after squaring a binomial

Also the group did not completely apply the chain rule to the function $e^{2\sin x}$. Not all the layers were peeled and all the group members did not detect this. They therefore were in the Intra- stage of the Triad since they focused on the function as a single entity.

Differentiate:

$$\sin(x + y) = e^{y^2 + 2x}$$

Figure 7. Task 3.

Table 3 summarizes the analysis of task 3 using the responses presented by the groups in this task.

Table 3
Analysis of task 3

	Incorrect responses	Partially correct	Completely correct	Chain rule preference	Connection to other concepts
Number of groups	6	1	3	7	5
% groups	50	8,3	25	35	41,7

The third task required students to differentiate sin(x+y)=e$^{y\text{^}2+2x}$ implicitly using the chain rule. 41,7% of the groups introduced natural logarithms on both sides of the equation before differentiating. They explained that they connected the relationships of exponentials in the right hand side function with logarithms which would get rid of the exponent. In this way they ended up with simple expressions on both sides and thus allowed them, to use the *straight* form technique of chain rule differentiation (see Figure 8).

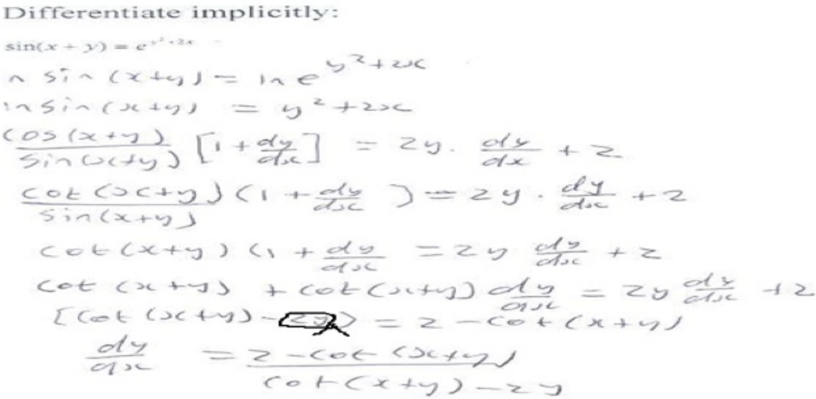

Figure 8. Differentiation using natural logarithms

Their calculations indicated a full understanding of the use of the chain rule except for omitting dx in the second step from the bottom of the solution. They operated in the Trans- stage of the triad since they could reflect on relationships between various objects from previous stages. They displayed coherence of understanding of differentiation rules and composition of functions.

25% of the groups presented responses of full construction of mental images of the chain rule and a connection between The other 35% of the groups applied the chain rule directly using the *straight* form technique and then processed the resulting function to get the derivative. Two of the responses indicated a transition from an operational to a structural mode of thinking since they brought the concept of the chain rule into existence and used it with caution, and preferred it over other methods of differentiation (see Figure 9).

Differentiate implicitly:

$\sin(x+y) = e^{y^2+2x}$

$$\cos(x+y)\cdot\left(1+\frac{dy}{dx}\right) = e^{y^2+2x}\cdot 2y\cdot\frac{dy}{dx}+2$$

$$\cos(x+y) + \cos x + y\frac{dy}{dx} = 2ye^{y^2+2x}\frac{dy}{dx}+2$$

$$\cos(x+y)\frac{dy}{dx} - 2ye^{y^2+2x}\frac{dy}{dx} = 2-\cos(x+y)$$

$$\frac{dy}{dx}\left(\cos(x+y) - 2ye^{y^2+2x}\right. = 2-\cos(x+y)$$

$$\frac{dy}{dx} = \frac{2-\cos(x+y)}{\cos(x+y)-2ye^{y^2+2x}}$$

Figure 9. Straight form technique used in differentiation

Differentiate:

$$y = \sqrt[3]{\frac{x(x+2)}{(x^2+1)}}$$

Figure 10. Task 4.

Table 4 summarizes the analysis of task 4 using the responses presented by the groups in this task.

Table 4
Analysis of task 4

	Incorrect responses	Partially correct	Completely correct	Chain rule preference	Connection to other concepts
Number of groups	3	4	2	1	8
% groups	25	33,3	16,7	100	66,7

Generally, one of two strategies was employed by students. The first form technique called for a specific connection between application of natural logarithms and differentiation.

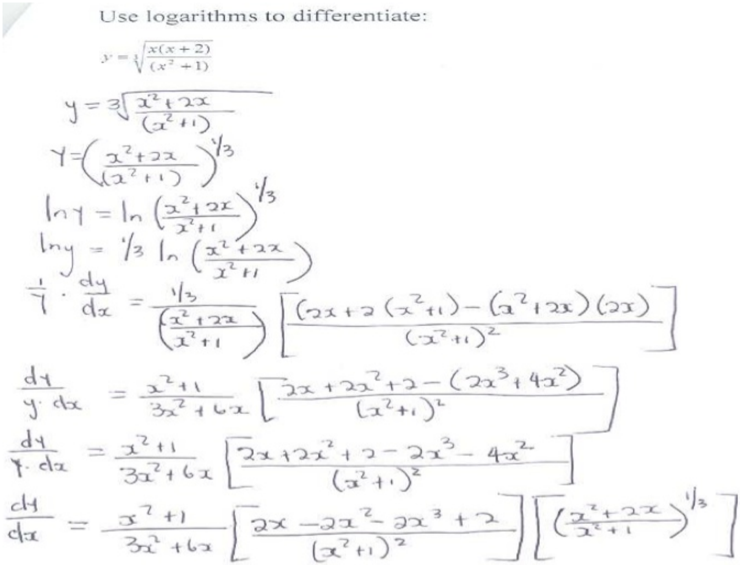

Figure 11. Group 3's response on logarithmic differentiation

16,7% of the groups displayed a coherent collection of the logarithmic rules and differentiation. Those groups were operating in the Trans-stage since they reflected on the explicit structure of the chain rule and were also able to operate on the mental constructions which made up their collection. Those students presented responses showing internal processes for manipulating logarithmic objects. Their schema enabled them to understand, organize, deal with and make sense out of application of the product rule, quotient, logarithmic rules and the chain rule. The other three groups could not apply logarithmic rules correctly and as such could not process the differentiation of the given task. This is illustrated in Figure 11 where students resolved the surd form of the function correctly and took natural logarithms both sides of the equation. The interpretation of logarithms was then incorrect since a bracket was left out in step three of the response. Thus the function differentiated was not the originally given one. Even in their process of differentiation some brackets were still left out when they should have been there.

The response illustrated in Figure 12 indicates that the derivative of the last term, $-\ln(x^2+1)$ in step four was recorded as $1/(x^2+1)\times 2$ instead of $1/(x^2+1)\times 2x$. In the next step the subtraction sign had been left out and then restored back again in the following one. The students in this group's actions indicated that they knew which steps to follow when differentiating. Their mental manipulations did not react to external cues of basic algebraic manipulations and as such transformation was not complete and their actions were not interiorized. Those students did not recognize the relationships between application of natural logarithms and algebraic manipulations resulting in multiplications when they were due and subtractions where appropriate. They perceived differentiation as a separate entities and even the rules applied were not remembered correctly. These were operating in the Intra- stage of the Triad.

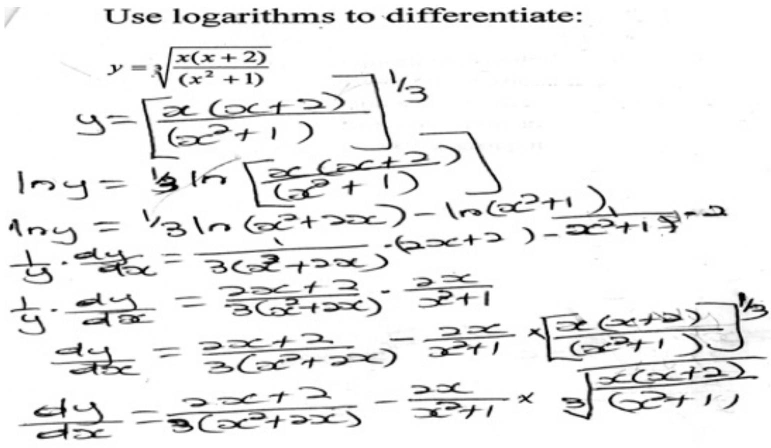

Figure 12. Incorrect application of chain rule in differentiation

The other group employed the straight form technique after converting the surd form to its exponential form. However, they did not then utilize the product and quotient rules appropriately. Their actions were not interiorized with regards to logarithms and this had an impact on applying the chain rule in the given task. Their mental images could not be related to the string of symbols forming the expression, since they could not interpret both the symbols and or manipulations. Since calculations reflect the active part of mental constructions, the differentiation rules for these students were not perceived as entities on which actions could be made. Dubinsky (2010) asserts that in such cases the difficulty does not depend on the nature of the formal expressions, but rather in the loss of the connections between the expressions and the situation instructions.

Conclusion

The researchers noticed that students in some groups would first copy a task in the worksheet onto their books. They would then work on it as individuals after which they compared their answers. Students argued and agreed upon certain responses. Individuals justified how they arrived at their responses. This way they taught each other and gave verbal descriptions of actions taken in their own words. They then repeated the actions many times with different tasks in their books and in the worksheet. Thus the worksheet helped the students interiorise the actions.

All groups did not use the *Leibniz* technique when differentiating the loaded trigonometric functions in all four tasks. Explanations given from interviewed group representatives indicated that this technique was complicated and would involve a long series of multiplication and substitutions of functions before and after differentiation.

A common error where students recorded the derivative of *cos x* correctly as *–sin x* but left out the brackets to end up with a different function from the one that was given for differentiation, was observed. Such students' actions of differentiation are detached from the basic algebraic operational signs. The multiplication sign left out indicates the absence of links between actions and procedures. Knowing the derivative of a particular function is not an indication of conceptual understanding since the relationships constructed internally were not connected to existing ideas. This understanding should also involve the knowledge and application of mathematical ideas and procedures related to basic arithmetic facts.

It was also noticed that most students in different groups were operating in the Intra- stage of the Triad. They had a collection of rules of differentiation with no recognition of relationships between them. Those students were helped by others who reflected on using the chain rule by applying the input by other students to group dynamics. The latter group had created an object of the chain rule. At the same time they applied actions on differentiation and as such the process of differentiating using the chain rule was encapsulated to form an object.

A possible modification to the proposed genetic decomposition was made. The student recognizes and applies the chain rule to specific situations using either the *straight*, *link* or *Leibniz* form techniques. This would then help the student to think of an interiorised process of differentiation in reverse and to construct a new process by reversing the existing one. Instruction on the conceptual understanding of the chain rule should incorporate all three different techniques.

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