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Testing Quasi-random Versus Pseudorandom Numbers on Bond Option Pricing

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Abstract

We investigate the impact of choosing quasi-random (deterministic) numbers versus pseudorandom numbers on the pricing of zero-coupon bond options. We compare quasi-Monte Carlo (QMC) simulations using Sobol, Faure, Niederreiter and Halton sequences to Monte Carlo (MC) simulations with pseudorandom generators such as congruential generation and Mersenne twister. We benchmark MC/QMC methods to the Cox-Ingersoll-Ross (1985) closed form solution. We examine bond option prices when the U.S. yield curve experiences different shapes -normal, inverse, flat and humped- and experiences a volatile environment or not. We show that one form of Halton sequence improves significantly bond options pricing when the shape of the yield curve is normal (85% of the time), whether the interest rate environment is volatile or not. We base our findings on 2,707 U.S. Treasury yield curves over the 2001-2012 period. Market participants in need of selecting an adequate number generator for pricing bond options with MC/QMC methods will find our paper appealing.

Keywords:

Faure, Halton, Sobol, Niederreiter sequences, Quasi-Monte Carlo simulation, Monte Carlo simulation, Bond option pricing, Cox-Ingersoll-Ross model.

JEL classification:

C52; C63; E43; G13.

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Contrastando números quasi-aleatorios frente a números pseudoaleatorios **en la valoración de opciones sobre bonos**

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Resumen

En este artículo se investiga el impacto que tiene el elegir números cuasi-aleatorios (deterministicos) o números pseudoaleatorios sobre la valoración de opciones sobre bonos cupón cero. Se comparan simulaciones cuasi-Monte Carlo utilizando secuencias de Sobol, Faure, Niederreiter y Halton con simulaciones Monte Carlo utilizando generadores pseudoaleatorios tales como los congruenciales y el Mersenne twister. Los métodos Monte Carlo y cuasi Monte Carlo se comparan con la solución en forma cerrada de Cox-Ingersoll-Ross. Se examinan los precios de las acciones sobre bonos con diferentes formas de la curva de tipos americana: normal, inversa, plana y encorvada, y ello tanto en un entorno de volatilidad o no. Se muestra que una forma de la secuencia de Halton mejora significativamente la valoración de opciones obre bonos cuando de la curva de tipos es normal (85% de las veces), independientemente de que el entorno de los tipos de interés sea volátil o no. Nuestras conclusiones se basan en 2.707 curvas de tipos del Bono Americano en el periodo 2001-2012. Los participantes en el mercado que necesitan un generador de números adecuado para la valoración de opciones sobre bonos con métodos Monte Carlo y cuasi Monte Carlo encontrarán atractivo este artículo.

Palabras clave:

Secuencias de Faure, Halton, Sobol, Niederreiter, Simulación cuasi-Monte Carlo, Simulación Monte Carlo, Valoración de opciones sobre bonos, Modelo Cox-Ingersoll-Ross.

1. Introduction

We present a methodological framework to test quasi-random numbers (Halton, Sobol, Faure and Niederreiter) and pseudorandom numbers (congruential generation, Mersenne twister) applied to Monte Carlo (MC) and quasi-Monte Carlo (QMC) simulations when pricing bond option. 'A quasi-random sequence, is "less random" than a pseudorandom number sequence, but can be more useful...because low discrepancy sequences tend to sample space "more uniformly" than random numbers. Algorithms that use such sequences may have superior convergence' (Burkardt, 2012). This is the basic assumption that we test on bond option pricing: are quasi-random sequences superior to pseudorandom sequences? More specifically, we test the pricing of a 2-year European[1] call option on a 5-year zero-coupon bond. The analytical solution of Cox-Ingersoll-Ross (1985) is well-known and will therefore constitute our benchmark. We price the option daily over an 11-year period (2001-2012), in different economic environments, the yield curve experiencing different shapes — normal, humped, flat or inverted — and interest rates being volatile or not. Closed-form solutions are quite rare considering the variety of derivatives and the abundance of exotic features that may be embedded in these products such as barriers: when the analytical solution is missing, especially when derivative products depart from plainness, numerical solutions such as MC/QMC methods may become helpful. Simulated trajectories take their source from a number generator: it is therefore essential to understand the nature and potential of generators underlying simulations.

Section 2 will review the literature concerning quasi-random and pseudorandom numbers. Section 3 will present the methodology in four steps. Section 4 will present the results and section 5 will wrap up our findings.

2. Literature Review

Random number generators may be classified as quasi-random (also called low-discrepancy sequences, see section 2.1 below) and pseudorandom detailed in section 2.2. In the last two decades (Faure and Lemieux, 2010), QMC methods spread out in finance to compensate limits of pseudorandom numbers. Among pioneers using Sobol and Faure sequences, we may cite Tezuka (1993), Traub and Paskov (1995) and Joy *et al.* (1996). Traub and Paskov compared MC method (using pseudorandom) with QMC method (using quasi-random numbers) when calculating a collateralized mortgage obligation, involving the numerical approximation of a number of integrals in 360 dimensions. Traub and Paskov concluded that QMC always beat MC methods. At this time, contemporary specialists in number theory believed that integrals of

[1] *Since the underlying security, a discount bond, makes no payments during the life of the option, the analysis of Merton (1973) implies that premature exercise is never optimal, and, hence American and European calls have the same value' (CIR, 1985).*

dimension ≥ 12 should not be solved by QMC, therefore Traub and Paskov's results were received with circumspection. The theory was improved by Papageorgiou (2001, 2003). Nowadays, QMC methods are extensively used to price derivative products. However, these methods have still limits for high dimensional integrals.

2.1. Low-discrepancy sequences

'Discrepancy is a measure of how inhomogeneous a set of d-dimensional vectors $\{r_i\}$ is distributed in the unit hypercube. If we generate a set of N multivariate draws $\{r_i\}$ from a selected uniform number generation method of dimensionality d, these N vectors describe the coordinates of points in the d-dimensional unit hypercube $[0,1]^d$. A sequence in $[0,1]^d$ is called a low-discrepancy sequence if for all $N>1$ the first N points in the sequence satisfy

$$D_N^{(d)} \leq c(d)\frac{(\ln N)^d}{N} \tag{1}$$

For some constant $c(d)$ that is only a function of d' (Jäckel, 2002).

Among low-discrepancy sequences, we select Faure, Halton, Sobol and Niederreiter due to their success amid authors: for example, Bratley and Fox (1988) and Faure and Lemieux (2010) have been promoters of these low-discrepancy sequences. Jäckel shows that through pairwise projections, 'low-discrepancy number generators tend to lose their quality of homogeneous coverage as the dimensionality increases' (Jäckel, 2002, pp. 88-91). Based on this observation, we choose a low-dimensionality for quasi-random sequences. However, we test Sobol and Niederreiter sequences for higher dimensionality.

Finally, we remind that quasi- and pseudo-random sequences generate numbers between $[0,1]$. Therefore, we convert these numbers by computing the inverse of the normal cumulative distribution function of the standard Normal distribution. The resulting series feeds the Wiener process in Equation 6. We apply the conversion to all quasi- and pseudo-random numbers discussed in this paper.

2.1.1. Faure sequences

With other pioneers, Joy *et al.* (1996) introduced QMC methods in finance. These authors referred to several types of quasi-random sequences such as Halton, Sobol and Faure. Since Fox (1986) explained the advantages of Faure sequences over the other two, Joy *et al.* (1996) focused their study on Faure sequences to price with QMC methods various derivative products, including European plain-vanilla equity options, exotic options such as options on the geometric mean of a portfolio, basket options, Asian options and swaptions. They proved the superiority of Faure sequences over pseudorandom generators. However, a limitation of the Faure sequence basic form has been identified: it 'suffers from correlations between different dimensions. These correlations result in poorly distributed two-dimensional projections' (Vandewoestyne *et al.,* 2010). These authors developed 'a randomly scrambled version of the Faure sequence, analyzed various scrambling methods and finally proposed a new nonlinear scrambling method, which has similarities with inversive congruential methods for pseudo-random number generation'. In our paper, we generate a Faure sequence of 3 dimensions in base 3. We use a default seed that is computed by the algorithm as the most suitable for the generation of the sequence. We do not choose to scramble the sequence. We borrow the Matlab algorithm of Faure sequence from Burkardt (2012), adapted from Algorithm 647 (Fox, 1986).

2.1.2. Halton sequences

Halton (1960) proposed the following sequence in one dimension. The j^{th} number H_j in the sequence is obtained using a two-step methodology: '1) Convert j in a number in base b, where b is some prime, for example j =17 in base 3 is equal to 122. 2) Reverse the digits and put a radix point (ie a decimal point base b) in front of the sequence. Our example returns 0.221 in base 3. The result is H_j. To get a sequence of n-tuples in n-space, we choose each component a Halton sequence with a different prime base b. Usually, the first n primes are used. The intuition behind Halton sequence is that every time the number of digits in j increases by one place, j's digit-reversed fraction becomes a factor of b finer-meshed' (Press *et al.,* 1992). As expected from a low-discrepancy sequence, the later produces points of finer and finer Cartesian grids, with a maximal spread-out order on each grid. Wang and Hickernell (2000) proposed a new method for randomizing the Halton sequence by randomizing the starting point of the sequence and therefore called the sequences 'random-start Halton'. Faure proposed to apply scrambling to 'digital (0, s)-sequences (in an arbitrary prime base $b \geq s$), by multiplying on the left the upper triangular generator matrices by non singular lower triangular matrices whose entries are randomly chosen in the set of digits $Z_b = \{0, 1, \ldots, b-1\}$. This methodology may be applied to any kind of digital sequences, like Halton or Niederreiter sequences' (Faure, 2006). As exposed in Faure and Lemieux, two lines of research have revived the interest in Halton sequences: 1) one has been to generalize Halton sequences by including permutations since basic Halton sequences are 'inadequate to integrate functions in moderate to large dimensions, in which case (t, s)-sequences such as the Sobol' sequences are usually preferred' (Faure and Lemieux, 2008). For example, Tamura (2006) proposed a randomization structure by Coordinate-wise and Digit-wise Permutations (CDP) that proves to give excellent results regardless of the classical low-discrepancy sequences (Faure, Halton, Sobol and Niederreiter sequences). 2) The second line explained in Faure and Lemieux (2008) has been the improvement by Atanassov (2004) in the upper bounds for the discrepancy of Halton sequences. Atanassov found clever generalizations of Halton sequences by means of permutations that are even asymptotically better than Niederreiter-Xing (1995, 1996) sequences in high dimensions. Unfortunately, the good asymptotic behavior was offset by remaining terms and was not sensitive to different choices of permutations of Atanassov. Thus, the reasons of the actual success of Halton and Faure sequences must be found in specific selections of good scramblings by means of tailor-made permutations detailed in Faure and Lemieux (2010).

In our paper, we generate a straightforward Halton sequence of one dimension in base 3 with a seed = 0 and a step = 1,000. Then, we generate a sequence in bases 2, 3 and 5 with a seed = 0 and a step = 1,000,000. We do not add randomization or permutation to the algorithm. We simply use a routine that 'selects elements of a "leaped" subsequence of the Halton sequence. The subsequence elements are indexed by a quantity called Step. The Step^{-th} subsequence element is simply the Halton sequence element with index Seed(1:Dim) + Step * Leap(1:Dim) with a default leap of 1' (Burkardt, 2012). Again, we borrow the Matlab algorithm of Halton sequence from Burkardt, adapted from Algorithm 247 (Halton and Smith, 1964) and Kocis and Whiten (1997).

2.1.3. Sobol sequences

Sobol and Levitan (1976) and Sobol (1977) proposed an original low-discrepancy sequence of 'binary fractions of length w bits, from a set of w special binary fractions, V_i, $i = 1, 2, ..., w$, called *direction numbers.* The j^{th} number X_j is generated by XORing together (bitwise exclusive or) the set of V_i's satisfying the criterion on i, the i^{th} bit of j is nonzero. As j increments, in other words, different ones of the V_i's flash in and out of X_j on different time scales. V_1 alternates between being present and absent most quickly, while V_k goes from present to absent (or vice versa) only every 2^{k-1} steps. Antonov and Saleev (1980) show that instead of using the bits of the integer j to select direction numbers, one could just use the bits of the Gray code of j, $G(j)$. Now $G(j)$ and $G(j+1)$ differ in exactly one bit position (Gray, 1953 and Black, 2011). A consequence is that the $j+1^{st}$ Sobol-Antonov-Saleev number can be obtained from the j^{-th} by *XOR*ing it with *a single* V_i, namely with i the position of the rightmost zero bit in j. It makes the calculation of the sequence very efficient. V_i are generated using the three following steps:

1) Each different Sobol's sequence (or component of an n-dimensional sequence) is based on a different primitive polynomial over the integers modulo 2, that is, a polynomial whose coefficients are either 0 or 1, and which cannot be factored (using modulo 2 integer arithmetic) into polynomial of lower order. Define P as such polynomial of degree q:

$$P = x^q + a_1 x^{q-1} + a_2 x^{q-2} + \ldots + a_{q-1} + 1 \tag{2}$$

2) Define a sequence of integers M_i by the q-term recurrence relation:

$$M_i = 2a_1 M_{i-1} \oplus 2^2 a_2 M_{i-2} \oplus \ldots \oplus 2^{q-1} a_{q-1} M_{i-q+1} \oplus \left(2^q M_{i-q} \oplus M_{i-q}\right) \tag{3}$$

Here bitwise XOR is denoted by $\oplus$. The starting values for this recurrence are that $M_1, \ldots, M_q$ can be arbitrary odd integers less than $2, \ldots, 2^q$, respectively.

3) Then, the direction numbers V_i are given by:

$$V_i = \frac{M_i}{2^i} \qquad i = 1, \ldots, w \qquad (4)$$ ‘ (Press *et al.,* 1992).

Sobol and Shukhman reaffirmed that ‘in contrast to random points that may cluster, quasi-random points keep their distance’ (Sobol and Shukhman, 2007). These authors propose a way to compute distances between quasi-random points of Sobol sequences. They extend their findings to Halton and Faure sequences. For practical implementation, we generate a Sobol sequence of 3 dimensions, with a seed = 0. Again, we base our choice of low-dimensionality of Sobol sequence on Jäckel’s (2002) remark and we increase the dimension to 10, with a seed = 10,000 to test for higher dimensionality. The Matlab algorithm of Sobol sequence is borrowed from Burkardt (2012), which is adapted from Algorithm 647 (Fox, 1986), Algorithm 659 (Bratley and Fox, 1988) and Antonov and Saleev’s ideas (1980).

2.1.4. Niederreiter sequences

Niederreiter (1992) presented a general framework for low-discrepancy sequences. Among the different algorithms he proposed (1988, 1996), only one has been implemented (Bratley *et al.,* 1994). The Niederreiter generator is similar to the Sobol’s one, based on arithmetic modulo m. However, the generator employs *irreducible* rather than *primitive* polynomials. Theoretically, the Niederreiter generator is supposed to be superior to Sobol’s in the limit (Jäckel, 2002). For practical implementation, we generate a Niederreiter sequence of 3 dimensions, base 2, with a seed = 0, then we increase the dimension to 10, with a seed = 10,000. We borrow the Matlab algorithm of Niederreiter sequence from Burkardt (2012), adapted from Algorithm 738 (Bratley *et al.,* 1994).

2.2. Pseudorandom numbers

Knuth argued that ‘every number random generator will fail in at least one application’ (Knuth 1997, 2012) due to the algebraic nature of its generation. It ‘always exists a high-dimensional embedding space R^d such that vector draws v whose elements are sequential draws from a one-dimensional number generation engine can appear as systematically aligned in a lower dimensional manifold’ (Jäckel, 2002). This is why computer-generated random numbers are called pseudorandom numbers (Sobol, 1994).

As a general rule regarding any pseudorandom number generator, the higher the number of simulations, the better as a result of the central limit theorem. We look at two pseudorandom number generators: congruential generator and Mersenne twister (Matsumoto and Nishimura, 1998).

2.2.1. *Congruential generator*

Park and Miller (1988) surveyed a large number of random number generators. The most popular, that we test in this paper, is the linear congruential generator, which is defined by a recurrence relation:

$$Y_{n+1} \equiv (aY_n + c) \qquad (\mathrm{mod}\ m) \tag{5}$$

with the following constant integers: Y_n is the sequence of pseudorandom values, $m > 0$ the modulus, $0 < a < m$ the multiplier, $0 \leq c < m$ the increment, $0 \leq Y_0 < m$ the seed or start value. If $c = 0$, the generator is called a multiplicative congruential method. If $c \neq 0$, the generator is called a mixed congruential method. The linear congruential generator has 'the advantage of being very fast, requiring only a few operations per call, this why it is so popular. It has the inconvenient of not being free of sequential correlation on successive calls' (Press *et al.,* 1992).

2.2.2. *Mersenne twister generator*

It is a popular generator proposed by Matsumoto and Nishimura (1998). The period of the sequence is a Mersenne number which is a prime number that can be written as $2^n - 1$ for some $n \in N$, and that belongs to the class of Twisted Generalized Feedback Shift Register (GFSR) sequence generator (Matsumoto and Kurita, 1992). The advantage is to have equidistribution properties in a minimum of 623 dimensions. An inconvenient is that 'for all random number sequences exists an embedding dimensionality in which, in the right projection, all sample points appear to lie in hyperplanes. It can have fatal consequences for a MC calculation if the problem that is evaluated just so happens to be susceptible to the used sequence's regularity' (Jäckel, 2002).

3. Methodology

We test quasi- and pseudo-random number generators when applied to the pricing of a 2-year European option on a 5-year zero-coupon bond, using MC and QMC methods. Since we can compute the 'exact' price of this option with the closed-form solution of Cox-Ingersoll-Ross (CIR, 1985), this 'exact' price will serve as benchmark of option prices obtained by MC/QMC simulations from the different number generators. We compute the option price over the 2,705 days of the database that contains U.S. yields curves of government bills and bonds from 2001 to 2012 (refer to section 3.5). Our methodology presents 4 steps.

3.1. Step 1

U.S. yield curves of government bills and bonds of the database must be bootstrapped in order to obtain zero-coupon yield curves that help computing premiums of 2-year European options on a 5-year zero-coupon bond. Starting with the curve of known bills and bonds yields, the bootstrapping technique solves for unknown zero-coupon yields using an arbitrage theory. Then, we convert the zero-coupon yields in continuously compounding rates, since the CIR solution deals with continuous rates and since MC/QMC methods, involving the stochastic CIR model (Equation 6 below), assumes also continuous rates.

3.2. Step 2

We interpolate the observed daily yield curve with a 20-point cubic spline interpolation (de Boor, 1978). This is a must for Kladivko's (2007) methodology that needs data points at regular time interval. The CIR (1985) model ensures mean reversion of interest rate towards the long-term average μ, with speed of adjustment α positive. $\sigma\sqrt{r}$ avoids the possibility of negative interest rates for all positive values of α and μ:

$$dr = \alpha(\mu - r)dt + \sigma\sqrt{r}\ dz_t \tag{6}$$

Kladivko's (2007) methodology aims at calibrating Equation 6 with the daily observed yield curve to find parameters α, μ and σ by maximizing the log-likelihood function (Equation 7) of the CIR process:

$$\ln L(\theta) = (N{-}1)\ln c + \sum_{i=1}^{N-1}\left\{u_{t_i} + v_{t_{i+1}} + 0.5q\ln\left(\frac{v_{t_{i+1}}}{u_{t_i}}\right) + \ln\left\{Iq\left(2\sqrt{u_{t_i}v_{t_{i+1}}}\right)\right\}\right\} \tag{7}$$

Where $u_{t_i} = cr_{t_i}e^{-\alpha\Delta t}$, $v_{t_{i+1}} = cr_{t_{i+1}}$

We find maximum likelihood estimates $\hat{\theta}$ of parameter vector θ maximizing the log-likelihood function (7) over its parameter space:

$$\hat{\theta} \equiv (\hat{\alpha}, \hat{\mu}, \hat{\sigma}) = \arg\ max \ln L(\theta) \tag{8}$$

Since the logarithmic function is monotonically increasing, maximizing the log-likelihood function also maximizes the likelihood function.

3.3. Step 3

We price a 2-year European call option on a 5-year zero-coupon bond with the CIR (1985) analytical solution. The inputs are the exercise price that we fix at 0.45 for the whole sample, the zero-coupon bond face value of $1, the 2 and 5-year maturities, the instantaneous rate and the three parameters α, μ and σ calibrated at step 2. We note that all these inputs (including α, μ and σ) are identical to both CIR analytical solution and MC/QMC simulations that make the CIR closed-form solution a valid benchmark for MC/QMC methods. The proxy of the instantaneous rate that we input in both analytical and numerical methods is the daily observed yield of the 1-month U.S. T-bill, with continuous compounding.

3.4. Step 4

We price a 2-year European call option on a 5-year zero-coupon bond with MC/QMC simulations of the CIR (1985) model (Equation 6). The inputs are identical to step 3. We simulate equation 6 using different number generators: Sobol, Faure, Halton and Niedderreiter sequences for quasi-random, congruential generation and Mersenne twister for pseudorandom numbers.

During the simulation, we make the time step dt varying with dt = [0.0833,0.1667, 0.25,0.5,1,1,2,2,3,10], respecting the maturity of the securities that constitute the U.S. yield curve at 1-, 3-, 6-month, 1-,2-,3-,5-,7-,10- and 20-year constant maturity. Thus, we simulate the instantaneous rate from 0 to 5 years with 10 to 10,000 simulations. Once the zero-coupon bond yield curve is simulated, we compute ${}_2r_5$ the forward rate of 3-year in 2-year, from ${}_2r_0$ the 2-year and ${}_0r_5$ the 5-year spot rates:

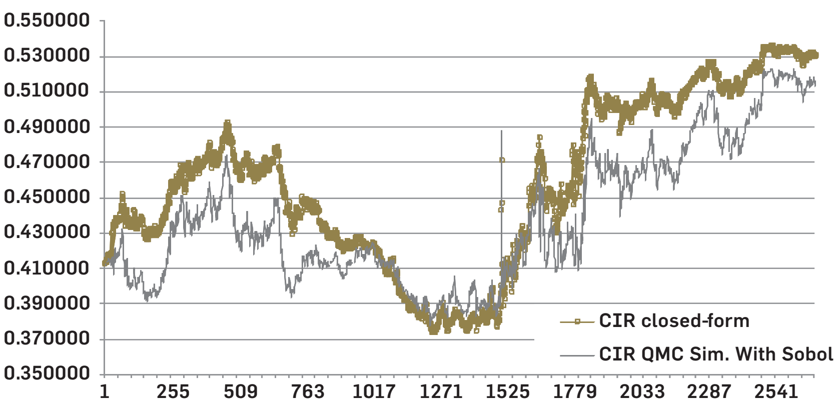
$$ {}_2r_5 = \frac{({}_0r_5\,x\,5 - {}_0r_2\,x\,2)}{3} \tag{9} $$

From ${}_2r_5$ of Equation 9, we compute the price of a 5-year pure discount bond in 2 years with \$1 face value. Finally, we compute the payoff of a 2-year European call option at maturity with an exercise price of 0.45 that we discount at time zero using Equation 10:

$$ Call = \left(1\,x\,e^{(-{}_2r_5\,x\,3)} - 0.45\right)^{+} x\,e^{(-{}_0r_2\,x\,2)} \tag{10} $$

Readers may wonder why we did not choose an option covering a wider range of the yield curve, e.g. a 2-year option on a 10- or 20-year zero-coupon bond. The answer is illustrated in Figures 1 and 2.

Figure 1. Price Series of a 2-year European call option on a 5-year zero-coupon bond with an exercise price = 0.45, over a sample of 2,705 yield curves, using the CIR (1985) closed form solution and a QMC method (1,000 simulations, Sobol sequence of 3 dimensions, with a seed = 0). Mean Square Error = 0.000692.



SOURCE: THE AUTHORS

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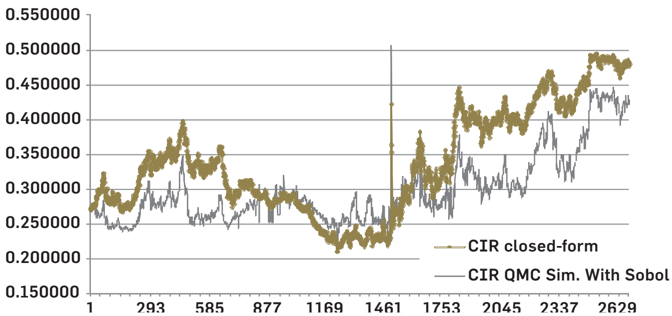
Figure 2. **Price Series of a 2-year European call option on a 10-year zero-coupon bond with an exercise price = 0.45, over a sample of 2,705 yield curves, using the CIR (1985) closed form solution and a QMC method (1,000 simulations, Sobol sequence of 3 dimensions, with a seed = 0). Mean Square Error = 0.003198.**



SOURCE: THE AUTHORS

The longer the bond maturity, the wider is the gap between option prices obtained from numerical and analytical solutions. The Mean Square Error (MSE) defined by Equation 11 is 0.000692 over the sample for an option on a 5-year bond when it is 0.003198 on a 10-year bond, almost 5 times higher.

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(Call\ price\ analitycal_i - Call\ price\ numerical_i)^2 \qquad (11)$$

The wider gap may be explained by the randomness inherent to MC/QMC methods, the degree of randomness increasing with maturity, but our intuition points to the CIR analytical method that used the instantaneous short term interest-rate as input. The longer the maturity, the wider the gap between the short term rate and the forward rate used to discount the bond. Therefore, we assume that the CIR model becomes inaccurate with maturity. We did not find anything to support our assumption in the literature. In order to investigate the problem, we replaced the 1-month T-bill yield as proxy of the instantaneous short rate in the CIR analytical solution by the 2-year bond yield, the latter being more representative of the average yield of the yield curve. When we priced a 2-year European call option on a 10-year zero-coupon bond over the 2,705 day-sample, we found that the gap measured with the MSE between the CIR analytical solution and the QMC method (using Sobol sequence of 3 dimensions, with a seed = 0 and 1,000 simulations), was significantly lower with this new proxy, with a MSE decreasing from 0.003198 with the 1-month yield to 0.001558 using the 2-year bond yield (more than 50% lower). We may deduce that our assumption is correct.

3.5. Database

The database includes market yields of U.S. Treasury securities (bills and notes) at 1-, 3-, 6-month, 1-,2-,3-,5-,7-,10- and 20-year constant maturity, quoted on investment basis yields on actively traded non-inflation-indexed issues adjusted to constant maturities. The U.S. yield curves of 2,705 days, extending from July 31, 2001 to May 24, 2012, are obtained from the Federal Reserve website[2]. Since the 30-year Treasury constant maturity series was discontinued on February 18, 2002, and reintroduced on February 9, 2006, we discard the 30-year maturity. We divide the database in four sub samples-normal, humped, flat and inverted yield curve. We present the statistics regarding the four types of yield curves in Table 2, using the criteria presented in Table 1.

Table 1. Classification of the U.S. yield curve in four occurrences: inverted, flat, humped and normal.

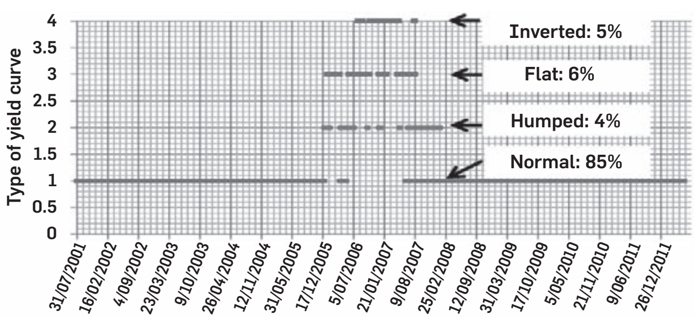
Type of curve:	Inverted	Else:	Flat	Else:	Humped	Else:	Normal
Criteria:	1-month rate is higher than 20-year rate		All rates remain in a range of 50 basis points		6-month rate is higher than 5-year rate		Remaining yield curves

Table 2. Counting occurrences among 2,705 observed U.S. yield curves from July. 31, 2001 to May 24, 2012.

Type of curve:	Normal	Humped	Flat	Inverted	Total
Number of occurrences:	2,298	104	168	135	2,705
% of occurrences:	85	4	6	5	100

Figure 3 illustrates the repartition of the four types of yield curve — normal, humped, flat and inverted — over the sample.

Figure 3. Plotting type of yield curve versus time on a sample ranging between July, 31 2001 and May 24, 2012: non normal curves concentrate between Dec. 27, 2005 and January 17, 2008.



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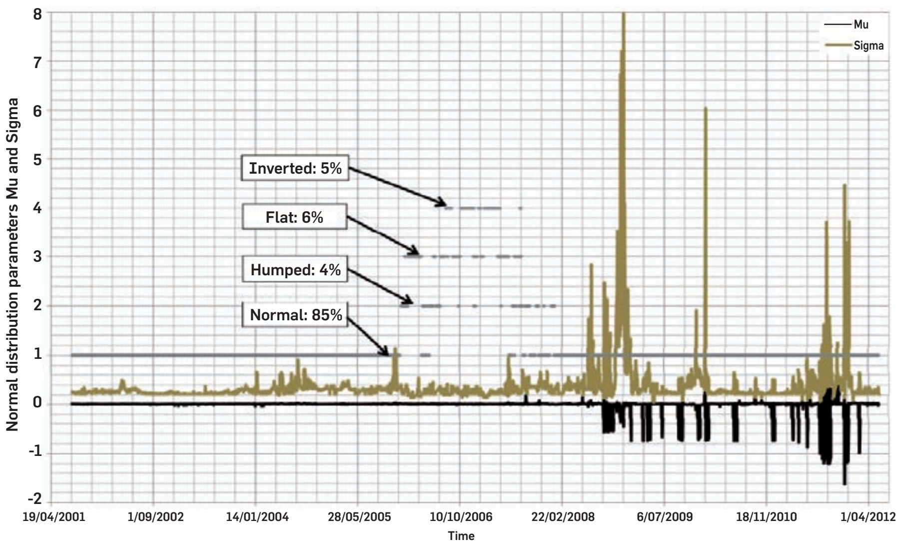
[2] *http://www.federalreserve.gov/releases/h15/data.htm. Accessed on January, 28 2013. Method for constructing yield curves used in this paper: 'Yields on Treasury nominal securities at "constant maturity" are interpolated by the U.S. Treasury from the daily yield curve for non-inflation-indexed Treasury securities. This curve, which relates the yield on a security to its time to maturity, is based on the closing market bid yields on actively traded Treasury securities in the over-the-counter market. These market yields are calculated from composites of quotations obtained by the Federal Reserve Bank of New York. The constant maturity yield values are read from the yield curve at fixed maturities, currently 1, 3, and 6 months and 1, 2, 3, 5, 7, 10, 20, and 30 years. This method provides a yield for a 10-year maturity, for example, even if no outstanding security has exactly 10 years remaining to maturity' (footnote on the Federal Reserve website).*

In addition to the type of yield curve, we test generators when the interest rate environment is volatile or not. To identify a volatile period, we interpolate daily observed yield curves with a 500-point cubic spline interpolation. Then, we extract a distribution of innovation terms from Equation 12. Innovation terms are deduced from equation 6 by rearranging the terms of the equation:

$$\varepsilon = \frac{dr - \alpha(\mu - r)dt}{\sigma\sqrt{r}\sqrt{dt}} \qquad (12)$$

Finally, we fit the Normal distribution to every yield curve distribution of the 2,705 days of our sample by calibrating μ and σ. We obtain Figure 4 below. Periods of our sample with a high σ will be defined as volatile.

Figure 4. **Variability of the Normal distribution parameters over the sample of 2,705 yield curves when the distribution of observed innovation terms has been fitted to the Normal distribution. Long-term averages of Mu = -0.00437 and Sigma =0.34738.**



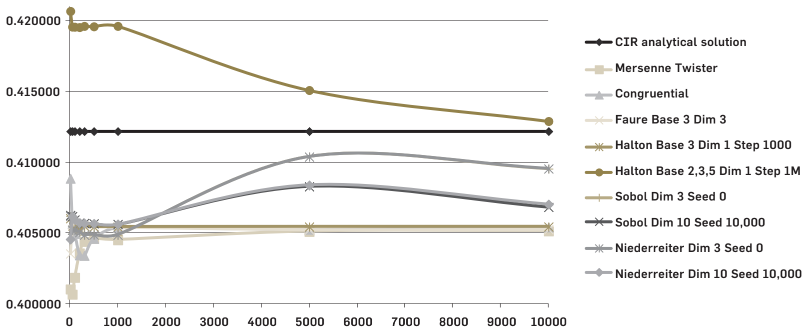
SOURCE: THE AUTHORS

4. Results

QMC and MC methods are tested with the Mean Square Error criteria (MSE, Equation 11) over the period of 2,705 days from July 31, 2001 to May 24, 2012. The benchmark is the CIR (1985) analytical model. We price a 2-year European call option on a 5-year discount bond. Figure 5 illustrates convergence of prices obtained with QMC and MC methods towards the price of 0.4122 obtained with the CIR analytical model on Aug.

21, 2007. We observe the progressive convergence of price of the QMC method using Halton sequence with bases 2, 3 and 5 and a step equal to 1,000,000. Niederreiter sequence with 3 dimensions and a seed equal to zero performs second best.

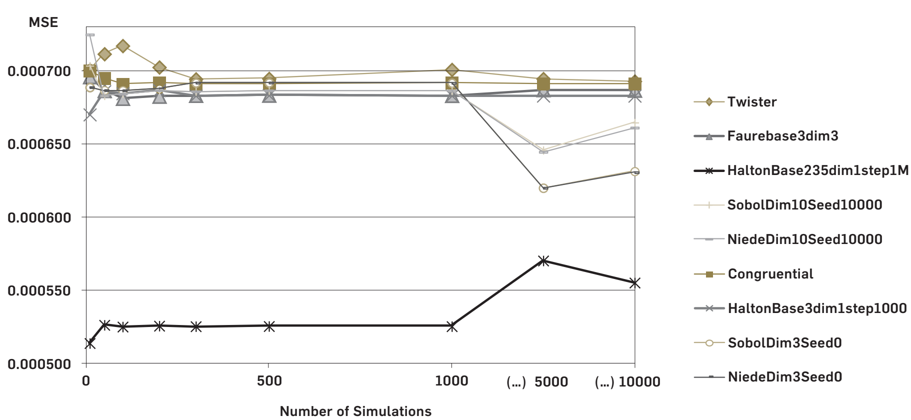
Figure 5. Illustrating convergence of prices of a 2-year call option on 5-year discount bond with an increasing number of simulations (10 to 10,000), using MC and QMC methods. The benchmark is the CIR (1985) analytical solution. Day: Aug. 21, 2007.



SOURCE: THE AUTHORS

We need of course to extend our results to the entire database of 2,705 days. Figure 6 illustrates the MSE as a function of number of simulations (from 10 to 10,000) for the whole sample. The lower the MSE, the better: Halton sequence with bases 2, 3 and 5 and a step equal to 1,000,000 outperforms all other pseudorandom numbers and quasi-random sequences with a MSE averaging 0.000533.

Figure 6. Mean Square Error (MSE) function of Number of Simulations using MC and QMC methods. The benchmark is the CIR (1985) analytical model. MSE computed over the sample of 2,705 yield curves.



SOURCE: THE AUTHORS

Table 3 gathers the inputs of Figure 6. It shows that, beside the outlier, Halton sequence with bases 2, 3 and 5, the other generators perform pretty much the same way.

Table 3. **Mean Square Error (MSE) function of Number of Simulations using MC and QMC methods. The benchmark is the CIR (1985) analytical model. MSE computed over a sample of 2,705 yield curves.**

Number of simulations	Mersenne Twister	Congruential	Faure Base 3 Dim 3	Halton Base 3 Dim 1 Step 1000	Halton Base2,3,5 Dim 1 Step 1M	Sobol Dim 3 Seed 0	Sobol Dim 10 Seed 10,000	Niederreiter Dim 3 Seed 0	Niederreiter Dim 10 Seed 10,000
10	0.000701	0.000701	0.000696	0.000670	0.000514	0.000689	0.000704	0.000689	0.000725
50	0.000712	0.000695	0.000687	0.000686	0.000526	0.000687	0.000683	0.000687	0.000683
100	0.000718	0.000691	0.000681	0.000685	0.000525	0.000687	0.000685	0.000687	0.000685
200	0.000703	0.000692	0.000683	0.000687	0.000526	0.000688	0.000687	0.000688	0.000687
300	0.000695	0.000691	0.000683	0.000683	0.000525	0.000692	0.000686	0.000692	0.000686
500	0.000695	0.000692	0.000684	0.000684	0.000526	0.000692	0.000687	0.000692	0.000687
1000	0.000701	0.000692	0.000683	0.000683	0.000525	0.000692	0.000687	0.000692	0.000687
5000	0.000695	0.000692	0.000687	0.000683	0.000570	0.000620	0.000646	0.000620	0.000645
10000	0.000693	0.000692	0.000687	0.000683	0.000555	0.000632	0.000665	0.000631	0.000661

First, we must say that whatever the numbers of simulations, MSEs in Table 3 do not vary much from 10 to 10,000 simulations, meaning that over the sample of 2,705 days, MSEs do not show convergence (does not decrease when the number of simulations increases), except for Sobol and Niederrreiter sequences that show a decrease in MSEs at five- and ten-thousand simulations. However, Table 3 shows the slight advantage of convergence of quasi-random sequences over pseudo-random for simulations between 10 and 1,000 where quasi-random sequences converge more quickly. Second, we confirm Jäckel's (2002) conclusion stating that 'low-discrepancy number generators tend to lose their quality of homogeneous coverage as the dimensionality increases.' Comparing Sobol and Niederreiter sequences with dimensions 3 and 10, Table 3 and Figure 6 show that sequences with dimension 3 outperform the ones with dimension 10.

Table 4 presents the computation and testing of MSEs resulting from QMC and MC methods with 10,000 simulations. We look at the whole sample of 2,705 days but also at six sub-samples: when the yield curve experiences different shapes — normal, humped, flat or inverted — and when interest rates are volatile or not.

Table 4. Testing the Mean Square Error (MSE) for Equality of Means using MC and QMC methods with 10,000 simulations. The benchmark is the CIR (1985) analytical model. Impact of the type of curve (normal, humped, flat, inverted) and the volatile environment on the MSE.

Type of Random Numbers Generator (Pseudo- and Quasi-)	Mersenne Twister	Congruential	Faure Base 3 Dim 3	Halton Base 3 Dim 1 Step 1000	Halton Base 2,3,5 Dim 1 Step 1M	Sobol Dim 3 Seed 0	Sobol Dim 10 Seed 10,000	Niederreiter Dim 3 Seed 0	Niederreiter Dim 10 Seed 10,000	Test for Equality of Means Between Series: Type of curve MSE All (Standard error) Anova F-statistic p-value
Line 1: MSE - (2,705 obs.) (Standard error)	0.000693 (0.0000113)	0.000692 (0.0000113)	0.000687 (0.0000112)	0.000683 (0.0000112)	0.000555 (0.00000911)	0.000632 (0.0000103)	0.000665 (0.0000109)	0.000631 (0.0000103)	0.000661 (0.0000108)	0.000656 (0.00000359) 17.17751 0.0000
L 2: MSE - Normal curve (2,298 obs.) (Standard error)	0.000804 (0.0000119)	0.000803 (0.0000119)	0.000797 (0.0000118)	0.000793 (0.0000117)	0.000634 (0.00000982)	0.000729 (0.000011)	0.00077 (0.0000115)	0.000728 (0.0000109)	0.000766 (0.0000114)	0.000758 (0.00000379) 23.57399 0.0000
L 3: MSE - Humped curve (104 obs.) (Standard error)	0.000018 (0.00000198)	0.0000179 (0.00000198)	0.0000182 (0.00000197)	0.0000184 (0.00000197)	0.0000593 (0.00000366)	0.0000311 (0.00000228)	0.0000223 (0.00000202)	0.0000313 (0.00000228)	0.0000224 (0.00000201)	0.0000265 (0.000000867) 33.92590 0.0000
L 4: MSE - Flat curve (168 obs.) (Standard error)	0.0000313 (0.00000111)	0.0000312 (0.00000112)	0.0000314 (0.00000111)	0.0000316 (0.0000011)	0.0000725 (0.00000136)	0.0000468 (0.00000116)	0.0000368 (0.00000112)	0.000047 (0.00000116)	0.0000368 (0.00000112)	0.0000406 (0.000000505) 137.3176 0.0000
L 5: MSE - Inverted curve (135 obs.) (Standard error)	0.000146 (0.00000827)	0.000146 (0.00000829)	0.000147 (0.0000083)	0.000147 (0.00000831)	0.000202 (0.00001)	0.000169 (0.00000896)	0.000155 (0.00000855)	0.000169 (0.00000897)	0.000155 (0.00000856)	0.000159 (0.00000293) 4.405494 0.0000
										Test for Equality of Means Between Series: Volatile versus non-volatile environment - MSE All (Standard error) Anova F-statistic p-value
L 6: MSE - Non-volatile environment (2,297 obs.) (Standard error)	0.000649 (0.0000119)	0.000648 (0.0000118)	0.000643 (0.0000118)	0.00064 (0.0000117)	0.000519 (0.00000955)	0.000591 (0.0000108)	0.000622 (0.0000114)	0.00059 (0.0000108)	0.000619 (0.0000114)	0.000613 (0.00000377) 13.81115 0.0000
L 7: MSE - Volatile environment (408 obs.) (Standard error)	0.000942 (0.0000313)	0.00094 (0.0000312)	0.000934 (0.0000311)	0.000928 (0.0000309)	0.000758 (0.0000253)	0.000861 (0.0000284)	0.000905 (0.00003)	0.00086 (0.0000284)	0.0009 (0.0000298)	0.000892 (0.00000992) 3.982511 0.0001

Over the whole sample (Line 1, Table 4), MSEs are statistically different. We identify three groups from the worst MSE (highest MSE) to the best (lowest MSE): 1) the group of Mersenne Twister, Congruential, Faure Base 3 Dim 3 and Halton Base 3 Dim 1; 2) the group of Sobol and Niederreiter sequences and finally 3) the outlier, Halton sequence with bases 2, 3 and 5. The first conclusion that we may draw is that pseudo-random numbers and quasi-random Faure and Halton sequences in their basic forms (base 3, dim 1 or 3) are not statistically different in terms of pricing power. Of course, this conclusion applies only to our example, the pricing of a 2-year European call option on a 5-year discount bond. This result contradicts the conclusion of Fox (1986) who explained the advantages of Faure sequences over Halton and Sobol. The second conclusion is that Sobol and Niederreiter sequences may be considered as second generation sequences since they offer significantly better results than basic Faure and basic Halton, being considered as first generation. Except Fox (1986), the literature confirms this general perception of ranking. At this stage, the Niederreiter generator is supposed to be superior to Sobol's in the limit (Jäckel, 2002). However, Table 4 shows that these two generators are not statistically different. The third conclusion is that quasi-random sequences must be tailor-made, "cooked" in order to offer better results: by changing the parameters of Halton sequence from base 3 dimension 1 and a step equal to 1 000 to bases 2, 3 and 5 dim 1 and a step equal to 1,000,000, we drastically improve the MSE result from 0.000683 to 0.000555.

Finally, Table 4 illustrates the MSE results when the yield curve experiences different shapes — normal, humped, flat or inverted — (Lines 2 to 5) and when interest rates are volatile or not (Lines 6 and 7). With a normal curve (Line 2), the situation is the same as for the whole sample (Line 1). Although MSEs are higher in this subsample, Halton sequence with bases 2, 3 and 5 is best. The contradictory result is for humped, flat and inverted curves (Lines 3 to 5): Halton sequence with bases 2, 3 and 5 is significantly worst in these 3 scenarios, whereas the group of Mersenne Twister, Congruential, Faure Base 3 Dim 3 and Halton Base 3 Dim 1 is best. In addition, either volatile or non-volatile the environment, Halton sequence with bases 2, 3 and 5 is significantly best.

5. Conclusion

We test quasi- and pseudo-random number generators applied to the pricing of a 2-year European call option on a 5-year zero-coupon bond, using MC and QMC methods. We compare QMC methods using Sobol, Faure, Niederreiter and Halton sequences to MC methods with pseudorandom generators such as congruential generation and Mersenne twister. We benchmark MC/QMC simulation to the Cox-Ingersoll-Ross (1985) closed form solution. We examine bond option prices when the U.S. yield curve experiences different shapes — normal, inverse, flat and humped — and experiences a volatile environment or not. We show that one form of Halton sequence (with bases 2, 3 and 5 and a step equal to 1 000 000) improves significantly bond options pricing when the shape of the yield curve is normal only (85% of the time) and whatever the volatility of the interest rate environment. We base our findings on 2,707 U.S. Treasury yield curves on the 2001-2012 period. In addition, our marginal results are that: 1) pseudorandom numbers (Mersenne Twister and Congruential Generation) and quasi-random Faure and Halton sequences in their basic forms (base 3, dim 1 or 3) are not statistically different in terms of pricing accuracy; 2) basic Sobol and Niederreiter sequences offer significantly better results than basic Faure, basic Halton and pseudo random numbers generators. 3) Niederreiter generator is supposed to be superior to Sobol's: our paper shows that they are statistically equal.

Reviewing the literature and results, we identify one limit of low-discrepancy sequences: they offer faster convergence with QMC methods but in order to obtain better results they must be tailor-made, "cooked": it is therefore difficult to generalize a given solution to a given problem such as pricing a given derivative product. Our evolved Halton sequence with bases 2, 3 and 5 may not be valid for pricing equity option for example.

Over the past twenty years, authors have regularly presented new developments on low-discrepancy sequences, proving their effectiveness in special situations, for example Traub and Paskov (1995) found that QMC always beat MC methods when valuing a collateralized mortgage obligation. We believe that a conservative approach should be to avoid generalizing the ability of low-discrepancy sequences in finance even if nowadays quasi-MC methods are extensively used to price derivative products. Their limit is related to the setting of parameters such as dimension, base, seed, step, adding or not permutations or scrambling, which results in multiple combinations. Since a given problem requires a special choice of sequence and parameters, it is presumptuous to generalize the ability and power of low-discrepancy sequences in finance. There will always be a lag between theory and empirical research on this topic.

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