A two logic qubits formalism as a robust quantum computer initialization protocol for the processing of quantum error correction codes

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(Received 9 October 20010; accepted 10 January 2011)

Abstract

A two logic qubits formalism, built from the direct product of a single qubit state with its dual $(|q\rangle \times |q*\rangle)$, is introduced. Natural phase shifting of a quantum state is represented by an unitary operation acting on a single qubit state. Thus, the respective initialization state for the transference of information (exchange protocol) is sustained on three maximally entangled Bell states. This state serves for building a simple dual error correction code. Through the use of present formalism it is also shown that quantum errors coming from a uniform external magnetic act as the identity on the information carrying system while change the spin orientation of the syndrome. In other words, information is encoded in such a way that it is protected from errors indefinitely, without intervention.

Key words: Unitary; initialization; quantum; error; correction; code; syndrome; noiseless.

Resumen

Es presentado un formalismo de dos qubits lógicos formado a partir del producto directo de un estado qubit con su dual $(|q\rangle \times |q*\rangle)$. El desplazamiento de fase natural de un estado cuántico es representado por una operación unitaria actuando sobre un estado de un solo qubit, así el respectivo protocolo de inicialización está sustentado sobre tres estados máximamente entrelazados de Bell. Este sirve para construir un código dual simple de corrección de errores. A través del uso del presente formalismo se demuestra también que errores cuánticos provenientes de un campo magnético uniforme externo actúan como la identidad sobre el portador de la información mientras que cambian la orientación de espín del síndrome. En otras palabras, la información es codificada de tal manera que está protegida de errores indefinidamente, sin intervención.

Palabras clave: Unitario; inicialización; cuántica; error; corrección; código; síndrome; silencioso.

PACS: 03.67.Lx; 03.65.Yz; 03.67.-a **ISSN 1870-9095**

I. INTRODUCTION

Uncertainty of phase of a quantum state, together with superposition principle, are to a great extent responsible for the power of quantum mechanics. As it is well known, phase indeterminacy is in principle irremovable [2] and leads to a non-absolut meaning of information. The indeterminacy can manifest itself in a variety of ways due to the interaction of the system with the environment or while initializing the quantum register [3]. The point is that the algorithms of quantum computation assume that the state of the quantum register has its phase uncertainty lumped together, so that it can be ignored [4]. That is, it is assumed that the qubit state acquires a global phase: $e^{i\phi}(\alpha|0\rangle + \beta|1\rangle)$, where α and β are complex numbers ($|\alpha|^2+|\beta|^2 = 1$) and $|0\rangle, |1\rangle$ are the basic qubits. This situation corresponds to an ideal initialization, a more realistic situations may not permit it to be lumped together. The phase uncertainty in the qubit state is twice instead i.e. $e^{i\phi 1}\alpha|0\rangle + e^{i\phi 2}\beta|1\rangle$. The above means that random phase shifts, without disentanglement[1] of the states can also cause serious problems in quantum computation [3]. These random phase shifts are unitary operations of the form:

$$\frac{\mathbf{1}}{\sqrt{|\boldsymbol{a}|^2+|\boldsymbol{b}|^2}}\begin{bmatrix} \boldsymbol{a}^* & \boldsymbol{b} \\ \boldsymbol{b} & -\boldsymbol{a} \end{bmatrix}, \tag{1}$$

where a^* and b^* are phases. As it has already been observed in [3], the representation of a unitary transformation in terms of a sequence of small-degree gate will introduce random phase shifts at each gate that will have an effect similar to random phases in a register. This makes very difficult to implement a quantum computing. The purpose of this letter is to introduce, from a pedagogical point of

[1]Quantum entanglement is a quantum mechanical phenomenon in which the quantum states of two or more objects are linked together so that one object can no longer be adequately described without full mention of its counterpart even though the individual objects may be spatially separated [1].

view, a simple two qubits formalism which leads to a robust initialization of the quantum computer based on maximally entangled Bell states. Due that the initialization state is itself a relationship which serves for the transference of information, this is indeed a computational protocol. From this initial state, two quantum correction codes are derived. One of them is so simple and the other is not trivial. The non trivial code arises when a uniform external magnetic field interacts with two spin - $^1/_2$ particles. As a consequence of this, it is shown that the respective time evolution operator acts as the identity on the logic qubit but that this changes the orientation of the syndrome[1]. In this way we achieve error protection without intervention, regardless of the current state. With the above, we intend to clear up basic concepts of quantum error correction codes to beginners which are often source of confusion and which lead to misunderstandings. The point is that the standard treatment on the subject done in classical books [4], does not illustrate enough the concept since they do not include specific examples.

The preparation of the initial state employed in this work is a mathematical result which eventually could be susceptible of being experimentally implemented. The idea is then to let the system naturally to take its phases and then to prepare it through appropriate unitary transformations until the two qubits initial state is suitable for building a quantum computer. The first basic assumption is that a single qubit state represented by $|\mathrm{q}\rangle$ admit their dual counterpart $|q^*\rangle$.The second assumption is that the two qubit initial state for a formally thought quantum computer can be built from $|\mathrm{q}\rangle \otimes |q^*\rangle$. In order to gain a deeper understanding of our procedure it is convenient to observe that if we denote the set of $N = 2^n$ logic registers by $\{n_0, n_1, n_2, \ldots n_{N-1}\}$ where $n_i = 0, 1$, then the initialization protocol consists in to place all the N states in a superposition where each basis state is equally probable. The way in which this is traditionally implemented is to place a bit, say a 0, in each cell and then to apply the following transformation [3],

$$\mathrm{M} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{2}$$

to each bit, transforming it into the superposition with the amplitudes $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. It is argued within the traditional approach that the physical object that carries the qubit, be it a photon, an electron, or an atom or a molecule is already physically present at its location. As a consequence of this, the accustomed initialization procedure consist in assuming that the system relaxes itself until each cell get a general random phase in the form [3],

$$|\phi\rangle = \frac{e^{i\theta_{11}}}{\sqrt{2}}\, |0\rangle + \frac{e^{i\theta_{12}}}{\sqrt{2}}\, |1\rangle, \tag{3}$$

According with this, the state N register would be then

$$|\psi\rangle = \frac{e^{i\theta_1}}{\sqrt{N}}\, |0 \ldots 0\rangle + \frac{e^{i\theta_2}}{\sqrt{N}}\, |0 \ldots 1\rangle + \cdots + \frac{e^{i\theta_2}}{\sqrt{N}} |1 \ldots 1\rangle. \tag{4}$$

Each of the above 2^n states have the same probability of occurrence. Observe that Eq. (4) follows after applying a unitary transformation of the type of Eq. (1) to each register by separate in the string of N entangled qubits. From the experimental point of view, the state ψ of Eq. (4) is achieved if each of its N qubit component states is prepared independently of the others. One must observe that the associated phases are unknown and so it is impossible to use amplitude amplification method. As it has already been pointed out in Ref. [3] phase rotation for a case where the phases are randomly distributed will be meaningless.

According with our approach, it will be assumed that the physical qubit does not relax from an isolated state and that instead it acquires a random phase from the very preparation. According with Eq. (1), this fact is reflected in the unitary operation

$$u = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\varphi} & e^{i\xi} \\ e^{-i\xi} & -e^{-i\varphi} \end{bmatrix}, \qquad 5)$$

acting on the cell 0. A quantum mechanics realistic single qubit initial state should be then

$$|\mathrm{q}\rangle = u|0\rangle = \frac{1}{\sqrt{2}} [e^{i\varphi}|0\rangle + \ e^{-i\xi}|1\rangle], \tag{6}$$

and not $|0\rangle$ as it is usually assumed. With this, we heve then that $|q^*\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\varphi}|0\rangle + \ e^{+i\xi}|1\rangle\right]$. On the other hand, we are interested in building an initial two qubits logical register from the product of two physical qubits, that is

$$|\psi\rangle_0 = |q^*\rangle \times |\mathrm{q}\rangle = \frac{1}{2}\ [|00\rangle + |11\rangle + \cos\theta\, (|01\rangle + 10 + i\sin\theta(10 - 01)], \tag{7}$$

Where $\theta \doteq \varphi - \xi$ and $cos^2\theta + sin^2\theta = 1$.That the state $|\psi\rangle_0$ of the above equation is suitable for an initialization protocol can be seen from the fact that each one of the four states $|00\rangle, |01\rangle, |10\rangle$, and $|11\rangle$ have the same probability of occurrence. As we will see lines below, it is possible to write $|\psi\rangle_0$ in terms of maximally entangled Bell states. At this stage it is worth to point out that Yu and Eberly [5] studied the entanglement dynamics of these Bell states which are given by

$$|\psi\rangle^{Bell}_{robust} = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |\psi\rangle^{Bell}_{fragile} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \tag{8}$$

when they are coupled to a heath bath. They found that the "robust state" lives in a Decoherence-Free [2] Subspace [6, 8] by which its entanglement is preserved. On the other hand, the entanglement of the qubits initially in the "fragile state"

[1] In order to illustrate the term syndrome see the pedagogical example of the three qubit flip code discussed in [4] pp. 427.

[3] In quantum mechanics, quantum decoherence is the mechanism by which quantum systems interact with their environments to exhibit probabilistically additive behavior. Quantum decoherence gives the appearance of wave function collapse. Decoherence occurs when a system interacts with its environment in a thermodinamically irreversible way [4].

decays to zero [9]. In terms of $|\psi\rangle^{Bell}_{fragile}$ and $|\psi\rangle^{Bell}_{robust}$ the initial logic state $|\psi\rangle_0$ of Eq. (7) is written as

$$|\psi\rangle_0 = \frac{1}{\sqrt{2}}\left[\left|\psi^{\text{Bell}}_{\text{fragile}}\right\rangle + \cos\theta\left|\psi^{\text{Bell}}_{\text{robust}}\right\rangle + i\sin\theta\left|\psi^{\text{Bell}}_{3}\right\rangle\right), \tag{9}$$

where

$$\left|\psi^{\text{Bell}}_{3}\right\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}. \tag{10}$$

In spite of the fact that $|\psi\rangle_0$ is rewritten now in a different way, it is implicitly understood that this state is suitable for initialization. What Eq. (9) indicates is that the natural random shift of Eq. (5), acquired in a natural way by a single qubit state, induces a consistent initialization in terms of Bell entangled states. In order to test the consistency of our approach, we shall give two different applications of Eqs. (7) and (9) to the theory of quantum error detection and correction codes.

II. $|\psi\rangle_0$ AS AN ERROR SYNDROME CAUSED BY DECOHERENCE3

One application of the present formalism is to the two-qubit simple error correction code which we present to continuation. For this purpose we shall assume that the bit flip error occurs only in the dual qubit $|q*\rangle$ with probability $\mathcal{P}$. That is, the error is detectable by the code as $(\sigma_x^{(1)}|q*\rangle) \times |q\rangle$ where $\sigma_x^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the Pauli operator. We also neglect the probability of error on two qubits. Furthermore, we will assume that $|\psi\rangle_0$ is the syndrome itself. Thus, according with this approach the logic qubit $|0\rangle$ will be protected when the initial state of Eq. (7) is transmitted. Under bit flip on the first qubit, the resulting code is composed by the states $\{|\psi\rangle_0, |\psi\rangle_1\}$, where $|\psi\rangle_0$ is given by Eq. (7) and

$$|\psi\rangle_1 = \frac{1}{\sqrt{2}}\left[\left|\psi^{\text{Bell}}_{\text{robust}}\right\rangle + \cos\theta\left|\psi^{\text{Bell}}_{\text{fragile}}\right\rangle - i\sin\theta\left|\psi^{\text{Bell}}_{3}\right\rangle\right] \tag{11}$$

The quantum correction code is contained in the new set of probabilities received after transmission of each individual states $\left|\psi^{\text{Bell}}_{\text{fragile}}\right\rangle, \left|\psi^{\text{Bell}}_{\text{robust}}\right\rangle$ and $\left|\psi^{\text{Bell}}_{3}\right\rangle$. Before transmission they have probabilities $1/2, cos^2\,\theta/2$ and $sin^2\,\theta/2$ respectively. After transmission they have instead probabilities $cos^2\,\theta/2, 1/2$, and $sin^2\,\theta/2$.

On the other hand the respective square magnitude of the fidelity is

$$|F|^2 = |_1\langle\psi|\psi\rangle_0|^2 = \frac{1}{4}(sin^2\theta + 2\cos\theta)^2\,. \tag{12}$$

In the Fig. 1 it is sketched $|\mathrm{F}|^2$ as a function of θ. From this figure it is concluded that the fidelity is maximal and equal to one when $\theta = 0, \pi, 2\pi$. From Eq. (7) this means that the term proportional to $\left|\psi^{\text{Bell}}_{3}\right\rangle$ vanishes. Therefore, the fidelity should be maximal only when the state $\{|\psi\rangle_0$ is a mixture of the states $\left|\psi^{\text{Bell}}_{\text{fragile}}\right\rangle$ and $\left|\psi^{\text{Bell}}_{\text{robust}}\right\rangle$ exclusively.

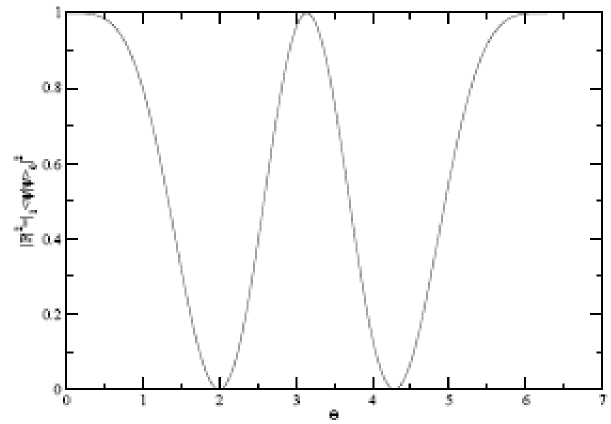

FIGURE 1: $|\boldsymbol{F}|^2$ of Eq. (12) as a function of $\boldsymbol{\theta}$.

III. PROTECTING QUANTUM INFORMATION INDEFINITELY AGAINST ERRORS THROUGH THE USE OF TWO SPIN $-1/2$ PARTICLES

We are encoding information in such a way that it is protected from errors indefinitely, without intervention. An information-carrying subsystem of qubits with this property is called "noiseless". In order to do the above we assume that an uniform external magnetic field B is acting along the x-direction on a system composed by two spin $-1/2$ particles. This situation arises, for example, in Nuclear Magnetic Resonance with spin $-1/2$ nuclei [10]. We will also assume that the corresponding noiseless subsystem is composed by one qubit. The respective subsystem identification involves a four-dimensional subspace which is defined by the following identification

$$\left|\psi^{\text{Bell}}_{\text{fragile}}\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \leftrightarrow |q\rangle|0\rangle,$$

$$\left|\psi^{\text{Bell}}_{\text{robust}}\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \leftrightarrow |q'\rangle|0\rangle,$$

$$\left|\psi^{\text{Bell}}_{3}\right\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) \leftrightarrow |q'\rangle|1\rangle,$$

and

$$\left|\psi_4\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \leftrightarrow |q\rangle|1\rangle, \tag{13}$$

where q and q' are two orthonormal projections of the first qubit ($\langle q|q'\rangle = \delta_{q,q'}$). As we will see lines below they define the syndrome of this quantum error correction code.

The uniform external field B acting along x-axis causes the spin to rotate according to an evolution of the form

$$\left|\psi_t\right\rangle = \mathrm{e}^{-iB\sigma_x t/2}|\psi\rangle. \tag{14}$$

Due that $\sigma_x|0\rangle = |1\rangle$ and $\sigma_x|1\rangle = |0\rangle$, this field induces bit flip errors on the qubit.

Now consider the physical system defined by two spin $-{}^1/_2$ particles with bit flip errors acting identically on each particle. The evolution caused by the uniform field is given by

$$|\psi_t\rangle_{12} = e^{-iB\sigma_x^{(1)}t/2}e^{-iB\sigma_x^{(2)}t/2}|\psi_t\rangle_{12}\ ,$$

$$= e^{-iBJ_xt/2}\,|\psi_t\rangle_{12}, \qquad (15)$$

where

$$J_x = \sigma_x^{(1)} + \sigma_x^{(2)}.$$

It is straightforward to check that the evolution operator $e^{-iBJ_xt/2}$, flips the orientation of the syndrome spin leaving untouched the logic qubit. Indeed, if we use Eqs. (13) and (14) it follows that

$$\frac{1}{2}J_x|q\rangle|0\rangle = \frac{1}{2}\left(\sigma_x^{(1)} + \sigma_x^{(2)}\right)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= \frac{1}{2\sqrt{2}}(|10\rangle + |01\rangle + |01\rangle + |10\rangle)$$

$$= |q'\rangle|0\rangle. \qquad (16)$$

likewise

$$\tfrac{1}{2}J_x|q'\rangle|0\rangle = |q\rangle|0\rangle, \qquad (17)$$

$$\tfrac{1}{2}J_x|q'\rangle|1\rangle = 0, \qquad (18)$$

and

$$\tfrac{1}{2}J_x|q\rangle|1\rangle = 0, \qquad (19)$$

From Eqs. (16)-(19) we obtain

$$|q\rangle|0\rangle_t = \cos(Bt)|q\rangle|0\rangle + \sin(Bt)|q'\rangle|0\rangle,$$

$$|q'\rangle|0\rangle_t = \cos(Bt)|q'\rangle|0\rangle + \sin(Bt)|q\rangle|0\rangle,$$

$$|q'\rangle|1\rangle_t = |q'\rangle|1\rangle,$$

$$|q\rangle|1\rangle_t = |q\rangle|1\rangle. \qquad (20)$$

From the above equation it is concluded that the errors coming from the external magnetic field act as the identity on the information carrying system. The time evolution operator rotates the syndrome leaving untouched the logic qubit. Thus, the subsystem is indefinitely noiseless.

IV. CONCLUSIONS

We have built a consistent initialization protocol state based in three maximally entangled Bell states. This is derived from the direct product from a qubit state and its dual counterpart. The present procedure relies on the assumption that the qubits do not relax from an isolated state and that instead they acquire a random phase from the very preparation. With our approach an efficient non-classical solution for secure communications is assured. This is exemplified by building two different quantum correction codes. The first example is a simple code while the second is a non trivial one. The second example of a quantum correction code given here has to do with environmental noise, which is due to the incomplete isolation of the system from the rest of the world. It was proved that these errors are controlled due to reliably measure the quantum system and the good choice for the initial state.

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