A Pseudo Ordinary Differential Equation for the Hysteretic Damper

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Abstract
Damping plays an important role in science and engineering. A mathematically simple, but physically realizable hysteretic damping model is still pending. In this letter, starting from the transfer function, a pseudo ordinary differential equation was derived by augmenting the differential order. This is a standard linear ODE, involving only differential terms, but integrals.

Keywords: Damping model; Hysteretic damping; Ordinary differential equation.

Resumen
La amortiguación juega un papel importante en la ciencia y en la ingeniería. Aún está pendiente un modelo de amortiguamiento histerético matemáticamente simple, pero físicamente realizable. En esta carta, a partir de la función de transferencia, derivamos una pseudo ecuación diferencial ordinaria al aumentar el orden diferencial. Se trata de una EDO lineal estándar, sólo con términos diferenciales, pero integrales.

Palabras clave: Modelo amortiguado, amortiguación histerética, ecuaciones diferenciales ordinarias.

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I. INTRODUCTION

Coulomb damping is introduced in a physical textbook as the simplest friction model, but it is nonlinear. In contrast, an engineering textbook prefers to a linear model. A linear viscous damping model is the simplest mathematically. To account for the frequency dependent property of a realistic energy dissipating mechanism, a general-purposed damping model assumes that the energy loss per cycle varies versus the vibration frequency [1].

Experiments have shown that the simplest form, a frequency independent model, could cover the damping property of many materials. This frequency independent model, or "rate-independent" damping model, has alternative names such as linear hysteretic damping, structural damping, material damping, complex stiffness, and internal damping. While the rate-independent damping model looks simple in the frequency domain, it has an unusual characteristic in the time domain which has puzzled scientists for a long time [2, 3, 4, 5, 6, 7]. The characteristic in question is the model has a non-causal response to the impulse before the impulse is applied to the system. Another issue is the equivalent ordinary differential equation (ODE), which has also fascinated scientists for a long time. The current consensus is this has been solved by using integro-differential equations [8].

A pseudo ODE was derived by augmenting the differential order in this letter. This is a standard linear ODE, involving only differential terms, but integrals.

II. ODE FOR HYSTERETIC DAMPER

A single-degree-of-freedom (SDOF) vibration with the linear hysteretic damper has a frequency response function as

$$H(j\omega) = \frac{1}{m(j\omega)^2 + k(1 + j\eta \operatorname{sign}\omega)}, \tag{1}$$

where m and k are the system mass and stiffness, respectively. $\eta>0$ is the loss-factor.

Assume that the excitation and response are $f(t)$ and $x(t)$, and their Fourier transform are $F(j\omega)$ and $X(j\omega)$, respectively. In light of the linear system theory,

$$X(j\omega) = H(j\omega)F(j\omega) = \frac{F(j\omega)}{m(j\omega)^2 + k(1 + j\eta \operatorname{sign}\omega)}, \tag{2}$$

that is

$$m(j\omega)^2 X(j\omega) + kX(j\omega) + j\eta \text{ sig } \omega X(j\omega) = F(j\omega) \,. \quad (3)$$

The Hilbert transform H [$x(t)$] is defined as

$$\hat{x}(t) = \mathcal{H}[x(t)] = \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau \,, \quad (4)$$

and in the frequency domain, Eq. (4) is equivalent to

$$\hat{X}(\omega) = j\text{sign}(\omega) X(\omega) \,, \quad (5)$$

here $\hat{X}(\omega)$ is the Fourier transform of $\hat{x}(t)$. In light of Eqs. (4) and (5), and the differential property, the time domain equivalent of Eq. (3) is

$$mx'' + k\eta\hat{x} + kx = f(t) \,. \quad (6)$$

Eq. (6) contains an integral, and is not a standard ODE. According to the Hilbert transform definition, we have

$$\hat{\hat{x}}(t) = \mathcal{H}[\mathcal{H}[x(t)]] = -x(t) \,, \quad (7)$$

$$\mathcal{H}[\frac{d\,x(t)}{dt}] = \frac{d\mathcal{H}[x(t)]}{dt} \,. \quad (8)$$

Eq. (7) is easily comprehended from Eq. (5). Eq. (8) is due to the linear property of the differential operation and Hilbert transform. $\hat{x}(t)$ can be solved from Eq. (6) as follows

$$\hat{x} = \frac{f(t) - mx'' - kx}{k\eta} \,. \quad (9)$$

Applying the Hilbert transform to both sides of Eq. (6) leads to (combining with Eq. (7) and Eq. (8))

$$m\hat{x}'' - k\eta x + k\hat{x} = \hat{f}(t)\sqrt{b^2 - 4ac} \,. \quad (10)$$

Thus, we have

$$\hat{x}'' = \frac{\hat{f}(t) + k\eta x - k\hat{x}}{m} \,. \quad (11)$$

Substituting Eq. (9) into Eq. (11) leads to

$$\hat{x}'' = \frac{\eta \hat{f}(t) - f(t) + k(\eta^2 + 1)x + mx''}{m\eta} \,. \quad (12)$$

Applying differential operations twice upon Eq. (6) leads to

$$mx^{(4)} + k\eta\hat{x}'' + kx'' = f''(t) \quad (13)$$

Substituting Eq. (12) into Eq. (13) yields

$$m^2 x^{(4)} + 2kmx'' + k^2(\eta^2 + 1)x = mf''(t) + k[f(t) - \eta\hat{f}(t)] \,. \quad (14)$$

That is a standard ODE with an augmented order.

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