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BRIEF ATLAS OF OFFSET CURVES

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Dedicado a la memoria de Mirian Andrés Gómez

Resumen. En este artículo se recuerda algunas de las definiciones básicas y las principales propiedades de las *offsets* de curvas algebraicas planas (véase [23]), así como un algoritmo para parametrizar las componentes de género cero de la curva *offset* (véase [4]). Además, se presenta un breve atlas en el que se obtiene la *offset* de varias curvas algebraicas planas, se analiza su racionalidad y se dan parametrizaciones.

Abstract. In this paper, we recall some of the basic definitions and main properties on offsets to algebraic plane curves (see [23]), as well as an algorithm for parametrizing the genus zero components of an offset (see [4]). In addition, we present a brief atlas where the offset of several algebraic plane curves are obtained, its rationality analyzed, and parametrizations are provided.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero (say $\mathbb{K} = \mathbb{C}$), and $\mathcal{C}$ an irreducible curve in $\mathbb{K}^2$; in general, this theory can be developed for irreducible hypersurfaces in $\mathbb{K}^n$, see e.g. [4], [23]. The offset curve (or parallel curve) to $\mathcal{C}$ at distance d is *essentially* the envelope of the system of spheres centered at the points of $\mathcal{C}$ with fixed radius d (see Fig. 1 and, for a formal definition, see Section 2). In particular, if $\mathcal{C}$ is parametrized by $\mathcal{P}(t) \in \mathbb{K}(t)^2$, the offset to $\mathcal{C}$ corresponds to the Zariski closure of the set in $\mathbb{K}^2$ generated by the formula

$$\mathcal{P}(t) \pm d \, \frac{\mathcal{N}(t)}{\|\mathcal{N}(t)\|}$$

where $\mathcal{N}(t)$ is the normal vector to $\mathcal{C}$ associated with $\mathcal{P}(t)$. For instance, if $\mathcal{C}$ is the parabola of equation $y_2 = y_1^2$, that can be parametrized as (t, t^2), then the offset at distance d is the Zariski closure of

$$\left\{ (t, t^2) \pm \frac{d}{\sqrt{1+4t^2}}(-2t, 1) \;\middle|\; t \in \mathbb{C} \setminus \left\{ \pm \frac{\sqrt{-1}}{2} \right\} \right\}.$$

The term "parallel" was apparently introduced by Leibniz in [13] for the case of plane curves. Also, in elementary texts on differential geometry (see [6]) or in some books on algebraic geometry (see for instance [7], [8], [19]) some elementary

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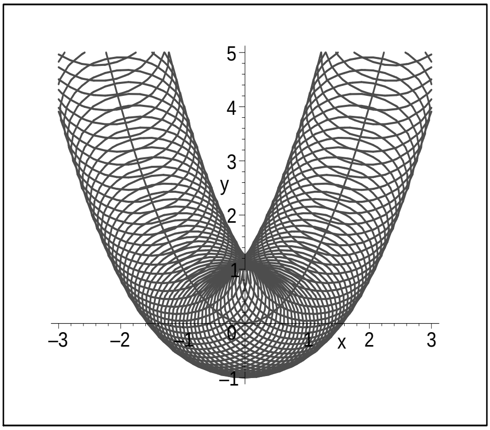

FIGURE 1. Construction of the offset of the parabola.

aspects of parallel curves are studied. Nevertheless, in the 1980s, CAGD (Computer Aided Geometric Design) community started to be interested on the topic, and they began to address problems related to offsets to curves and surfaces, due to the important role that offsets play in practical applications as tolerance analysis, geometric control, robot path-planning and numerical-control machining problems, etc. [11], [12]. As a consequence of this applicability, many interesting questions directly related to algebraic geometry have been addressed (see, e.g. [1], [2], [3], [4], [5], [9], [10], [14], [15], [16], [17], [18], [21], [22], [23], [24]) and, currently, the study of offsets continue being an active research area.

In this paper, we recall some of the basic definitions and main properties on offsets to algebraic plane curves, and we present a brief atlas where the offset of several algebraic plane curves are obtained, and its rationality analyzed. Furthermore, in case of offset genus zero, a rational parametrization is computed. The examples presented in this paper have been executed with the computer algebra system Maple, and with the package for constructive algebraic geometry CASA.

2. Basic Notions

In $\mathbb{K}^2$ we consider the symmetric bilinear form $B((x_1,x_2),(y_1,y_2)) = x_1y_1 + x_2y_2$, which induces a metric vector space with light cone $\mathfrak{L}$ of isotropy given as (see [20])

$$\mathfrak{L} = \{(x_1,x_2) \in \mathbb{K}^2 \mid x_1^2 + x_2^2 = 0\}.$$

In this context, the circle of center $(a_1,a_2) \in \mathbb{K}^2$ and radius $d \in \mathbb{K}$ is the plane curve defined by $(x_1 - a_1)^2 + (x_2 - a_2)^2 = d^2$. We will say that the distance between the points $\bar{x}$, $\bar{y} \in \mathbb{K}^2$ is $d \in \mathbb{K}$ if $\bar{y}$ is on the circle of center $\bar{x}$ and radius d. Notice that the "distance" is hence defined up to multiplication by ± 1. On the other hand, if $\bar{x} \notin \mathfrak{L}$ we denote by $\|\bar{x}\|$ any of the elements in $\mathbb{K}$ such that $\|\bar{x}\|^2 = B(\bar{x},\bar{x})$, and if $\bar{x} \in \mathfrak{L}$, then $\|\bar{x}\| = 0$. We usually work with both solutions of $\|\bar{x}\|^2 = B(\bar{x},\bar{x})$. For this reason we use the notation $\pm\|\bar{x}\|$.

In this situation, let $\mathcal{C}$ be the affine irreducible plane curve defined by the irreducible polynomial $f(\bar{y}) \in \mathbb{K}[\bar{y}]$, $\bar{y} = (y_1, y_2)$, let $d \in \mathbb{K}^*$ be a non-zero field element, $\bar{x} = (x_1, x_2)$, and $f_i = \frac{\partial f}{\partial y_i}$. We assume that $\mathcal{C}$ is none of the two lines defining $\mathfrak{L}$.

In order to get a formal definition of the offset, one introduces the following incidence diagram

$$\begin{array}{ccc} & \mathfrak{B}(\mathcal{C}, d) \subset \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \pi_1(\mathfrak{B}(\mathcal{C}, d)) \subset \mathbb{K}^2 & & \mathcal{C} \subset \mathbb{K}^2 \end{array} \qquad \text{(Incidence Diagram)}$$

where the offset incidence variety is

$$\mathfrak{B}(\mathcal{C}, d) = \left\{ (\bar{x}, \bar{y}, \lambda) \in \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \Bigg/ \begin{array}{l} f(\bar{y}) = 0 \\ \bar{x} = \bar{y} + \lambda(f_1(\bar{y}), f_2(\bar{y})) \\ (x_1 - y_1)^2 + (x_2 - y_2)^2 = d^2 \end{array} \right\}$$

and

$$\begin{array}{llll} \pi_1 : & \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} & \longrightarrow \mathbb{K}^2, \\ & (\bar{x}, \bar{y}, \lambda) & \longmapsto \bar{x} \end{array} \qquad \begin{array}{lll} \pi_2 : & \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} & \longrightarrow \mathbb{K}^2 \\ & (\bar{x}, \bar{y}, \lambda) & \longmapsto \bar{y}. \end{array}$$

Then, we define the **offset of $\mathcal{C}$ at distance** d as the algebraic Zariski closure in $\mathbb{K}^2$ of $\pi_1(\mathfrak{B}(\mathcal{C}, d))$, and we denote it by $\mathcal{O}_d(\mathcal{C})$; i.e.

$$\mathcal{O}_d(\mathcal{C}) = \overline{\pi_1(\mathfrak{B}(\mathcal{C}, d))}.$$

In the above definition, we have considered only irreducible curves. Note that the same reasoning can be done for reducible curves, introducing the offset as the union of the offset of the irreducible components.

Since we have assumed that $\mathcal{C}$ is none of the lines of isotropy, then $\mathcal{O}_d(\mathcal{C})$ is never empty. Furthermore, with the exception of $\mathcal{C}$ being a circle and d its radius, $\mathcal{O}_d(\mathcal{C})$ has at most two components, all of the them of dimension 1 (see [23]); i.e. with the exception of that particular circle, each component of $\mathcal{O}_d(\mathcal{C})$ is an algebraic curve.

Another important property of offsets relates the normal vectors to $\mathcal{C}$ and to its offsets (see [23]). More precisely: let $P \in \mathcal{C}$, and let $Q \in \mathcal{O}_d(\mathcal{C})$ be any of the two points on the offset generated by P, then it holds that the normal vectors to $\mathcal{C}$ at P and the normal vectors to $\mathcal{O}_d(\mathcal{C})$ at Q are parallel.

In order to compute the offset, one can proceed as follows: Let I be ideal in $\mathbb{K}[\bar{x}, \bar{y}, \lambda]$ generated by the polynomials defining $\mathfrak{B}(\mathcal{C}, d)$. Then, by the Closure Theorem (see [8] p. 122), one has that $\mathcal{O}_d(\mathcal{C}) = V(I \cap \mathbb{K}[\bar{x}])$. Hence elimination theory techniques, such as Gröbner bases, provide the offset.

Reasoning as in Section 2 in [22], one may introduce the notion of generic offset. Let us consider d as a new variable. Now, $\mathfrak{B}(\mathcal{C}, d)$ is seen as an algebraic set in

$\mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \times \mathbb{K}$; we denote it by $\mathfrak{B}(\mathcal{C}, d)_G$. Then, the **generic offset** is defined as

$$\mathcal{O}_d(\mathcal{C})_G = \overline{\pi_1(\mathfrak{B}(\mathcal{C}, d)_G)}.$$

Now, if I_G is the ideal in $\mathbb{K}[x, y, \lambda, d]$ generated by the polynomials defining $\mathfrak{B}(\mathcal{C}, d)_G$, by the Closure Theorem (see [8] p. 122), one has that $\mathcal{O}_d(\mathcal{C})_G = V(I_G \cap \mathbb{K}[x, d])$. Moreover, reasoning as in Theorem 6 in [22] which is a direct consequence of Exercise 7, p. 283 in [8], one gets that for almost all values of $d \in \mathbb{K}^*$ the generic offset specializes properly.

3. Parametrizing Offsets

In this Section we summarize the results on the rationality of the offsets to curves, presented in [4], by deriving an algorithm for parametrizing offsets.

The rationality of the components of the offsets is characterized by means of the existence of parametrizations of the curve whose normal vector has rational norm, and by means of the rationality of the components of an associated curve, that is usually simpler than the offset, as shown in the examples. As a consequence, one deduces that offsets to rational curves behave as follows: they are either reducible with two rational components (double rationality), or rational, or irreducible and not rational.

For this purpose, we first need to introduce two new concepts: rational Pythagorean hodographs and the curve of reparametrization. Let

$$\mathcal{P}(t) = (P_1(t), P_2(t)) \in \mathbb{K}(\bar{t})^2$$

be a rational parametrization of $\mathcal{C}$. Then, $\mathcal{P}(t)$ is RPH (**Rational Pythagorean Hodograph**) if its normal vector $\mathcal{N}(t) = (N_1(t), N_2(t))$ satisfies that

$$N_1(t)^2 + N_2(t)^2 = m(t)^2,$$

with $m(t) \in \mathbb{K}(t)$. For short we will express this fact writing $\|\mathcal{N}(t)\| \in \mathbb{K}(t)$. On the other hand, we define the **reparametrizing curve of $\mathcal{O}_d(\mathcal{C})$ associated with $\mathcal{P}(t)$** as the curve generated by the primitive part with respect to x_2 of the numerator of

$$x_2^2\, P_1^{'}(x_1) - P_1^{'}(x_1) + 2\, x_2\, P_2^{'}(x_1),$$

where $P_i^{'}$ denotes the derivative of P_i. In the following, we denote by $\mathcal{G}_{\mathcal{P}}(\mathcal{C})$ the reparametrizing hypersurface of $\mathcal{O}_d(\mathcal{C})$ associated with $\mathcal{P}(t)$.

Summarizing the results in [4], one can outline the following algorithm for offsets.

Algorithm: offset parametrization

- Given: a proper rational parametrization $\mathcal{P}(t)$ of the plane curve $\mathcal{C}$ in $\mathbb{K}^2$.
- Decide: whether the components of $\mathcal{O}_d(\mathcal{C})$ are rational.
- Determine: (in the affirmative case) a rational parametrization of each component of $\mathcal{O}_d(\mathcal{C})$.

1. [Normal vector computation] Compute the normal vector $\mathcal{N}(t)$ of $\mathcal{P}(\bar{t})$
2. [Checking RPH] If $||\mathcal{N}(t)|| \in \mathbb{K}(\bar{t})$ then return $\ll \mathcal{O}_d(\mathcal{C})$ has two rational components parametrized by $\mathcal{P}(t) \pm \frac{d}{||\mathcal{N}(t)||}\mathcal{N}(t) \gg$.

3. [Determination of the reparametrizing curve] Determine $\mathcal{G}_{\mathcal{P}}(\mathcal{C})$, and decide whether $\mathcal{G}_{\mathcal{P}}(\mathcal{C})$ is rational.
4. [Determination of the reparametrizing curve and parametrization] IF $\mathcal{G}_{\mathcal{P}}(\mathcal{C})$ is not rational THEN RETURN $\ll$ no component of $\mathcal{O}_d(\mathcal{C})$ is rational $\gg$ ELSE
   4.1. Determine a rational parametrization $\mathcal{R}(t) = (\tilde{R}(t), R(t))$ of $\mathcal{G}_{\mathcal{P}}(\mathcal{C})$.
   4.2. RETURN $\ll$ $\mathcal{O}_d(\mathcal{C})$ is a rational curve parametrized by

$$\mathcal{Q}(t) = \mathcal{P}(\tilde{R}(t)) + \frac{2\,d\,R(t)}{N_2(\tilde{R}(t))(R(t)^2+1)}\mathcal{N}(\tilde{R}(t)),$$

   where $\mathcal{N} = (N_1, N_2) \gg$.

4. Atlas of Offsets Curves

In this section we apply the previous algorithm to analyze the rationality of the offset curve of several classical rational curves, and in the case of rationality we compute a rational parametrization of the offset. In addition, we also compute the implicit equation of the corresponding generic offset. The rational curves included in the atlas are:

1. *The Circle* (Example 1),
2. *The Parabola* (Example 2),
3. *The Hyperbola* (Example 3),
4. *The Ellipse* (Example 4),
5. *The Cardioid* (Example 5),
6. *The Three-leafed Rose* (Example 6),
7. *The Trisectrix of Maclaurin* (Example 7),
8. *The Folium of Descartes* (Example 8),
9. *The Tacnode* (Example 9),
10. *The Conchoid of de Sluze* (Example 10),
11. *The Epitrochoid* (Example 11),
12. *The Ramphoid Cusp* (Example 12),
13. *The Lemniscata of Bernoulli* (Example 13),
14. *Cuspidal curves* (Example 14)

Example 1. (Offset of the Circle).
Let $\mathcal{C}$ be the circle defined by

$$y_1^2 + y_2^2 - r^2.$$

1. *Implicit equation: the implicit equation of the generic offset is*

$$(x_1^2 + x_2^2 - (d+r)^2)(x_1^2 + x_2^2 - (d-r)^2)$$

2. *Rationality character: Double rational (i.e. the offset has two rational components)*
3. *Offset parametrization: its components are parametrized by*

$$\left(\frac{(d \pm r)2t}{t^2+1}, \frac{(d \pm r)(t^2-1)}{t^2+1}\right)$$

4. <u>Remark:</u> *Note that if* $d = r$ *one component of the offset degenerates to a point, namely the center. Indeed, let* I *be the ideal generated by the polynomials defining* $\mathfrak{B}(\mathcal{C}, d)$*, with* $d = r$*. Then, computing a Gröbner basis, one sees that*

   $I \cap \mathbb{C}[\bar{x}] =< -x_2\left(-{x_2}^2 + 4\,r^2 - {x_1}^2\right), -x_1\left(-{x_2}^2 + 4\,r^2 - {x_1}^2\right) >,$

   and hence $\mathcal{O}_r(\mathcal{C})$ *is the circle of equation* $x_1^2 + x_2^2 = 4r^2$ *union the origin.*

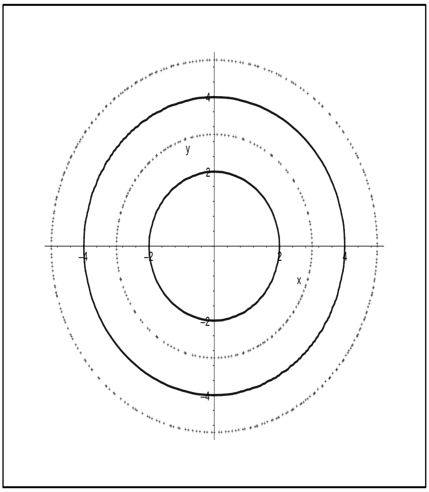

FIGURE 2. Offset of the unit circle at $d = 3$ and $d = 4$ (untraced curves).

Example 2. (Offset of the Parabola). *Let* $\mathcal{C}$ *be the parabola defined by*

$$y_2 - ay_1^2,\ a \neq 0.$$

1. <u>Implicit equation:</u> *The implicit equation of the generic offset is*
   $-d^2+{x_2}^2+8\,x_2d^2a-2\,x_2{x_1}^2a-8\,{x_2}^3a-8\,d^4a^2+{x_1}^4a^2+16\,{x_2}^4a^2-8\,d^2{x_2}^2a^2+32\,{x_1}^2{x_2}^2a^2-20\,{x_1}^2d^2a^2+8\,{x_1}^2x_2d^2a^3-32\,{x_1}^2d^2{x_2}^2a^4+16\,{x_1}^6a^4-32\,{x_2}^3a^3d^2-32\,{x_2}^3a^3{x_1}^2-40\,{x_1}^4x_2a^3+16\,{x_1}^4{x_2}^2a^4+48\,{x_1}^2d^4a^4-48\,{x_1}^4d^2a^4+32\,x_2d^4a^3+16\,d^4a^4{x_2}^2-16\,d^6a^4$
2. <u>Rationality character:</u> *rational*
3. <u>Offset parametrization:</u> *the generic offset can be parametrized as*

$$\left(\frac{\left(t^2-1\right)\left(-t^2-1+4\,dat\right)}{4at\left(t^2+1\right)}, \frac{t^6-t^4-t^2+1+32\,dt^3a}{16at^2\left(t^2+1\right)}\right)$$

4. <u>Details of the computation:</u>
   - $\mathcal{P}(t) = (t, at^2)$ *and* $\mathcal{N}(t) = (-2at, 1)$.
   - $G_{\mathcal{P}}$ *is defined by* $x_2^2 - 1 + 4x_2ax_1$.
   - $\mathcal{R} = (-\frac{t^2-1}{4at}, t)$.

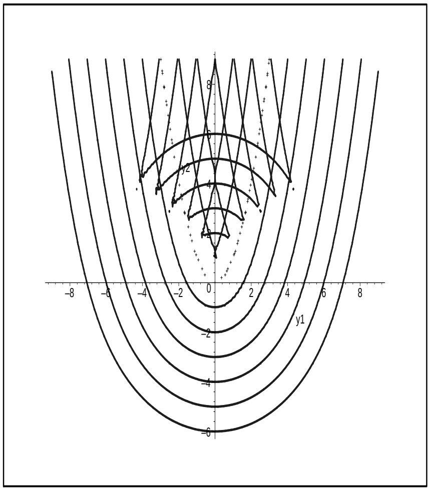

FIGURE 3. Parabola (untraced curve) and the offsets for several distances.

Example 3. (*Offset of the Hyperbola*).
Let $\mathcal{C}$ be the Hyperbola defined over $\mathbb{C}$ by the equation

$$\frac{{y_1}^2}{a^2} - \frac{{y_2}^2}{b^2} - 1, \quad a > 0,\ b > 0, a \neq b.$$

1. *Implicit equation: The implicit equation of the generic offset is*
$$(-2a^2b^6-2a^6b^2+a^8+b^8-6b^4a^4)d^4+(b^8+6b^4a^4+6a^2b^6){x_1}^4+(-6a^6b^2-4a^2b^6-8b^4a^4-2b^8)d^2{x_1}^2+(-2a^8-8b^4a^4-4a^6b^2-6a^2b^6){x_2}^2d^2+(-6b^4a^2-2a^4b^2-2a^6){x_2}^4{x_1}^2+(-4a^4-2a^2b^2)d^2{x_2}^6+(b^4+6a^4+6a^2b^2)d^4{x_2}^4+(-2a^2b^2-4b^4){x_1}^6d^2+(a^8+6a^6b^2+6b^4a^4){x_2}^4+(-2b^8a^2+2a^6b^4-2a^4b^6+2a^8b^2)d^2+(-2b^8a^2-4a^6b^4-6a^4b^6){x_1}^2+(6a^6b^4+4a^4b^6+2a^8b^2){x_2}^2+(a^4+b^4+2a^2b^2)d^8+(2b^4a^2-2a^4b^2+2b^6-2a^6)d^6+(-2b^6-4b^4a^2){x_1}^6+(-6b^4a^2+4a^6+6a^4b^2-4b^6){x_1}^2d^2{x_2}^2+(-2a^2b^2-2b^4-6a^4)d^2{x_1}^2{x_2}^4+(-10a^4b^2-6b^4a^2-6a^6){x_2}^4d^2+(2a^6+4a^4b^2){x_2}^6+b^8a^4+2b^6a^6+a^8b^4+(10a^2b^2+6a^4+6b^4){x_1}^2d^4{x_2}^2+(-6b^4-2a^4-2a^2b^2){x_1}^4d^2{x_2}^2+(-4a^2b^2+a^4+b^4){x_1}^4{x_2}^4+b^4{x_1}^8+a^4{x_2}^8+(-6a^2b^2-2b^4-4a^4){x_2}^2d^6+(2b^4-2a^2b^2){x_1}^6{x_2}^2+(-6a^2b^2-2a^4-4b^4)d^6{x_1}^2+(6a^6+2b^6+8a^4b^2+4b^4a^2){x_2}^2d^4+(-6a^6b^2-6a^2b^6-10b^4a^4){x_1}^2{x_2}^2+(2a^4-2a^2b^2){x_1}^2{x_2}^6+(a^4+6b^4+6a^2b^2)d^4{x_1}^4+(-6b^6-4a^4b^2-2a^6-8b^4a^2){x_1}^2d^4+(6b^6+6a^4b^2+10b^4a^2){x_1}^4d^2+(2b^4a^2+6a^4b^2+2b^6){x_1}^4{x_2}^2$$

2. *Rationality character: irreducible and non rational*

3. <u>*Details of the computation:*</u>
   - $\mathcal{P}(t) = (\frac{a\left(a^2+t^2b^2\right)}{-t^2b^2+a^2}, -2\,\frac{ab^2t}{-t^2b^2+a^2})$ *and* $\mathcal{N}(t) = (\frac{2b^2a(a^2+t^2b^2)}{(-t^2b^2+a^2)^2}, \frac{4b^2a^3t}{(-t^2b^2+a^2)^2})$.
   - $G_{\mathcal{P}}$ *is defined by* $x_2^2a^2x_1 - x_1a^2 - a^2x_2 - b^2x_1^2x_2$.

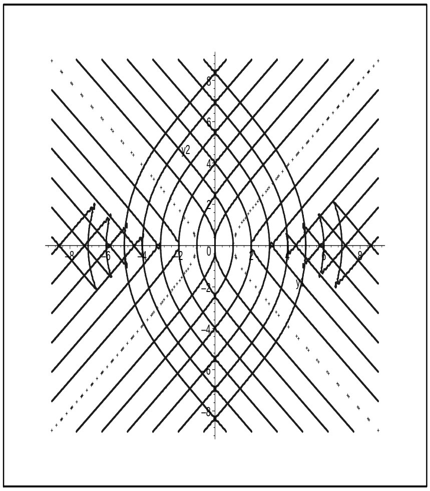

FIGURE 4. Hyperbola (untraced curve) and the Offsets of the Hyperbola for several distances.

Example 4. (Offset of the Ellipse). *Let $\mathcal{C}$ be the ellipse defined by*

$$\frac{{y_1}^2}{a^2} + \frac{{y_2}^2}{b^2} - 1, \quad a > 0,\ b > 0, a \neq b.$$

1. <u>*Implicit equation:*</u> *The implicit equation of the generic offset is*

$$\begin{aligned}
&(4\,a^2\,b^2 + a^4 + b^4)\,{x_1}^4\,{x_2}^4 + (a^4 - 6\,a^2\,b^2 + 6\,b^4)\,{x_1}^4\,d^4 + (10\,b^4\,a^2 - 6\,a^4\,b^2 - \\
&6\,b^6)\,{x_1}^4\,d^2 + (-2\,b^8\,a^2 + 6\,b^6\,a^4 - 4\,a^6\,b^4)\,{x_1}^2 + (-2\,a^8\,b^2 + 6\,a^6\,b^4 - 4\,b^6\,a^4)\,{x_2}^2 + \\
&(-2\,b^8\,a^2 - 2\,a^8\,b^2 + 2\,b^6\,a^4 + 2\,a^6\,b^4)\,d^2 + (2\,b^4\,a^2 - 2\,b^6 - 6\,a^4\,b^2)\,{x_1}^4\,{x_2}^2 + \\
&(2\,a^6\,b^2 - 6\,b^4\,a^4 + 2\,a^2\,b^6 + b^8 + a^8)\,d^4 + (-2\,a^2\,b^2 + b^4 + a^4)\,d^8 + (-2\,b^6 + 6\,a^6 - \\
&8\,a^4\,b^2 + 4\,b^4\,a^2)\,{x_2}^2\,d^4 + (-2\,a^6 - 8\,b^4\,a^2 + 6\,b^6 + 4\,a^4\,b^2)\,{x_1}^2\,d^4 + (-10\,b^4\,a^4 + \\
&6\,a^2\,b^6 + 6\,a^6\,b^2)\,{x_1}^2\,{x_2}^2 + a^8\,b^4 - 2\,a^6\,b^6 + b^8\,a^4 + (2\,a^4 + 2\,a^2\,b^2)\,{x_1}^2\,{x_2}^6 + (6\,a^4 + \\
&b^4 - 6\,a^2\,b^2)\,d^4\,{x_2}^4 + (-2\,a^4 + 6\,a^2\,b^2 - 4\,b^4)\,{x_1}^2\,d^6 + (6\,a^2\,b^2 - 2\,b^4 - 4\,a^4)\,{x_2}^2\,d^6 + \\
&(-10\,a^2\,b^2 + 6\,b^4 + 6\,a^4)\,{x_2}^2\,{x_1}^2\,d^4 + (6\,b^4\,a^4 + b^8 - 6\,a^2\,b^6)\,{x_1}^4 + (6\,b^4\,a^4 + \\
&a^8 - 6\,a^6\,b^2)\,{x_2}^4 + (-4\,a^4\,b^2 + 2\,a^6)\,{x_2}^6 + (-6\,a^4 + 2\,a^2\,b^2 - 2\,b^4)\,{x_1}^2\,{x_2}^4\,d^2 + \\
&(2\,a^2\,b^2 + 2\,b^4)\,{x_1}^6\,{x_2}^2 + (-6\,b^4\,a^2 + 10\,a^4\,b^2 - 6\,a^6)\,d^2\,{x_2}^4 + (2\,a^2\,b^2 - 4\,a^4)\,d^2\,{x_2}^6 + \\
&(-2\,a^8 + 4\,a^6\,b^2 + 6\,a^2\,b^6 - 8\,b^4\,a^4)\,{x_2}^2\,d^2 + (2\,a^2\,b^2 - 6\,b^4 - 2\,a^4)\,{x_1}^4\,{x_2}^2\,d^2 + \\
&b^4\,{x_1}^8 + a^4\,{x_2}^8 + (-4\,b^4\,a^2 + 2\,b^6)\,{x_1}^6 + (2\,b^4\,a^2 - 2\,b^6 + 2\,a^4\,b^2 - 2\,a^6)\,d^6 + \\
&(-6\,a^4\,b^2 - 6\,b^4\,a^2 + 4\,b^6 + 4\,a^6)\,{x_1}^2\,{x_2}^2\,d^2 + (-2\,a^6 - 6\,b^4\,a^2 + 2\,a^4\,b^2)\,{x_1}^2\,{x_2}^4 + \\
&(6\,a^6\,b^2 + 4\,a^2\,b^6 - 8\,b^4\,a^4 - 2\,b^8)\,{x_1}^2\,d^2 + (-4\,b^4 + 2\,a^2\,b^2)\,{x_1}^6\,d^2
\end{aligned}$$

2. *Rationality character: irreducible and non rational*
3. *Details of the computation:*
   - $\mathcal{P}(t) = (\frac{a(t^2-1)}{t^2+1}, \frac{2\,bt}{t^2+1})$ *and* $\mathcal{N}(t) = (\frac{2b(-1+t^2)}{(1+t^2)^2}, \frac{-4at}{(1+t^2)^2})$.
   - $G_{\mathcal{P}}$ *is defined by* $-x_2^2 a x_1 + a x_1 + x_2 b - x_2 b x_1^2$.

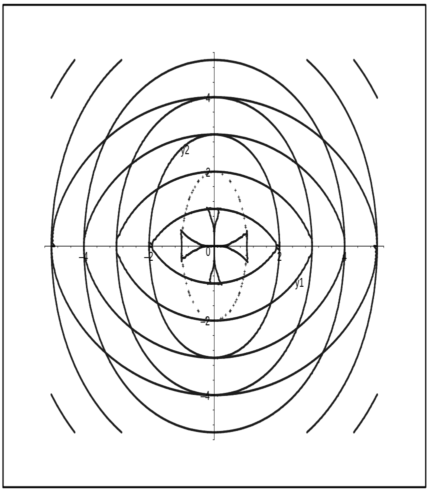

FIGURE 5. Ellipse (untraced curve) and the Offsets of the Ellipse for several distances.

Example 5. (Offset of the Cardioid). *Let $\mathcal{C}$ be the Cardioid defined by*

$$(y_1^2 + 4\,y_2 + y_2^2)^2 - 16\,(y_1^2 + y_2^2).$$

1. *Implicit equation: The implicit equation of the generic offset is* $24\,{x_2}^2\,{x_1}^2\,d^4 - 36\,{x_2}^2\,{x_1}^4\,d^2 - 36\,{x_2}^4\,{x_1}^2\,d^2 + 16\,{x_2}^6\,{x_1}^2 + 12\,{x_2}^4\,d^4 - 4\,d^6\,{x_1}^2 + 12\,d^4\,{x_1}^4 + 4\,{x_1}^8 + 24\,{x_2}^4\,{x_1}^4 + 16\,{x_2}^2\,{x_1}^6 - 12\,d^2\,{x_1}^6 + 192\,{x_1}^2\,{x_2}^5 + 64\,x_2\,{x_1}^6 - 144\,d^2\,{x_2}^5 - 128\,{x_1}^6 - 12\,{x_2}^6\,d^2 - 1024\,d^2\,{x_2}^3 - 1024\,{x_2}^3\,{x_1}^2 - 768\,d^2\,{x_2}^4 + 528\,d^4\,{x_2}^2 + 336\,d^4\,{x_1}^2 - 192\,d^2\,{x_1}^4 - 256\,x_2\,d^2\,{x_1}^2 - 1024\,x_2\,{x_1}^4 - 960\,d^2\,{x_2}^2\,{x_1}^2 + 192\,{x_1}^4\,{x_2}^3 - 4\,d^6\,{x_2}^2 - 16\,d^6\,x_2 - 16\,d^6 + 384\,{x_2}^4\,{x_1}^2 + 1024\,d^4 - 2048\,d^2\,{x_1}^2 + 1024\,{x_1}^4 + 1280\,x_2\,d^4 + 96\,d^4\,{x_2}^3 - 144\,d^2\,x_2\,{x_1}^4 + 96\,d^4\,x_2\,{x_1}^2 + 256\,{x_2}^6 + 4\,{x_2}^8 + 64\,{x_2}^7 - 288\,d^2\,{x_2}^3\,{x_1}^2$
2. *Rationality character: rational*
3. *Offset parametrization: the generic offset can be parametrized as*

$$\left(\frac{(-9+t^2)\,(d\,t^6 - 117\,d\,t^4 + 3456\,t^3 - 1053\,d\,t^2 + 729\,d)}{(243\,t^2 + 27\,t^4 + t^6 + 729)\,(t^2+9)},\right.$$

$$\left.-\frac{18\,(d\,t^6 - 16\,t^5 - 21\,d\,t^4 + 864\,t^3 - 189\,d\,t^2 - 1296\,t + 729\,d)\,t}{(243\,t^2 + 27\,t^4 + t^6 + 729)\,(t^2+9)}\right)$$

4. <u>*Details of the computation:*</u>
   - $\mathcal{P}(t) = (\frac{-1024\,t^3}{256\,t^4+32\,t^2+1}, \frac{-2048\,t^4+128\,t^2}{256\,t^4+32\,t^2+1})$ *and* $\mathcal{N}(t) = (\frac{256\,t\,(48\,t^2-1)}{(16\,t^2+1)\,(256\,t^4+32\,t^2+1)}, \frac{1024\,t^2\,(16\,t^2-3)}{(16\,t^2+1)\,(256\,t^4+32\,t^2+1)})$.
   - $G_{\mathcal{P}}$ *is defined by* $32\,x_2^2\,x_1^3 - 6\,x_2^2\,x_1 - 32\,x_1^3 + 6\,x_1 - 48\,x_2\,x_1^2 + x_2$.
   - $\mathcal{R} = (\frac{-3\,t}{2\,(-9+t^2)}, \frac{-t\,(-27+t^2)}{9\,(-3+t^2)})$.

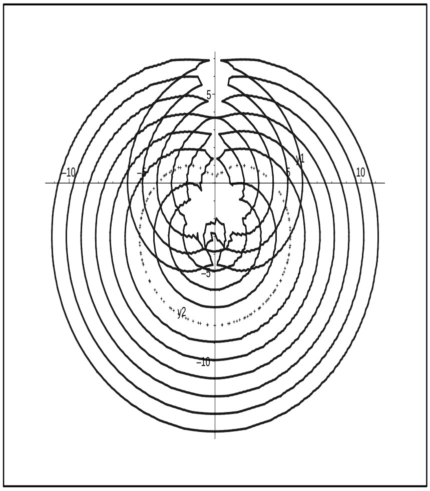

FIGURE 6. Cardioid (untraced curve) and the Offsets of the Cardioid for several distances.

Example 6. (Offset of the Three-leafed Rose). *Let $\mathcal{C}$ be the Three-leafed Rose defined by*

$$(y_1^2 + y_2^2)^2 + r\,y_1\,(3\,y_2^2 - y_1^2) = 0, \quad r \in \mathbb{C},\ r \neq 0.$$

1. <u>*Implicit equation:*</u> *The implicit equation of the generic offset for* $r = 1$ *is* $-63191384064x_1x_2^6d^2 + 32212254720d^4x_2^8x_1^2 + 106451435520x_1^4d^8 - 163980509184d^6x_2^2x_1^2 + 128043712512d^4x_1x_2^8 + 18345885696x_2^2d^4 - 48318382080x_1x_2^{10}d^2 + 23187161088x_1^4d^4 + 95806291968x_1^2d^8 + 106904420352x_2^6d^4x_1 - 17179869184x_1^6x_2^2d^6 - 17179869184x_1^2x_2^6d^6 + 3221225472x_1^2x_2^4d^8 + 3221225472x_1^4x_2^2d^8 + 94220845056x_1x_2^4d^8 + 62813896704x_1^3x_2^2d^8 + 67947724800x_2^2x_1d^8 + 3623878656x_1^{10} + 764411904x_1^6 + 764411904x_1^8 + 3057647616x_1^6x_2^2 + 4586471424x_1^4x_2^4 - 14495514624x_1^8x_2^2 -$

$7247757312x_1^6x_2^4+43486543872x_1^4x_2^6-3567255552x_1^6d^2-22649241600x_1^8d^2+$
$7644119040x_1^4x_2^2d^2 - 76101451776x_1^4d^2x_2^4 + 88785027072x_1^6d^2x_2^2 -$
$4586471424x_1^4x_2^2 - 6879707136x_1^4d^2 + 6879707136x_1^2x_2^4 + 3057647616x_1^2x_2^6 -$
$13759414272x_1^2x_2^2d^2 - 22932357120x_1^2x_2^4d^2 + 95806291968x_2^2d^8 -$
$137933881344d^6x_2^2x_1+764411904x_2^8+18345885696x_1^2d^4-57076088832x_1^2d^6+$
$64424509440d^4x_1^6x_2^4 + 32212254720d^4x_1^8x_2^2 - 1528823808d^2x_2^6 -$
$6879707136d^2x_2^4 - 21743271936x_1d^{10}x_2^2 + 46374322176x_1^2x_2^2d^4 -$
$156732751872x_1d^6x_2^4 - 104488501248x_1^3d^6x_2^2 + 23187161088d^4x_2^4 -$
$57076088832d^6x_2^2 + 256087425024x_1^5x_2^4d^4 - 52881784832x_1^5d^6x_2^2 -$
$177167400960x_1^3x_2^8d^2 + 10737418240x_1^6x_2^6 + 178174033920x_1^3x_2^4d^4 +$
$16106127360x_1^9d^2x_2^2 + 64424509440d^4x_1^4x_2^6 + 37580963840x_1^8x_2^6 +$
$10737418240x_1^9x_2^4 + 22548578304x_1^{10}x_2^4 - 15300820992d^2x_1^{10} +$
$7851737088d^2x_1^9 - 12079595520d^2x_2^{10} - 1811939328x_1^9 - 3221225472x_1^{10}x_2^2 +$
$21063794688x_1^7d^2 + 7644119040x_1^5d^2 - 190253629440x_2^6x_1^2d^2 -$
$2717908992x_2^8d^2 - 6115295232x_1^3d^4 - 81990254592d^6x_2^4 -$
$65229815808x_1^5d^4 + 26046627840d^4x_2^6 + 41901096960x_1^6d^4 -$
$81990254592x_1^4d^6 + 341449900032d^4x_1^3x_2^6 - 16986931200x_1^4x_2^2d^4 +$
$220830105600x_1^2x_2^4d^4 + 18345885696x_2^2x_1d^4 + 130459631616x_1^3x_2^2d^4 +$
$195689447424x_1x_2^4d^4 + 35634806784x_1^5d^4x_2^2 + 7516192768x_1^{12}x_2^2 +$
$22548578304x_1^4x_2^8 - 158645354496x_2^6d^6x_1 - 2147483648x_1^{1}3 +$
$1073741824x_1^{12} - 15288238080x_1^3x_2^2d^2 - 22932357120x_1x_2^4d^2 -$
$105318973440x_1^3x_2^4d^2 + 16307453952x_1^7x_2^2 - 48922361856x_1^5x_2^4 +$
$48922361856x_1^3x_2^6 + 32614907904x_2^8x_1^2 + 10871635968x_1^7x_2^4 +$
$25367150592x_1^5x_2^6+19931332608x_1^3x_2^8+5435817984x_1x_2^{10}-12230590464d^6-$
$1811939328x_1^{11} + 1073741824x_1^{1}4 - 4294967296x_1^8d^6 + 6442450944d^4x_1^{10} -$
$42681237504d^4x_1^9 + 6442450944d^4x_2^{10} - 225485783040x_1^5x_2^6d^2 +$
$1528823808x_2^2x_1^5 + 7644119040x_2^4x_1^3 + 4586471424x_2^6x_1 - 35634806784x_1^7d^4 -$
$22649241600x_1^3d^8 - 31406948352x_1^5d^8 + 1073741824d^8x_2^6 +$
$1073741824x_1^6d^8 - 4294967296x_2^8d^6 - 264408924160x_2^4x_1^3d^6 +$
$36691771392d^8 + 12230590464d^{1}2 + 52244250624x_1^5d^6 + 45977960448x_1^3d^6 +$
$7247757312x_1^3d^{10} + 52881784832x_1^7d^6 + 16106127360x_1^{11}d^2 +$
$212902871040x_1^2x_2^2d^8 - 47110422528x_1^5x_2^4d^2 + 328564998144x_1^4d^4x_2^4 +$
$196494753792x_1^6d^4x_2^2 - 25769803776x_2^4x_1^4d^6 - 1811939328x_1^9x_2^2 -$
$21063794688x_1^5d^2x_2^2-1528823808x_1^7-36691771392d^{10}-4294967296x_1^{11}x_2^2-$
$6442450944x_1^8x_2^4 - 25769803776d^2x_1^{10}x_2^2 - 85899345920d^2x_1^6x_2^6 -$
$64424509440d^2x_1^4x_2^8 + 241591910400x_2^6x_1^2d^4 + 56371445760x_1^8d^4 +$
$106451435520d^8x_2^4 - 57076088832d^{10}x_2^2 - 57076088832d^{10}x_1^2 +$
$53150220288x_2^8d^4-103750303744d^6x_1^6-4294967296d^2x_1^{12}+1073741824x_2^{1}4-$
$25769803776d^2x_1^2x_2^{10} - 301587234816d^6x_1^4x_2^2 - 317693362176d^6x_1^2x_2^4 -$
$102676561920d^6x_2^6 - 4294967296d^2x_2^{12} - 64424509440d^2x_1^8x_2^4 -$
$159450660864d^2x_1^4x_2^6 - 62813896704d^2x_1^3x_2^6 - 23555211264d^2x_1x_2^8 -$
$89389006848d^2x_2^8x_1^2 - 96636764160x_1^7d^2x_2^4 - 47513075712d^2x_1^8x_2^2 -$
$114353504256d^2x_1^6x_2^4 + 6442450944x_1x_2^{12} + 7516192768x_1^2x_2^{12} +$
$30064771072x_1^3x_2^{10} + 22548578304x_1^4x_2^{10} + 53687091200x_1^5x_2^8 +$
$37580963840x_1^6x_2^8 + 42949672960x_1^7x_2^6 + 9663676416x_1^2x_2^{10}$

2. *Rationality character: irreducible and non rational*
3. *Details of the computation:*
   - $\mathcal{P}(t) = (\frac{(1-3\,t^2)}{(1+t^2)^2}, \frac{(1-3\,t^2)\,t}{(1+t^2)^2})$ *and* $\mathcal{N}(t) = (\frac{-(-12\,t^2+3\,t^4+1)}{(1+t^2)^3}, \frac{2\,t\,(-5+3\,t^2)}{(1+t^2)^3})$.
   - $G_{\mathcal{P}}$ *is defined by* $-5\,x_2^2\,x_1 + 3\,x_2^2\,x_1^3 + 5\,x_1 - 3\,x_1^3 - 12\,x_2\,x_1^2 + 3\,x_2\,x_1^4 + x_2$.

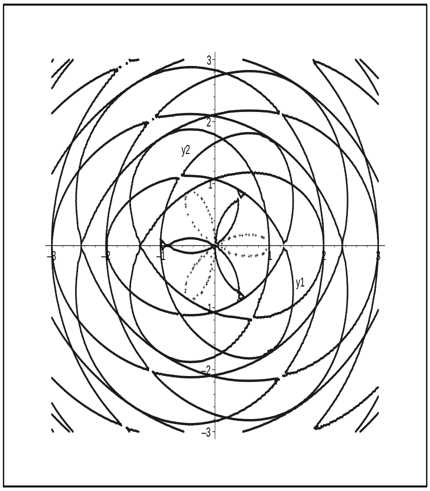

FIGURE 7. Three-leafed Rose (untraced curve) and the Offsets of the Three-leafed Rose for several distances.

Example 7. (Offset of the Trisectrix of Maclaurin). *Let $\mathcal{C}$ be the Trisectrix of Maclaurin defined by*

$$y_1\,(y_1^2 + y_2^2) - a\,(y_2^2 - 3\,y_1^2), \quad a \in \mathbb{C},\ a \neq 0.$$

1. *Implicit equation: The implicit equation of the generic offset for $a = 1$ is* $-8x_1x_2^6d^2+296d^4x_1^3-8d^2x_1^7-840x_2^2d^4-1296x_1^4+864x_1^2x_2^2-144x_2^4+210x_1^4d^4+4320x_1^3d^2-1728x_1^5+192x_1x_2^4-648x_1^6+47x_1^8+44x_1^6x_2^2-54x_1^4x_2^4-6x_1^6x_2^4-4x_1^4x_2^6-164x_1^6d^2-84x_1^4x_2^2d^2+18x_1^4d^2x_2^4-8x_1x_2^2d^6+5d^2x_1^8-1080x_1^4x_2^2+1440x_1^4d^2+296x_1^2x_2^4-52x_1^2x_2^6+2048x_1^2x_2^2d^2+132x_1^2x_2^4d^2+3456x_1^2d^2-40x_2^6-x_2^8-116x_1^2d^6+52d^2x_2^6+480d^2x_2^4+36x_1^2x_2^2d^4-78d^4x_2^4+4d^6x_2^2+16d^6x_1^2x_2^2-2304d^4-24x_1^5x_2^2d^2-216x_1x_2^2d^4+12d^4x_1^5-136d^6x_1+24d^4x_1^3x_2^2-8d^6x_1^3+2d^8x_1-184d^2x_1^5-2688d^4x_1+8x_2^6x_1^2d^2+x_2^8d^2-4d^4x_2^6-18x_1^2x_2^4d^4+12x_1x_2^4d^4-4d^8x_2^2-5d^8x_1^2+10d^6x_1^4+6d^6x_2^4+d^{10}+1632x_1d^2x_2^2-x_1^10+264x_1x_2^4d^2-24x_1^3x_2^4d^2+8x_1^3x_2^6+2x_1x_2^8-x_2^8x_1^2-760d^4x_1^2-10d^4x_1^6-376x_1^5x_2^2-312x_1^3x_2^4+88x_1x_2^6-32d^6+2x_1^9+1152x_2^2d^2+23d^8-4x_1^8x_2^2+24x_1^7+8x_1^7x_2^2+12x_1^5x_2^4-24d^4x_1^4x_2^2+16d^2x_1^6x_2^2+592x_1^3x_2^2d^2$

2. *Rationality character: irreducible and non rational*
3. *Details of the computation:*
   - $\mathcal{P}(t) = (\frac{(t^2-3)}{1+t^2}, \frac{t\,(3-t^2)}{1+t^2})$ *and* $\mathcal{N}(t) = (\frac{(6t^2+t^4-3)}{(1+t^2)^2}, \frac{8\,t}{(1+t^2)^2})$.
   - $G_{\mathcal{P}}$ *is defined by* $4x_1x_2^2 - 4x_1 + 6x_2x_1^2 + x_2x_1^4 - 3x_2$.

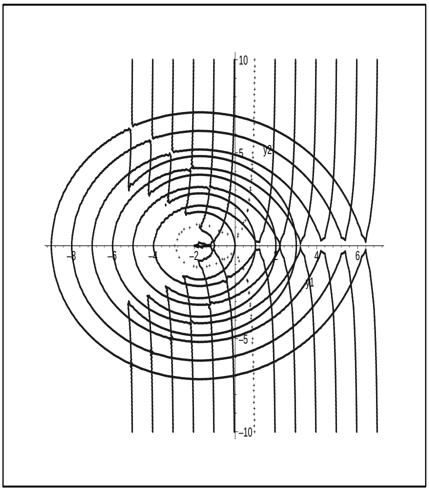

FIGURE 8. Trisectrix of Maclaurin (untraced curve) and the Offsets of Trisectriz of Maclaurin for several distances.

Example 8. (Offset of the Folium of Descartes). *Let $\mathcal{C}$ be the Folium of Descartes defined by*

$$y_1^3 + y_2^3 - 3\,a\,y_1\,y_2, \quad a \neq 0.$$

1. *Implicit equation: The implicit equation of the generic offset for $a = 1$ is*

$-248714388x_1d^8 + 38263752d^8x_2^5 - 248714388x_1x_2^6d^2 + 6377292d^2x_2x_1^{11} +$
$19131876x_1^4d^8 - 841802544d^6x_2^2x_1^2 + 191318760x_2^2d^4 + 86093442x_1^2x_2^2 -$
$35075106x_1^4d^4 - 420901272x_1d^6 - 229582512x_1d^4 + 172186884x_1^2d^8 +$
$108413964d^4x_2^5 + 1256326524d^4x_1^4x_2 - 229582512x_1^6x_2 - 229582512x_1^2x_2^3 -$
$229582512x_1^3x_2^2 - 57395628x_1x_2^4 + 44641044x_1^{10} + 38263752d^4x_1^2x_2 +$
$9565938x_1^6 + 34012224x_1^8 - 714256704x_1^4x_2^3 + 741891636x_1^6x_2^2 +$
$1109648808x_1^4x_2^4 - 114791256x_1^8x_2^2 - 153055008x_1^6x_2^3 - 6377292x_1^6x_2^4 +$
$178564176x_1^6x_2^5 - 6377292x_1^4x_2^6 + 382637520x_1^4x_2^7 - 563327460x_1^6d^2 -$
$165809592x_1^8d^2 - 25509168x_1^{10}x_2^3 - 38263752x_1^8x_2^5 - 867311712d^4x_1x_2^3 -$
$867311712d^4x_1^3x_2 - 605842740x_1^4x_2^2d^2 + 669615660d^4x_1x_2 -$
$1135157976x_1^4x_2^3d^2 - 51018336x_1^4d^2x_2^4 - 44641044x_1^6d^2x_2^3 - 184941468x_1^4d^2x_2^5 +$
$165809592x_1^6d^2x_2^2 - 25509168x_1^6x_2^7 - 6377292x_1^4x_2^9 + 459165024x_1^4x_2^2 -$
$353939706x_1^4d^2 + 459165024x_1^2x_2^4 - 918330048x_1^2x_2^5 + 741891636x_1^2x_2^6 +$
$1128780684x_1^2x_2^3d^2 - 765275040x_1^2x_2^2d^2 - 605842740x_1^2x_2^4d^2 + 172186884x_2^2d^8 -$
$86093442x_1^2d^2 + 12754584x_2^7x_1^7 + 8503056x_2^9x_1^5 + 2125764x_2^{11}x_1^3 +$
$1428513408d^4x_1^2x_2^3 + 86093442d^4 + 9565938x_2^6 - 25509168x_2^7 +$
$34012224x_2^8 + 191318760x_1^2d^4 + 459165024x_1^2d^6 + 229582512x_2^3d^2 -$
$563327460d^2x_2^6 - 353939706d^2x_2^4 + 593088156d^2x_2^5 - 1071385056x_1^2x_2^2d^4 -$
$35075106d^4x_2^4 + 459165024d^6x_2^2 - 25509168x_1^5x_2^7 - 25509168x_2x_1^{11} -$
$146677716d^4x_2^3 + 6377292x_1d^12 + 76527504d^8x_1^2x_2^3 - 51018336x_1^3x_2^9 -$
$25509168x_1x_2^{11} - 44641044x_1^6x_2^6 + 5314410x_1^8x_2^6 - 6377292x_1^9x_2^4 +$
$6377292x_1^{10}x_2^4 - 9565938d^2x_1^{10} + 14880348d^2x_1^9 - 9565938d^2x_2^{10} -$
$420901272d^6x_2 - 886443588d^4x_1^5x_2 - 51018336x_1^9 + 76527504d^2x_2^5x_1^6 -$
$337996476d^4x_2^5x_1^2 + 204073344x_1^7x_2^3d^2 + 9565938x_1^{10}x_2^2 + 25509168x_1^{10}x_2 +$
$2125764x_1^{11}x_2^3 + 76527504x_1^8x_2^3d^2 - 624974616x_1^4d^4x_2^3 - 57395628x_1^4x_2 -$
$822670668x_2^3x_1d^2 - 1403004240d^4x_1^3x_2^3 + 280600848x_1^7d^2 + 593088156x_1^5d^2 +$
$165809592x_2^6x_1^2d^2 - 165809592x_2^8d^2 - 146677716x_1^3d^4 - 242337096d^6x_2^4 +$
$108413964x_1^5d^4 - 89282088x_1^3d^6 + 165809592d^4x_2^6 + 165809592x_1^6d^4 -$
$242337096x_1^4d^6 + 414523980x_1^4x_2^2d^4 + 414523980x_1^2x_2^4d^4 + 38263752x_2^2x_1d^4 +$
$1428513408x_1^3x_2^2d^4 + 1256326524x_1x_2^4d^4 + 4251528x_1^{12}x_2^2 - 6377292x_1^{12}x_2 +$
$51018336x_1^9x_2 - 12754584x_1^4x_2^8 + 286978140x_2^2x_1d^2 + 44641044x_2^{10} +$
$1128780684x_1^3x_2^2d^2 + 612220032x_1x_2^4d^2 - 1135157976x_1^3x_2^4d^2 -$
$127545840x_1^7x_2^2 - 216827928x_1^5x_2^4 - 153055008x_1^3x_2^6 - 318864600x_1x_2^8 -$
$114791256x_2^8x_1^2 + 382637520x_1^7x_2^4 + 178564176x_1^5x_2^6 + 25509168x_1x_2^{10} -$
$886443588d^4x_2^5x_1 - 8503056x_1^{11} + 1062882x_1^14 - 280600848x_1x_2^5d^2 +$
$289103904x_1^3x_2^3d^2 - 51018336x_2^9 - 8503056x_2^{11} + 153055008x_2^5x_1 +$
$707879412x_1^3x_2^3 - 216827928x_2^5x_1^4 - 127545840x_2^7x_1^2 + 382637520x_2^7x_1 +$
$374134464x_2^5x_1^3 + 374134464x_2^3x_1^5 - 242337096x_1^7x_2^3 - 688747536x_1^5x_2^5 -$
$242337096x_1^3x_2^7 + 51018336x_1x_2^9 + 170061120x_1^2x_2^9 + 280600848d^2x_2^7 -$
$401769396x_1^2x_2^5d^2 - 86093442x_2^2d^2 - 918330048x_2^2x_1^5 - 714256704x_2^4x_1^3 -$
$229582512x_2^6x_1 + 153055008d^6 + 105225318d^8 + 25509168d^{10} - 3188646d^12 -$
$2125764d^14 + 200884698x_1^2x_2^2d^8 + 44641044x_1x_2^6d^4 - 102036672d^6x_1^6x_2^2 -$
$184941468x_1^5x_2^4d^2 + 31886460d^8x_1^4x_2 - 146677716d^8x_1^2x_2 - 178564176d^8x_1x_2^3 -$
$178564176d^8x_1^3x_2 + 440033148d^8x_1x_2 + 1192553604x_1^4d^4x_2^4 - 51018336x_1^4d^6x_2^3$

$$
\begin{aligned}
&-51018336x_1^6d^4x_2^3+656861076x_1^6d^4x_2^2+153055008x_1^5x_2+170061120x_1^9x_2^2-\\
&401769396x_1^5d^2x_2^2+229582512x_1^3d^2-25509168x_1^7-25509168x_1^7x_2^5+\\
&1836660096x_1^3x_2^5d^2+586710864x_1x_2^7d^2+1836660096x_1^5d^2x_2^3-\\
&51018336x_1^9x_2^3-318864600x_1^8x_2+14880348x_2^9d^2-235959804x_2^7x_1^2d^2+\\
&25509168x_2^7x_1^4d^2+127545840x_2^9x_1d^2+204073344x_2^7x_1^3d^2+8503056x_1^9x_2^5+\\
&25509168d^4x_2^7+153055008x_1^5x_2^5d^2-440033148d^6x_2^2x_1+382637520x_1^7x_2-\\
&12754584x_1^8x_2^4+25509168d^2x_1^{10}x_2-25509168d^2x_1^{10}x_2^2-34012224d^2x_1^6x_2^6-\\
&35075106d^2x_1^4x_2^8-63772920d^8x_1^5x_2+182815704x_1^3x_2^5d^6-114791256x_1^7x_2^3d^4+\\
&23383404x_1^9x_2^3d^2+182815704x_1^5d^6x_2^3-125420076d^8x_1^3x_2^3+656861076x_2^6x_1^2d^4+\\
&25509168d^2x_1x_2^{10}+76527504d^2x_1^3x_2^8+76527504d^2x_1^5x_2^6+25509168d^2x_1^7x_2^4+\\
&95659380x_1^4x_2^2d^8+95659380x_1^2x_2^4d^8-146677716x_2^2x_1d^8+76527504x_1^3x_2^2d^8+\\
&31886460x_1x_2^4d^8-624974616x_1^3x_2^4d^4-63772920d^8x_2^5x_1+35075106x_1^8d^4+\\
&11691702x_1^2d^{1}2+19131876d^8x_2^4+11691702d^{1}2x_2^2+85030560d^8x_2^3+\\
&22320522d^4x_1^{10}+6377292d^4x_1^9+22320522d^4x_2^{10}-28697814d^{10}x_2^4-\\
&25509168d^{10}x_2^3+3188646d^{10}x_2^2-44641044d^{1}0x_2-44641044d^{10}x_1+\\
&3188646d^{10}x_1^2-25509168d^{10}x_1^3-28697814d^{10}x_1^4+6377292d^{1}2x_2+\\
&25509168x_1^7d^4+35075106x_2^8d^4+85030560x_1^3d^8+38263752x_1^5d^8+\\
&41452398d^8x_2^6+41452398x_1^6d^8-89282088d^6x_2^3-72275976d^6x_1^5-\\
&44641044d^6x_1^6-72275976d^6x_2^5-7440174d^2x_1^{12}+1062882x_2^{1}4+\\
&38263752x_1^7x_2^5d^2-25509168d^2x_1^2x_2^{10}-127545840d^6x_1^4x_2^4-631351908d^6x_1^4x_2^2-\\
&631351908d^6x_1^2x_2^4-102036672d^6x_1^2x_2^6+299732724d^6x_1^4x_2-\\
&76527504d^6x_2^2x_1^5-51018336d^6x_2^4x_1^3+280600848d^6x_1^5x_2+63772920d^6x_1^7x_2+\\
&89282088d^4x_1^4x_2^6-31886460d^4x_1^9x_2+25509168d^4x_1^7x_2^2-51018336d^4x_1^3x_2^6-\\
&31886460d^4x_1x_2^8+66961566d^4x_2^8x_1^2-31886460d^4x_1^8x_2-255091680d^4x_1^7x_2+\\
&31886460d^{10}x_1x_2^3-25509168d^{10}x_1x_2^2+51018336d^{10}x_1x_2-\\
&51018336d^{10}x_1^2x_2^2-25509168d^{10}x_1^2x_2+31886460d^{10}x_1^3x_2-6377292d^{1}2x_1x_2-\\
&191318760d^6x_1^3x_2-440033148d^6x_1^2x_2+44641044d^4x_1^6x_2+66961566d^4x_1^8x_2^2+\\
&89282088d^4x_1^6x_2^4-38263752d^6x_1^8-44641044d^6x_2^6-38263752d^6x_2^8-\\
&25509168d^6x_1^7-25509168x_2^7d^6+6377292x_2^9d^4-248714388d^8x_2-\\
&7440174d^2x_2^{12}+584585100d^6x_2^3x_1^2-191318760d^6x_2^3x_1+584585100x_1^3d^6x_2^2+\\
&299732724x_1d^6x_2^4-337996476x_1^5d^4x_2^2-35075106d^2x_1^8x_2^4-382637520x_1^3x_2^5d^4-\\
&255091680x_1x_2^7d^4+408146688x_1^3d^6x_2^3+280600848x_1d^6x_2^5-\\
&382637520x_1^5d^4x_2^3+25509168x_2^7x_1^2d^4-76527504x_2^5d^6x_1^2+6377292x_2^{11}x_1d^2+\\
&38263752x_2^7x_1^5d^2+23383404x_2^9x_1^3d^2-31886460x_2^9x_1d^4-114791256x_2^7x_1^3d^4+\\
&63772920x_2^7d^6x_1-165809592x_1^5x_2^5d^4-567578988d^2x_1^4x_2^6+612220032d^2x_1^4x_2+\\
&127545840d^2x_1^9x_2-235959804d^2x_1^7x_2^2-44641044d^2x_1^3x_2^6-178564176d^2x_1x_2^8-\\
&239148450d^2x_2^8x_1^2-280600848d^2x_1^5x_2-178564176d^2x_1^8x_2+\\
&586710864d^2x_1^7x_2+765275040d^6x_1x_2-229582512d^4x_2+286978140d^2x_1^2x_2-\\
&248714388d^2x_1^6x_2-239148450d^2x_1^8x_2^2-567578988d^2x_1^6x_2^4-6377292x_1x_2^{12}+\\
&4251528x_1^2x_2^{12}-25509168x_1^3x_2^{10}+6377292x_1^4x_2^{10}-38263752x_1^5x_2^8+\\
&5314410x_1^6x_2^8-25509168x_1^7x_2^6-822670668d^2x_1^3x_2+9565938x_1^2x_2^{10}
\end{aligned}
$$

2. Rationality character: *irreducible and non rational*
3. Details of the computation:
   - $\mathcal{P}(t)=(\frac{3\,t}{1+t^3},\frac{3\,t^2}{1+t^3})$ *and* $\mathcal{N}(t)=(\frac{3\,t\,(-2+t^3)}{(1+t^3)^2},\frac{-3\,(-1+2\,t^3)}{(1+t^3)^2})$.

- $G_{\mathcal{P}}$ *is defined by* $x_2^2 - 2\,x_2^2\,x_1^3 - 1 + 2\,x_1^3 + 4\,x_2\,x_1 - 2\,x_2\,x_1^4$.

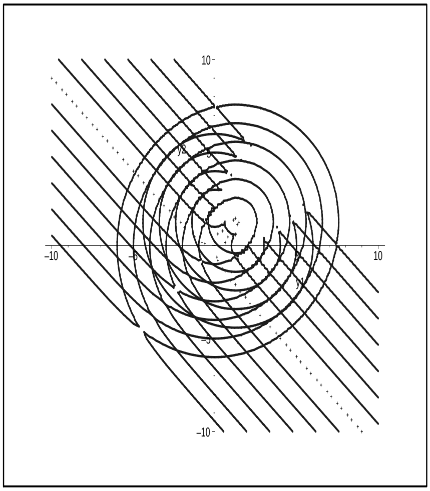

FIGURE 9. Folium of Descartes (untraced curve) and the Offsets of the Folium of Descartes for several distances.

Example 9. (Offset of the Tacnode). *Let* $\mathcal{C}$ *be the Tacnode defined by*

$$2\,y_1^4 - 3\,y_1^2\,y_2 + y_2^2 - 2\,y_2^3 + y_2^4.$$

1. *Implicit equation: The implicit equation of the generic offset is a polynomial of degree* 20 *with* 493 *terms (we do not write here for space reasons)*
2. *Rationality character: irreducible and non rational*
3. *Details of the computation:*
   - $\mathcal{P}(t) = (\frac{18\,t^4+21\,t^3-7\,t-2}{18\,t^4+48\,t^3+64\,t^2+40\,t+9}, \frac{36\,t^4+84\,t^3+73\,t^2+28\,t+4}{18\,t^4+48\,t^3+64\,t^2+40\,t+9})$ *and*

$$\mathcal{N}(t) = \left(\frac{-2\,(108\,t^6+990\,t^5+2340\,t^4+2520\,t^3+1410\,t^2+401\,t+46)}{(18\,t^4+48\,t^3+64\,t^2+40\,t+9)^2}, \frac{486\,t^6+2304\,t^5+3882\,t^4+3144\,t^3+1303\,t^2+256\,t+17}{(18\,t^4+48\,t^3+64\,t^2+40\,t+9)^2}\right).$$

   - $G_{\mathcal{P}}$ *is defined by* $486x_2^2x_1^6+2304x_2^2x_1^5+3882x_2^2x_1^4+3144x_2^2x_1^3+1303x_2^2x_1^2+256x_2^2x_1+17x_2^2- 486x_1^6 - 2304x_1^5 - 3882x_1^4 - 3144x_1^3 - 1303x_1^2 - 256x_1 - 17 + 432x_2x_1^6 + 3960x_2x_1^5 + 9360x_2x_1^4 + 10080x_2x_1^3 + 5640x_2x_1^2 + 1604x_2x_1 + 184x_2$..

Example 10. (Offset of the Conchoid of de Sluze). *Let* $\mathcal{C}$ *be the Conchoid of de Sluze defined by*

$$(y_1 - 1)\,(y_1^2 + y_2^2) + y_1^2.$$

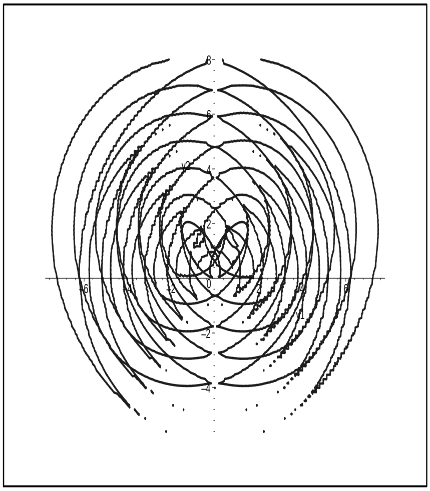

FIGURE 10. Tacnnode (untraced curve) and the Offsets of the Tacnode for several distances.

1. *Implicit equation: The implicit equation of the generic offset is*
$6\,d^4\,{x_1}^4 - d^2\,{x_2}^6 - 32\,d^2\,{x_2}^2 + 3\,{x_1}^6\,{x_2}^2 + 3\,{x_1}^4\,{x_2}^4 + {x_1}^2\,{x_2}^6 - 12\,{x_2}^4\,{x_1}^3 - 2\,{x_2}^6\,x_1 - 18\,{x_2}^2\,{x_1}^5 + {x_1}^8 - 18\,d^4\,{x_2}^2\,x_1 - 9\,d^2\,{x_1}^4\,{x_2}^2 - 6\,d^2\,{x_1}^2\,{x_2}^4 - 32\,{x_1}^3\,{x_2}^2 + 9\,d^4\,{x_2}^2\,{x_1}^2 + 16\,{x_1}^6 + d^8 - 3\,d^6\,{x_2}^2 + 3\,d^4\,{x_2}^4 - 40\,x_1\,{x_2}^4 + 16\,{x_2}^4 + {x_2}^6 - 4\,d^6\,{x_1}^2 + 48\,{x_1}^4\,{x_2}^2 + 33\,{x_1}^2\,{x_2}^4 + 24\,d^2\,{x_1}^5 + 12\,d^4\,{x_2}^2 - 60\,d^2\,{x_2}^2\,{x_1}^2 - 4\,{x_1}^6\,d^2 + 16\,d^4 - 8\,{x_1}^7 + 8\,d^6\,x_1 - 24\,d^4\,{x_1}^3 + 8\,{x_2}^2\,d^2\,x_1 - 32\,{x_1}^3\,d^2 + 32\,d^4\,x_1 - 24\,d^2\,{x_1}^4 - 21\,d^2\,{x_2}^4 + 8\,d^6 + 12\,{x_2}^4\,d^2\,x_1 + 36\,{x_2}^2\,{x_1}^3\,d^2$
2. *Rationality character: rational*
3. *Offset parametrization: the generic offset can be parametrized as*

$$\left(-\frac{t\,(d\,t^5 - t^5 - 6\,t^4 + 6\,d\,t^4 + 15\,d\,t^3 - 14\,t^3 + 20\,d\,t^2 - 16\,t^2 + 15\,d\,t - 8\,t + 6\,d)}{2 + 15\,t^4 + 20\,t^3 + 15\,t^2 + 6\,t^5 + 6\,t + t^6},\right.$$
$$\left.-\frac{t^8 + 8\,t^7 + 26\,t^6 + 44\,t^5 + 2\,d\,t^4 + 40\,t^4 + 16\,t^3 + 8\,d\,t^3 + 12\,d\,t^2 + 8\,d\,t + 2\,d}{(2 + 15\,t^4 + 20\,t^3 + 15\,t^2 + 6\,t^5 + 6\,t + t^6)\,(1 + t)}\right)$$

4. *Details of the computation:*
   - $\mathcal{P}(t) = (1 - \frac{1}{1+t^2}, (1 - \frac{1}{1+t^2})\,t)$ *and* $\mathcal{N}(t) = (-t^2\,\frac{3+t^2}{(1+t^2)^2}, \frac{2\,t}{(1+t^2)^2})$.
   - $G_{\mathcal{P}}$ *is defined by* $x_2^2 - 1 + 3\,x_2\,x_1 + x_2\,x_1^3$.
   - $\mathcal{R} = (\frac{-t\,(t+2)}{t+1}, \frac{-1}{3\,t^2+3\,t+1+t^3})$.

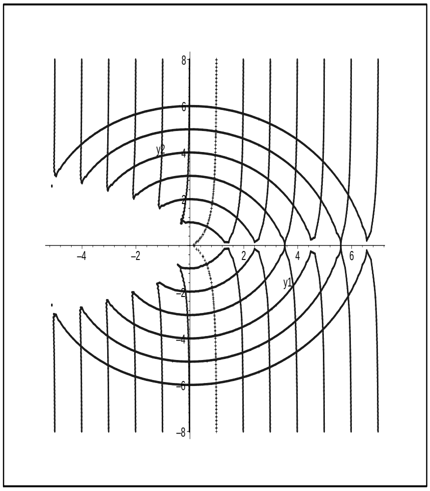

FIGURE 11. Conchoid of de Sluze (untraced curve) and the Offsets of Conchoid of de Sluze for several distances.

Example 11. (Offset of the Epitrochoid). *Let $\mathcal{C}$ be the defined by*

$$y_2^4 + 2\,y_1^2\,y_2^2 - 34\,y_2^2 + y_1^4 - 34\,y_1^2 + 96\,y_1 - 63.$$

1. *Implicit equation: The implicit equation of the generic offset is* $63504-288792x_1+72x_1x_2^6d^2-64800d^2-72d^6x_2^2x_1^2+542025x_1^2+104265x_2^2-316128x_1x_2^2+16758x_2^2d^4-537312x_1^3+294652x_1^4+349688x_1^2x_2^2+55036x_2^4+916x_1^4d^4+178632x_1d^2+216x_1d^6-14472x_1d^4+9x_1^2d^8-80400x_1^5-160800x_1^3x_2^2-80400x_1x_2^4+9x_1^{10}+3574x_1^6-596x_1^8-2384x_1^6x_2^2-3576x_1^4x_2^4+45x_1^8x_2^2+90x_1^6x_2^4+90x_1^4x_2^6+436x_1^6d^2-36x_1^8d^2+1308x_1^4x_2^2d^2-216x_1^4d^2x_2^4-144x_1^6d^2x_2^2+15330x_1^4x_2^2-34460x_1^4d^2+19938x_1^2x_2^4-2384x_1^2x_2^6-57400x_1^2x_2^2d^2+1308x_1^2x_2^4d^2+9x_2^2d^8-200916x_1^2d^2+1296d^4+8182x_2^6-596x_2^8+17334x_1^2d^4-756x_1^2d^6+436d^2x_2^6-22940d^2x_2^4+1832x_1^2x_2^2d^4+916d^4x_2^4-756d^6x_2^2-24x_1^9+72x_1^7d^2+3000x_1^5d^2-144x_2^6x_1^2d^2-36x_2^8d^2-6576x_1^3d^4-36d^6x_2^4-72x_1^5d^4+24x_1^3d^6+54d^4x_2^6+54x_1^6d^4-36x_1^4d^6+162x_1^4x_2^2d^4+162x_1^2x_2^4d^4-6576x_2^2x_1d^4-144x_1^3x_2^2d^4-72x_1x_2^4d^4+117048x_2^2x_1d^2+9x_2^{10}+6000x_1^3x_2^2d^2+3000x_1x_2^4d^2+216x_1^3x_2^4d^2-96x_1^7x_2^2-144x_1^5x_2^4-96x_1^3x_2^6-24x_1x_2^8+45x_2^8x_1^2-78804x_2^2d^2+10080x_2^2x_1^5+10080x_2^4x_1^3+3360x_2^6x_1+216x_1^5d^2x_2^2+117048x_1^3d^2+3360x_1^7+24d^6x_2^2x_1$
2. *Rationality character: irreducible and non rational*

3. *Details of the computation:*
   - $\mathcal{P}(t) = (\frac{-7\,t^4+288\,t^2+256}{t^4+32\,t^2+256}, \frac{-80\,t^3+256\,t}{t^4+32\,t^2+256})$ *and*
     $\mathcal{N}(t) = (\frac{-16\,(5\,t^4-288\,t^2+256)}{(t^2+16)\,(t^4+32\,t^2+256)}, \frac{-1024\,t\,(t^2-8)}{(t^2+16)\,(t^4+32\,t^2+256)})$.
   - $G_{\mathcal{P}}$ *is defined by*
     $32\,x_2^2\,x_1^3 - 256\,x_2^2\,x_1 - 32\,x_1^3 + 256\,x_1 - 5\,x_2\,x_1^4 + 288\,x_2\,x_1^2 - 256\,x_2$.

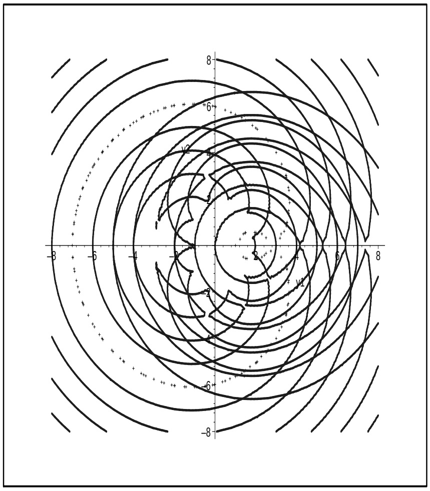

FIGURE 12. Epitrochoid (untraced curve) and the Offsets of the Epitrochoid for several distances.

Example 12. (Offset of the Ramphoid Cusp). *Let $\mathcal{C}$ be the Ramphoid Cusp defined by*

$$y_1^4 + y_1^2\,y_2^2 - 2\,y_1^2\,y_2 - y_1\,y_2^2 + y_2^2.$$

1. *Implicit equation: The implicit equation of the generic offset is a polynomial of degree* 20 *with* 933 *terms (we do not write here for space reasons)*
2. *Rationality character: irreducible and non rational*
3. *Details of the computation:*
   - $\mathcal{P}(t) = (\frac{t^2-2\,t+1}{2\,t^2+2}, \frac{t^4-4\,t^3+6\,t^2-4\,t+1}{6\,t^4+8\,t^2+2})$ *and*
     $\mathcal{N}(t) = (\frac{-2\,(-7\,t^5-2\,t^3+3\,t^6+5\,t^4+t^2+t-1)}{(3\,t^4+4\,t^2+1)^2}, \frac{t^2-1}{(t^2+1)^2})$.
   - $G_{\mathcal{P}}$ *is defined by*
     $9\,x_2^2\,x_1^5 - 9\,x_1^5 + 12\,x_2\,x_1^5 - 16\,x_2\,x_1^4 + 9\,x_2^2\,x_1^4 - 9\,x_1^4 + 4\,x_2\,x_1^3 - 6\,x_1^3 + 6\,x_2^2\,x_1^3 - 4\,x_2\,x_1^2 - 6\,x_1^2 + 6\,x_2^2\,x_1^2 + x_2^2\,x_1 - x_1 + 4\,x_2 + x_2^2 - 1$.

Example 13. (Offset of the Lemniscata of Bernoulli). *Let $\mathcal{C}$ be the Lemniscata of Bernoulli defined by*

$$(y_1^2 + y_2^2)^2 - 2a^2(y_1^2 - y_2^2).$$

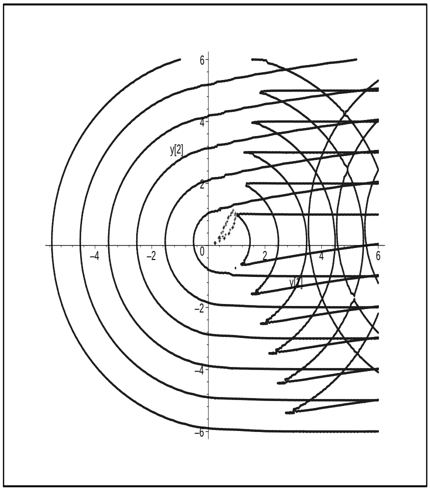

FIGURE 13. Ramphoid Cusp (untraced curve) and the Offsets of the Ramphoid Cusp for several distances.

1. *Implicit equation: The implicit equation of the generic offset is*

$$
\begin{aligned}
&-6\,d^{10}\,x_1{}^2-20\,d^6\,x_1{}^6+13\,a^4\,x_1{}^8+16\,a^8\,d^4+4\,a^8\,x_1{}^4-12\,a^6\,x_1{}^6+6\,a^2\,x_2{}^{10}+\\
&44\,a^6\,x_1{}^4\,d^2-16\,a^8\,x_1{}^2\,d^2-46\,a^4\,d^2\,x_1{}^6+24\,a^2\,d^2\,x_1{}^8-18\,a^2\,x_2{}^2\,x_1{}^8+18\,a^2\,x_2{}^8\,x_1{}^2+\\
&12\,a^2\,x_2{}^6\,x_1{}^4-12\,a^2\,x_2{}^4\,x_1{}^6+12\,a^6\,x_2{}^6+4\,a^8\,x_2{}^4+13\,a^4\,x_2{}^8-6\,d^8\,a^2\,x_1{}^2-\\
&36\,d^4\,x_1{}^6\,a^2-4\,d^6\,a^4\,x_1{}^2-36\,a^2\,x_2{}^2\,d^4\,x_1{}^4+6\,x_1{}^{10}\,x_2{}^2+20\,x_1{}^6\,x_2{}^6-30\,x_1{}^2\,x_2{}^8\,d^2-\\
&60\,x_1{}^6\,x_2{}^4\,d^2-60\,x_1{}^4\,x_2{}^6\,d^2-6\,x_1{}^{10}\,a^2-30\,x_1{}^8\,d^2\,x_2{}^2+x_2{}^{12}+4\,a^4\,x_2{}^2\,x_1{}^6-\\
&18\,a^4\,x_2{}^4\,x_1{}^4+4\,a^4\,x_2{}^6\,x_1{}^2-16\,a^8\,x_2{}^2\,d^2-4\,a^4\,x_2{}^2\,d^6-8\,a^8\,x_2{}^2\,x_1{}^2-20\,a^6\,x_2{}^4\,x_1{}^2+\\
&20\,a^6\,x_2{}^2\,x_1{}^4+24\,d^6\,x_1{}^4\,a^2-40\,d^4\,a^6\,x_1{}^2+45\,d^4\,a^4\,x_1{}^4+90\,x_1{}^4\,d^4\,x_2{}^4+\\
&60\,x_2{}^6\,x_1{}^2\,d^4-6\,x_2{}^{10}\,d^2+x_1{}^{12}-42\,a^4\,x_2{}^4\,d^2\,x_1{}^2-42\,a^4\,x_2{}^2\,d^2\,x_1{}^4+36\,a^2\,x_2{}^4\,d^4\,x_1{}^2+\\
&42\,a^4\,x_2{}^2\,d^4\,x_1{}^2-24\,a^2\,x_2{}^8\,d^2+36\,a^2\,x_2{}^6\,d^4+45\,a^4\,x_2{}^4\,d^4-24\,a^2\,x_2{}^4\,d^6+\\
&40\,a^6\,x_2{}^2\,d^4+48\,a^2\,x_2{}^2\,d^2\,x_1{}^6+d^{12}-48\,a^2\,x_2{}^6\,d^2\,x_1{}^2+15\,d^4\,x_1{}^8-20\,d^6\,x_2{}^6+\\
&60\,x_1{}^6\,d^4\,x_2{}^2+15\,d^8\,x_1{}^4+6\,x_1{}^2\,x_2{}^{10}+15\,x_1{}^4\,x_2{}^8+15\,d^4\,x_2{}^8+6\,a^2\,x_2{}^2\,d^8-\\
&44\,a^6\,x_2{}^4\,d^2-46\,a^4\,x_2{}^6\,d^2+30\,d^8\,x_2{}^2\,x_1{}^2+15\,x_2{}^4\,x_1{}^8+15\,d^8\,x_2{}^4-6\,d^{10}\,x_2{}^2-\\
&60\,d^6\,x_2{}^4\,x_1{}^2-8\,d^8\,a^4-60\,d^6\,x_1{}^4\,x_2{}^2-6\,d^2\,x_1{}^{10}
\end{aligned}
$$

2. *Rationality character: irreducible and non rational*
3. *Details of the computation:*
   - $\mathcal{P}(t) = \left(\frac{a\sqrt{2}\,(t+t^3)}{1+t^4}, \frac{a\sqrt{2}\,(t-t^3)}{1+t^4}\right)$ *and* $\mathcal{N}(t) = \left(-\frac{a\sqrt{2}\left(1-3\,t^4-3\,t^2+t^6\right)}{(1+t^4)^2}, -\frac{a\sqrt{2}\left(-1+3\,t^4-3\,t^2+t^6\right)}{(1+t^4)^2}\right)$.
   - $G_{\mathcal{P}}$ *is defined by* $x_2^2 - 3x_2^2x_1^4 + 3x_2^2x_1^2 - x_1^6x_2^2 - 1 + 3x_1^4 - 3x_1^2 + x_1^6 + 2x_2 - 6x_2x_1^4 - 6x_2x_1^2 + 2x_1^6x_2$.

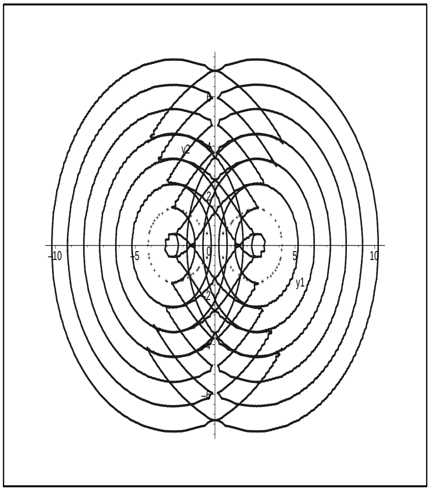

FIGURE 14. Lemniscata of Bernoulli (untraced curve) and the Offsets of the Lemniscata of Bernoulli for several distances.

Example 14. (Offset of Cuspidal Curves). *Let $\mathcal{F}$ be a family of rational affine curves, called Cuspidal Curves, defined by*

$$b_1^{n-1}\,(y_1-a_0)^n - a_1^n\,(y_2-b_0)^{n-1}, \quad a_1\neq 0, b_1\neq 0, \quad n>1$$

1. *Rationality character: rational*
2. *Offset parametrization: the generic offset can be parametrized as*

$$\left(\mathcal{P}(\varphi)+\frac{2\,d\,t}{(n-1)\,a_1\,\varphi^{n-2}\,(t^2+1)}\mathcal{N}(\varphi)\right), \quad \textit{with } \varphi=\frac{a_1(1-n)(t^2-1)}{2b_1 n\,t}$$

3. *Details of the computation:*
   - $\mathcal{P}(t)=(a_1t^{n-1}+a_0, b_1t^n+b_0)$ *and* $\mathcal{N}(t)=(-nb_1t^{n-1},(n-1)a_1t^{n-2})$.
   - $G_{\mathcal{P}}$ *is defined by* $2\,n\,b_1\,x_1\,x_2+(n-1)\,a_1\,(x_2^2-1)$.
   - $\mathcal{R}=(\frac{a_1(1-n)(t^2-1)}{2b_1 n\,t},t)$.
4. *Remark: Parabolas* $\{(t,kt^2)\}$, *and the cubics* $\{(t^2,kt^3)\}$ *belong to this family.*
5. *Particular case of a cuspidal curve of degree 5: we consider the cubics* $y_1^5-y_2^4$ *paramentrized by* $\{(t^4,t^5)\}$. *The implicit equation of the generic offset is*

$-300781250x_2^4x_1^5d^2 + 65536x_2^8 - 1600000x_2^8x_1 + 9765625x_2^8x_1^2 - 262144x_2^6d^2 + 3200000x_2^4x_1^6 - 19531250x_2^6x_1^5 + 6272000x_2^6x_1d^2 - 1228800x_2^4x_1^3d^2 - 19531250x_2^4x_1^7 + 9765625x_2^{10} + 97656250x_2^4x_1^3d^4 - 131072x_1^5d^4 - 41015625x_2^8d^2 + 30200000x_2^4x_1^4d^2 - 36250000x_2^6x_1^2d^2 + 393216x_2^4d^4 - 50200000x_2^4d^6 - 195312500x_2^6x_1^3d^2 + 66250000x_2^6d^4 - 97656250x_2^6d^4x_1 - 58593750x_1^{10}d^2 + 64800000x_2^4x_1^2d^4 - 9792000x_2^4d^4x_1 + 9765625d^{12} - 1600000d^{10} + 409600x_1^3d^6 - 2048000x_1d^8 + 65536x_1^{10} + 19328000x_1^6x_2^2d^2 + 65536d^8 + 9765625x_1^{10}x_2^2 - 409600x_1^8d^2 + 9765625x_1^{12} - 786432x_1^5x_2^2d^2 - 1600000x_1^{11} + 222656250x_2^4d^6x_1 - 48828125x_1^8x_2^2d^2 + 48828125x_2^2x_1^2d^8 - 150000000x_2^2x_1d^8 + 97656250x_1^6x_2^2d^4 - 97656250x_1^4x_2^2d^6 + 819200x_1^3x_2^2d^4 - 18400000x_1^4x_2^2d^4 - 120000000x_1^7x_2^2d^2 + 90000000x_1^5x_2^2d^4 + 180000000x_1^3x_2^2d^6 - 59200000x_1^2x_2^2d^6 + 7168000x_1x_2^2d^6 + 4288000x_1^6d^4 - 11200000x_1^4d^6 + 20800000x_1^2d^8 + 9800000x_1^9d^2 - 44800000x_1^7d^4 + 91600000x_1^5d^6 - 80000000x_1^3d^8 + 25000000x_1d^{10} - 58593750x_1^2d^{10} - 195312500x_1^6d^6 + 146484375x_1^4d^8 + 16800000x_2^2d^8 - 262144x_2^2d^6 + 146484375x_1^8d^4 - 9765625x_2^2d^{10} - 131072x_2^4x_1^5$,

and the generic offset can be parametrized as

$$\left(\frac{(16\,t^8 - 32\,t^6 + 32\,t^2 - 16 + 625\,d\,t^4)\,(t^2-1)}{625\,t^4\,(t^2+1)},\right.$$
$$\left.\frac{2\,(-16\,t^{12} + 64\,t^{10} - 80\,t^8 + 80\,t^4 - 64\,t^2 + 16 + 3125\,d\,t^6)}{3125\,t^5\,(t^2+1)}\right)$$

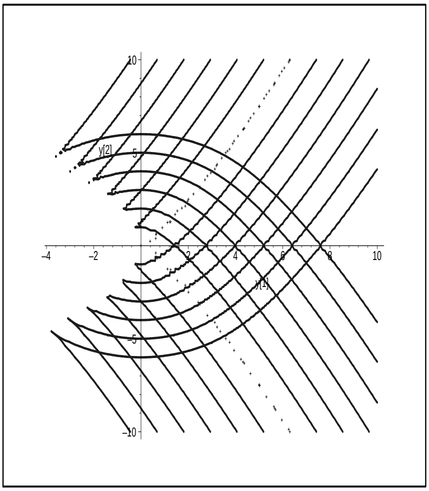

FIGURE 15. Cuspidal curve (t^4, t^5) (untraced curve) and the Offsets of the curve for several distances.

REFERENCES

[1] J. G. ALCAZAR. Good Global Behavior of Offsets to Plane Algebraic Curves. *Journal of Symbolic Computation* **43**, 659–680, 2008.

[2] J. G. ALCAZAR. Good Local Behavior of Offsets to Regularly Parametrized Surfaces. *Journal of Symbolic Computation* **43**, 845–857, 2008.

[3] J. G. Alcazar, R. Sendra. Local Shape of Offsets to Algebraic Curves. *Journal of Symbolic Computation* **42**, 338–351, 2007.

[4] E. Arrondo, J. Sendra, J. R. Sendra. Parametric Generalized Offsets to Hypersurfaces. *Journal of Symbolic Computation* **23**(2–3), 267–285, 1997.

[5] E. Arrondo, J. Sendra, J. R. Sendra. Genus Formula for Generalized Offset Curves. *Journal of Pure and Applied Algebra* **136**(3), 199–209, 1999.

[6] J. W. Bruce, P. J. Giblin. *Curves and Singularities.* Cambridge University Press, 1984.

[7] E. Brieskorn, H. Knörrer. *Plane Algebraic Curves.* Birkhäuser Verlag, 1986.

[8] D. Cox, J. Little, D. O'Shea. *Ideals, Varieties, and Algorithms.* Springer-Verlag, New York, 1997.

[9] R. T. Farouki, C. A. Neff. Analytic Properties of Plane Offset Curves. *Comput. Aided Geom. Des.* **7**, 83–99, 1990.

[10] R. T. Farouki, C. A. Neff. Algebraic Properties of Plane Offset Curves. *Comput. Aided Geom. Des.* **7**, 100–127, 1990.

[11] J. Hoschek, D. Lasser. *Fundamentals of Computer Aided Geometric Design.* A.K. Peters Wellesley MA., Ltd. 1993.

[12] C. M. Hoffmann. *Geometric and Solid Modeling.* Morgan Kaufmann Publishers, Inc. 1993.

[13] G. W. Leibniz. Generalia de natura linearum, anguloque contactus et osculi povocationibis aliisque cognatis et eorum usibus nonnullis. *Acta Eruditorum*, 1692.

[14] W. Lü Offset-Rational Parametric Plane Curves. *Comput. Aided Geom. Des.* **12**, 601–617, 1995.

[15] M. Peternell. Rational two-parameter families of spheres and rational offset surfaces. *Journal of Symbolic Computation.* To appear. 2009.

[16] M. Peternell, H. Pottmann. A Laguerre geometric approach to rational offsets. *Computer Aided Geometric Design* **15**, 223–249, 1998.

[17] H. Pottmann, W. Lü, B. Ravani. Rational Ruled Surfaces and their Offsets. *Graphical Models and Image Processing* **58**, 544–552, 1996.

[18] H. Pottmann. Rational Curves and Surfaces with Rational Offsets. *Comput. Aided Geom. Des.* **12**, 175–192, 1995.

[19] G. Salmon. *A Treatise an the Higher Plane Curves.* Chelsea, New York ,1960.

[20] E. Snapper, E. Troyer. *Metric Affine Geometry.* Academic Press, 1971.

[21] F. San Segundo, J. R. Sendra. Degree Formulae for Offset Curves. *Journal of Pure and Applied Algebra* **195**, 301–335, 2005.

[22] F. San Segundo, J. R. Sendra. Partial Degree Formulae for Plane Offset Curves. *Journal of Symbolic Computation* **44**, 635–654, 2009.

[23] J. Sendra, J. R. Sendra. Algebraic Analysis of Offsets to Hypersurfaces. *Mathematische Zeitschrift* **234**, 697–719, 2000.

[24] J. Sendra, J. R. Sendra. Rationality Analysis and Direct Parametrization of Generalized Offsets to Quadrics. *Applicable algebra in Engineering, Communication and Computing* **11**, 111–139, 2000.

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