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Metaphysical Notes Concerning Hilbert and His Studies on Non-Euclidean and Non-Archimedean Geometries

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RESUMEN

Este artículo pretende dejar claro que el camino por el que Hilbert demuestra la independencia de los axiomas en sus *Fundamentos de geometría* no implica ni que los axiomas sean puramente convencionales, ni que la geometría euclidiana haya quedado completamente superada, ni que la mente humana pueda liberarse de los principios más básicos. Sostiene que la obra de Hilbert sirve, en cambio, para mostrar que la geometría estudia una parte de la realidad, que el espacio no debe concebirse del modo absoluto como lo hicieron Newton y Kant, y que los axiomas tienen fuerza de tales sólo dentro del género al que se refieren sus términos.

ABSTRACT

This paper intends to show that the way in which Hilbert's *Foundations of Geometry* demonstrates the independence of geometric axioms implies neither that geometry and its axioms are merely conventional nor that Euclidian geometry has been absolutely defeated nor that the human mind can be freed even from the most basic theoretical principles. This paper holds that Hilbert's work is useful, instead, to show that geometry studies a part of reality, that space cannot be conceived in the absolute way in which Newton and Kant conceived it, and that the axioms play their theoretical role if and only if the subject matter of the discourse includes the nature to which the terms of each axiom refer.

In his work *Grundlagen der Geometrie*, Hilbert attempted to prove the independence of Geometry's axioms through the analysis of different geometrical constructions that prescind from one or another of them. Thus he sought to elucidate that the axioms cannot be deduced from one another [Hilbert (1992), p. 32]. For instance, "Axiom III 5 [congruence of triangles] cannot be deduced from the other Axioms I [incidence], II [order], III 1-4 [congruence], IV [parallels] and V [continuity] by logical inference" [Hilbert (1992), p. 39].[1] How did he know this? Because he proved that it is possible to define the construction of segments in such a way that from it one can build a whole consistent geometry to which axiom III 5 is not applied, even though all the other axioms are applicable [Hilbert (1992), pp. 39-41, § 11]. In this paper we will examine some passages in which Hilbert accomplished this very job as regards the parallels axiom and Archimedes' axiom (along with axiom III 5, which is taken in a restricted form), and we will see that in a determined geometrical construction, at least, omitting Archimedes' axiom (V 1) leads to the inapplicability of the axiom that the whole is greater than any of its parts.

Our goal is to determine, through reflection, and in the light of the two particular problematic topics already mentioned a) if this Hilbertian strategy implies, as some believe and claim, that mathematics is a mere "construct" which contains arbitrarily chosen principles, and b) if our intellect can evolve so that the axioms seen traditionally as the most sacred and even a whole geometrical construction with its axioms and theorems alike (in particular the so called "Euclidean") can be superseded.

We will consider in this paper mainly the *Grundlagen der Geometrie* and other contemporary works by Hilbert. We will leave aside, especially, the difficult question regarding the extent to which the works of Hilbert on the theory of relativity affected his conception of the relationship between Euclidian geometry and the space of our experience. However, some of the remarks that will follow could shed light on the adequacy of Hilbert's very Kantian attempt to transform relativistic physics into a fully mathematical discipline.[2]

The present paper has been divided in three parts: (I) prefatory remarks,[3] (II) non-Euclidian geometries and (III) non-Archimedean geometries.

I

The strength and truth of axioms or principles cannot be proved. Our sole task here, thus, is to meditate on the mathematician's activity in order to excavate its meaning. This way, perhaps, the non-arbitrariness of the axioms, their self-evidence, and the connection of their strength with the natures to which their terms refer can be manifested to our mind.

The non-arbitrariness of the axioms can be manifested in two ways. First, not every axiomatic system stands as relevant for mathematics [Corry (2002), p. 31. Aleksandrov et al (1965), pp. 264-265]. There are axioms that emanate from the nature of the particular kind of space or quantity (discrete or continuous) that is subject of study in a particular construction or are implied by it. Only those axioms can survive a rational examination. Second, according to what Hilbert in person taught:

> The edifice of science is not raised like a dwelling, in which the foundations are first firmly laid and only then one proceeds to construct and to enlarge the rooms. Science prefers to secure as soon as possible comfortable spaces to wander around and only subsequently, when signs appear here and there that the loose foundations are not able to sustain the expansion of the rooms, it sets

> about supporting and fortifying them. This is not a weakness, but rather the right and healthy path of development.[4]

Hilbert was "in search of the adequate [axiomatic] system for each of the known and sufficiently elaborated theories, and not the other way around" [Corry (2002), p. 39]. However, as stated above, in the work *Grundlagen der Geometrie*, he intended also to show that no axiom can be derived from any other. To this end, he employed the strategy of building axiomatic systems in which one of the axioms was not postulated, so that it was kept out of the system precisely because it could not be derived from the others. In such constructions he employed artificial or conventional postulates with the particular purpose of studying the well-established axioms and the relationships between them. We should keep in mind this second purpose in order to explain some of the paradoxes that are to be found in Hilbert's work.[5]

The German author always thought not only that the axioms are not arbitrary but also that there was a relationship between the axiomatic system of geometry and physical reality. However, when he tried to explain such a relationship, he fell into inaccuracies and even contradictions.

In exploring the origin of the principles of physics and mathematics, Hilbert spoke about "observation" and "intuition." These actions truly intervene in the origin of theoretical principles. The way in which Hilbert understood them, however, made it difficult for him to give an account of the kind of relationship that such actions establish between reality and the mind. For this reason, the very notion of "truth" was a puzzle for Hilbert. He, indeed, wrote a question mark following the word "truth" in a letter to Frege dated December 29th 1899 [Hilbert and Frege (1980), p. 13].[6] Hilbert, thus, thought that the axioms have their origin in experience and intuition but, once formulated, their concepts would be separated from experience and intuition,[7] in such a way that it would become difficult to assert their relationship to reality. In the same letter already cited, he seemed to think that the axioms could be utterly arbitrary and that truth and even existence would mean only (logical) consistence [Hilbert and Frege (1980), p. 12]. In the same place, he also stated that the axioms could be applied to a diversity of realities as long as those realities satisfied the axioms. But, he added, if the system had been developed enough, as was the Maxwellian theory of electricity, it would take a lot of evil-will to try to apply the axioms to different phenomena [Hilbert and Frege (1980), p. 13].

Why was it so difficult for Hilbert to explain the basic experiences underlying the axioms of the science that he practiced all his life to a point in which he was unable to elucidate their relationship to reality? The answer might be connected to several facts. First of all, he neglected the distinction between physics and mathematics. Moreover, to reflect metaphysically on the correspondence between reality and the formulae of their own sciences is a task neither for mathematicians as such (Hilbert was before anything else a mathematician) nor for physicists as such.[8] Until the XVIII century European scientists had a solid classical education, thanks to which they were able to metaphysically reflect on their disciplines. But at the end of the XVIII century such tradition was impaired and knowledge was fragmented. Perhaps here lies the cause behind Hilbert's perplexities concerning the foundations of science:[9] he was trying to restore the unity of knowledge, but this had been broken long before and at his time the task was far from easy.[10] Finally, he was under the influence of the philosophy of Kant.

While Huygens and Newton knew very well that physics (even mathematical physics) and mathematics (even geometry) cannot use the same method, Hilbert seems to have completely disregarded this basic truth. He wanted to enclose physical sciences within mathematical sciences and, to this end, he tried to confer on physical sciences an axiomatic abstraction and a firmness of results similar to those of geometry. At the same time, he conceived geometry as a natural science [Hilbert (1902), pp. 442 and 454-455. Corry (1997), pp. 104-109]. He overlooked, thus, a fundamental maxim concerning the physic-mathematical sciences:

> Reason is employed in another way, not as furnishing a sufficient proof of a principle [not as demonstrating the principle by its causes], but as confirming an already postulated principle, by showing the congruity of its effects. Thus in astrology the theory of eccentrics and epicycles [proper to the Ptolemaic system which, since it was more mathematized than the Aristotelian, rested more upon hypothesis] is postulated, because thereby the sensible appearances of the heavenly movements can be saved. Not, however, as if this proof were sufficient, for perhaps some other hypothesis might save them [Aquinas (1956), I, q. 32, a. 1, ad 2m; (1947), I, q. 32, a. 1, ad 2m. Crombie (1959^1), p. 89].

In geometry one can build proper demonstrations but in mathematical physics this is impossible. Geometry can derive its theorems from the axioms in a deductive way. Mathematical physics can only imagine hypotheses trying to "save the phenomena" and testing them in experience. Huygens acknowledged this weakness of mathematical physics. In the Preface to his *Traité de la Lumière*, he asserted that his theory of light tried to draw proofs for hypothetical principles from their consequences and, for this very reason, such a theory could not be so firm as geometry [Crombie (1959^2), pp. 326-327]. Kant, instead, missed this distinction because he conceived mechanics as an *a priori* science and Euclidean space as *the* space of our sensible perceptions (organized by the two forms of our sensibility) [Körner (1960), pp. 138-139 y 140-141].

Perhaps due to the wide influence of Kant in German academic life, Hilbert did not exploit some very subtle Aristotelian distinctions concerning the origin of principles and the way to use them, both in mathematics and in physics.[11] Kant was correct, of course, in pointing out the active role of human mind in obtaining the axioms. These are not "given" to reason by the senses. But it should be noted that, as we have shown in a previous paper, this active character of the intellect can be better explained with the Aristotelian notion of agent intellect than with the Kantian notion of *a priori* forms of sensibility [Casanova (2003)].[12] According to the philosopher from Stagyra, principles belong to the theoretical virtue that he called "intellect" (*noûs*), and they are formed from the analysis of basic notions belonging to each of these two disciplines: discrete and continuous quantity, on the one hand, and sensible essences, on the other. Such notions have their origin in induction, in the sense experience of the genera that constitute their respective subject matters, illuminated by the agent intellect and grasped by the possible intellect [Aristotle (1956), III 4-6; (1964) II 19; (1894), VI 6]. However, the common origin of those notions in sense experience adopts a different form in each discipline. In mathematics the notions result from abstraction of the form "quantity," whereas in physics they result from abstraction of the whole from concrete matter [Aquinas (1972), Lesson II, q. 1, a. 1]. Physics must always fix its eyes in experience, while mathematics can proceed in a more abstract way, with a peculiar style with which we will concern ourselves later in this paper. To neither of these two sciences belongs the task of reflecting on the sense in which their notions or judgments relate to reality. Because neither of them reflects on *what is* their respective genus. This reflection belongs to first philosophy or metaphysics.

Despite all his perplexities, and without the heat of the epistolary discussion with Frege, Hilbert's predominant opinion even in 1919 was the following:

> [Mathematics] has nothing to do with arbitrariness. Mathematics is in no sense like a game, in which certain tasks are determined by arbitrarily established rules. Rather, it is a conceptual system guided by internal necessity, that can only be so, and never otherwise [Hilbert (1919-1920), p. 14; Corry (1997), p. 116].

Moreover, according to Hilbert, even though pure mathematics concerns necessary truths, its origin can be found (at least in part) in experience, observation and description of the concrete objects of finite arithmetic and geometry.[13] We can use again Hilbert's own words:

> […] while the creative power of pure reason is at work, the outer world again comes into play, forces upon us new questions from actual experience, opens up new branches of mathematics [which encompasses physics, according to Hilbert], and while we seek to conquer these new fields of knowledge for the realm of pure thought, we often find the answers to old unsolved problems and thus at the same time advance most successfully the old theories. And it seems to me that the numerous and surprising analogies and that apparently prearranged

> harmony which the mathematician so often perceives in the questions, methods and ideas of the various branches of his science, have their origin in this ever-recurring interplay between thought and experience [Hilbert (1902), p. 440].[14]

These prefatory remarks are sufficient to see that some changes in Hilbert's philosophical approach to geometry need to be introduced, in order to account for the reality and/or veracity of the axioms. In reference to non-Euclidean geometries we have to abandon the conception of Euclidean geometry as being the only one that corresponds to the space of perception, on the one hand, and, on the other, we have to keep in mind that, even though the axioms of geometry have their origin in experience, the abstract character of mathematics makes room for different ways of analysis of experience. In reference to the particular non-Archimedean geometry that we will study, it should become clear that Hilbert's (at least apparent) claim of the supersession of the principle that "the whole is greater than any of its parts" disregards an axiom's exclusive applicability to a subject matter in which the genera to which the terms of the axiom refer are present. Thus, we are pushed to attempt, from a philosophical perspective, a different explanation of the reality and/or veracity both of the axioms and of the Euclidean and non-Euclidean geometries. We hope to aid in establishing precisely the basic truths towards which the great mathematician was striving.

II

When Hilbert in § 10 of his *Grundlage der Geometrie*, through his method, lays aside the axiom of parallels in order to build a non-Euclidean geometry, he mentions a real space, a sphere. This fact seems, at first glance, surprising. How can one overlook a spatial axiom, apply all other axioms and new postulates to a real space and still obtain a geometry that can be "true"? Could it be that geometry does not have anything to do with reality, but is instead a pure mental construction? Everything seems to point in this direction because even the most venerable Euclidean building was constructed on the basis of an axiom that we can overlook in a completely artificial way.

We know, however, that Hilbert does not think that geometry is completely artificial, but he thinks instead that it is a kind of description or understanding of the natural world [Körner (1960), p. 88. Corry (2002), p. 35]. How can we explain this seeming paradox?

One could attempt a first explanation saying that the sphere (the space which is analyzed in a non-Euclidean geometry) is conceived as being included in a wider Euclidean space, a kind of "absolute space," Newtonian or Kantian. In this way, non-Euclidean geometry can make sense because its propositions can be transformed and, thus, understood by or in a Euclidean geometry.

Reflection on the Aristotelian and Leibnizian conceptions of place and space, however, allows one to understand that such an artificial explanation is not necessary. Space is not an absolute entity, but a set of relations–real or of reason–between the extreme parts of bodies [Aristotle (1950), IV, 1-9. Leibniz (1989), pp. 114-115, 120-121, 128-134 y 165-166]. There is no absolute space in which one can find a sphere, but a diversity of spaces that can be analyzed into their elements in different ways.

If one postulates a spherical surface and defines each of the points on it as any two extremes of each diameter that can be traced in each of the longest circles or equators that can be found on the sphere; and if each line is the perimeter of one of those equators, no parallel will go through a point external to a line. Any line that goes through an external point sections the first line in a point (such as it was defined). This spherical surface, thus, would not be under the Euclidean axiom of parallels, or Hilbertian Axiom IV: "Let *a* be any line and A a point not on it. Then there is at most one line in the plane, determined by *a* and A, that passes through A and does not intersect *a*" [Hilbert (1992), p. 25].

Euclidean geometry, then, is still true, even though there are non-Euclidean geometries that are also true.[15] Euclid accomplished demonstrations properly speaking and, thus, they were true. How can we then explain that a variety of geometries are possible?

Geometry is abstract, according to both Aquinas and Aristotle. The notion of "quantity" (continuous or discrete) came to our minds through experience, but it was abstracted while we were children. For this reason we can forget its origin and think that such notion is innate, as do Plato, Leibniz and Kant. In mathematical demonstrations one has to suppose some basic notions included within the genus "quantity" as the subject of the "passions" or predicates that are to be demonstrated. Among those basic notions one finds "unity," "point," "line," "surface," "space," etc. After they are supposed along with the meaning of their names, then one can construct other things, like triangles and squares. Later, one searches for demonstrations of further things, such as the addition of all the inner angles of a triangle equaling 180 degrees, or the diagonal being incommensurable with the sides of the square, etc. Each step yields the subject of the following steps (because, for example, once the triangle has been constructed, one can prove that the subject "triangle" has inner angles that sum to 180 degrees). And every step supposes that the basic notions exist and what is meant by the words with which we name them. These basic notions, and the genus "quantity" to which they belong, do not exist in themselves, but in the genus "substance," which mathematics does not consider [Aquinas (1955), book I, lesson II, §§ 17-19. Aristotle (1957), VI 1].[16]

Because of its abstract character, mathematics does not have as one of its tasks to determine in what sense mathematical research is real. For this reason, moreover, as Aquinas teaches, mathematics can look like a mere "construct," "because its demonstrations are presented as if they were craft-operations, like this: *On a given straight line, may an equilateral triangle be built*" [Aquinas (1955) § 17]. For this reason, finally, demonstrations can take different paths. This does not mean, however, that they do not represent anything real and are completely arbitrary.[17] Euclidean demonstrations are as firmly standing today as they always were, even though we know today that they do not exhaust the explanation of space. And they are standing precisely because, as any true demonstration, they explain an effect as a necessary consequence of a well-known (formal) cause.[18]

On this point, our perspective for understanding the nature of mathematics is better than those of Newton and Kant. After the development of mathematics and physics during the XIX and XX centuries, we can see the relationship between mathematics and experience more easily. We also can understand more easily —with Aristotle and Leibniz— that space is relative. We can finally see, within the spirit of Aquinas' theory of mathematics but beyond its letter, that constructions of geometry alternative to that of Euclid are possible.

In order to explain how Euclidean and non-Euclidean geometries can be both true, one has to consider then that the axiom of parallels comes into play when the kind of space with which one is dealing is "plane" or is analyzed as plane. If one is dealing with a different kind of space or analysis, this principle is void. As any other axiom, it is conditioned to the presence of the natures to which its precise terms refer. If those natures are not present in the subject matter with which a demonstration is concerned, then the axiom cannot be applied. In order to determine this presence, the ambiguity of language in which so many sophistical objections are grounded must be avoided [Aristotle, (1958), 1].

In the paper "On the Hypotheses which Lie at the Bases of Geometry," by Riemann, one finds a strong ratification of what has been asserted in the last paragraphs. Geometry assumes as things given the notion of space and the first principles for the construction in space, while giving merely nominal definitions of every primitive notion. She leaves in darkness, thus, on the one hand, the relations among these assumptions and, on the other, the problem concerning the necessity or even the possibility of those connections. Even though one can construct a magnitude extended in more than three dimensions, space is a particular kind of magnitude extended only in three dimensions. Geometry needs to be nourished by experience because one has to uncover the simplest "matters of fact" from which one can build the measure-relations of space. In this enterprise there is not only one possible path because the "matters of fact" that are sufficient to determine the measure-relations can be organized in a diversity of systems of which the most important is that which Euclid has laid down as a foundation. Those "matters of fact," thus, must be assumed as hypotheses [Riemann (1973), pp. 107-122].

On this account Hilbert's meta-mathematical reflections are insufficient. Kantian influence could be behind his assertion that Euclidean geometry is the only one that corresponds to our spatial experience, even though he makes clear that this topic belongs not to geometry but to logic-mathematical investigations. According to Hilbert, non-Euclidean and non-Archimedean geometries are arbitrary creations and represent an extension of the word "geometry," similar to the extension of arithmetic represented by complex numbers. He asserts, however, that some objects "behave" in accordance with non-Euclidean or non-Archimedean geometry. In Corry's exposition there are references only to those objects that would behave in accordance with non-Euclidean geometry, such as the paths of light [Corry (1997), 128-129]. In reference to the character of being an extension of the word "geometry," as regards non-Euclidean three-dimensional geometries, it seems to me that Hilbert is wrong. As regards non-Archimedean geometries, perhaps he is correct, as we will explore below.

Despite our disagreement, there is an observation that Hilbert makes in 1905 that confirms what we have stated here. He thinks that his observation has to do with geometrical axioms in general, but, in fact, above all else it has to do with non-Euclidean geometries. Axioms can be chosen more or less arbitrarily. One can begin defining some entities such as point, line and plane. But one could also begin by defining different entities. Not anyone, however, nor with the only restriction of consistency. According to Corry, one cannot begin by defining chairs, tables and beer-mugs. Instead, according to Hilbert, we have to begin by defining beings that are close to the intuitive geometrical matters of fact such as circles and spheres, from which the adequate axioms can be built in such a way that they would not contradict the usual intuitive geometry.[19]

III

Non-Archimedean geometries are those in which the fundamental rules exclude Archimedes' axiom (V 1): "If AB and CD are any segments, then there exists a number *n* such that *n* segments CD constructed contiguously from A, along the ray from A through B, will pass beyond the point B" [Hilbert (1992), p. 26]. They are concerned, thus, with a non-continuum object of investigation.

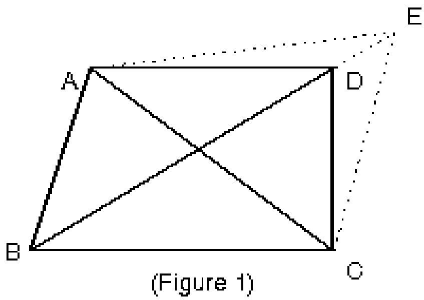
In Appendix II of *Foundations of Geometry*, "The Theorem on the Equality of the Base Angles of an Isosceles Triangle," one can find an instance of non-Archimedean geometry [Hilbert (1992), pp. 113-132]. There one has the construction of a geometry that a) uses all the axioms I-IV except axiom III 5 of triangle congruence (which is applied in a restricted way: only equipositioned triangles would be congruent[20]), b) excludes Archimedes' axiom [Hilbert (1992), pp. 114-115] and c) arbitrarily defines the order of a set of numbers, and a set of graphic representations, rotations, mappings, projections and of comparison or measurement of segments (one segment is rotated on the other in order to perform the comparison) [Hilbert (1992), pp. 115-120]. In establishing the axioms and rules a) and c), Hilbert makes use of notions such as "point," "line," "plane," "angle," "triangle," "parallel," etc. However, the restriction of axiom III 5, the exclusion of axiom V 1, and the use of the definitions, once applied, result in a subject of research that is something different from area or volume. In fact, with the mentioned postulates "the concept of area loses its meaning" [Hilbert (1992), p. 127]. That is to say, the subject matter that can be studied with such an axiomatic system is not any more the magnitude extended in three dimensions that we call "space." In this subject of research, the full version of III 5 cannot be proved. This gives Hilbert insights about the independence of axioms and about "the logical connection of the theorem of the isosceles triangle with the other elementary theorems of plane geometry, in particular, with the theory of area" [Hilbert (1992), p. 115]. In this subject of research, moreover, theorem 29, proposition 39 of Book I of Euclid's *Elements*, and the axiom that the whole is greater than any of its parts are not valid.

Let us examine briefly the content of theorem 29, proposition 39, of Book I of Euclid's *Elements*. After this, let us examine also the manner in which Hilbert shows that the theorem is not demonstrable in the particular non-Archimedean geometry that he is building, and the consequences that he draws from the fact that this theorem is not valid.

Such theorem establishes that two equal triangles constructed on the same base and on the same side of it are built between the same parallels. According to Hilbert, that is to say, both such triangles have equal altitudes [Hilbert (1992), p. 128]. For this demonstration, Euclid makes use of the principle according to which the whole is greater than any of its parts. Let us look to the theorem:

> Let the equal triangles ABC, DBC be on the same base BC, and on the same side of it: they shall be between the same parallels.
> Join AD. AD shall be parallel to BC. For if it is not, through A draw AE parallel to BC, meeting BD at E, and join EC.
> Then the triangle ABC is equal to the triangle EBC, because they are on the same base BC, and between the same parallels BC, AE [and because triangles with the same base, and between the same parallels, are equal: Proposition 37]. But the triangle ABC is equal to the triangle DBC [, by hypothesis]. Therefore, also the triangle DBC is equal to the triangle EBC, the greater [the whole] to the less [the part]; which is impossible. Therefore, AE is not parallel to BC.

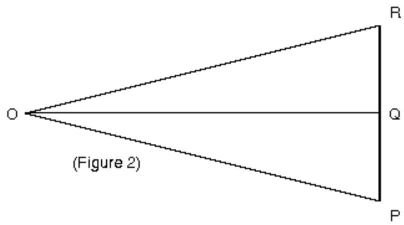
In the same manner it can be shown, that no other straight line through A but AD is parallel to BC; therefore AD is parallel to BC.
Wherefore, *equal triangles*, etc. Q.E.D. [Euclid (1948), p. 42 and (1920), pp. 94-95]. (See figure 1)



(Figure 1)

In the body of his work [Hilbert (1992), p. 64, § 19], long before the passages where the non-Archimedean geometry that we are studying here is built, Hilbert asserts that it is possible to construct a geometry with axioms I-IV, setting aside axiom III 5 (which is assumed in a restricted version) and in which theorem 48 of his own *Foundations of Geometry* (equivalent to Euclidean theorem 29) would not be valid. As a consequence, the fundamental axiom "the whole is greater than any of its parts" would not be valid in such geometry either. He then refers his reader to Appendix II, from page 127 on.

What we discover on pages 127 ff. is that Hilbert applies the non-Archimedean geometry of Appendix II to the following instance:[21] Consider the right triangle OQP, and another right triangle with the same base OQ, and a third vertex R that constitutes the reflection image of the point P with respect to the segment OQ. That is to say, R lies on the same line perpendicular to OQ on which lies P and at the same distance from OQ that P, but in an opposite side. It should happen that the two segments OP and OR have the same length. But within the geometry defined in Appendix II this does not happen. In order to compare both segments, in accordance with the rules defined previously, one rotates OP onto the axis x (on which the segment OQ lies, the point O of which lies on the origin), so that its extension would be that of the ray that goes from the origin to a new point [Hilbert (1992), p. 123]. The same thing is done with the segment OR, applying the rules defined in Appendix II. The length of OP is found to be different from the length of OR. From this finding Hilbert concludes that, "The hypotenuses of two symmetrically situated right triangles with coinciding legs are different and hence the images of segments under a reflection in a line are not necessarily equal to those in the original figure" [Hilbert (1992), p. 126].[22] (See figure 2)



(Figure 2)

Hilbert asserts that in his non-Archimedean geometry Pythagoras' theorem is valid because Euclid only made use of equipositioned triangles in demonstrating it, so that applying the restricted version of axiom III 5 one can prove Pythagoras' theorem [Hilbert (1992), p. 127]. If one applies Pythagoras' theorem to those triangles in figure 2, then with the hypotenuses of both triangles, one can form rectangles of equal sides (squares), which would be equicomplementable because one would be able to carry them to each other by congruent mappings. But, because the hypotenuses are not equal, the theorem according to which a rectangle decomposed into triangles cannot be filled again completely if one of the triangles is left apart is not valid [Bernays (1992), p. 220]. Thus, the concepts of equicomplementability and equal area are not equivalent in the context of Appendix II. "Equicomplementability" means precisely a relationship between two figures such that they can be filled exactly by the same [smaller] geometrical figures. But in this new context of Appendix II one says that two geometrical figures are equicomplementable if they can be carried to each other by congruent mappings. Thus, the square constructed on OP would be equicomplementable with the one constructed on OR, even if the second could fit within the first [for being smaller] [Bernays (1992), p. 219].[23]

Theorem 29 of Euclid is proved on page 68 by showing that if two equicomplementable triangles–that is to say, *in that context*, two triangles with the same area–have the same base, they also have the same altitude. But in the context of Appendix II rectangles and triangles with different sides or bases and altitudes could be equicomplementable. Theorem 29 of Euclid is proved on page 68 by "using the concept of area" [Hilbert (1992), p. 128], a concept that now has been excluded, that "loses its meaning without the broader form (III 5) of the axiom of triangle congruence" [Hilbert (1992), p. 127], because one and the same triangle can have different areas according to which side is chosen as its base for the calculation. This can be seen in the very triangle of our example, OQR, and following the rules established in Appendix II in order to calculate the magnitude of the segments [Hilbert (1992), p. 127]. Thus, Theorem 29 of Euclid would not be valid in Appendix II, precisely because here the principle according to which the whole is *greater* than any of its parts cannot be applied.

But, can we stretch this conclusion in order to claim that the human mind can come to free itself from an axiom that has always been seen as fundamental and that seems presupposed in any understanding of extended area? Can one state that geometry can be constructed in a completely arbitrary way, to the point of rejecting some of its fundamental axioms? With the purpose of answering these questions in the light of Hilbert's works, one must consider the end that the author is seeking when he writes these pages of the appendix, the applicability of this geometry to physical reality and its mathematical reception, plus the internal meaning of what Hilbert does.

As regards the end that Hilbert seeks, one can say that it is multiple. First, he shows that even if one presupposes the restricted form of axiom III 5, one cannot prove the broader form without the axioms of continuity (V 1 and V 3). At least in an implicit way, he establishes also that the Archimedean axiom of continuity is independent from the others. Thus, it cannot be derived from them if it is not postulated explicitly [Hilbert (1992), p. 41]. He, finally, "shed new light on the logical connection of the theorem [of the equality of the base angles] of the isosceles triangle with the other elementary theorems of plane geometry, in particular, with the theory of area" [Hilbert (1992), pp. 114-115].

Arbitrary definitions, designed with these purposes, should not lead us to conclude that geometry is or can be completely conventional. Hilbert wants to reach knowledge concerning "well established and elaborated mathematical entities," "in retrospective" [Corry (1997), p. 115]. Through the example of Appendix II, he shows neither that "guashi equals balls" nor that "beer-mugs equal chairs," but precisely the relationships between axioms and theorems well established in accepted theories. Apparently arbitrary definitions, then, accomplished an end relevant for geometry, even if they could not be received within an accepted mathematical or physical theory.

In some passages, Hilbert asserts that non-Archimedean geometries are an extension of geometry (in the case with which we are dealing, such an extension leads, through a system of conventions, to the cessation of any talk of area). Something similar had happened with complex numbers, states Hilbert, because their first appearance surpassed the axiomatic of arithmetic, forcing its enlargement. We can argue that in some sense this had happened even before with negative numbers. The analysis of fundamental notions can lead one to postulate new notions that are useful to better understand the genus subject to study. In the case of negative numbers, the meaning of the addition and its applicability to a diversity of physical or moral entities, like debts, seems clear. In other cases the meaning is less clear. The applicability to real beings (not to mere beings of reason), however, leads one to think that the new notions have to do with reality, at least in the sense that they are entities of reason that allow one to better know or to formulate reality. Something similar occurs in natural language with the use of negative adverbs or of conjunctions. These words are beings of reason that often are not the similarity of any real being, but without them we would not be able to understand or to express real beings. In Leo Corry's exposition of the axiomatization of geometry by Hilbert, although some instances of application of non-Euclidean geometries are suggested, there lacks any example of application of non-Archimedean geometries.

I ignore whether or not this geometry of the appendix has had some true reception in mathematical theory. I know, instead, that even Hilbert himself thinks that the Archimedean axiom is necessary for the application of mathematics to any measurement of physical quantities because without that axiom quantities would not be comparable with one another. Astronomy is based precisely on the commensurability of celestial and terrestrial dimensions; and atomic physics, on the applicability of the division of our macroscopic measurements to the microscopic world. I know also that Hilbert thinks that this necessity could be understood as a result of his research that has demonstrated the independence of the axiom of continuity and, thus, its central character both in mathematical as well as in physical theories, which theories could not substitute the axiom of triangle congruence for the axiom of continuity.[24]

What we have asserted allows us to open a brief parenthesis. If a mathematical theory is relevant, if it genuinely flows from the genus subject to science, it can receive applications unforeseen at the time of its formulation. Why? Because, as Aristotle and Aquinas would have told us, quantity is the accident through which all the other accidents join the sensible substance. Due to the fundamental character of quantity, even the qualities of sensible beings have quantitative dimensions that can be submitted to the study of mathematical physics [Aquinas (1955), book I, lesson 2, § 17; (1956), III, q. 77, a 2, c. Maritain (1995), p. 152].[25] Aristotle and Aquinas clearly knew this fact as regards the subject matters of astronomy, optics, music, and mechanics. (They attempted, however, to understand the essences to the extent allowed to human acumen. They did not limit their research to the quantitative expression of some of the essences' properties).[26] In this Aristotelian manner it is easy to give an account of the "ever-recurring interplay between thought and experience" as Hilbert poses it [Hilbert (1902), p. 440], or the applicability of algebra to physics.

Even if this geometry of the appendix has been received by mathematical science, even if it flows from some kind of discrete quantity, it cannot prove the basis of thinking that the human mind can be liberated from the axiom according to which a whole extension must be greater than a partial extension. Instead, this geometry only can be a grounding to consider that any axiom is applicable only when the meanings of the terms that constitute its content are present in the object under study.[27] If one is not concerned with the concept of extended area properly speaking, the axiom might be inapplicable. Indeed, in Appendix II one is not comparing a whole with its parts, but rather segments and geometrical figures with the result of mappings and rotations that are performed on them. Neither is one comparing the area of a triangle with the (smaller) area of the same triangle. But one is comparing the multiplication of quantities that, according to more or less arbitrary conventions, correspond to the value of the base or the altitude of a triangle. One is comparing also a segment (OR) with another (OP), through a rotation, and concluding that the two segments which in the context of areas should have the same magnitude, in the context of this special geometry have different magnitudes because the very notion of magnitude has been modified by arbitrary definitions of the rules of rotation and mapping and of the ordination of numbers.[28]

We should consider an additional point. In the definition of rules that leads us to consider a reality different from area, because such rules make use of concepts that have to do with extended area (but precisely in order to point towards a new kind of abstracted reality), one has to presuppose the axiom according to which the whole is greater than any of its parts and perhaps even the axiom of continuity. Indeed, without these axioms words like "number," "equal," "added," "subtracted," "multiplied," "divided," "greater," "smaller," "sine," "co-sine," "point," "line" would have no meaning. Hilbert establishes the non-Archimedean system of numbers that works as the basis of the appendix precisely making use of those words [Hilbert (1992), pp. 115-117]. This observation does not lessen the independence of the axioms. What Hilbert attempts to prove is proven. That is to say, the axiom of continuity cannot be obtained unless it is postulated as an axiom and without the axiom of continuity the axiom of triangle congruence cannot be proved.

I think that these and similar philosophical reflections could be applied to all the other non-Pythagorean geometries which Hilbert considers in his work *Grundlagen der Geometrie*.[29]

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NOTES

[1] Let us check some examples: I- Axioms of incidence. 1- "For every two points A, B there exists a line *a* that contains each of the points A, B." (ibid., p. 3) II-

Axioms of order. 1- "If a point B lies between a point A and a point C, then the points A, B, C are three distinct points of a line, and B then also lies between C and A." (ibid., p. 5) III- Axioms of congruence. 1- "If A, B are two points on a line *a*, and A' is a point on *a'*, then it is always possible to find a point B' on a given side of the line *a'* through A', such that the segment AB is congruent or equal to the segment A'B'." (ibid., p. 10) 5- If two sides of a triangle are congruent to two sides of another triangle and the correspondent angles that those sides form are also congruent, then the other two angles of each of those triangles are also congruent to the other two angles of the other triangle (ibid., p. 12). IV- Axiom of parallels, which will be explained in the text. V- Axioms of continuity. 1- "If AB and CD are any segments, then there exists a number *n* such that *n* segments CD constructed contiguously from A, along the ray from A through B, will pass beyond the point B." (ibid., p. 26)

[2] I owe the awareness about this aspect of Hilbert's work to fruitful conversations with Don Howard and Katherine Brading.

[3] These remarks will be brief for two reasons: the space allowed in journals for papers is limited, on the one hand; and, on the other, I will rely on my previous research concerning the nature of mathematics. I published some results of it in the paper "Sobre la realidad de las matemáticas." *Areté* XV, n.º 1 (2003), pp. 35-62.

[4] Manuscript of a course taught in Göttingen in 1905, quoted by Corry (1997), p. 130.

[5] As Stephan Körner noted, another purpose of Hilbert's work was to show the consistency of geometry's axiomatic system [Körner (1960) pp. 75-84]. This purpose was severely impaired by Gödel's works. Gödel's second theorem, for instance, demostrated that the consistency of a formal system is not provable within the system [Körner (1960) p. 95].

[6] This hesitation concerning the concept of "truth" is, perhaps, another manifestation of the Kantian influence.

[7] See the manuscript of the course of 1905, nn. 36-37, quoted by Corry (1997), p. 127. See also, Hilbert, *Foundations of Geometry*, p. 2, where the author holds that geometrical axioms are connected to our spatial intuition. Even though their author does not even guess so, these texts bear a strong similarity with Aristotelian theory: mathematics according to such theory is the fruit of a form (quantity) abstracted from experience. We are able to reflect on that experience by postulating as hypotheses the fundamental notions in order to construct demonstrations. This construction, moreover, can be done in an inventive way, through dialectics, or in a systematic way, as Euclid did later.

[8] Competence in science does not guarantee competence in the philosophy of science.

[9] Stephan Körner in *The Philosophy of Mathematics. An Introduction Essay*, also points out –as we do– the air of paradox that surrounds Hilbert's work as regards the origin of axioms. See pp. 98-106, but especially page 98.

[10] A contemporary statement by Max Weber shows that the situation in social science and the sciences of the spirit was very much alike. [Mitzman (1970), p. 209]

[11] The influence of Kant in Hilbert is quite clear [Körner (1960), pp. 72-74]. In a letter to Schumacher written on November 1st of 1844, Gauss rightly praised Aristotle as the only Philosopher able to give definitions in accordance with science. Among those who gave wrong definitions, he counts Kant for whom, however, he al-

ways felt some respect. (He did not keep such respect for the Idealist Philosophers posterior to Kant, among whom we find Hegel. Gauss indeed named the philosophy of Hegel *insania*, in a letter to Schumacher) [Dunnington (1955), pp. 313-317].

[12] Gauss knew, for example, that the notion of space of the geometers has to do with experience, even though it is not a mirror-like image of the world. He knew, thus, that in knowledge our reason is active. For this motive he appreciated the *Critique of Pure Reason*. He thought, however, that the Kantian theory of space was very wrong. See the opinions of Gauss on philosophers and philosophy, and fragments of his letters, in Dunnington [(1955), pp. 313-317].

[13] See Stephan Körner [(1960), pp. 72-74 and 98]. According to this author, the other part of the origin of the axioms belongs to an analysis of the (Kantian) Ideas of transfinite arithmetic. The consistency of both parts of mathematics, finite and transfinite, can be proven through the construction of formal systems.

[14] Leo Corry [(1997), p. 120] claims that in this passage Hilbert holds a Leibinizian preestablished harmony. I think the text is clear in denying it. Concerning this point as regards the applicability of mathematics to physics according to Hilbert, see Stephan Körner [(1960), p. 88].

[15] Notice that we are dealing specifically with geometries that are non-Euclidean because they are not plane and not concerning ourselves with those non-Euclidean geometries which are so because they contain more than three dimensions. These other geometries are metaphorical extensions to non-spatial relations of the real world [Casanova (2003), p. 37]. Jacques Maritain, quoting Aquinas, who in his turn quoted Ptolemy, has shown that real dimensions are determined by the number of perpendiculars that can meet at a line on a point in physical space [Maritain (1995), p. 43].

[16] There are in mathematics non-constructive demonstrations, of course, but these are always based upon a context that has been constructed previously.

[17] The explanation contained in this and the previous paragraph is an exact response to the problems that, according to Stephan Körner, raise the "existence propositions" of mathematics [(1960), pp. 174-176]. And the explanation answers to such problems within a realistic, Aristotelian conception of mathematics.

[18] In the case of Euclidean demonstrations the cause is only formal, of course. When the theory of relativity fused physics and geometry, it built a conceptual system with which one can "save the phenomena." Such a system, however, is far from both geometrical and physical entities. Saunders Mac Lane holds, like us, that geometry is not *a priori*. He also holds that geometry is abstract and for this reason Euclidean geometry was completely unaffected by the deviation of light-rays [(1986), p. 411].

[19] See Manuscript of the course of 1905, n. 39 [Corry (1997), p. 128]. In the light of these considerations, and others made earlier, one can grasp that the exposition and critique that A. D'Abro made of Hilbert [D'Abro (1959), pp. 191-213] was mistaken or shallow in many respects, but perhaps the published texts of Hilbert were confusing enough to allow room for this kind of reading. Thus, 1) D'Abro held that according to Hilbert axiomatic systems do not define their terms and, as a consequence, they can be applied to any reality, because mathematics does not refer to any particular reality but to relations. This would explain why systems can be applied to physics and why Hilbert found an equivalence between geometry and arithmetic (see ibid., p. 197). To this I reply that the application of geometry to physics does not presuppose that the geometrical axioms have to do with mere relations and do not have to

do with abstract quantity. Moreover, Hilbert himself thought that geometry (not arithmetic) was a natural science. 2) D' Abro underlined too much the opposition between Poincaré and Hilbert, as if the latter did not hold that intuitions have a place in mathematics and that axiomatization is only an exercise posterior to the finding of mathematical truths (*ibid*., pp. 191-213; in particular, pp. 198 and 202-204). D' Abro underlined the opposition so much, in fact, that he came to conclude that theoretical physicists do not need to pay any attention to the nature of mathematics, even though they ground their research on a mathematical scheme, because the nature of mathematics is an obscure issue (see ibid., p. 212). To which I reply that what indeed happens is that these issues concerning the nature of mathematics and the relationships between mathematics and physics are the job of philosophy, not of physics.

[20] This means that the angle ABC would not equal the angle CAB [Hilbert (1992), p. 113].

[21] The instance is constructed in Hilbert (1992), pp. 125-126, but the relevant consequences for our present purpose are drawn from this instance later, *ibid*., pp. 127-128.

[22] Figure 2 is taken from page 125.

[23] Frankly, I do not fully understand how this can be true according to what was stated by Hilbert [(1992), pp. 121-122], because OP and OR are not congruent. But this does not affect my reasoning because if there were a mistake in Hilbert's and Bernay's projections, my conclusions would be valid and even stronger: the principle according to which a whole extension is greater than any of its parts holds when the notion of extended area is at stake.

[24] Hilbert also thinks, however, that the Archimedean axiom of continuity has to be tested by experiments, in a way similar to that with which Gauss tried to prove the theorem of the addition of the internal angles of a triangle. See manuscript of the course of 1905, quoted by Corry [(1997), pp. 125-126]. Indeed, due to the abstraction of geometry, the theorem of the internal angles of a triangle cannot be proved experimentally but through analysis of a plane space. If we have a different kind of space, we will have to analyze this other kind in a different way. And if such is the case of astronomy, then this science will have to use a geometry that analyzes the other kinds of space. I consider that Hilbert is wrong also as regards the experimental proof of the Archimedean axiom of continuity. Continuity is a notion that is previous to that of the addition of the internal angles of a triangle. Faced with the paradoxes of Zeno, Aristotle realized a metaphysical analysis of common experience that seems sufficient to persuade us of the strength of the axiom.

[25] Even a mammal's being depends in some way on the contraction and expansion of a small quantity of sensible extension which we call "heart." In Modern Philosophy, quantity became the *res extensa* of Descartes, the primary qualities of Locke or the space of Kant, which is the Cartesian *res extensa* but without the *res* [Kant (2001), p. 30]. Gottlob Frege held an alternative view about the object of mathematics, as is well known. But he also stated that he had no objections to the Newtonian way of conceiving that object [(1968, pp. 25-26], and the Newtonian way is the same as the Aristotelian, at least in the case of arithmetic.

[26] I think that if we remove neo-Pythagoreanism from the assertions of Niehls Bohr concerning the complementarity of methods of approximation to biological reality, we would see that the different ways of approach, the one that reduces biological

processes to their physic-mathematical component and the one that considers the relationship of such processes and the living being as a whole, are compatible and they even need each other. Without a vision of the whole, the living being, physic-mathematical analysis of one of the processes would lose its meaning.

[27] Aristotle and Aquinas knew this pre-requisite for the applicability of the axioms. This knowledge is implicit in *Metaphysics* IV 3, 1005b15-16. The knowledge of any *being* implies the (presence in the soul of the) principle of non-contradiction. It is also implicit in *Posterior Analytics* I, 1 and 7; II, 19. In this last text, the grasping of the universal (term) causes the principle.

[28] In every instance of a supposed superseding of a fundamental principle within physics or mathematics that I have been able to examine, I have found something similar to what is said in the text. Either one was dealing with a pseudo-principle (like the Kantian or Laplacian of causality), or one had not understood in a correct way the principle or the context to which it was applied. a) Thus, for example, when one claims that an infinite contained in another equals the continent (so that the whole would not be greater than any of its parts), one overlooks that in the very notion of infinite one finds that it cannot be a "part" of another thing, even though one can say that there are some infinites greater than others. It is obvious that between the integer 1 and the integer 2 there are infinite rational and irrational numbers and that, however, this infinite is contained in another infinite (the one that contains every real number). But the first infinite of this example cannot be called a part of the second. To understand this, think that for the same reason a point cannot be called a "part" of the line on which it lies, even if it is contained in the line, as Aristotle showed in book VI of his *Physics*. b) In a similar way, the supposed violation of the principle of *tertio excluso* by quantum mechanics as it is presented by Weizsäcker has to do with a conception of reality completely non-Aristotelian. In an Aristotelian conception, an intermediate state of the kind to which Weizsäcker refers is perfectly possible. Between being in act a statue of Hermes and not being at all, there can be another state: being a statue of Hermes in potency. There one would not find a violation of the Aristotelian principle of *tertio excluso*. This coincides with the description Heisenberg does of the ontology that underlies Weizsäcker's logic, "[...] If one considers the word 'state' as describing some potentiality rather than a reality–one may even simply replace the term 'state' by the term 'potentiality'–then the concept 'coexisting potentialities' is quite plausible, since one potentiality may involve or overlap other potentialities" [Heisenberg (1962), p. 185]. Quine added in 1970 and 1986 that when one asserts that a principle of classical logic, like that of *tertio excluso*, has been superseded, what happens is that one "changes the subject," by changing the meaning of logical connectors (conjunctions, disjunctions, negations, etc.), in order to fulfill a goal (in quantum mechanics or in intuitionist mathematics) that can be fulfilled without changing the traditional meanings [Quine (1986), pp. 80-86]. I paid attention to this text thanks to Martin Kurd and J. A. Cover [(1998), pp. 380-381], who use the edition of 1970. The change of subject to which Quine refers is what happens with the principle of the whole being greater than any of its parts in the work of Hilbert: after the definitions of Appendix II, the concept of area is no longer the object of the mathematical investigations. Donald Gillies agrees with Quine in that according to him "quantum logic" was not successful in the solution of the problems of microphysics: see Gillies [(1998), pp. 317 and 319]. Gillies presupposes, however, that there can be other cases in which this change

of logic can be useful and mentions the instance of non-monotonic logics of artificial intelligence, as if they were a violation of Aristotelian logic. Gillies ignores that such non-monotonic logics often are, in fact and in many ways, very close to Aristotle's dialectic or topical method.

[29] The geometry of Appendix II is in some sense Pythagorean (because it accepts Pythagoras' theorem, as we have shown already), and in some sense it is not because the addition of two sides of a triangle would not necessarily be greater than the third side [Hilbert (1992), p. 128].

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