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On Nicely Smooth Banach Spaces

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1. Introduction

We work with *real* Banach spaces. We will denote by $B(X)$, $S(X)$ and $B[x, r]$ respectively the closed unit ball, the unit sphere and the closed ball of radius $r > 0$ around $x \in X$. We will identify any element $x \in X$ with its canonical image in X^{**}. All subspaces we usually consider are *norm* closed.

Definition 1.1. (a) We say $A \subseteq B(X^*)$ is a norming set for X if $\|x\| = \sup\{x^*(x) : x^* \in A\}$, for all $x \in X$. A closed subspace $F \subseteq X^*$ is a norming subspace if $B(F)$ is a norming set for X.

(b) A Banach space X is

(i) *nicely smooth* if X^* contains no proper norming subspace;

(ii) has the *Ball Generated Property* (BGP) if every closed bounded convex set in X is ball-generated, i.e., intersection of finite union of balls;

(iii) has *Property* (II) if every closed bounded convex set in X is the intersection of closed convex hulls of finite union of balls, or equivalently, w*-points of continuity (w*-PCs) of $B(X^*)$ are norm dense in $S(X^*)$ [5];

(iv) has the *Mazur Intersection Property* (MIP) (or, Property (I)) if every closed bounded convex set in X is intersection of balls, or equivalently, w*-denting points of $B(X^*)$ are norm dense in $S(X^*)$ [8].

Clearly, the MIP implies both Property (II) and the BGP. In this work, we obtain a sufficient condition for the BGP, and show that Property (II) implies the BGP, which, in turn, implies nice smoothness.

Notice that if a separable space is nicely smooth then it has separable dual. (The converse is false, since a dual space is nicely smooth if and only if it is reflexive.) It follows that if nice smoothness was inherited by subspaces, a nicely smooth space would necessarily be Asplund. However, recent work of Jiménez Sevilla and Moreno [15] shows that this is not true. More precisely, they showed that any Banach space can be isomorphically embedded in a Banach space with the MIP, a property stronger than nice smoothness.

In this work, we obtain some necessary and/or sufficient conditions for a space to be nicely smooth, and show that they are all equivalent for separable or Asplund spaces. These sharpen known results. We observe that every equivalent renorming of a space is nicely smooth if and only if it is reflexive. We also show that if X is nicely smooth, $X \subseteq E \subseteq X^{**}$ and E has the finite-infinite intersection property ($\infty.f$.IP) (Definition 2.15), then $E = X^{**}$. In particular, X is nicely smooth with $\infty.f$.IP if and only if X is reflexive. Coming to stability results, we show that the class of nicely smooth spaces is stable under c_0 and ℓ_p sums ($1 < p < \infty$) and also under finite ℓ_1 sums. We show that while nice smoothness is not a three space property, existence of a nicely smooth renorming is a three space property. We show that the Bochner L_p spaces ($1 < p < \infty$) are nicely smooth if and only if X is both nicely smooth and Asplund. And for a compact Hausdorff space K, $C(K, X)$ is nicely smooth if and only if K is finite and X is nicely smooth. We also study nice smoothness of certain operator and tensor product spaces.

2. Main Results

We record the following corollary of the proof of the Hahn-Banach Theorem (see e.g., [18, Section 48]) for future use.

Lemma 2.1. *Let X be a normed linear space and let Y be a subspace. Suppose $x_0 \notin Y$ and $y^* \in S(Y^*)$. Then*

$$\sup\{y^*(y) - \|x_0 - y\| : y \in Y\} \leq \inf\{y^*(y) + \|x_0 - y\| : y \in Y\}$$

and α lies between these two numbers if and only if there exists a Hahn-Banach (i.e., norm preserving) extension x^* of y^* with $x^*(x_0) = \alpha$.

For a Banach space X, let us define

$$\mathcal{O}(X) = \{x^{**} \in X^{**} : \|x^{**} + x\| \geq \|x\| \text{ for all } x \in X\}\,.$$

And we have the following characterization:

Proposition 2.2. *For a Banach space X, the following are equivalent:*

(a) *X is nicely smooth.*

(b) *For all $x^{**} \in X^{**}$,*

$$\bigcap_{x \in X} B[x, \|x^{**} - x\|] = \{x^{**}\}\,.$$

(c) $\mathcal{O}(X) = \{0\}$.

(d) *For all nonzero $x^{**} \in X^{**}$, there exists $x^* \in S(X^*)$ such that every Hahn-Banach extension of x^* to X^{**} takes nonzero value at x^{**}.*

(e) *Every norming set $A \subseteq B(X^*)$ separates points of X^{**}.*

Proof. Equivalence of (a) and (b) is [12, Lemma 2.4].

(a) ⇔ (c) [10, Lemma I.1] shows that F is a norming subspace of X^* if and only if $F^{\perp} \subseteq \mathcal{O}(X)$. Hence the result.

(c) ⇔ (d) Each of the statements below is clearly equivalent to the next:

(i) $x^{**} \in \mathcal{O}(X)$,

(ii) $\|x\| \leq \|x^{**} - x\|$ for all $x \in X$,

(iii) for every $x^* \in S(X^*)$,

$$\sup\{x^*(x) - \|x^{**} - x\| : x \in X\} \leq 0 \leq \inf\{x^*(x) + \|x^{**} - x\| : x \in X\}\,,$$

(iv) every $x^* \in S(X^*)$ has a Hahn-Banach extension x^{***} with $x^{***}(x^{**}) = 0$ (Lemma 2.1).

Hence the result.

(a) ⇔ (e) Since any norming set spans a norming subspace, this is clear. ∎

We note a characterization of the BGP.

THEOREM 2.3. *A Banach space X has the BGP if and only if for every $x^* \in X^*$ and $\varepsilon > 0$, there exists w*-slices $S_1, S_2, \dots, S_n$ of $B(X^*)$ such that for any $(x_1^*, x_2^*, \dots, x_n^*) \in \prod_{i=1}^n S_i$, there are scalars $a_1, a_2, \dots, a_n$ such that $\|x^* - \sum_{i=1}^n a_i x_i^*\| \leq \varepsilon$.*

Proof. Observe that X has the BGP if and only if every $x^* \in X^*$ is ball-continuous on $B(X)$ [11, Theorem 8.3]. Now the result follows from the characterization of such functionals obtained in [5, Theorem 2]. ∎

This leads to the following, more tractable sufficient condition for the BGP.

DEFINITION 2.4. A point x_0^* in a convex set $K \subseteq X^*$ is called a w*-small combination of slices (w*-SCS) point of K, if for every $\varepsilon > 0$, there exist w*-slices $S_1, S_2, \dots, S_n$ of K, and a *convex* combination $S = \sum_{i=1}^n \lambda_i S_i$ such that $x_0^* \in S$ and $\operatorname{diam}(S) < \varepsilon$.

PROPOSITION 2.5. *If X^* is the closed linear span of the w*-SCS points of $B(X^*)$, then X has the BGP.*

Proof. Let $x^* \in X^*$ and $\varepsilon > 0$. Since the set of w*-SCS points of $B(X^*)$ is symmetric and spans X^*, there exist w*-SCS points $x_1^*, x_2^*, \dots, x_n^*$ of $B(X^*)$, and *positive* scalars $a_1, a_2, \dots, a_n$ such that $\|x^* - \sum_{i=1}^n a_i x_i^*\| \leq \varepsilon/2$. By definition of w*-SCS points, for each $i = 1, 2, \dots, n$, there exist w*-slices $S_{i1}, S_{i2}, \dots, S_{in_i}$ of $B(X^*)$, and a convex combination $S_i = \sum_{k=1}^{n_i} \lambda_{ik} S_{ik}$ such that $x_i^* \in S_i$ and $\operatorname{diam}(S_i) < \varepsilon/(2\sum_{i=1}^n a_i)$. Now, for any $(x_{ik}^*) \in \prod_{i=1}^n \prod_{k=1}^{n_i} S_{ik}$,

$$\begin{aligned}\left\| x^* - \sum_{i=1}^n \sum_{k=1}^{n_i} a_i \lambda_{ik} x_{ik}^* \right\| &\leq \left\| x^* - \sum_{i=1}^n a_i x_i^* \right\| + \sum_{i=1}^n a_i \left\| x_i^* - \sum_{k=1}^n \lambda_{ik} x_{ik}^* \right\| \\ &\leq \varepsilon/2 + \sum_{i=1}^n a_i \operatorname{diam}(S_i) \leq \varepsilon\,.\end{aligned}$$

Hence by Theorem 2.3, X has the BGP. ∎

COROLLARY 2.6. *Property* (II) *implies the* BGP, *which, in turn, implies nicely smooth.*

Proof. Since X has Property (II), w*-PCs of $B(X^*)$ are norm dense in $S(X^*)$, and a w*-PC is necessarily a w*-SCS point (this follows from Bourgain's Lemma, see e.g., [17, Lemma 1.5]). Thus, Property (II) implies the BGP.

That the BGP implies nicely smooth is proved in [11, Theorem 8.3]. But here is an elementary proof.

Let F be a norming subspace of X^*. Then $B(X)$ is $\sigma(X, F)$-closed, so that every ball-generated set is also $\sigma(X, F)$-closed. But if every closed bounded convex set is $\sigma(X, F)$-closed, then $F = X^*$. ∎

We now obtain a localization of [5, Theorem 3.1] and [9, Lemma 6].

PROPOSITION 2.7. *Let X be a Banach space. Let $x_0^* \in S(X^*)$ and $x_0^{**} \in X^{**}$. The following are equivalent:*

(a) *every Hahn-Banach extension of x_0^* to X^{**} takes the same value at x_0^{**};*

(b) *x_0^{**}, considered as a function on $(B(X^*), w^*)$, is continuous at x_0^*;*

(c) $\sup\{x_0^*(x) - \|x_0^{**} - x\| : x \in X\} = \inf\{x_0^*(x) + \|x_0^{**} - x\| : x \in X\}$;

(d) *for any $\alpha \in \mathbb{R}$, if $x_0^{**}(x_0^*) \neq \alpha$, then there exists a ball B^{**} in X^{**} with centre in X such that $x_0^{**} \in B^{**}$ and B^{**} and x_0^{**} lies in the same open half space determined by x_0^* and α.*

Proof. (a) ⇔ (b) This is a natural localization of the proof of [13, Lemma III.2.14]. We omit the details.

(a) ⇔ (c) Follows from Lemma 2.1.

(c) ⇒ (d) Suppose $x_0^{**}(x_0^*) > \alpha$. By Lemma 2.1, $\inf\{x_0^*(x) + \|x_0^{**} - x\| : x \in X\} > \alpha$. By (c), it follows that $\sup\{x_0^*(x) - \|x_0^{**} - x\| : x \in X\} > \alpha$. So, there exists $x \in X$ such that $x_0^*(x) - \|x_0^{**} - x\| > \alpha$. Put $B^{**} = B^{**}[x, \|x_0^{**} - x\|]$. Clearly, $x_0^{**} \in B^{**}$, and $\inf x_0^*(B^{**}) = x_0^*(x) - \|x_0^{**} - x\| > \alpha$. Similarly for $x_0^{**}(x_0^*) < \alpha$.

(d) ⇒ (c) Suppose $x_0^{**}(x_0^*) > \alpha$. By (d), there exists a ball $B^{**} = B^{**}[x, r]$ in X^{**} such that $x_0^{**} \in B^{**}$ and $\inf x_0^*(B^{**}) > \alpha$. This implies $\|x_0^{**} - x\| \leq r$ and $\inf x_0^*(B^{**}) = x_0^*(x) - r > \alpha$. It follows that $x_0^*(x) - \|x_0^{**} - x\| > \alpha$, whence $\sup\{x_0^*(x) - \|x_0^{**} - x\| : x \in X\} > \alpha$. Since α was arbitrary, it follows that $\sup\{x_0^*(x) - \|x_0^{**} - x\| : x \in X\} \geq x_0^{**}(x_0^*)$. Similarly, $\inf\{x_0^*(x) + \|x_0^{**} - x\| : x \in X\} \leq x_0^{**}(x_0^*)$. The result now follows from Lemma 2.1. ∎

COROLLARY 2.8. [5, Theorem 3.1] *For a Banach space X and $f_0 \in S_{X^*}$, the following are equivalent:*

(i) *f_0 is a w*-w PC of B_{X^*};*

(ii) *f_0 has a unique Hahn-Banach extension in X^{***};*

(iii) *for any* $x_0^{**} \in X^{**}$ *and* $\alpha \in \mathbb{R}$, *if* $f_0(x_0^{**}) \neq \alpha$, *then there exists a ball* B^{**} *in* X^{**} *with centre in* X *such that* $x_0^{**} \in B^{**}$ *and* B^{**} *and* x_0^{**} *lies in the same open half space determined by* f_0 *and* α.

We now identify some necessary and some sufficient conditions for a space to be nicely smooth.

Definition 2.9. For $x \in S(X)$, let $D(x) = \{f \in S(X^*) : f(x) = 1\}$. The set valued map D is called the duality map and any selection of D is called a support mapping.

Theorem 2.10. *For a Banach space* X, *consider the following statements:*

(a) X^* *is the closed linear span of the w*-weak PCs of* $B(X^*)$.

(b) *Any two distinct points in* X^{**} *are separated by disjoint closed balls having centre in* X.

(b_1) *For every* $x^{**} \in X^{**}$, *the points of w*-continuity of* x^{**} *in* $S(X^*)$ *separates points of* X^{**}.

(b_2) *For every nonzero* $x^{**} \in X^{**}$, *there is a point of w*-continuity* $x^* \in S(X^*)$ *of* x^{**} *such that* $x^{**}(x^*) \neq 0$.

(c) X *is nicely smooth.*

(d) *For every norm dense set* $A \subseteq S(X)$ *and every support mapping* ϕ, *the set* $\phi(A)$ *separates points of* X^{**}.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) *and* (a) $\Rightarrow$ (b_1) $\Rightarrow$ (b_2) $\Rightarrow$ (c) $\Rightarrow$ (d). *Moreover, if* X *is an Asplund space (or, separable), all the above conditions are equivalent, and equivalent to each of the following:*

(e) X^* *is the closed linear span of the w*-strongly exposed points of* $B(X^*)$.

(f) X^* *is the closed linear span of the w*-denting points of* $B(X^*)$.

(g) X^* *is the closed linear span of the w*-SCS points of* $B(X^*)$.

(h) X *has the* BGP.

Proof. (a) $\Rightarrow$ (b) Let $x_0^{**} \neq y_0^{**}$. By (a), there exists a w*-w PC $x_0^* \in B_{X^*}$, such that $(x_0^{**} - y_0^{**})(x_0^*) > 0$. Let $\alpha \in \mathbb{R}$ be such that

$$x_0^{**}(x_0^*) > \alpha > y_0^{**}(x_0^*)\,.$$

Now applying Corollary 2.8, it follows that there exists a ball B_1^{**} with centre in X with $x_0^{**} \in B_1^{**}$ and $\inf x_0^*(B_1^{**}) > \alpha$. And there exists a ball B_2^{**} with centre in X such that $y_0^{**} \in B_2^{**}$ and $\sup x_0^*(B_2^{**}) < \alpha$. Clearly, $B_1^{**} \cap B_2^{**} = \emptyset$.

(b) $\Rightarrow$ (c) Clearly, (b) implies condition (b) of Proposition 2.2.

(a) $\Rightarrow$ (b_1) $\Rightarrow$ (b_2) follows from definitions.

(b_2) $\Rightarrow$ (c) By (b_2), for every nonzero $x^{**} \in X^{**}$, there is a point of w*-continuity $x^* \in S(X^*)$ of x^{**} such that $x^{**}(x^*) \neq 0$. By Proposition 2.7, every Hahn-Banach extension of x^* to X^{**} takes the same value at x^{**}. The result now follows from Proposition 2.2(d).

(c) $\Rightarrow$ (d) We simply observe that $\phi(A)$ is a norming set for X.

Clearly, even without X being Asplund, (e) $\Rightarrow$ (f) $\Rightarrow$ (g) $\Rightarrow$ (h) $\Rightarrow$ (c), and (f) $\Rightarrow$ (a). Now if X is Asplund (if X is separable, (d) implies X^* is separable), then for $A = \{x \in S(X) : \text{the norm is Fréchet differentiable at } x\}$, and any support mapping ϕ, $\phi(A) = \{\text{w*-strongly exposed points of } B(X^*)\}$. Hence, (d) $\Rightarrow$ (e). ∎

Remark 2.11. (a) If in the first part, we simply assume that the w*-weak PCs of $B(X^*)$ form a norming set, then (a) – (c) are equivalent. And under the stronger assumption that the set

$$\{x \in S(X) : D(x) \text{ intersects the w*-weak PCs of } B(X^*)\}$$

is dense in $S(X)$, (a) – (d) are equivalent. This happens if X is Asplund.

Can any of the implications be reversed in general?

(b) In [9, Lemma 5], Godefroy proves (a) $\Rightarrow$ (c) by actually showing (b_1) $\Rightarrow$ (c). (b_2) is a weaker sufficient condition.

(c) In [4, Theorem 7], it is observed that (f) $\Rightarrow$ (h). (g) is a weaker sufficient condition.

PROPOSITION 2.12. *If every separable subspace of X is nicely smooth, then X has the BGP, and hence, is nicely smooth.*

Proof. Since for separable spaces nice smoothness is equivalent to the BGP, the result follows from the characterization of ball-continuous functionals obtained in [11, Theorem 2.4 and 2.5]. ∎

THEOREM 2.13. *A Banach space X is reflexive if and only if every equivalent renorming is nicely smooth.*

Proof. The converse being trivial, suppose X is not reflexive. Let $x^{**} \in X^{**} \setminus X$ and let $F = \{x^* \in X^* : x^{**}(x^*) = 0\}$. Define a new norm on X by

$$\|x\|_1 = \sup\{x^*(x) : x^* \in B(F)\} \quad \text{for } x \in X\,.$$

It follows from the proof of [11, Theorem 8.2] that $\|\cdot\|_1$ is an equivalent norm on X with F as a proper norming subspace. ∎

Remark 2.14. (a) In [14] the authors showed that X is reflexive if and only if for any equivalent norm, X is Hahn-Banach smooth and has ANP-III. This was strengthened in [3, Corollary 2.5] to just Hahn-Banach smooth. The above is an even stronger result with even easier proof.

(b) In [11, Theorem 8.1 and 8.2], it is shown that a subset $W \subseteq X$ is ball-generated for every equivalent renorming if and only if it is weakly compact. The global analogue of this local result would read: every equivalent renorming of X has the BGP if and only if X is reflexive. Thus in the global version, our result is somewhat stronger, except for separable spaces, though, as the proof shows, it is implicit in [11, Theorem 8.2].

Definition 2.15. A Banach space X is said to have the finite-infinite intersection property ($\infty.f$.IP) if every family of closed balls in X with empty intersection contains a finite subfamily with empty intersection. It is well known that all dual spaces and their 1-complemented subspaces have $\infty.f$.IP

Theorem 2.16. *X is nicely smooth with $\infty.f$.IP if and only if X is reflexive.*

Proof. Sufficiency is obvious.

For necessity, recall from [11, Theorem 2.8] that X has $\infty.f$.IP if and only if $X^{**} = X + \mathcal{O}(X)$. Since X is nicely smooth, $\mathcal{O}(X) = \{0\}$ and consequently, X is reflexive. ∎

Remark 2.17. Since Hahn-Banach smooth spaces (respectively, spaces with Property (II)) are nicely smooth, [3, Theorem 2.11] (respectively, [3, Theorem 3.4]) follows as an immediate corollary with a simpler proof. Indeed, we have stronger results.

THEOREM 2.18. *If X is nicely smooth, $X \subseteq E \subseteq X^{**}$ and E has the $\infty.f$.IP, then $E = X^{**}$.*

Proof. Using the Principle of Local Reflexivity and w*-compactness of dual balls, it is easy to see that E has $\infty.f$.IP if and only if any family of closed balls with centres in E that intersects in E^{**} also intersects in E.

Now if $X \subseteq E \subset X^{**}$, let $x_0^{**} \in X^{**} \setminus E$. Consider the family

$$\mathcal{B} = \{B[x, \|x_0^{**} - x\|] : x \in X\}.$$

Clearly, they intersect at $x_0^{**} \in X^{**} \subseteq E^{**}$. And since X is nicely smooth, by Proposition 2.2 (b), the intersection in X^{**} is singleton $\{x_0^{**}\}$. But then, this family cannot intersect in E. ∎

Remark 2.19. Can one strengthen this to show that if X is nicely smooth, E has $\infty.f$.IP and $X \subseteq E$, then $X^{**} \subseteq E$? Recall that c_0 is nicely smooth, and it is known [16] that if $c_0 \subseteq E$ and E is 1-complemented in a dual space, then ℓ_∞ is isomorphic to a subspace of E. Our question is isometric.

In the special case of $X = c_0$, we can prove:

PROPOSITION 2.20. *If $c_0 \subseteq E$ and E has $\infty.f$.IP, then ℓ_∞ is a quotient of E.*

Proof. Since $c_0 \subseteq E$, $\ell_\infty \subseteq E^{**}$. Let $F = E \cap \ell_\infty$. Since ℓ_∞ is "injective", let $T : E \to \ell_\infty$ be the norm preserving extension of the inclusion map from F to ℓ_∞. Note that T is identity on c_0. We will prove that T is onto.

Let $x_0^{**} \in \ell_\infty$. Let

$$\mathcal{B} = \{B[x, \|x_0^{**} - x\|] : x \in c_0\}.$$

As before, this family intersects in E^{**}, and hence in E. And since $\|T\| = 1$, if $e \in E$ belongs to this intersection, so does Te. But the intersection in ℓ_∞ is singleton $\{x_0^{**}\}$, since c_0 is nicely smooth. Thus, $Te = x_0^{**}$. ∎

And in general, we have the result under stronger assumptions on both X and E:

PROPOSITION 2.21. *If X has Property (II), E is a dual space and $X \subseteq E$, then $X^{**} \subseteq E$.*

Proof. Let $E = Y^*$. Let $i : X \to Y^*$ be the inclusion map. Then $i^* : Y^{**} \to X^*$ is onto. We will show that $i^*|_Y : Y \to X^*$ is onto. It will follow that X^* is a quotient of Y and hence, X^{**} is a subspace of $Y^* = E$.

To show that $i^*|_Y : Y \to X^*$ is onto, it suffices to check that $\overline{i^*(B(Y))} = B(X^*)$. Since X has Property (II), it suffices to check that any w*-PC $x^* \in B(X^*)$ belongs to $\overline{i^*(B(Y))}$. Let $\Lambda \in Y^{**}$ be a norm preserving extension of x^* to Y^*. Let $\{y_\alpha\} \subseteq B(Y)$ such that $y_\alpha \to \Lambda$ in the w*-topology of $B(Y^{**})$. Then $y_\alpha|_X \to x^*$ in the w*-topology of $B(X^*)$. Since x^* is a w*-PC, $y_\alpha|_X \to x^*$ in norm, i.e., $x^* \in \overline{i^*(B(Y))}$. ∎

3. Stability Results

THEOREM 3.1. *Let $\{X_\alpha\}_{\alpha\in\Gamma}$ be a family of Banach spaces. Then $X = \bigoplus_{\ell_p} X_\alpha$ $(1 < p < \infty)$ is nicely smooth if and only if for each $\alpha \in \Gamma$, X_α is nicely smooth.*

Proof. We show that $\mathcal{O}(X) = \{0\}$ if and only if for every $\alpha \in \Gamma$, $\mathcal{O}(X_\alpha) = \{0\}$.

Now, $X = \bigoplus_{\ell_p} X_\alpha$ implies $X^{**} = \bigoplus_{\ell_p} X_\alpha^{**}$, and $x^{**} \in \mathcal{O}(X)$ if and only if

$$\begin{aligned} \|x^{**} + x\|_p &\geq \|x\|_p \text{ for all } x \in X \\ \iff \sum_{\alpha\in\Gamma} \|x_\alpha^{**} + x_\alpha\|^p &\geq \sum_{\alpha\in\Gamma} \|x_\alpha\|^p \text{ for all } x \in X\,. \end{aligned}$$

It is immediate that if for every $\alpha \in \Gamma$, $x_\alpha^{**} \in \mathcal{O}(X_\alpha)$, then $x^{**} \in \mathcal{O}(X)$. And hence, $\mathcal{O}(X) = \{0\}$ implies for every $\alpha \in \Gamma$, $\mathcal{O}(X_\alpha) = \{0\}$.

Conversely, suppose for every $\alpha \in \Gamma$, $\mathcal{O}(X_\alpha) = \{0\}$. Let $x^{**} \in X^{**} \setminus \{0\}$. Let $\alpha_0 \in \Gamma$ be such that $x_{\alpha_0}^{**} \neq 0$. Then $x_{\alpha_0}^{**} \notin \mathcal{O}(X_{\alpha_0})$. Hence, there exists $x_{\alpha_0} \in X_{\alpha_0}$ such that $\|x_{\alpha_0}^{**} + x_{\alpha_0}\| < \|x_{\alpha_0}\|$. Choose $\varepsilon > 0$ such that $\|x_{\alpha_0}^{**} + x_{\alpha_0}\|^p + \varepsilon < \|x_{\alpha_0}\|^p$. Then there exists a finite $\Gamma_0 \subseteq \{\alpha \in \Gamma : x_\alpha^{**} \neq 0\}$ such that $\alpha_0 \in \Gamma_0$ and $\sum_{\alpha\notin\Gamma_0} \|x_\alpha^{**}\|^p < \varepsilon$. If $\alpha \in \Gamma_0$, then $x_\alpha^{**} \notin \mathcal{O}(X_\alpha)$. Hence, there exists $x_\alpha \in X_\alpha$ such that $\|x_\alpha^{**} + x_\alpha\| < \|x_\alpha\|$. Define $y \in X$ by

$$y_\alpha = \begin{cases} x_\alpha & \text{if} \quad \alpha \in \Gamma_0 \\ 0 & \text{otherwise} \quad . \end{cases}$$

Then we have,

$$\begin{aligned}
\|x^{**}+y\|_p^p &= \sum_{\alpha\in\Gamma}\|x^{**}_\alpha+y_a\|^p \\
&= \sum_{\substack{\alpha\in\Gamma_0\\ \alpha\neq\alpha_0}}\|x^{**}_\alpha+x_a\|^p+\|x^{**}_{\alpha_0}+x_{\alpha_0}\|^p+\sum_{\alpha\notin\Gamma_0}\|x^{**}_\alpha\|^p \\
&< \sum_{\substack{\alpha\in\Gamma_0\\ \alpha\neq\alpha_0}}\|x_a\|^p+\|x^{**}_{\alpha_0}+x_{\alpha_0}\|^p+\varepsilon \\
&< \sum_{\alpha\in\Gamma_0}\|x_\alpha\|^p=\|y\|_p^p
\end{aligned}$$

which shows that $x^{**}\notin\mathcal{O}(X)$. ∎

Remark 3.2. The above argument also works for finite ℓ_1 (or ℓ_∞) sums and shows that if X is the ℓ_1 (or ℓ_∞) sum of $X_1, X_2, \dots, X_n$, then X is nicely smooth if and only if for every coordinate space X_i is so.

However, if Γ is infinite, $X=\bigoplus_{\ell_1}X_\alpha$ is never nicely smooth as $\bigoplus_{c_0}X^*_\alpha$ is a proper norming subspace of $X^*=\bigoplus_{\ell_\infty}X^*_\alpha$.

A similar argument also shows that nice smoothness is not stable under infinite ℓ_∞ sums.

We now show that nice smoothness is stable under c_0 sums.

THEOREM 3.3. *Let $\{X_\alpha\}_{\alpha\in\Gamma}$ be a family of Banach spaces. Then $X=\bigoplus_{c_0}X_\alpha$ is nicely smooth if and only if for each $\alpha\in\Gamma$, X_α is nicely smooth.*

Proof. As before, we will show that $\mathcal{O}(X)=\{0\}$ if and only if for every $\alpha\in\Gamma$, $\mathcal{O}(X_\alpha)=\{0\}$.

Necessity is similar to that in Theorem 3.1.

Conversely, suppose for every $\alpha\in\Gamma$, $\mathcal{O}(X_\alpha)=\{0\}$. And let $x^{**}\in X^{**}\setminus\{0\}$. Let $\alpha_0\in\Gamma$ be such that $x^{**}_{\alpha_0}\neq 0$. Then $x^{**}_{\alpha_0}\notin\mathcal{O}(X_{\alpha_0})$. Hence, there exists $x_{\alpha_0}\in X_{\alpha_0}$ such that $\|x^{**}_{\alpha_0}+x_{\alpha_0}\|<\|x_{\alpha_0}\|$. The triangle inequality shows that for any $\lambda\geq 1$, $\|x^{**}_{\alpha_0}+\lambda x_{\alpha_0}\|<\|\lambda x_{\alpha_0}\|$. Thus, replacing x_{α_0} by λx_{α_0} for some $\lambda\geq 1$, if necessary, we may assume $\|x_{\alpha_0}\|>\|x^{**}\|_\infty$.

Define $y\in X$ by

$$y_\alpha=\begin{cases} x_{\alpha_0} & \text{if} \quad \alpha=\alpha_0\\ 0 & \text{otherwise} \end{cases}.$$

Then,

$$\|x^{**} + y\|_\infty = \max\{\sup\{\|x^{**}_\alpha\|_{\alpha\neq\alpha_0}\}, \|x^{**}_{\alpha_0} + x_{\alpha_0}\|\} < \|x_{\alpha_0}\| = \|y\|_\infty$$

whence $x^{**} \notin \mathcal{O}(X)$. ∎

Corollary 3.4. *Nice smoothness is not a three space property.*

Proof. Let $X = c$, the space of all convergent sequences with the sup norm. Recall that $c^* = \ell_1$ and that ℓ_1 acts on c as

$$\langle \boldsymbol{a}, \boldsymbol{x}\rangle = a_0 \lim x_n + \sum_{n=0}^{\infty} a_{n+1}x_n, \quad \boldsymbol{a} = \{a_n\}_{n=0}^{\infty} \in \ell_1,\ \boldsymbol{x} = \{x_n\}_{n=0}^{\infty} \in c\,.$$

It follows that $\{\boldsymbol{a} \in \ell_1 : a_0 = 0\}$ is a proper norming subspace for c.

Put $Y = c_0$. Then, by Theorem 3.3, Y is nicely smooth and $\dim(X/Y) = 1$, so that X/Y is also nicely smooth. But, by above, X is not nicely smooth. ∎

Remark 3.5. Since nice smoothness is not an isomorphic property, perhaps a more pertinent question here would be whether having a nicely smooth renorming is a three space property. The answer in this case is yes.

Theorem 3.6. *Let X be a Banach space. Let Y be a subspace of X. If both Y and X/Y have a nicely smooth renorming, then so does X.*

Proof. Observe that a Banach space X has a nicely smooth renorming if and only if for any proper subspace $M \subseteq X^*$, the norm interior of the w*-closure of its unit ball is empty.

Now suppose Y and X/Y both have a nicely smooth renorming. Suppose there exists a subspace $M \subseteq X^*$ such that $\overline{B(M)}^{w^*}$ has nonempty interior, i.e., there exists $m \in M$ and $\varepsilon > 0$ such that $B(m, \varepsilon) \subseteq \overline{B(M)}^{w^*}$. We will show that M is not proper.

Consider the inclusion map $i : Y \to X$. Then $i^* : X^* \to Y^*$ is the natural quotient map and is w*-continuous. It follows that $i^*(B(m, \varepsilon)) \subseteq \overline{B(i^*(M))}^{w^*}$. Therefore, $i^*(M) \subseteq Y^*$ is a subspace the w*-closure of whose unit ball has nonempty interior. Because of our assumption on Y, we have $i^*(M) = Y^*$, and hence, $X^* = M + Y^\perp$.

Put $N = M \cap Y^\perp$. The natural isomorphism between X^*/M and $Y^\perp/N$ shows that the relative norm interior of $\overline{B(N)}^{w^*}$ is also nonempty. Since

$Y^\perp = (X/Y)^*$, our assumption on X/Y implies that $M \cap Y^\perp = Y^\perp$, i.e., M contains $Y^\perp$.

Thus, $X^* = M + Y^\perp = M$. ∎

Remark 3.7. Since finite ℓ_1 sums of infinite dimensional Banach spaces fail Property (II) [3, Proposition 3.7], the space $c_0 \bigoplus_{\ell_1} c_0$ produces an example of a nicely smooth space, which being Asplund, also has the BGP, but lacks Property (II).

Recall that a closed subspace $M \subseteq X$ is said to be an M-summand if there is a projection P on X with range M such that $\|x\| = \max\{\|Px\|, \|x - Px\|\}$ for all $x \in X$.

Proposition 3.8. *The* BGP *is inherited by M-summands.*

Proof. We follow the arguments of [1, Proposition 2].

Let Y be an M-summand in X with P the corresponding projection. Let K be a closed bounded convex set in Y. Since X has the BGP,

$$K = \bigcap_{i \in I} \bigcup_{k=1}^{n_i} B[x_{ik}, r_{ik}] \,,$$

where for each i and k, $K \cap B[x_{ik}, r_{ik}] \neq \emptyset$.

Given i and k, let $x \in K \cap B[x_{ik}, r_{ik}] \subseteq Y$, then $\|x - x_{ik}\| \leq r_{ik}$, so that $\|x_{ik} - Px_{ik}\| = \|(x - x_{ik}) - P(x - x_{ik})\| \leq \|x - x_{ik}\| \leq r_{ik}$.

Claim : $K = \bigcap_{i \in I} \bigcup_{k=1}^{n_i} B_Y[Px_{ik}, r_{ik}]$. $(*)$

Since $\|P\| = 1$, we have

$$K = P(K) \subseteq P(\bigcap_{i \in I} \bigcup_{k=1}^{n_i} B[x_{ik}, r_{ik}]) \subseteq \bigcap_{i \in I} \bigcup_{k=1}^{n_i} B_Y[Px_{ik}, r_{ik}] \,.$$

Conversely, if x is in the RHS of $(*)$, for all $i \in I$, there exists k such that $\|x - x_{ik}\| = \max\{\|x - Px_{ik}\|, \|x_{ik} - Px_{ik}\|\} \leq r_{ik}$, as $\|x_{ik} - Px_{ik}\| \leq r_{ik}$. Thus, $x \in \bigcap_{i \in I} \bigcup_{k=1}^{n_i} B[x_{ik}, r_{ik}] = K$. ∎

Theorem 3.9. *Let X be a Banach space, μ denote the Lebesgue measure on $[0, 1]$ and $1 < p < \infty$. The following are equivalent:*

(a) *$L_p(\mu, X)$ has* BGP.

(b) *$L_p(\mu, X)$ is nicely smooth.*

(c) *X is nicely smooth and Asplund.*

Proof. Clearly (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c) Since $L^q(\mu, X^*)$ is always a norming subspace of $L^p(\mu, X)^*$, $\frac{1}{p} + \frac{1}{q} = 1$, and they coincide if and only if X^* has the RNP with respect to μ [7, Chapter IV], (b) implies X^* has the RNP, or, X is Asplund. Also for any norming subspace $F \subseteq X^*$, $L^q(\mu, F)$ is a norming subspace of $L^p(\mu, X)^*$. Hence, (b) also implies X is nicely smooth.

(c) $\Rightarrow$ (a) If X is nicely smooth and Asplund, by Theorem 2.10, X^* is the closed linear span of the w*-denting points of $B(X^*)$. And it suffices to show that $L^p(\mu, X)^* = L^q(\mu, X^*)$ is the closed linear span of the w*-denting points of $B(L^q(\mu, X^*))$.

Let $F = \sum_{i=1}^n \alpha_i x_i^* \chi_{A_i}$ with $x_i^* \in S(X^*)$ for all $i = 1, 2, \ldots, n$ be a simple function in $S(L^q(\mu, X^*))$. Let $\varepsilon > 0$. Now, for each $i = 1, 2, \ldots, n$, there exists $\lambda_{ik} \in \mathbb{R}$, and x_{ik}^*, w*-denting points of $B(X^*)$, $k = 1, 2, \ldots, N$, such that $\|x_i^* - \sum_{k=1}^N \lambda_{ik} x_{ik}^*\| < \varepsilon$. For $k = 1, 2, \ldots, N$. Define

$$F_k = \sum_{i=1}^n \alpha_i \lambda_{ik} x_{ik}^* \chi_{A_i} .$$

Since each x_{ik}^* is a w*-denting point of $B(X^*)$, for each k, $F_k/\|F_k\|$ is a w*-denting point of $B(L^q(\mu, X^*))$ [1, Lemma 10]. And,

$$\begin{aligned}
\left\| F - \sum_{k=1}^N F_k \right\|_q^q &= \left\| \sum_{i=1}^n \alpha_i x_i^* \chi_{A_i} - \sum_{k=1}^N \sum_{i=1}^n \alpha_i \lambda_{ik} x_{ik}^* \chi_{A_i} \right\|_q^q \\
&= \sum_{i=1}^n |\alpha_i|^q \left\| x_i^* - \sum_{k=1}^N \lambda_{ik} x_{ik}^* \right\|^q \mu(A_i) \\
&< \sum_{i=1}^n \varepsilon^q |\alpha_i|^q \mu(A_i) \le \varepsilon^q \|F\|_q^q \le \varepsilon .
\end{aligned}$$

∎

The analogues of the following results for Property (II) were obtained in [3].

Proposition 3.10. *Let K be a compact Hausdorff space, then $C(K, X)$ is nicely smooth if and only if K is finite and X is nicely smooth.*

Proof. For a compact Hausdorff space K and a Banach space X, the set

$$A = \{\delta(k) \otimes x^* : k \in K,\, x^* \in S(X^*)\} \subseteq B(C(K,X)^*)$$

is a norming set for $C(K,X)$. So, if $C(K,X)$ is nicely smooth, $C(K,X)^* = \overline{span}(A)$. It follows that K admits no nonatomic measure, whence K is scattered. Now, let K_1 denote the set of isolated points of K. Then K_1 is dense in K, so, the set

$$A_1 = \{\delta(k) \otimes x^* : k \in K_1,\, x^* \in S(X^*)\}$$

is also norming. Thus, $C(K,X)^* = \overline{span}(A_1)$. But if $k \in K \setminus K_1$, then for any $x^* \in S(X^*)$, $\delta(k) \otimes x^* \notin \overline{span}(A_1)$. Hence, $K = K_1$, whence K must be finite. And if $k_0 \in K_1$, $x \to \chi_{\{k_0\}} x$ is an isometric embedding of X into $C(K,X)$ as an M-summand. Thus, X is nicely smooth.

The converse is immediate from Theorem 3.3. ∎

Remark 3.11. (a) It is immediate that for $C(K)$ spaces Property (II), the BGP and nice smoothness (indeed, any of the conditions of Theorem 2.10) are equivalent, and are equivalent to reflexivity.

(b) It follows from the above and [3, Theorem 3.9] that $C(K,X)$ has Property (II) if and only if K is finite and X has Property (II). Only the special case of $C(K)$ is noted in [3].

Proposition 3.12. *Let X be a Banach space such that there exists a bounded net $\{K_\alpha\}$ of compact operators such that $K_\alpha \longrightarrow Id$ in the weak operator topology. If $\mathcal{L}(X)$ is nicely smooth, then X is finite dimensional.*

Proof. For $x \in X$, $x^* \in X^*$, let $x \otimes x^*$ denote the functional defined on $\mathcal{L}(X)$ by $(x \otimes x^*)(T) = x^*(T(x))$. Then $\|x \otimes x^*\| = \|x\|\|x^*\|$. And, since $\|T\| = \sup\{x^*(T(x) : \|x^*\| = 1,\, \|x\| = 1\} = \sup\{(x \otimes x^*)(T) : \|x^*\| = 1,\, \|x\| = 1\}$, it follows that $A = \{x \otimes x^* : \|x^*\| = 1,\, \|x\| = 1\}$ is a norming set, and hence, $\mathcal{L}(X)^* = \overline{span}(A)$.

Claim: $K_\alpha \longrightarrow Id$ weakly.

Since $\{K_\alpha\}$ is bounded, it suffices to check that $K_\alpha \longrightarrow Id$ on A, i.e., to check $x^*(K_\alpha(x)) \longrightarrow x^*(x)$ for all $\|x\| = 1$, $\|x^*\| = 1$. But, $K_\alpha(x) \longrightarrow x$ weakly, hence the claim.

Thus, Id is a compact operator, so that X is finite dimensional. ∎

Proposition 3.13. *For a compact Hausdorff space K, $\mathcal{L}(X, C(K))$ is nicely smooth if and only if $\mathcal{K}(X, C(K))$ is nicely smooth if and only if X is reflexive and K is finite.*

Proof. Suppose $\mathcal{L}(X, C(K))$ is nicely smooth. By definition of the norm, $A = \{\delta(k) \otimes x : x \in B(X), k \in K\}$ is a norming set for $\mathcal{L}(X, C(K))$, and hence, $\mathcal{L}(X, C(K))^* = \overline{span}(A)$. It follows that $\mathcal{L}(X, C(K)) = \mathcal{K}(X, C(K))$ and that $\mathcal{K}(X, C(K))$ is nicely smooth.

Now, from the easily established identification, $\mathcal{K}(X, C(K)) = C(K, X^*)$ and Proposition 3.10, it follows that $\mathcal{K}(X, C(K))$ is nicely smooth if and only if K is finite and X^* is nicely smooth, which, in turn, is equivalent to K is finite and X is reflexive.

Also, if K is finite, $C(K)$ is finite dimensional, so that $\mathcal{L}(X, C(K)) = \mathcal{K}(X, C(K))$. This completes the proof. ∎

Coming to general tensor product spaces, the proof of [12, Theorem 5.2] combined with Theorem 2.10 actually shows that:

THEOREM 3.14. *If X, Y are nicely smooth Asplund spaces, then $X \otimes_\varepsilon Y$ is nicely smooth.*

We prove the converse for general Banach spaces.

THEOREM 3.15. *Let X, Y be Banach spaces such that $X \otimes_\varepsilon Y$ is nicely smooth. Then both X and Y are nicely smooth.*

Proof. Let M and N be norming subspaces of X^* and Y^* respectively. Then $B(M)$ and $B(N)$ are norming sets for X and Y respectively. Hence, $\overline{co}^{w^*}(B(M)) = B(X^*)$ and $\overline{co}^{w^*}(B(N)) = B(Y^*)$. Thus

$$B(X^*) \otimes B(Y^*) = \overline{co}^{w^*}(B(M)) \otimes \overline{co}^{w^*}(B(N)) .$$

By definition of the injective norm, $B(X^*) \otimes B(Y^*)$ is a norming set for $X \otimes_\varepsilon Y$. Thus it follows that $co(B(M)) \otimes co(B(N))$ is a norming set for $X \otimes_\varepsilon Y$. Since $co(B(M) \otimes B(N)) \supseteq co(B(M)) \otimes co(B(N))$, it follows that $co(B(M) \otimes B(N))$ and hence $B(M) \otimes B(N)$ is a norming set for $X \otimes_\varepsilon Y$. And since this space is nicely smooth,

$$(X \otimes_\varepsilon Y)^* = \overline{span}(B(M) \otimes B(N)) .$$

Suppose $x^* \in X^*$, then for any $y^* \in S(Y^*)$ and $\varepsilon > 0$, there exist $f_i \in B(M)$, $e_i \in B(N)$ and $\lambda_i \in \mathbb{R}$ such that

$$\|x^* \otimes y^* - \sum_{i=1}^{n} \lambda_i f_i \otimes e_i\| < \varepsilon .$$

Applying to elementary tensors, this implies

$$\left|(x^* \otimes y^* - \sum_{i=1}^{n} \lambda_i f_i \otimes e_i)(x \otimes y)\right| < \varepsilon \|x\| \|y\| \quad \text{for all } x \in X,\, y \in Y$$

$$\implies \quad \left|x^*(x) y^*(y) - \sum_{i=1}^{n} \lambda_i f_i(x) e_i(y)\right| < \varepsilon \|x\| \|y\| \quad \text{for all } x \in X,\, y \in Y$$

$$\implies \quad \left\|x^*(x) y^* - \sum_{i=1}^{n} \lambda_i f_i(x) e_i\right\| < \varepsilon \|x\| \quad \text{for all } x \in X\,.$$

Let $x \in E = \bigcap ker f_i$. Then, $\|x^*(x)y^*\| < \varepsilon\|x\|$, i.e., $|x^*(x)| < \varepsilon\|x\|$. That is, $\|x^*|_E\| < \varepsilon$. This implies $d(x^*, span\{f_i\}) < \varepsilon$.

It follows that $x^* \in M$ and hence X is nicely smooth. Similarly for Y. ∎

It seems difficult to obtain analogues of Theorems 3.14 and 3.15 for the projective tensor product. However, we have the following

Proposition 3.16. *Suppose X, Y are Banach spaces such that X^* has the approximation property and $\mathcal{L}(X, Y^*) = \mathcal{K}(X, Y^*)$, i.e., any bounded linear operator from X to Y^* is compact. Then the following are equivalent:*

(a) *$\mathcal{K}(X, Y^*)$ is nicely smooth.*

(b) *X, Y are reflexive (and hence nicely smooth).*

(c) *$X \otimes_\pi Y$ is reflexive (and hence nicely smooth).*

Proof. (a) ⇒ (b) Since X^* has the approximation property,

$$\mathcal{K}(X, Y^*) = X^* \otimes_\varepsilon Y^*$$

and it follows from Theorem 3.15 that X^* and Y^* are nicely smooth, and therefore, X and Y are reflexive.

(b) ⇒ (c) This is a well-known result of Holub (see [7]).

(c) ⇒ (a) X and Y being closed subspaces of the reflexive space $X \otimes_\pi Y$ are themselves reflexive and from

$$\mathcal{K}(X, Y^*)^* = (X \otimes_\pi Y)^{**} = X \otimes_\pi Y$$

it follows that $\mathcal{K}(X, Y^*)$ is reflexive, and hence, nicely smooth. ∎

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