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Banach Spaces, à la Recherche du Temps Perdu*

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What follows is the opening conference of the late night seminar at the III Conference on Banach spaces held at Jarandilla de la Vera, Cáceres. Maybe the reader should not take everything what follows too seriously: after all, it was designed for a friendly seminar, late in the night, talking about things around a table shared by whisky, preprints and almonds. Maybe the reader should not completely discard it. Be as it may, it seems to me by now that everything arrives in the nick of time.

A twisted sum of two quasi-Banach spaces Y and Z is a quasi-Banach space X admitting Y as a subspace in such a way that the quotient X/Y is Z. For instance, the product space $Y \oplus X$ is a twisted sum of Y and X, called (for good reasons) the trivial twisted sum. The theory of twisted sums strives for elucidating the structure of a twisted sum in terms of properties of the factor spaces. Three aspects of this general problem are:

1) *Three-space problems.* Does a twisted sum have a certain property P whenever the factor spaces have P? Most of what I know is contained into [8]. A warning, I shall usually shorten "three-space" to 3SP.

2) *Twisted properties.* Which properties enjoy a twisted sum a space Y having a property P and another space Z whith a nother property Q? These properties have been considered in [9], and called $P - by - Q$ properties.

3) *Existence problems.* When nontrivial twisted sums of two given spaces exist? There are cases in which only trivial sums are possible; for instance, every twisted sum of c_0 and a separable space is trivial (Sobczyk) and every twisted sum of a Banach space complemented in its bidual (ultrasummand) and a $\mathcal{L}_1$-space is trivial as well (Lindenstrauss). To say that all twisted sums

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between two given spaces A and B are trivial we write $\mathrm{Ext}(A, B) = 0$ appealing to homological algebra notation. So, most results involving the study of complemented or uncomplemented subspaces can be expressed in this way. For instance, $\mathrm{Ext}(c_0, l_2(\Gamma)) \neq 0$ means the existence of "a" Johnson–Lidenstrauss counterexample for the 3SP problem for the property WCG (Weakly compactly generated). To give an idea of how nontrivial questions or ideas can emerge, here's one open problem that appears explicitly posed in [15]:

QUESTION. Characterize those Banach spaces Z such that $\mathrm{Ext}(l_2, Z) = 0$.

Our purpose in this talk is to present the basic ideas leading towards a right understanding of those problems.

.- *I'm Winston Wolf, I solve problems.*
.- *Good, 'cause we got one.*
.- *Let's get down to brass tacks, gentlemen. If I was informed correctly, the clock is ticking.*

(from *Pulp Fiction*)

1. Du côté de chez Algebra

We shall denote QBan the category of quasi-Banach spaces, in which the objects are quasi-Banach spaces and the arrows are linear continuous applications. The subcategory in which the objects are Banach spaces shall be denoted Ban. An exact sequence in QBan or Ban is a diagram $0 \to Y \to X \to X \to 0$ in the category with the property that the kernel of each arrow coincides with the image of the preceding. The open mapping theorem guarantees that Y is a subspace of X such that the corresponding quotient X/Y is Z. Two exact sequences $0 \to Y \to X \to Z \to 0$ and $0 \to Y \to X_1 \to Z \to 0$ are called equivalent if there exists an arrow $T : X \to X_1$ making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

An exact sequence is said to split if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$. Given two objects Y and Z in QBan (resp. in Ban) then $\mathrm{Ext}_{\mathrm{QBan}}(Y, Z)$ (resp. $\mathrm{Ext}_{\mathrm{Ban}}(Y, Z)$) denotes the vector space (although we are not very much interested in the vector space structure at this moment) of equivalence classes of exact sequences $0 \to Y \to X \to Z \to 0$ in QBan (resp. Ban). A rather delicate point is that if Y and Z are Banach spaces then $\mathrm{Ext}_{\mathrm{QBan}}(Y, Z)$ may be different of $\mathrm{Ext}_{\mathrm{Ban}}(Y, Z)$: this is Ribe's example [18] of an exact sequence of quasi-Banach spaces $0 \to \mathbb{R} \to E \to l_1 \to 0$ that does not split.

If one is going to do things the way of algebra, there are two rules to follow.

RULE 1: THINGS HAVE TO BE DONE BY DRAWING. And what is a drawing? It is a *commutative* diagram formed by objects (points, spaces...) and arrows (operators...). Rule 2 (to arrive in due time) shall say more: starting and ending with 0. For instance, instead of saying: Y is a subspace of a Banach space X and the quotient space X/Y is Z, one says: an exact sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$$

in the category Ban; after all, the open mapping theorem is there to warrant that the meaning of *exact sequence* in the category of Banach spaces is exactly the starting statement.

Allright, nice; so what? I mean, it may sound surprising that one can expect to do by mere drawing what takes a lot of work to do by "Banach space methods". But the key is that what one does is not mere drawing. The categorical theory is not simply a way to say "different" the same things, it is more a way to say things correctly (thus saying more and better than with classical methods). For instance, the kernel of an operator $T : X \to Z$ is classically defined as the space $\{x \in X : Tx = 0\}$. The algebra says that the kernel of an arrow $\cdot \xrightarrow{T} \cdot$ is another arrow $\cdot \xrightarrow{k} \cdot \xrightarrow{T} \cdot$ such that $Tk = 0$; and such that when for some arrow v one has $Tv = 0$ then v factorizes through k (i.e., there is an arrow u such that $ku = v$). Thus, there is more information encoded in the algebraic notion of kernel than in the Banach space term. In this way, each arrow one draws contains a lot of information, and this is what allows one to do things by drawing.

For instance, we could make some fuss about proving that if you have a subspace Y of X then

$$\frac{X^{**}/X}{Y^{**}/Y} = \frac{X^{**}/Y^{**}}{X/Y}.$$

Nevertheless, the algebra is clear at this point. Keep listening.

In what followsm we'll write quite freely $\mathrm{Ker}\, T$ and its dual notion $\mathrm{coker}\, T$; however, one is not always sure to be staying inside the category of Banach spaces. The idea is that if $T : X \to X$ then $\operatorname{coker} T$ should be $X/\overline{\operatorname{Im} T}$; hence one needs that $\operatorname{Im} T$ be closed. Into embeddings or quotient maps are good types of operators to apply this.

RULE 2: DRAWINGS MUST BE COMPLETE. A drawing is complete when it starts with 0 and ends with 0. Otherwise, the principle says two things: that the drawings *can* be completed and that they *have to* be completed.

Thus, if one has a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & Y_1 & \longrightarrow & X_1 & \longrightarrow & Z_1 & \longrightarrow & 0 \end{array}$$

with exact rows then one has to complete the diagram until

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ker}\alpha & \longrightarrow & \operatorname{Ker}\beta & \longrightarrow & \operatorname{Ker}\gamma & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & Y_1 & \longrightarrow & X_1 & \longrightarrow & Z_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \operatorname{coker}\alpha & \longrightarrow & \operatorname{coker}\beta & \longrightarrow & \operatorname{coker}\gamma & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

which is commutative with exact rows and columns. But still this is not complete: have you seen the open end at the upmost right corner? And the open start at the lowest left corner? True, in our applications it shall be expected that α, β and γ shall be either embeddings or quotient maps, so that either Ker or coker shall be 0, which is the most interesting case for us. But for the sake of completeness let us present the general case: *This* is the complete diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker}\alpha & \longrightarrow & \operatorname{Ker}\beta & \longrightarrow & \operatorname{Ker}\gamma & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & \overset{\omega}{\text{———}} & \downarrow{\scriptstyle\beta} & \text{———} & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & Y_1 & \longrightarrow & X_1 & \longrightarrow & Z_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & \longrightarrow & \operatorname{coker}\alpha & \longrightarrow & \operatorname{coker}\beta & \longrightarrow & \operatorname{coker}\gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

thus, there exists a "connecting map" $\omega : \operatorname{Ker}\gamma \to \operatorname{coker}\alpha$ making exact the "long sequence"

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker}\alpha & & \operatorname{coker}\alpha & \longrightarrow & \operatorname{coker}\beta & & \\
 & & \downarrow & & \uparrow{\scriptstyle\omega} & & \downarrow & & \\
 & & \operatorname{Ker}\beta & \longrightarrow & \operatorname{Ker}\gamma & & \operatorname{coker}\gamma & \longrightarrow & 0.
\end{array}$$

As we have already said, our first applications shall be the diagrams in which the connecting maps are either embeddings or quotient maps. Applying this to diagrams in which the starting maps are into isomorphisms then the kernels are 0 and the result says that the sequence of cokernels (quotient spaces) is exact. And conversely, applying this to diagrams in which the starting maps are quotient maps, what one gets is that the sequence of kernels is also exact. Thus, completing the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y^{**} & \longrightarrow & X^{**} & \longrightarrow & Z^{**} & \longrightarrow & 0
\end{array}$$

one obtains the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y^{**} & \longrightarrow & X^{**} & \longrightarrow & Z^{**} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y^{**}/Y & \longrightarrow & X^{**}/X & \longrightarrow & Z^{**}/Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The exactness of the right column and lower row is the desired isomorphism.

Especially interesting is what happens when one has a "one-dimensional" diagram say

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$$

and introduces "open-ends" in a "second dimension". Thus, if one has a drawing such as

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & X & \xrightarrow{q} & Z & \longrightarrow & 0 \\
 & & & & & & \uparrow{\scriptstyle T} & & \\
 & & & & & & M & &
\end{array}$$

then our principles say that the diagram has to be completed now as a two dimensional diagram; i.e., it has to be completed, in the two directions, until obtaining a full diagram starting and ending with 0 both in vertical and horizontal lines. This makes a new "universal" object, called Pull-back, appear to get the diagram (when working with Banach spaces we shall assume that

T is either a quotient or an embedding map, as usual); but in general one has:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & \operatorname{coker} u & = & \operatorname{coker} T & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X & \xrightarrow{q} & Z & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle u} & & \uparrow{\scriptstyle T} & & \\
0 & \longrightarrow & Y & \longrightarrow & PB & \xrightarrow{v} & M & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & \operatorname{Ker} u & = & \operatorname{Ker} T & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Look how a new object appeared: the space PB is called the *pull-back* of q and T. In the category of quasi-Banach spaces pull-backs exist: it is the closed subspace of $X \oplus M$ defined as $PB = \{(x, m) \in X \oplus M : qx = Tm\}$. The arrows $v : PB \to M$ and $u : PB \to X$ are the restrictions to PB of the canonical projections. The inclusion $Y \to PB$ is given by $y \to (y, 0)$.

The dual notion of pull-back is that of *push-out*. It appears to be able to complete drawings such as

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle T} & & & & & & \\
 & & M & & & & & &
\end{array}
$$

when T is either a quotient map or an embedding up to

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{Ker} T & = & \operatorname{Ker} v & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle v} & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & PO & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{coker} T & = & \operatorname{coker} v & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

The new object is called push-out. Push-outs exist in the category of quasi-Banach spaces. The push-out space PO is he quotient space $M \oplus X/\overline{\Delta}$ where $\Delta = \{(Ty, y) \in M \oplus X\}$. The arrows $M \to PO$ and $X \to PO$ are the composition of the natural injections into $M \oplus X$ with the quotient map. The quotient map $PO \to Z$ is given by $(m, x) + \Delta y \to qx$.

.- "I guess you take life pretty seriously."
.- "Now and then. Why?"
She laughed very gently and said goodbye. I sat there for a while taking life seriously.

(from *The Long Goodbye*; Raymond Chandler)

2. À l'ombre des jeunes transformations en fleur

Up to here we have two theories: one (say, intrinsic Banach space theory) in which one describes things the way they are found: subspaces, quotients, duals, etc. Another one, we are starting to describe, that uses the homological algebra in an essential way. But, even accepting the algebraic language as description of things, what not many people seems to be aware is that one still has freedom to choose which kind of "description" is going to use for the objects considered. To be more concrete, let us agree that we have decided to use the algebraic approach, so that we are interested in the object $\operatorname{Ext}(A,B)$. Maybe it is not the last word to say that $\operatorname{Ext}(A,B)$ are the equivalence classes of exact sequences starting in A and ending in B modulus the equivalence relation previously described. Maybe there are better ways to describe that object.

Of course, homological algebra has a very precise name for this process: it is called a *natural transformation*. Assume that we have found a "better description" for $\operatorname{Ext}(A,B)$; say, $Q(B,A)$. This means that the same dependence and properties of $\operatorname{Ext}(\cdot,\cdot)$ should be shared by $Q(\cdot,\cdot)$. Up to here, simple: $\operatorname{Ext}(\cdot,\cdot)$ is a *functor* and thus $Q(\cdot,\cdot)$ must also be a functor [observe, by the way, how this language is (well, can be) much more precise than the standard: a functor is an object with well established properties; and, moreover, it is essential to determine between which categories acts; in other words, we are also asking about the structure of $\operatorname{Ext}(\cdot,\cdot)$!]. Well, a natural transformation $\tau : \operatorname{Ext} \to Q$ between the two functors (descriptions) we have is a correspondence assigning to each object (in this case, to each couple A, B an arrow $\tau_{A,B} : \operatorname{Ext}(A,B) \to Q(B,A)$ in such a way that this correspondence respects arrows; precisely, that if one has an arrow $f : A \to A_1$ and $g : B \to B_1$ then the square

$$\begin{array}{ccc} \operatorname{Ext}(A,B) & \xrightarrow{\tau_{A,B}} & Q(B,A) \\ {\scriptstyle \operatorname{Ext}(f,g)}\downarrow & & \downarrow{\scriptstyle \operatorname{Ext}(g,f)} \\ \operatorname{Ext}(A_1,B_1) & \xrightarrow{\tau_{A_1,B_1}} & Q(B_1,A_1) \end{array}$$

is commutative. This is the precise meaning of "having a different description for the same object".

And it turns out that there exists a third description provided by Kalton [12] and Kalton and Peck [13]; a far simpler and better description indeed. The idea is to describe exact sequences of quasi-Banach spaces using functions; a special type of functions called *quasi-linear* maps. The fact is that this new theory is so well shaped and adapted to the problem that anybody should suspect its categorical underlying. Just in time! It is proved in [5] that there exist natural transformations between the "three" theories. But we shall not enter into that today.

I never can be tied to raw, new things,
For I first saw the light in an old town,

(*Background,* H.P. Lovecraft)

3. Du côté des espaces de Banach

The algebraic tools are not only different ways to say things that have been previously known. But, even in the case of known results, they provide clean proofs where only proofs existed. For instance, the following is a well-known result in Banach space theory having an involved proof (see [17, Thm. 2.f.8 and Remark 1]).

Proposition. *Let* $q_1 : l_1 \to Z$ *and* $q_2 : l_1 \to Z$ *be two quotient maps onto* Z. *If the kernels* $\operatorname{Ker} q_1$ *and* $\operatorname{Ker} q_2$ *are infinite dimensional then they are isomorphic.*

However, its proof can be reduced to the knowledge of a basic result:

Lemma. *An infinite dimensional closed subspace of* l_1 *contains a complemented copy of* l_1.

as follows: consider the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \operatorname{Ker} q_1 & \longrightarrow & l_1 & \xrightarrow{q_1} & Z & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle q_2} & & \\
0 & \longrightarrow & \operatorname{Ker} q_1 & \longrightarrow & PB & \xrightarrow{q_1} & l_1 & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \operatorname{Ker} q_2 & = & \operatorname{Ker} q_2 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since l_1 is l_1, the middle column and row split. Hence $PB = \operatorname{Ker} q_1 \oplus l_1 = \operatorname{Ker} q_2 \oplus l_1$ and, by the lemma, $\operatorname{Ker} q_1 = A_1 \oplus l_1$ and $\operatorname{Ker} q_2 = A_2 \oplus l_1$. Therefore

$$\operatorname{Ker} q_1 = A_1 \oplus l_1 = A_1 \oplus l_1 \oplus l_1 = \operatorname{Ker} q_1 \oplus l_1 = PB$$

and also $\operatorname{Ker} q_2 = PB$, so that $\operatorname{Ker} q_1 = \operatorname{Ker} q_2$. ∎

And, of course, we also have the dual result (see [17, Thm. 2.f.12]):

PROPOSITION. *Let $i : A \to \ell_\infty$ and $j : A \to \ell_\infty$ be two isomorphic embeddings into ℓ_∞. If the quotient spaces $\ell_\infty/i(A)$ and $\ell_\infty/j(A)$ contain ℓ_∞ then they are isomorphic.*

Proof. Consider the push-out diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & \ell_\infty & \longrightarrow & \ell_\infty/i(A) & \longrightarrow & 0 \\
 & & {\scriptstyle j}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \ell_\infty & \longrightarrow & PO & \longrightarrow & \ell_\infty/i(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \ell_\infty/j(A) & = & \ell_\infty/j(A) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since the middle row and column split, $PO = \ell_\infty \oplus \ell_\infty/i(A) = \ell_\infty \oplus \ell_\infty/j(A)$. Since, say, $\ell_\infty/j(A)$ contains ℓ_∞ then also $\ell_\infty/j(A) = \ell_\infty \oplus B$, and therefore $PO = \ell_\infty \oplus \ell_\infty/j(A) = \ell_\infty \oplus \ell_\infty \oplus B = \ell_\infty \oplus B = \ell_\infty/j(A)$. Analogously $PO = \ell_\infty/i(A)$, so that $\ell_\infty/j(A) = \ell_\infty/i(A)$. ∎

Since "not containing ℓ_∞" is a 3SP property (see [8]) the hypothesis is strictly weaker than the usual "A is separable".

When I consider everything that grows
Holds in perfection but a little moment.
That this huge stage presenteth nought but shows
Whereon the stars in secret influence comment

(*Sonnet 15,* W. Shakespeare)

4. LA STRUCTURE PRISIONNIERE

Let $0 \to Y \to X \to Z \to 0$ be an exact sequence of Banach spaces. Assume that Z is separable. In the category of Banach spaces one always have a "fundamental" exact sequence: the projective presentation, namely

$$0 \longrightarrow K \xrightarrow{i} l_1 \xrightarrow{q} Z \longrightarrow 0.$$

(we have already seen that it does not matter which quotient map has been chosen). Moreover, the two sequences are closely connected: there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & l_1 & \stackrel{q}{\longrightarrow} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

since q can be lifted to X. Hence, each exact sequence $0 \to Y \to X \to Z \to 0$ gives an operator $\phi : K \to Y$. Conversely, given an operator $\phi : K \to Y$ one obtains an exact sequence $0 \to Y \to X \to Z \to 0$ making the push-out of $\{\phi, i\}$. To determine the form that the equivalence relation adopts, one only has to observe that if ϕ can be extended to a map $\Phi : l_1 \to Y$ then the push-out sequence is trivial. The rest is mere comprobation: the equivalence relation between elements of $L(K, Y)$ that corresponds to the equivalence relation between exact sequences is:

$$\phi \sim \psi \Leftrightarrow \phi - \psi \quad \text{can be extended to an operator} \quad l_1 \to Y.$$

In other words,

$$\mathrm{Ext}(Y, Z) = L(K, Y)/L(l_1, Y).$$

Sound queer? It shouldn't.

Though leaves are many, the root is one;
Through all the lying days of my youth
I swayed my leaves and flowers in the sun;
Now I may wither into the truth

(*The coming of wisdom with time,*
W.B. Yeats)

Actually, we are imprisoned on the iron claws of the second principle: drawings must be complete; since when one starts with $0 \to K \to l_1 \to Z \to 0$ and applies the functor $L(\cdot, Y)$ one gets

$$0 \longrightarrow L(Z, Y) \longrightarrow L(l_1, Y) \longrightarrow L(K, Y)$$

as an incomplete diagram since the functor $L(\cdot, Y)$ is not exact at the right end: operators $K \to Y$ are not necessarily extendable to operators $l_1 \to Y$. But drawings *have to* be complete. The preceding arguments show that

$$0 \longrightarrow L(Z, Y) \longrightarrow L(l_1, Y) \longrightarrow L(K, Y) \longrightarrow \mathrm{Ext}(Y, Z) \longrightarrow 0$$

is the complete diagram; hence the last arrow is surjective and we obtain the desired isomorphism. We shall see, when the proper time comes, that this last sequence is nothing but a particular instance of the long homology sequence (which has become shorter by the presence of the projective space l_1).

5. Le problème disparu

By the time being, answering question 1 is very easy. Recall that the question was: characterize those Banach spaces Z such that $\mathrm{Ext}(l_2, Z) = 0$. We know that if $0 \to K \to l_1 \to Z \to 0$ is a projective presentation of Z the condition necessary and sufficient to make all exact sequences $0 \to l_2 \to X \to Z \to 0$ split is that all operators $K \to l_2$ can be extended to operators $l_1 \to l_2$.

This is the time of their lives for Banach spacers: operators $l_1 \to l_2$ are 1-summing (which, for our purposes means that they factorize through an L_∞ space), hence all operators $K \to l_2$ have to be 1-summing; more yet, 1-summing operators can be extended to any superspace; thus if all operators $K \to l_2$ are 1-summing then all are extendable. So,

$$\mathrm{Ext}(l_2, Z) = 0 \iff L(K_Z, Y) = \Pi_1(K_Z, Y).$$

Erect and sublime, for one moment of time
In the next, that wild figure they saw
(As if stung by a spasm) plunge into a chasm,
While they listened in awe.

(*The Hunting of the Snark,* L. Carroll)

6. L'algebre retrouvée

Since the functor $L(\cdot, A)$ is not exact, when we apply it to an exact sequence $0 \to Y \to X \to Z \to 0$ one obtains

$$0 \longrightarrow L(Z, A) \longrightarrow L(X, A) \longrightarrow L(Y, A)$$

a drawing that must be completed. This is the

Theorem. (*Long sequence in the first variable) Let* $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ *be an exact sequence in QBan, and let* A *be a quasi-Banach space. There exists an exact sequence*

$$\begin{array}{l} 0 \longrightarrow L(Z,A) \xrightarrow{q^*} L(X,A) \xrightarrow{j^*} L(Y,A) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \omega \\ \qquad\qquad\qquad\qquad\qquad \mathrm{Ext}(A,Z) \xrightarrow{\alpha} \mathrm{Ext}(A,X) \xrightarrow{\beta} \mathrm{Ext}(A,Y). \end{array}$$

Observe that the previous case was obtained when X was a projective object and then $\mathrm{Ext}(A,X)=0$ (that's why the sequence ended with $\mathrm{Ext}(A,Z)$!) Observe also that the rightmost end is open, so the sequence *cannot* end here and must continue: that's why the name "long sequence". Finally, observe that by duality there must also be a long homology sequence in the first variable.

Even for a talk like this it comes the time of proving something.

Proof of Theorem. The exactness in the first three terms of the sequence is easy and well-known; the operators j^* and q^* are simple composition with, respectively, j and q. We start the proof defining the maps ω, α and β.

If $T : Y \to A$ is an operator then $\omega(T)$ is the push-out sequence generated by the couple $\{j, T\}$ as in the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{j} & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T} & & \downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & PO & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

With some notational abuse let us call F to an exact sequence $0 \to A \to F \to Z \to 0$. The map α is then defined as follows: if $E \in \mathrm{Ext}(A,Z)$ is an exact sequence $0 \to A \to E \to Z \to 0$ then $\alpha(E)$ is the lower exact sequence corresponding to the pull-back square

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle q} & & \\ 0 & \longrightarrow & A & \longrightarrow & PB & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

If $F \in \mathrm{Ext}(A,X)$ denotes the exact sequence $0 \to A \to F \to X \to 0$ then $\beta(F)$ is the lower exact sequence corresponding to the pull-back square

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & F & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle i} & & \\
0 & \longrightarrow & A & \longrightarrow & PB & \longrightarrow & Y & \longrightarrow & 0
\end{array}$$

It is clear that ω, α and β are group homomorphisms.

$\operatorname{Im}\omega = \operatorname{Ker}\alpha$: The containment $\operatorname{Im}\omega \subset \operatorname{Ker}\alpha$ is consequence of the fact that making push-out and then pull-back produces an equivalent sequence to that obtained making first pull-back and then push-out; now, the pull-back sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\ j\ } & X & \xrightarrow{\ q\ } & Z & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle q} & & \\
0 & \longrightarrow & Y & \longrightarrow & PB & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

splits: since $PB = \{(x, x') \in X \oplus X : qx = qx'$ the map $\pi(x, x') = (x - x', 0)$ is a linear continuous projection onto $Y = \{(y, 0) : y \in Y\}$.

We prove then that $\operatorname{Ker}\alpha \subset \operatorname{Im}\omega$. Let thus $E \in \operatorname{Ext}(A, Z)$ be an element in $\operatorname{Ker}\alpha$. This gives the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle q} & & \\
0 & \longrightarrow & A & \longrightarrow & A \oplus X & \longrightarrow & X & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & Y & = & Y & &
\end{array}$$

hence the upper row is the push-out sequence corresponding to the arrow $\pi_A i : Y \to A$ defined on the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \| & & & & \uparrow{\scriptstyle q} & & \\
 & & A & \xleftarrow{\ \pi_A\ } & A \oplus X & & X & & \\
 & & & & \uparrow{\scriptstyle i} & & \uparrow & & \\
 & & & & Y & = & Y & &
\end{array}$$

$\operatorname{Im}\alpha = \operatorname{Ker}\beta$: If one observes the diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \| & & & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & F & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & Y & \longrightarrow & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

it is clear that $\beta\alpha = 0$ since one is making pull-back with 0. Hence $\operatorname{Im}\alpha \subset \operatorname{Ker}\beta$. On the other hand, assume that $F \in \operatorname{Ker}\beta$. We have an exact sequence $0 \to A \to F \to X \to 0$ that becomes trivial when restricted to Y. Thus, if $G = \beta(F)$ then $G = A \oplus Y$, hence Y is a subspace of F. Since the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & F & \longrightarrow & X & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & Y & = & Y & \longrightarrow & 0
\end{array}$$

is commutative with exact rows, it can be completed with the exact row of the quotients yielding

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & F/Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & F & \longrightarrow & X & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & Y & = & Y & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0, & &
\end{array}$$

which shows that $F = \alpha(F/Y)$.

The look of a laurel tree birthered for May
or a sycamore bared for a new November
is as old and sad as my furtherest day -
What is it, what is it, I almost remember?

(*Temps perdu,* Dorothy Parker)

7. Par les moustaches de Plekszy-Gladtz!

Time is ripe to wonder what knowledge we obtain from the long homology sequence. We give some examples because we're about time.

PROPOSITION. *The property* $\mathrm{Ext}(A,\cdot) = 0$ *is a 3SP property.*

with this we mean that if we have an exact sequence $0 \to Y \to X \to Z \to 0$ in which $\mathrm{Ext}(A,Y) = 0$ and $\mathrm{Ext}(A,Z) = 0$ then necessarily $\mathrm{Ext}(A,X) = 0$ (courtesy of the long sequence).

We can give two important properties that adopt this form. Let us say that a Banach space X is a *Kalton-Pelczynski* space (after [15]) when $\mathrm{Ext}(l_2, X) = 0$ then

PROPOSITION. *To be a Kalton-Pelczynski space is a 3SP property.*

One more to go. A quasi-Banach space Z is said to be a K-space (following [12] if $\mathrm{Ext}_{\mathrm{QBan}}(\mathbb{R}, Z) = 0$. Then,

PROPOSITION. *To be a K-space is a 3SP property.*

This result can be used to obtain new examples of K- (and non-K-) spaces: for instance, every twisted sum of c_0 and $l_p(I)$ (including the Johnson-Lindenstrauss spaces) are K-spaces (see [3], [4]).

More possible applications yet? Lindenstrauss theorem (see [16]) asserts that if Y is complemented in its bidual and Z is a $\mathcal{L}_1$-space then $\mathrm{Ext}(Y,Z) = 0$. The Johnson-Lindenstrauss spaces are not complemented in their biduals but, as a consequence of Sobczyk's theorem

PROPOSITION. $\mathrm{Ext}(JL_p, Z) = 0$ *for each separable* $\mathcal{L}_1$*-space Z.*

I would suggest [4], [3] for pastime reading.

It was my thirtieth year to heaven
And I rose
In a rainy autumnn
And I walked abroad in a shower of all my days

(*Poem in October,* Dylan Thomas).

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