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On the Compactness of the Natural Tensor Product of Compact Operators

JESÚS M.F. CASTILLO

Dpto. de Matemáticas, Univ. de Extremadura, Avda. Elvas s/n
06071-*Badajoz, Spain, e-mail: castillo@ba.unex.es*

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INTRODUCTION

Let $T : A \longrightarrow X$ and $S : B \longrightarrow Y$ be continuous operators acting between Banach spaces. If α denotes a tensornorm defined on a tensor product space, i.e., a norm satisfying $\alpha(T \otimes S : A \otimes B \longrightarrow X \otimes Y) \leq \alpha(T : A \longrightarrow X)\alpha(S : B \longrightarrow Y)$, then the continuity of $T \otimes S$ follows easily from that of T and S. If, in addition, A and B are ℓ_p-spaces, and if T and S are compact, then the same happens with $T \otimes S$. Things are different when α is not a tensornorm.

When tensor products with an ℓ_p-space are considered, there exists, beyond the π and ϵ topology, a third topology which has a great interest regarding applications: the topology induced on $\ell_p \otimes X$ by the space $\ell_p(X)$ of absolutely p-summable sequences. This topology is termed natural topology, the norm that induces the natural topology is denoted Δ_p, and it is termed the natural norm. Clearly one has $\ell_p \hat{\otimes}_{\Delta_p} X = \ell_p(X)$ (see [3, Sec. 7]).

Unfortunately, the natural norm is not a tensornorm, and thus it may happen that the natural tensor product of two continuous operators is not continuous [3, p. 7.3, 7.5 and 7.6].

Here we want to prove that given $T : \ell_p \longrightarrow \ell_q$ and $S : X \longrightarrow Y$ compact operators, their natural tensor product is compact, provided it is continuous, in the following cases:

(i) $p > q$, or

(ii) $p \leq q$ and $Y = \ell_q$.

1. SOME LEMMAS

The following result can be found in [2]. For the sake of completeness we sketch the proof.

LEMMA 1. *Let (f_n) be a sequence in $\ell_p(X), 1 \leq p < +\infty$, (resp., $c_0(X)$). The sequence (f_n) is norm null if and only if the following two conditions are satisfied:*

(a) *Any continuous projection onto ℓ_p (resp., c_0) transforms (f_n) into a norm null sequence of ℓ_p (resp., c_0).*

(b) *Any continuous projection onto X transforms (f_n) into a norm null sequence of X.*

Proof. The necessity is clear. We prove that (b) implies that

$$\lim_{N\to\infty} \sup\left\{\sum_{k=N}^{\infty} \|f_n(k)\|_X^p : n \in \mathbb{N}\right\} = 0.$$

Otherwise, it is possible to find an $\epsilon > 0$ and two increasing sequences, (n_i) and (N_i), of integers such that

$$\sum_{k=N_i+1}^{k=N_{i+1}} \|f_{n_i}(k)\|^p > \epsilon.$$

Choose norm one elements $x^*(i,k) \in X^*$ such that $< x^*(i,k), f_{n_i}(k) >= \|f_{n_i}(k)\|$, and form the element $(y^*(k)) \in \ell_\infty(X^*)$ defined by $y^*(k) = x^*(i,k)$, $N_i < k \leq N_{i+1}$, and $y^*(k) = 0$ otherwise. The sequence $(y^*(k))$ defines a continuous projection $P : \ell_p(X) \longrightarrow \ell_p$ as follows: $P(g_n) = (< y^*(k), g_n(k) >)_k$. Since the set $\{P(f_n)\}_n$ is relatively compact in ℓ_p, one has:

$$\lim_{N\to\infty} \sup\left\{\sum_{k=N}^{\infty} |P(f_n)|^p : n \in \mathbb{N}\right\} = 0.$$

which is in contradiction with

$$\sup\left\{\sum_{k=N}^{\infty} |P(f_n)|^p : n \in \mathbb{N}\right\} \geq \sum_{k=N_i+1}^{k=N_{i+1}} \|f_{n_i}(k)\|^p > \epsilon.$$

The remainder of the proof is easy, and the case of $c_0(X)$, analogous. ∎

Remark. Let us call a complemented copy of X into $\ell_p(X)$ natural if it is obtained by means of one of the inclusions: $j_n : X \longrightarrow \ell_p(X)$, $j_n(x) = (0,\ldots,x,0,\ldots)$; a complemented copy of ℓ_p into $\ell_p(X)$ shall be termed natural if it is obtained trough an application of the form $I_p : \ell_p \longrightarrow \ell_p(X)$, $I_p(\xi) = (\xi_1 x_1, \xi_2 x_2, \ldots)$, for some sequence $(x_n) \subset X$ with $\|x_n\| = 1$. A natural projection onto a natural copy of X is a projection having the form: $P_k : \ell_p(X) \longrightarrow X$, where $P_k(x) = x_k$; analogously, a natural projection onto a natural copy of ℓ_p is a projection of the form $P(x) = (< x^*(k), x(k) >)$ with $\|x^*(k)\| = 1$. In the preceding proof it is sufficient to work with natural projections.

In what follows we will assume that $p > 1$; otherwise the results are elementary.

LEMMA 2. *Let $U : \ell_p(X) \longrightarrow Z$ be a weakly compact operator. Assume that its restrictions to all isomorphic copies of X and ℓ_p are compact operators. Then U is compact.*

Proof. The operator $U^* : Z^* \longrightarrow l_{p^*}(X^*)$ is weakly compact. By the Davis-Figiel-Johnson-Pelczynski factorization theorem, it is possible to assume in what follows that Z is reflexive. Given (x_n^*) a weakly null sequence in Z^*, $(U^*x_n^*)$ is such that its projection onto ℓ_{p^*} and X^* are norm null. Thus $(U^*x_n^*)$ is norm null, the operator U^* is compact, and so is U. ∎

Since $\ell_p\hat{\otimes}_{\Delta_p}X = \ell_p(X)$, the following result is straightforward (see [4]).

LEMMA 3. *Let $T : \ell_p \longrightarrow \ell_q$ and $S : X \longrightarrow Y$ be weakly compact operators. If $T\hat{\otimes}S : \ell_p\hat{\otimes}_{\Delta_p}X \longrightarrow \ell_q\hat{\otimes}_{\Delta_q}Y$ is continuous, then it is weakly compact.*

2. Main Results

PROPOSITION 4. *Let $T : \ell_p \longrightarrow \ell_q$ and $S : X \longrightarrow Y$ be compact operators. If $p > q$, then $T\hat{\otimes}S : \ell_p\hat{\otimes}_{\Delta_p}X \longrightarrow \ell_q\hat{\otimes}_{\Delta_q}Y$, assuming it is continuous, is compact.*

Proof. As before, $I_p : \ell_p \longrightarrow \ell_p\hat{\otimes}_{\Delta_p}X$ shall denote a natural inclusion, while $P_k : \ell_p(X) \longrightarrow X$ and $Q_k : \ell_q(Y) \longrightarrow Y$ shall denote the natural projections onto the k^{th} coordinate. If $y = (y_n^*)$ is a normalized sequence of elements of Y^*, $P_y : \ell_q(Y) \longrightarrow \ell_q$ denotes the natural projection that y defines. Finally, $i_q : \ell_q\hat{\otimes}_{\Delta_q}Y \longrightarrow \ell_q(Y)$ is the canonical inclusion.

By Lemma 3, $T\hat{\otimes}S$ is weakly compact. By Lemma 2 we only need to verify that the restrictions of $T\hat{\otimes}S$ to the natural copies of ℓ_p and X which are inside $\ell_p(X)$ are compact. For the copies of X it is evident, since the application $(T\hat{\otimes}S)j_n(x) = Te_n \otimes Sx$ is clearly compact.

For the copies of ℓ_p some extra work is needed, since a description of the map

$$i_q(T\hat{\otimes}S)I_q : \ell_p \longrightarrow \ell_p\hat{\otimes}_{\Delta_p}X \longrightarrow \ell_q\hat{\otimes}_{\Delta_q}Y \longrightarrow \ell_q(Y)$$

is not easy to obtain. It is enough, however, to verify that weakly null sequences of ℓ_p are transformed into norm null sequences of $\ell_p(X)$. To this end, let (ξ_n) be a weakly null sequence of ℓ_p. Regarding the lemmas, it is necessary to verify that, for all k and all norm one sequences (x_n) of X, $(P_k i_q(T\hat{\otimes}S)I_p(\xi_n))$ is a null sequence and that the sequence $((P_y I_p(\xi_n))$ is convergent to 0 for all sequences y.

Let us define the map $R : \ell_p \longrightarrow X$ by $R(e_n) = P_k((Te_n)x_n)$. The operator R is continuous since the sequence $P_k((Te_n)x_n)$ is weakly-p^*-summable:

$$\sum_n \|(Te_n)(k)x_n\|^{p^*} \leq \sum_n |(Te_n)(k)|^{p^*} < \infty.$$

Since $SR = P_k i_q(T\hat{\otimes}S)I_p(\xi_n)$, this part of the proof is complete. The part of ℓ_q is just an application of Pitt's lemma: if $p > q$, all continuous operators $\ell_p \longrightarrow \ell_q$ are compact (see [3]).

Remark. This result is an improvement of [1], where it has been proved that if all operators from ℓ_p into ℓ_q and all operators from X into Y are compact, then all operators from $\ell_p\hat{\otimes}_{\Delta_p}X$ into $\ell_q\hat{\otimes}_{\Delta_q}Y$ are compact.

In case that not all operators from ℓ_p into ℓ_q are compact, it is still possible to obtain a nice result for $Y = \ell_q$:

PROPOSITION 5. *Let $T : \ell_p \longrightarrow \ell_q$ and $S : X \longrightarrow \ell_q$ be compact operators, $p \leq q$. The operator $T\hat{\otimes}S : \ell_p\hat{\otimes}_{\Delta_p}X \longrightarrow \ell_q\hat{\otimes}_{\Delta_q}\ell_q$, assuming it is continuous, is compact.*

Proof. Consider the tensornorms d_p and g_p such as defined in [3, p. 17—18] by means of:

$$d_p(z) = \inf\left\{ \sup_{\|x^*\|\leq 1} \left(\sum_{k=1}^{k=n} |x^*(x_k)|^{p^*}\right)^{1/p^*} \left(\sum_{k=1}^{k=n} \|y_k\|^p\right)^{1/p}\right\}$$

$$g_p(z) = \inf\left\{ \sup_{\|y^*\|\leq 1} \left(\sum_{k=1}^{k=n} |y^*(y_k)|^{p^*}\right)^{1/p^*} \left(\sum_{k=1}^{k=n} \|x_k\|^p\right)^{1/p}\right\}$$

where the infimum is taken over all representations $z = \sum_{k=1}^{k=n} x_k \otimes y_k$. Their dual norms are noted d_p^* and g_p^*. Taking into account the so-called Chevet-Persson-Saphar inequalities [3, p. 186] $g_{q^*}^* \leq g_q$, $d_p \leq \Delta_p \leq g_{p^*}^*$, it is possible to factorize the original operator $T \otimes S$ in the form

$$\begin{array}{ccc}
\ell_p \otimes_{\Delta_p} X & \xrightarrow{T\otimes S} & \ell_q \otimes_{\Delta_q} \ell_q \\
\downarrow {\scriptstyle id\otimes id} & & {\scriptstyle id\otimes id} \uparrow \\
\ell_p \otimes_{d_p} X & & \ell_q \otimes_{g_{q^*}^*} \ell_q \\
\downarrow {\scriptstyle T\otimes S} & & \uparrow \\
\ell_q \otimes_{d_p} \ell_q & & {\scriptstyle id\otimes id} \Big| \\
\downarrow {\scriptstyle id\otimes id} & & \Big| \\
\ell_q \otimes_{d_q} \ell_q & = \!= \!= & \ell_q \otimes_{g_q} \ell_q
\end{array}$$

The proof is finished. ∎

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The content of this note corresponds to a question that A. Defant suggested some time ago during an informal meeting in his office at Oldenburg. At the same time he posed the question, he wrote down the proof of Proposition 5 on the blackboard, which, therefore, should be credited to him (excluding possible mistakes, for which I should be credited).

References

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